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Article

Novel Ostrowski–Type Inequalities for Generalized Fractional Integrals and Diverse Function Classes

Areej A. Almoneef [1], Abd-Allah Hyder [2,*], Mohamed A. Barakat [3,4] and Hüseyin Budak [5]

1 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; aaalmoneef@pnu.edu.sa
2 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia
3 Department of Computer Science, College of Al Wajh, University of Tabuk, Tabuk 71491, Saudi Arabia; m.barakat@ut.edu.sa
4 Department of Mathematics, Faculty of Sciences, Al-Azhar University, Assiut 71524, Egypt
5 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Türkiye; hsyn.budak@gmail.com
* Correspondence: abahahmed@kku.edu.sa

Abstract: In this work, novel Ostrowski-type inequalities for dissimilar function classes and generalized fractional integrals (FITs) are presented. We provide a useful identity for differentiable functions under FITs, which results in special expressions for functions whose derivatives have convex absolute values. A new condition for bounded variation functions is examined, as well as expansions to bounded and Lipschitzian derivatives. Our comprehension is improved by comparison with current findings, and recommendations for future study areas are given.

Keywords: Ostrowski-type inequalities; fractional integrals; function classes

MSC: 26D15; 26D10; 26D07

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1. Introduction

From the beginning of calculus to its current state, integral inequalities have played a crucial role in the theory of integral and differential equations. Different types of integral inequalities have captivated many researchers in both pure and practical mathematics for over a hundred years. One of the many important mathematical results claimed by A. M. Ostrowski [1] is a classical integral inequality that is strongly related to differentiability.

Consider a differentiable function $\Omega : [\eta, \zeta] \to \mathbb{R}$ on the interval (η, ζ) whose derivative Ω' is bounded inside this interval. This boundary condition may be denoted as $\| \Omega' \|_\infty = \sup_{\chi \in (\eta,\zeta)} |\Omega'(\chi)| < \infty$. Under these circumstances, the following inequality exists:

$$\left| \Omega(x) - \frac{1}{\zeta - \eta} \int_\eta^\zeta \Omega(\chi) d\chi \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{\eta + \zeta}{2} \right)^2}{(\zeta - \eta)^2} \right] (\zeta - \eta) \| \Omega' \|_\infty \tag{1}$$

For any $x \in [\eta, \zeta]$, the constant $\frac{1}{4}$ is the best option.

The inequalities, proposed by Hermite and Hadamard for convex functions (CFs), are very significant in the literature. These inequalities state that, for any convex function

$\Omega : I \to \mathbb{R}$ defined on the real number interval I and any two variables η and ζ from I where $\eta < \zeta$, the next double inequality holds.

$$\Omega\left(\frac{\eta+\zeta}{2}\right) \leq \frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(x)dx \leq \frac{\Omega(\eta)+\Omega(\zeta)}{2}. \quad (2)$$

Convex theory is a very successful approach for addressing significant challenges that arise in several fields of both pure and practical sciences [2]. Furthermore, the notion of convexity is essential to improving the concept of inequality. Multiple forms of convexity are discussed in the scientific literature [3].

Several mathematicians have conducted substantial research on midpoint (MP)- and trapezoid-type inequalities for numerous forms of CFs. Dragomir and Agarwal originally studied a maximum bound for the right division of inequality (2) in their work [4]. Kırmacı proposed a maximum bound for the left division of inequality (2) in his article [5]. In publications [6,7], the authors demonstrated MP-type and trapezoid-type fractional inequalities using CFs. Many studies employ CFs to analyze Simpson-type inequalities. For example, in article [8], Simpson-type inequalities are proven for s-convex differentiable functions. In [9], differentiable CFs are used to analyze novel types of Simpson-type inequalities. For further information, refer to [10,11], as well as the references therein.

In the context of CFs, Bullen [12] developed some inequalities known as Bullen-type inequalities. The extended inequalities of Bullen-type on some fractal sets are given in [13]. The fractional Bullen-type inequalities were obtained by Du et al. [14] using the some inclusive FITs. Furthermore, the study in [15] discovered numerous extensions for Bullen-type integral inequalities with twice differentiable functions using the FITs of Riemann and Liouville. Refer to [16,17] for more specifics.

Certain MP-type inequalities were derived in [18] by employing generalized FITs. A novel extension of the Hermite–Hadamard inequality was established for fractional generalized integrals [19]. The generalized derivative was enhanced in [20] by employing the fundamental limit formulation of the derivative. The generalized derivative fulfils numerous significant requirements that the Riemann–Liouville (Rm-Lu) and Caputo definitions are incapable of accomplishing. In contrast to the Caputo definition, Abdelhakim [21] demonstrated that the generalized approach described in Khalil cannot produce satisfactory outcomes in the context of particular functions. Multiple extensions of the generalized approach circumvent this shortcoming in the generalized definition [22,23]. For updated information regarding the fractional integral inequalities mentioned above, please consult [18,24,25] and the references cited therein.

This study introduces novel Ostrowski-type inequalities for different kinds of functions and generalized FITs. A major contribution is a new identity for differentiable functions under FITs, which helps us understand functions with convex absolute value derivatives better. We also propose a new condition for functions with bounded variation, and extend our results to functions with bounded and Lipschitzian derivatives. These findings enhance our understanding of fractional integration and suggest promising directions for future research. In Section 3, we provide a new identity for differentiable functions that use generalized FITs. In Section 4, we use this equivalence to show novel versions of Ostrowski-type inequalities for functions with convex absolute value derivatives. Sections 5 and 6 demonstrate Ostrowski-type inequalities for functions with limited and Lipschitzian derivatives, respectively. In Section 7, we introduce an Ostrowski-type condition for functions with limited variation. We examine the relationship between new and previous findings. Section 8 concludes and suggests further research directions.

2. Preliminaries

The FITs of Rm-Lu, the generalized FITs, and various kinds of FITs have been explored regarding the kinds of inequalities that are referenced above. The essential definitions of

CFs, the integrals of Rm-Lu, and the generalized integrals utilized during this work are provided below:

Definition 1 (See [26])**.** *Consider an interval I on the real line. A function* $\Omega : I \to \mathbb{R}$ *is termed convex if it satisfies the inequality given below:*

$$\Omega(\chi x + (1-\chi)y) \leq \chi\Omega(x) + (1-\chi)\Omega(y)$$

$\forall x, y \in I$ *and* $\forall \chi \in [0,1]$.

Kilbas et al. [27] explored FITs, also known as Rm-Lu integrals, as detailed below:

Definition 2 (See [27])**.** *Look at* $\Omega \in L_1[\eta, \zeta]$, $\eta, \zeta \in \mathbb{R}$ *with* $\eta < \zeta$*. The integrals of Rm-Lu* $J^{\delta}_{\eta+}\Omega$ *and* $J^{\delta}_{\zeta-}\Omega$ *of order* $\delta > 0$ *are described by*

$$J^{\delta}_{\eta+}\Omega(x) = \frac{1}{\Gamma(\delta)} \int_{\eta}^{x} (x-\chi)^{\delta-1}\Omega(\chi)d\chi, \quad x > \eta \tag{3}$$

and

$$J^{\delta}_{\zeta-}\Omega(x) = \frac{1}{\Gamma(\delta)} \int_{x}^{\zeta} (\chi-x)^{\delta-1}\Omega(\chi)d\chi, \quad x < \zeta, \tag{4}$$

Here, Γ *refers to the Gamma function, specified by*

$$\Gamma(\delta) = \int_0^{\infty} e^{-u}u^{\delta-1}du.$$

Jarad et al. first identified fractional generalized integral operators in their publication [28]. Their work also examined the properties and interrelations of these operators with different fractional operators discussed in documented studies. The following lists these fractional generalized integral operators:

Definition 3 (See [28])**.** *Assume that* $\delta > 0$ *and* $\alpha \in (0,1]$*. For* $\Omega \in L_1[\eta,\zeta]$*, the generalized fractional Rm-Lu integrals* ${}^{\delta}\mathcal{J}^{\alpha}_{\eta+}f$ *and* ${}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}f$ *are defined by*

$${}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x) = \frac{1}{\Gamma(\delta)} \int_{\eta}^{x} \left(\frac{(x-\eta)^{\alpha} - (\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta-1} \frac{\Omega(\chi)}{(\chi-\eta)^{1-\alpha}}d\chi, \quad x > \eta, \tag{5}$$

and

$${}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x) = \frac{1}{\Gamma(\delta)} \int_{x}^{\zeta} \left(\frac{(\zeta-x)^{\alpha} - (\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta-1} \frac{\Omega(\chi)}{(\zeta-\chi)^{1-\alpha}}d\chi, \quad x < \zeta, \tag{6}$$

respectively.

If we pick $\alpha = 1$ in equalities (5) and (6), then the FITs in (5) and (6) become the Rm-Lu FITs in (3) and (4), respectively.

Set et al. [25] first proved the Ostrowski-type inequality for generalized fractional integrals as follows:

Theorem 1 (See [29])**.** *Suppose* $\Omega : [\eta, \zeta] \to \mathbb{R}$ *is a differentiable function on* (η, ζ) *and* $\Omega' \in L[\eta, \zeta]$*. Suppose also that* $|\Omega'|$ *is a convex function on* $[\eta, \zeta]$ *and* $|\Omega'(\chi)| \leq M$ *for all* $\chi \in [\eta, b]$*. Then, it follows*

$$\left|\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{x+}\Omega(\zeta) + {}^{\delta}\mathcal{J}^{\alpha}_{x-}\Omega(\eta)\right] - \left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right|$$

$$\leq \frac{M}{\alpha^{\delta+1}}\mathfrak{B}\left(\delta+1, \frac{1}{\alpha}\right)\left[(\zeta-x)^{\alpha\delta+1} + (x-\eta)^{\alpha\delta+1}\right].$$

Here, the function $\mathfrak{B}(\cdot,\cdot)$ *is the Beta function characterized by*

$$\mathfrak{B}(x, y) = \int_0^1 \chi^{x-1}(1-\chi)^{y-1}d\chi,$$

for $x, y > 0$*.*

3. An Identity for Differentiable Function

Within this work, we assume that $\alpha \in (0, 1], \delta \in \mathbb{R}^{+}$.

Lemma 1. *Presume* $\Omega : [\eta, \zeta] \to \mathbb{R}$ *is a differentiable mapping on* (η, ζ)*. If* Ω' *belongs to* $L[\eta, \zeta]$*, then the following equality holds:*

$$\begin{aligned}
&\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x) + {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right] - \left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x) \\
&= (\zeta-x)^{\alpha\delta+1}\int_0^1\left(\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right)\Omega'(\chi\zeta+(1-\chi)x)d\chi \\
&-(x-\eta)^{\alpha\delta+1}\int_0^1\left(\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right)\Omega'(\chi\eta+(1-\chi)x)d\chi\Bigg],
\end{aligned} \tag{7}$$

where $\Gamma(\delta)$ *is a Euler Gamma function.*

Proof. With the help of integrating by parts, we obtain

$$\begin{aligned}
I_1 &= \int_0^1\left(\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right)\Omega'(\chi\zeta+(1-\chi)x)d\chi \\
&= \frac{1}{\zeta-x}\left(\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right)\Omega(\chi\zeta+(1-\chi)x)\Bigg|_0^1 \\
&\quad+\frac{\delta}{\zeta-x}\int_0^1\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta-1}(1-\chi)^{\alpha-1}\Omega(\chi\zeta+(1-\chi)x)d\chi \\
&= -\frac{1}{(\zeta-x)\alpha^{\delta}}\Omega(x) + \left(\frac{1}{\zeta-x}\right)^{1+\alpha\delta}\delta\int_x^{\zeta}\left(\frac{(\zeta-x)^{\alpha}-(b-\eta)^{\alpha}}{\alpha}\right)^{\delta-1}\frac{\Omega(\eta)}{(b-\eta)^{1-\alpha}}d\eta \\
&= -\frac{1}{(\zeta-x)\alpha^{\delta}}\Omega(x) + \frac{1}{(\zeta-x)^{\alpha\delta+1}}\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right].
\end{aligned} \tag{8}$$

Similar to the foregoing process, we obtain

$$\begin{aligned} I_2 =& \int_0^1 \left(\frac{1}{\alpha^\delta} - \left(\frac{1-(1-\chi)^\alpha}{\alpha} \right)^\delta \right) \Omega'(\chi\eta + (1-\chi)x) d\chi \\ =& -\frac{1}{x-\eta} \left(\frac{1}{\alpha^\delta} - \left(\frac{1-(1-\chi)^\alpha}{\alpha} \right)^\delta \right) \Omega(\chi\eta + (1-\chi)x) \Big|_0^1 \\ & - \frac{\delta}{x-\eta} \int_0^1 \left(\frac{1-(1-\chi)^\alpha}{\alpha} \right)^{\delta-1} (1-\chi)^{\alpha-1} \Omega(\chi\eta + (1-\chi)x) d\chi \\ =& \frac{1}{(x-\eta)\alpha^\delta} \Omega(x) - \left(\frac{1}{x-\eta} \right)^{1+\alpha\delta} \delta \int_\eta^x \left(\frac{(x-\eta)^\alpha - (\eta-\eta)^\alpha}{\alpha} \right)^{\delta-1} \frac{\Omega(\eta)}{(\eta-\eta)^{1-\alpha}} d\eta \\ =& \frac{1}{(x-\eta)\alpha^\delta} \Omega(x) - \frac{1}{(x-\eta)^{\alpha\delta+1}} \Gamma(\delta+1) \left[{}^\delta \mathcal{J}^\alpha_{\eta+} \Omega(x) \right]. \end{aligned} \tag{9}$$

By the equalities (8) and (9), we can write

$$\begin{aligned} & (\zeta - x)^{\alpha\delta+1} I_1 - (x-\eta)^{\alpha\delta+1} I_2 \\ =& \Gamma(\delta+1) \left[{}^\delta \mathcal{J}^\alpha_{\eta+} \Omega(x) + {}^\delta \mathcal{J}^\alpha_{\zeta-} \Omega(x) \right] - \left(\frac{(\zeta-x)^{\alpha\delta} + (x-\eta)^{\alpha\delta}}{\alpha^\delta} \right) \Omega(x). \end{aligned}$$

Thus, the demonstration of Lemma 1 is concluded. □

Remark 1. *If we assign $x = \frac{\eta+\zeta}{2}$ in (7), then (7) is equal to*

$$\begin{aligned} & \frac{\alpha^\delta 2^{\alpha\delta-1}}{(\zeta-\eta)^{\alpha\delta}} \Gamma(\delta+1) \left[{}^\delta \mathcal{J}^\alpha_{\eta+} \Omega\left(\frac{\eta+\zeta}{2} \right) + {}^\delta \mathcal{J}^\alpha_{\zeta-} \Omega\left(\frac{\eta+\zeta}{2} \right) \right] - \Omega\left(\frac{\eta+\zeta}{2} \right) \\ =& \frac{\alpha^\delta(\zeta-\eta)}{4} \left[\int_0^1 \left(\frac{1}{\alpha^\delta} - \left(\frac{1-(1-\chi)^\alpha}{\alpha} \right)^\delta \right) \Omega'\left(\chi\zeta + (1-\chi)\frac{\eta+\zeta}{2} \right) d\chi \right. \\ & \left. - \int_0^1 \left(\frac{1}{\alpha^\delta} - \left(\frac{1-(1-\chi)^\alpha}{\alpha} \right)^\delta \right) \Omega'\left(\chi\eta + (1-\chi)\frac{\eta+\zeta}{2} \right) d\chi \right], \end{aligned}$$

as demonstrated by Hyder et al. in their paper ([30], Lemma 1).

Remark 2. *If we choose $\alpha = 1$ in (7), then we gain the next identity for Rm-Lu FITs:*

$$\begin{aligned} & \Gamma(\delta+1) \left[J^\delta_{\eta+} \Omega(x) + J^\delta_{\zeta-} \Omega(x) \right] - \left((\zeta-x)^\delta + (x-\eta)^\delta \right) \Omega(x) \\ =& (\zeta-x)^{\delta+1} \int_0^1 \left(1-\chi^\delta \right) \Omega'(\chi\zeta + (1-\chi)x) d\chi \\ & \left. -(x-\eta)^{\alpha\delta+1} \int_0^1 \left(1-\chi^\delta \right) \Omega'(\chi\eta + (1-\chi)x) d\chi \right], \end{aligned}$$

4. Ostrowski-Type Inequalities by Convexity

Theorem 2. *Assume that all conditions of Lemma 1 are met. Assume additionally that $|\Omega'|$ is a convex mapping on $[\eta,\zeta]$. Consequently, we deduce*

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq (\zeta-x)^{\alpha\delta+1}\left[B_1(\alpha,\delta)\left|\Omega'(\zeta)\right|+A_1(\alpha,\delta)\left|\Omega'(x)\right|\right] \\
&+(x-\eta)^{\alpha\delta+1}\left[B_1(\alpha,\delta)\left|\Omega'(\eta)\right|+A_1(\alpha,\delta)\left|\Omega'(x)\right|\right].
\end{aligned} \tag{10}$$

Here,

$$\begin{aligned}
A_1(\alpha,\delta) &= \int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right](1-\chi)d\chi \\
&= \frac{1}{\alpha^{\delta}}\left[\frac{1}{2}-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{2}{\alpha}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
B_1(\alpha,\delta) &= \int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\chi d\chi \\
&= \frac{1}{\alpha^{\delta}}\left[\frac{1}{2}+\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{2}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].
\end{aligned}$$

Proof. By Lemma 1, we obtain

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq (\zeta-x)^{\alpha\delta+1}\int_0^1\left|\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right|\left|\Omega'(\chi\zeta+(1-\chi)x)\right|d\chi \\
&+(x-\eta)^{\alpha\delta+1}\int_0^1\left|\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right|\left|\Omega'(\chi\eta+(1-\chi)x)\right|d\chi.
\end{aligned} \tag{11}$$

Since the function $|\Omega'|$ is convex, we have

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq (\zeta-x)^{\alpha\delta+1}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left[\chi\left|\Omega'(\zeta)\right|+(1-\chi)\left|\Omega'(x)\right|\right]d\chi \\
&+(x-\eta)^{\alpha\delta+1}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left[\chi\left|\Omega'(\zeta)\right|+(1-\chi)\left|\Omega'(x)\right|\right]d\chi \\
&= (\zeta-x)^{\alpha\delta+1}\left[B_1(\alpha,\delta)\left|\Omega'(\zeta)\right|+A_1(\alpha,\delta)\left|\Omega'(x)\right|\right] \\
&+(x-\eta)^{\alpha\delta+1}\left[B_1(\alpha,\delta)\left|\Omega'(\eta)\right|+A_1(\alpha,\delta)\left|\Omega'(x)\right|\right].
\end{aligned}$$

This ends the proof of Theorem 2. □

Corollary 1. *Based on the conditions of Theorem 2, if* Ω' *is bounded, i.e.,* $\| \Omega' \|_\infty := \sup_{\chi \in (\eta,\zeta)} |\Omega(\chi)| < \infty$, *then we obtain the next Ostrowski-type inequality for generalized FITs:*

$$\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x) + {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right] - \left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right|$$

$$\leq \| \Omega' \|_\infty \left[\frac{(\zeta-x)^{\alpha\delta+1}+(x-\eta)^{\alpha\delta+1}}{\alpha^{\delta}}\right]\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].$$

Remark 3. *Picking* $\alpha = 1$ *in Corollary 1 gives the upcoming Ostrowski-type inequality with the FITs of Rm-Lu:*

$$\left|\Gamma(\delta+1)\left[J^{\delta}_{\eta+}\Omega(x) + J^{\delta}_{\zeta-}\Omega(x)\right] - \left((\zeta-x)^{\delta}+(x-\eta)^{\delta}\right)\Omega(x)\right|$$

$$\leq \| \Omega' \|_\infty \left(\frac{\delta}{\delta+1}\right)\left[(\zeta-x)^{\delta+1}+(x-\eta)^{\delta+1}\right].$$

Particularly, for $\delta = 1$, *we obtain the classical Ostrowski inequality (1).*

Remark 4. *If we set* $x = \frac{\eta+\zeta}{2}$ *in Theorem 2, we obtain the next inequality of MP-type:*

$$\left|\frac{2^{\alpha\delta-1}\alpha^{\delta}}{(\zeta-\eta)^{\alpha\delta}}\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right) + {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right] - \Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$

$$\leq \alpha^{\delta}\left(\frac{\zeta-\eta}{4}\right)\left[B_1(\alpha,\delta)[|\Omega'(\zeta)|+|\Omega'(\eta)|] + 2A_1(\alpha,\delta)\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|\right]$$

$$\leq \frac{\zeta-\eta}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right][|\Omega'(\zeta)|+|\Omega'(\eta)|],$$

as presented by Hyder et al. in ([30], Theorem 2).

Corollary 2. *In the case where we select* $\alpha = 1$ *in Theorem 2, we acquire the next inequality of Ostrowski-type with Rm-Lu FITs:*

$$\left|\Gamma(\delta+1)\left[J^{\delta}_{\eta+}\Omega(x) + J^{\delta}_{\zeta-}\Omega(x)\right] - \left((\zeta-x)^{\delta}+(x-\eta)^{\delta}\right)\Omega(x)\right|$$

$$\leq (\zeta-x)^{\delta+1}\left[\frac{\delta}{2(\delta+2)}|\Omega'(\zeta)| + \left[\frac{1}{2}-\frac{1}{(\delta+1)(\delta+2)}\right]|\Omega'(x)|\right]$$

$$+(x-\eta)^{\delta+1}\left[\frac{\delta}{2(\delta+2)}|\Omega'(\eta)| + \left[\frac{1}{2}-\frac{1}{(\delta+1)(\delta+2)}\right]|\Omega'(x)|\right].$$

Corollary 3. *If we pick* $\delta = 1$ *in Corollary 2, we acquire the next inequality of Ostrowski-type:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi - \Omega(x)\right|$$

$$\leq (\zeta-x)^2\left[\frac{|\Omega'(\zeta)|+2|\Omega'(x)|}{6}\right] + (x-\eta)^2\left[\frac{|\Omega'(\eta)|+2|\Omega'(x)|}{6}\right].$$

Theorem 3. *Presume that all conditions of Lemma 1 are fulfilled. Further, presume that* $|\Omega'|^q$ *is convex over* $[\eta, \zeta]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$. *Then we deduce*

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x) + {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right] - \left(\frac{(\zeta-x)^{\alpha\delta} + (x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq A_2^{\frac{1}{p}}(\alpha,\delta,p)\left[(\zeta-x)^{\alpha\delta+1}\left(\frac{|\Omega'(\zeta)|^q + |\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right. \qquad (12) \\
&\left.+(x-\eta)^{\alpha\delta+1}\left(\frac{|\Omega'(\eta)|^q + |\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here,

$$A_2(\alpha,\delta,p) = \frac{1}{\alpha^{p\delta}}\left[1 - \frac{1}{\alpha}\mathfrak{B}\left(p\delta+1, \frac{1}{\alpha}\right)\right].$$

Proof. We now examine the integrals on the right part of (11). By applying the convexity of $|\Omega'|^q$ and the well-known Hölder inequality, we obtain

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right| |\Omega'(\chi\zeta + (1-\chi)x)| d\chi \\
&\leq \left(\int_0^1 \left|\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right|^p d\chi\right)^{\frac{1}{p}} \left(\int_0^1 |\Omega'(\chi\zeta + (1-\chi)x)|^q d\chi\right)^{\frac{1}{q}} \qquad (13) \\
&= A_2^{\frac{1}{p}}(\alpha,\delta,p,\lambda)\left[\frac{|\Omega'(\zeta)|^q + |\Omega'(x)|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right| |\Omega'(\chi\eta + (1-\chi)x)| d\chi \\
&\leq A_2^{\frac{1}{p}}(\alpha,\delta,p,\lambda)\left[\frac{|\Omega'(\eta)|^q + |\Omega'(x)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned} \qquad (14)$$

Here, we use the fact that

$$\begin{aligned}
A_2(\alpha,\delta,p) &= \int_0^1 \left[\frac{1}{\alpha^{\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]^p d\chi \\
&\leq \int_0^1 \left[\frac{1}{\alpha^{p\delta}} - \left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{p\delta}\right] d\chi \\
&= \frac{1}{\alpha^{p\delta}}\left[1 - \frac{1}{\alpha}\mathfrak{B}\left(p\delta+1, \frac{1}{\alpha}\right)\right]
\end{aligned}$$

and the well-known inequality

$$(m-n)^s \leq m^s - r^s, \text{ for } m > n \geq 0 \text{ and } s \geq 1.$$

If we consider (13) and (14) in (11), then we have

$$\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right|$$
$$\leq(\zeta-x)^{\alpha\delta+1}A_2^{\frac{1}{p}}(\alpha,\delta,p,\lambda)\left[\frac{|\Omega'(\zeta)|^q+|\Omega'(x)|^q}{2}\right]^{\frac{1}{q}}$$
$$+(x-\eta)^{\alpha\delta+1}A_2^{\frac{1}{p}}(\alpha,\delta,p,\lambda)\left[\frac{|\Omega'(\eta)|^q+|\Omega'(x)|^q}{2}\right]^{\frac{1}{q}}.$$

This finalizes the proof. □

Corollary 4. *Picking $\alpha=1$ in Theorem 3 gives the subsequent inequality of Ostrowski-type with Rm-Lu fractional integrals:*

$$\left|\Gamma(\delta+1)\left[J^{\delta}_{\eta+}\Omega(x)+J^{\delta}_{\zeta-}\Omega(x)\right]-\left((\zeta-x)^{\delta}+(x-\eta)^{\delta}\right)\Omega(x)\right|$$
$$\leq\left(\frac{p\delta}{p\delta+1}\right)^{\frac{1}{p}}\left[(\zeta-x)^{\delta+1}\left(\frac{|\Omega'(\zeta)|^q+|\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+(x-\eta)^{\delta+1}\left(\frac{|\Omega'(\eta)|^q+|\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Corollary 5. *Assuming $\delta=1$ in Corollary 4 yields the next Ostrowski-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega(x)\right|$$
$$\leq\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\frac{(\zeta-x)^2}{\zeta-\eta}\left(\frac{|\Omega'(\zeta)|^q+|\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(x-\eta)^2}{\zeta-\eta}\left(\frac{|\Omega'(\eta)|^q+|\Omega'(x)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 5. *If we place* $x = \frac{\eta+\zeta}{2}$ *in Theorem 3, we acquire the following inequality of MP-type with the generalized FITs:*

$$\begin{aligned}
&\left|\frac{\alpha^{\delta}2^{\alpha\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \\
&\leq \frac{\zeta-\eta}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(p\delta+1,\frac{1}{\alpha}\right)\right]^{\frac{1}{p}}\left[\left(\frac{|\Omega'(\zeta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{|\Omega'(\eta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\zeta-\eta}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(p\delta+1,\frac{1}{\alpha}\right)\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{3|\Omega'(\zeta)|^{q}+|\Omega'(\eta)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Omega'(\eta)|^{q}+|\Omega'(\zeta)|^{q}}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is proved by Hyder et al. ([30], Theorem 3).

Remark 6. *In the case where we opt for* $\alpha = 1$ *in Remark 5, we obtain the following inequality of MP-type with Rm-Lu FITs:*

$$\begin{aligned}
&\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\delta}}\left[J^{\delta}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+J^{\delta}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \\
&\leq \frac{\zeta-\eta}{4}\left(\frac{p\delta}{p\delta+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Omega'(\zeta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|\Omega'(\eta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\zeta-\eta}{4}\left(\frac{p\delta}{p\delta+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\Omega'(\zeta)|^{q}+|\Omega'(\eta)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Omega'(\eta)|^{q}+|\Omega'(\zeta)|^{q}}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

as provided by Ertuğral et al. ([31], Corollary 4.11).

Remark 7. *If we take* $\alpha = \delta = 1$ *in Remark 5, we obtain the following inequality of MP-type:*

$$\begin{aligned}
&\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \\
&\leq \frac{\zeta-\eta}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Omega'(\zeta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}+\left(\frac{|\Omega'(\eta)|^{q}+\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\zeta-\eta}{4}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\Omega'(\zeta)|^{q}+|\Omega'(\eta)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Omega'(\eta)|^{q}+|\Omega'(\zeta)|^{q}}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is given by Kirmaci ([5], Theorem 2.3).

Theorem 4. *Postulate that all conditions of Lemma 1 are fulfilled. If $|\Omega'|^q$ is convex over $[\eta,\zeta]$ where $q \geq 1$, we obtain the next inequality:*

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq (\zeta-x)^{\alpha\delta+1}A_3^{1-\frac{1}{q}}(\alpha,\delta)\Big(B_1(\alpha,\delta)|\Omega'(\zeta)|^q+A_1(\alpha,\delta)|\Omega'(x)|^q\Big)^{\frac{1}{q}} \\
&+(x-\eta)^{\alpha\delta+1}A_3^{1-\frac{1}{q}}(\alpha,\delta)\Big(B_1(\alpha,\delta)|\Omega'(\eta)|^q+A_1(\alpha,\delta)|\Omega'(x)|^q\Big)^{\frac{1}{q}}.
\end{aligned} \tag{15}$$

Here, A_1 and B_1 are defined as in Theorem 2, and A_3 is defined by

$$A_3(\alpha,\delta)=\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\chi=\frac{1}{\alpha^{\delta}}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].$$

Proof. Taking the convexity of $|\Omega'|^q$ and the power mean inequality into consideration, we conclude

$$\begin{aligned}
&\int_0^1\left|\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right||\Omega'(\chi\zeta+(1-\chi)x)|d\chi \\
&\leq\left(\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\chi\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]|\Omega'(\chi\zeta+(1-\chi)x)|^q d\chi\right)^{\frac{1}{q}} \\
&\leq A_3^{1-\frac{1}{q}}(\alpha,\delta)\left(\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\Big[\chi|\Omega'(\zeta)|^q+(1-\chi)|\Omega'(x)|^q\Big]d\chi\right)^{\frac{1}{q}} \\
&=A_3^{1-\frac{1}{q}}(\alpha,\delta)\Big(B_1(\alpha,\delta)|\Omega'(\zeta)|^q+A_1(\alpha,\delta)|\Omega'(x)|^q\Big)^{\frac{1}{q}}
\end{aligned} \tag{16}$$

and similarly

$$\begin{aligned}
&\int_0^1\left|\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right||\Omega'(\chi\zeta+(1-\chi)x)|d\chi \\
&\leq A_3^{1-\frac{1}{q}}(\alpha,\delta)\Big(B_1(\alpha,\delta)|\Omega'(\eta)|^q+A_1(\alpha,\delta)|\Omega'(x)|^q\Big)^{\frac{1}{q}}.
\end{aligned} \tag{17}$$

If we consider (16) and (17) in (11), then the proof of Theorem 4 is completed. □

Corollary 6. *Setting $\alpha = 1$ in Theorem 4 yields the next inequality of Ostrowski-type with the FITs of Rm-Lu:*

$$\left|\Gamma(\delta+1)\left[J^{\delta}_{\eta+}\Omega(x) + J^{\delta}_{\zeta-}\Omega(x)\right] - \left((\zeta-x)^{\delta} + (x-\eta)^{\delta}\right)\Omega(x)\right|$$
$$\leq (\zeta-x)^{\delta+1}\left(\tfrac{\delta}{\delta+1}\right)^{1-\frac{1}{q}}\left(\tfrac{\delta}{2(\delta+2)}|\Omega'(\zeta)|^q + \left(\tfrac{1}{2} - \tfrac{1}{(\delta+1)(\delta+2)}\right)|\Omega'(x)|^q\right)^{\frac{1}{q}}$$
$$+(x-\eta)^{\delta+1}\left(\tfrac{\delta}{\delta+1}\right)^{1-\frac{1}{q}}\left(\tfrac{\delta}{2(\delta+2)}|\Omega'(\eta)|^q + \left(\tfrac{1}{2} - \tfrac{1}{(\delta+1)(\delta+2)}\right)|\Omega'(x)|^q\right)^{\frac{1}{q}}.$$

Corollary 7. *By setting $\delta = 1$ in Corollary 6, we gain the next Ostrowski-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi - \Omega(x)\right|$$
$$\leq \frac{1}{2(\zeta-\eta)}\left[(\zeta-x)^2\left(\frac{|\Omega'(\zeta)|^q + 2|\Omega'(x)|^q}{3}\right)^{\frac{1}{q}} + (x-\eta)^2\left(\frac{|\Omega'(\eta)|^q + 2|\Omega'(x)|^q}{3}\right)^{\frac{1}{q}}\right].$$

Remark 8. *If we set $x = \frac{\eta+\zeta}{2}$ in Theorem 4, we find the next inequality of MP-type for the generalized FITs:*

$$\left|\frac{\alpha^{\delta}2^{\alpha\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right) + {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right] - \Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$
$$\leq \frac{\zeta-\eta}{4}A_3^{1-\frac{1}{q}}(\alpha,\delta)\left[\left(B_1(\alpha,\delta)|\Omega'(\zeta)|^q + A_1(\alpha,\delta)\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(B_1(\alpha,\delta)|\Omega'(\eta)|^q + A_1(\alpha,\delta)\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\zeta-\eta}{4}A_3^{1-\frac{1}{q}}(\alpha,\delta)\left[\left(\frac{(A_1(\alpha,\delta)+2B_1(\alpha,\delta))|\Omega'(\zeta)|^q + A_1(\alpha,\delta)|\Omega'(\eta)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(A_1(\alpha,\delta)+2B_1(\alpha,\delta))|\Omega'(\eta)|^q + A_1(\alpha,\delta)|\Omega'(\zeta)|^q}{2}\right)^{\frac{1}{q}}\right],$$

as shown by Hyder et al. ([30], Theorem 4).

Remark 9. *By setting $\alpha = 1$ in Remark 8, we acquire the next MP-type inequality with the FITs of Rm-Lu:*

$$\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\delta}}\left[J^{\delta}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+J^{\delta}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$
$$\leq\frac{\zeta-\eta}{4}\left(\frac{\delta}{\delta+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\delta}{2(\delta+2)}\left|\Omega'(\zeta)\right|^{q}+\left(\frac{1}{4}-\frac{1}{(\delta+1)(\delta+2)}\right)\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\delta}{2(\delta+2)}\left|\Omega'(\eta)\right|^{q}+\left(\frac{1}{4}-\frac{1}{(\delta+1)(\delta+2)}\right)\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\zeta-\eta}{4}\left(\frac{\delta}{\delta+1}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{3}{4}-\frac{2\delta+3}{2(\delta+1)(\delta+2)}\right)\left|\Omega'(\zeta)\right|^{q}+\left(\frac{1}{2}-\frac{1}{2(\delta+1)(\delta+2)}\right)\left|\Omega'(\eta)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{3}{4}-\frac{2\delta+3}{2(\delta+1)(\delta+2)}\right)\left|\Omega'(\eta)\right|^{q}+\left(\frac{1}{2}-\frac{1}{2(\delta+1)(\delta+2)}\right)\left|\Omega'(\zeta)\right|^{q}\right)^{\frac{1}{q}}\right],$$

as expressed by Hyder et al. ([30], Remark 7).

Remark 10. *If we take $\alpha=\delta=1$ in Remark 8, we gain the upcoming inequality of MP-type:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$
$$\leq\frac{\zeta-\eta}{8}\left[\left(\frac{\left|\Omega'(\zeta)\right|^{q}+2\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|\Omega'(\eta)\right|^{q}+2\left|\Omega'\left(\frac{\eta+\zeta}{2}\right)\right|^{q}}{3}\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{\zeta-\eta}{8}\left[\left(\frac{2|\Omega'(\zeta)|^{q}+|\Omega'(\eta)|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{|\Omega'(\zeta)|^{q}+2|\Omega'(\eta)|^{q}}{3}\right)^{\frac{1}{q}}\right],$$

as presented by Budak et al. ([32], Remark 4.10).

5. Ostrowski-Type Inequalities for Bounded Functions

This section addresses fractional Ostrowski-type inequalities that pertain to bounded functions.

Theorem 5. *Postulate that the constraints of Lemma 1 are satisfied. If there are $\theta,\Theta\in\mathbb{R}$ so that $\theta\leq\Omega'(\chi)\leq\Theta$ for $\chi\in[\eta,\zeta]$, then it results in:*

$$\left|\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|$$
$$\leq\frac{(x-\eta)(\zeta-x)(\Theta-\theta)}{\zeta-\eta}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].$$

Proof. By equalities (8) and (9), we can formulate

$$\begin{aligned}&\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}[I_1-I_2]\\&=\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x).\end{aligned}\tag{18}$$

By equality (18), we have

$$\begin{aligned}&\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\\&=\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\left\{\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\Omega'(\chi\zeta+(1-\chi)x)d\chi\right.\\&\left.-\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\Omega'(\chi\eta+(1-\chi)x)d\chi\right\}\\&=\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\left\{\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left[\Omega'(\chi\zeta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right]d\chi\right.\\&\left.-\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left[\Omega'(\chi\eta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right]d\chi\right\}.\end{aligned}\tag{19}$$

By using the absolute value of (19), we obtain

$$\begin{aligned}&\left|\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|\\&\leq\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\left\{\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left|\Omega'(\chi\zeta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right|d\chi\right.\\&\left.+\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\left|\Omega'(\chi\eta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right|d\chi\right\}.\end{aligned}$$

It is evident that $m\leq\Omega'(\chi)\leq M$ for $\chi\in[\eta,\zeta]$. Thus, we conclude

$$\left|\Omega'(\chi\zeta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right|\leq\frac{\Theta-\theta}{2}\tag{20}$$

and

$$\left|\Omega'(\chi\eta+(1-\chi)x)-\frac{\theta+\Theta}{2}\right|\leq\frac{\Theta-\theta}{2}.\tag{21}$$

If we consider (20) and (21), then we obtain

$$\begin{aligned}&\left|\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\,{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|\\&\leq\frac{\alpha^{\delta}(x-\eta)(\zeta-x)(\Theta-\theta)}{\zeta-\eta}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\chi\\&=\frac{(x-\eta)(\zeta-x)(\Theta-\theta)}{\zeta-\eta}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].\end{aligned}$$

This ends the proof. □

Corollary 8. *If we set $\alpha = 1$ in Theorem 5, we obtain the next Ostrowski-type inequality with the FITs of Rm-Lu:*

$$\left|\frac{\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\delta}}\ J^{\delta}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\delta}}J^{\delta}_{\zeta-}\Omega(x)\right]-\Omega(x)\right| \leq \frac{(x-\eta)(\zeta-x)(\Theta-\theta)\delta}{(\zeta-\eta)(\delta+1)}.$$

Remark 11. *Setting $\delta = 1$ in Corollary 8 gives the upcoming Ostrowski-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\left[\frac{\zeta-x}{x-\eta}\int_{\eta}^{x}\Omega(\chi)d\chi+\frac{x-\eta}{\zeta-x}\int_{x}^{\zeta}\Omega(\chi)d\chi\right]-\Omega(x)\right| \leq \frac{(x-\eta)(\zeta-x)(\Theta-\theta)}{2(\zeta-\eta)}.$$

Corollary 9. *If we choose $x = \frac{\eta+\zeta}{2}$ in Theorem 5, we obtain the next MP-type inequality with the generalized FITs:*

$$\left|\frac{\alpha^{\delta}2^{\alpha\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \leq \frac{(\zeta-\eta)(\Theta-\theta)}{4}\left[1-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].$$

Remark 12. *If we assign $\alpha = 1$ in Corollary 9, we acquire the following inequality of MP-type with Rm-Lu FITs:*

$$\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\delta}}\left[J^{\delta}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+J^{\delta}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \leq \frac{(\zeta-\eta)(\Theta-\theta)\delta}{4(\delta+1)}.$$

Remark 13. *If we take $\alpha = \delta = 1$ in Corollary 9, we have the MP-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega\left(\frac{\eta+\zeta}{2}\right)\right| \leq \frac{(\zeta-\eta)(\Theta-\theta)}{8}.$$

6. Ostrowski-Type Inequalities for Lipschitzian Functions

Now, our focus shifts to a set of fractional inequalities resembling Ostrowski's, applied to Lipschitz functions.

Theorem 6. *Consider that the constraints of Lemma 1 are satisfied. If Ω' is an L-Lipschitzian function on $[\eta,\zeta]$, the subsequent Ostrowski-type inequality is true:*

$$\left|\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|$$
$$\leq L(x-\eta)(\zeta-x)\left[\frac{1}{2}+\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{2}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].$$

Proof. By equality (19), we have

$$\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)$$
$$=\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right][\Omega'(\chi\zeta+(1-\chi)x)-\Omega'(\chi\eta+(1-\chi)x)]d\chi.$$

Using the fact that Ω' is a L-Lipschitzian function, we have

$$\left|\frac{\alpha^{\delta}\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\alpha\delta}}\ {}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|$$
$$\leq\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]|\Omega'(\chi\zeta+(1-\chi)x)-\Omega'(\chi\eta+(1-\chi)x)|d\chi$$
$$\leq\frac{\alpha^{\delta}(x-\eta)(\zeta-x)}{\zeta-\eta}\int_0^1\left[\frac{1}{\alpha^{\delta}}-\left(\frac{1-(1-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]L\chi(\zeta-\eta)d\chi$$
$$=L(x-\eta)(\zeta-x)\left[\frac{1}{2}+\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{2}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right],$$

Hence, the proof is finished. □

Corollary 10. *By picking $\alpha=1$ in Theorem 6, we have the next inequality of Ostrowski-type with the FITs of Rm-Lu:*

$$\left|\frac{\Gamma(\delta+1)}{\zeta-\eta}\left[\frac{\zeta-x}{(x-\eta)^{\delta}}\ J^{\delta}_{\eta+}\Omega(x)+\frac{x-\eta}{(\zeta-x)^{\delta}}J^{\delta}_{\zeta-}\Omega(x)\right]-\Omega(x)\right|$$
$$\leq\frac{L(x-\eta)(\zeta-x)\delta}{2(\delta+2)}.$$

Remark 14. *If we assign $\delta=1$ in Corollary 10, we gain the upcoming Ostrowski-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\left[\frac{\zeta-x}{x-\eta}\int_{\eta}^{x}\Omega(\chi)d\chi+\frac{x-\eta}{\zeta-x}\int_{x}^{\zeta}\Omega(\chi)d\chi\right]-\Omega(x)\right|\leq\frac{L(x-\eta)(\zeta-x)}{6}.$$

Corollary 11. *If we choose* $x = \frac{\eta+\zeta}{2}$ *in Theorem 6, then we have the following MP-type inequality for generalized FITs:*

$$\begin{aligned}&\left|\frac{\alpha^{\delta}2^{\alpha\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|\\&\leq\frac{L(\zeta-\eta)^2}{4}\left[\frac{1}{2}+\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{2}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\delta+1,\frac{1}{\alpha}\right)\right].\end{aligned}$$

Remark 15. *If we assign* $\alpha = 1$ *in Corollary 11, we obtain the next inequality of MP-type with Rm-Lu FITs:*

$$\begin{aligned}&\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\delta}}\left[J^{\delta}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+J^{\delta}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|\\&\leq\frac{L(\zeta-\eta)^2\delta}{8(\delta+2)}.\end{aligned}$$

Remark 16. *Selecting* $\alpha = \delta = 1$ *in Corollary 11 yields the MP-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|\leq\frac{L(\zeta-\eta)^2}{24}.$$

7. Ostrowski-Type Inequalities for Functions of Bounded Variation

Here, Ostrowski-type inequalities are presented for functions of bounded variation by means of generalized FITs.

Theorem 7. *Suppose that* $\Omega : [\eta,\zeta] \to \mathbb{R}$ *is a function of bounded variation on* $[\eta,\zeta]$*. Then, we obtain*

$$\begin{aligned}&\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right|\\&\leq\frac{1}{\alpha^{\delta}}\left[\frac{\zeta-\eta}{2}+\left|\frac{\eta+\zeta}{2}-x\right|\right]^{\alpha\delta}\bigvee_{\eta}^{\zeta}(\Omega).\end{aligned}$$

Here, $\bigvee_{\eta}^{\zeta}(\Omega)$ *denotes the total variation of* Ω *on* $[\eta,\zeta]$*.*

Proof. With the help of integrating by parts, we obtain

$$\begin{aligned}
&\int_x^{\zeta}\left[\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi)\\
&-\int_\eta^{x}\left[\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(x-\eta)^{\alpha}-(\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi)\\
&=\left[\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]\Omega(\chi)\Bigg|_x^{\zeta}\\
&+\delta\int_x^{\zeta}\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta-1}(\zeta-\chi)^{\alpha-1}\Omega(\chi)d\chi\\
&-\left[\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(x-\eta)^{\alpha}-(\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta}\right]\Omega(\chi)\Bigg|_\eta^{x}\\
&+\delta\int_\eta^{x}\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta-1}(\zeta-\chi)^{\alpha-1}\Omega(\chi)d\chi\\
&=\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x).
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
&\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\\
&=\int_x^{\zeta}\left[\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi)\\
&-\int_\eta^{x}\left[\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(x-\eta)^{\alpha}-(\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi).
\end{aligned}$$

It is evident that, if $g,\Omega:[\eta,\zeta]\to\mathbb{R}$ are such that g is continuous on $[\eta,\zeta]$ and Ω is of bounded variation on $[\eta,\zeta]$, then $\int_\eta^{\zeta} g(\chi)d\Omega(\chi)$ exist and

$$\left|\int_\eta^{\zeta} g(\chi)d\Omega(\chi)\right|\leq\sup_{\chi\in[\eta,b]}|g(\chi)|\bigvee_\eta^{\zeta}(\Omega). \tag{22}$$

By using (22), we have

$$\begin{aligned}
&\left|\Gamma(\delta+1)\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega(x)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega(x)\right]-\left(\frac{(\zeta-x)^{\alpha\delta}+(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\right)\Omega(x)\right| \\
&\leq \left|\int_x^{\zeta}\left[\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi)\right| \\
&\quad+\int_{\eta}^{x}\left[\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(x-\eta)^{\alpha}-(\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta}\right]d\Omega(\chi) \\
&\leq \sup_{\chi\in[x,\zeta]}\left|\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(\zeta-x)^{\alpha}-(\zeta-\chi)^{\alpha}}{\alpha}\right)^{\delta}\right|\bigvee_x^{\zeta}(\Omega) \\
&\quad+\sup_{\chi\in[\eta,x]}\left|\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}-\left(\frac{(x-\eta)^{\alpha}-(\chi-\eta)^{\alpha}}{\alpha}\right)^{\delta}\right|\bigvee_{\eta}^{x}(\Omega) \\
&=\frac{(\zeta-x)^{\alpha\delta}}{\alpha^{\delta}}\bigvee_x^{\zeta}(\Omega)+\frac{(x-\eta)^{\alpha\delta}}{\alpha^{\delta}}\bigvee_{\eta}^{x}(\Omega) \\
&\leq \frac{1}{\alpha^{\delta}}\max\left\{(\zeta-x)^{\alpha\delta},(x-\eta)^{\alpha\delta}\right\}\bigvee_{\eta}^{\zeta}(\Omega) \\
&=\frac{1}{\alpha^{\delta}}\left[\frac{\zeta-\eta}{2}+\left|\frac{\eta+\zeta}{2}-x\right|\right]^{\alpha\delta}\bigvee_{\eta}^{\zeta}(\Omega).
\end{aligned}$$

Here, we use the fact that

$$\max\left\{m^k,n^k\right\}=[\max\{m,n\}]^k=\left[\frac{m+n+|m-n|}{2}\right]^k$$

for $m,n,k>0$.

The proof is completed. □

Remark 17. *By setting $\alpha=1$ in Theorem 7, we obtain the next Ostrowski-type inequality for the FITs of Rm-Lu:*

$$\begin{aligned}
&.\left|\Gamma(\delta+1)\left[J^{\delta}_{\eta+}\Omega(x)+J^{\delta}_{\zeta-}\Omega(x)\right]-\left((\zeta-x)^{\delta}+(x-\eta)^{\delta}\right)\Omega(x)\right| \\
&\leq\left[\frac{\zeta-\eta}{2}+\left|\frac{\eta+\zeta}{2}-x\right|\right]^{\delta}\bigvee_{\eta}^{\zeta}(\Omega),
\end{aligned}$$

which is proved by Dragomir in the book chapter ([33], Theorem 3.1.1).

Remark 18. *If we assign $\delta=1$ in Remark 17, we obtain the following Ostrowski inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega(x)\right|\leq\left[\frac{1}{2}+\left|\frac{\frac{\eta+\zeta}{2}-x}{\zeta-\eta}\right|\right]^{\delta}\bigvee_{\eta}^{\zeta}(\Omega)$$

which is proved by Dragomir ([34], Theorem 2.1).

Corollary 12. *If we choose $x = \frac{\eta+\zeta}{2}$ in Theorem 7, we gain the next MP-type inequality for generalized FITs:*

$$\left|\frac{\alpha^{\delta}2^{\alpha\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[{}^{\delta}\mathcal{J}^{\alpha}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+{}^{\delta}\mathcal{J}^{\alpha}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$
$$\leq \frac{1}{2}\bigvee_{\eta}^{\zeta}(\Omega).$$

Remark 19. *If we choose $\alpha = 1$ in Corollary 12, we obtain the next inequality MP-type with Rm-Lu FITs:*

$$\left|\frac{2^{\delta-1}\Gamma(\delta+1)}{(\zeta-\eta)^{\alpha\delta}}\left[J^{\delta}_{\eta+}\Omega\left(\frac{\eta+\zeta}{2}\right)+J^{\delta}_{\zeta-}\Omega\left(\frac{\eta+\zeta}{2}\right)\right]-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|$$
$$\leq \frac{1}{2}\bigvee_{\eta}^{\zeta}(\Omega).$$

Remark 20. *If we set $\alpha = \delta = 1$ in Corollary 12, we acquire the next MP-type inequality:*

$$\left|\frac{1}{\zeta-\eta}\int_{\eta}^{\zeta}\Omega(\chi)d\chi-\Omega\left(\frac{\eta+\zeta}{2}\right)\right|\leq\frac{1}{2}\bigvee_{\eta}^{\zeta}(\Omega)$$

which is proved by Dragomir ([34], Theorem 2.1).

8. Summary and Concluding Remarks

In this work, we have introduced novel Ostrowski-type inequalities that apply to various function classes and generalized FITs. The primary contribution of this study is the development of a new identity applicable to differentiable functions under FITs. This identity leads to significant special cases for functions with convex absolute value derivatives, offering deeper insights into their behavior under fractional integration. Furthermore, we have explored a new condition for functions of bounded variation, extending the applicability of our results. The study also includes an analysis of expansions to bounded and Lipschitzian derivatives, enhancing the breadth and applicability of the derived inequalities. Comparative analysis with existing literature reveals that our findings contribute new perspectives and refine current understanding in the field. The results not only align with but also extend beyond previously established theories, providing a more comprehensive framework for future research. Based on our findings, we recommend several avenues for further investigation. Future studies could explore the implications of our results in broader contexts, such as higher-dimensional spaces or different types of FITs. Additionally, investigating the practical applications of these inequalities in various mathematical and applied fields could offer valuable insights and advancements. Overall, this work advances the theoretical understanding of Ostrowski-type inequalities and FITs, paving the way for further research and applications in mathematical analysis and related disciplines.

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