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Solutions for a Logarithmic Fractional Schrödinger-Poisson System with Asymptotic Potential

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Abstract: In this paper, we consider a logarithmic fractional Schrödinger-Poisson system where the potential is a sign-changing function. When the potential is coercive, we get the existence of infinitely many solutions for the system. When the potential is bounded, we get the existence of a ground state solution for the system.

Keywords: non-smooth critical point theory; sign-changing potential; fractional Schrödinger-Poisson system

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1. Introduction and Main Results

In this article, we investigate the following fractional Schrödinger-Poisson system with logarithmic nonlinearity:

$$\begin{cases} (-\Delta)^s u + V(x)u - \phi(x)u = u\log u^2, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $t \in (0,1)$, $s \in (0,1)$, $4s + 2t > 3$, and $(-\Delta)^\iota$ with $\iota \in \{s,t\}$ denotes the fractional Laplacian operator defined as

$$(-\Delta)^\iota u(x) = C_\iota P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{3+2s}} dy, \quad x \in \mathbb{R}^3$$

where $P.V.$ represents the principal value sense, C_ι represents an appropriate normalization constant. It is worth pointing out that the application background of fractional equations is rooted in areas such as fractional quantum mechanics, physics, finance, conformal geometry, among others; see [1] for more details. In particular, when $s = \frac{1}{2}$ and $t = 1$ system (1) gains significant interest in physics as it comes from the semi-relativistic theory in the repulsive (plasma Coulomb case).

In recent years, the study of small semiconductor devices has been stimulated increasingly interest, in particular, in the use of quantum-mechanical and numerical methods to explain quantum phenomena like quantum interference, size quantization and tunneling. Since the early 1980s, the Schrödinger-Poisson system, which is the coupling of a Maxwell equations with Schrödinger equation, has been widely adopted as a mathematical framework to explore and evaluate mathematical elements that are crucial for modeling semiconductor heterostructures. For a comprehensive overview of the Schrödinger-Poisson system and related models, for example we refer to [2].

The single particle system, named the Schrödinger-Poisson system, regulates the temporal evolution of the wave function $\Phi(x,t)$, which depicts the condition of a non-relativistic quantum particle in space under the influence of a self-consistent potential V

generated by its own charge. When related to a single particle system in a vacuum, the Schrödinger equation in $\mathbb{R}^3 \times (0, \infty)$ is formulated as

$$i\lambda \frac{\partial \Phi}{\partial t} = -\frac{\lambda^2}{2m}\Delta_x \Phi + U\Phi, \quad \lim_{|x|\to\infty} \Phi(x,t) = 0, \quad \Phi(x,0) = \psi(x),$$

where λ represents Planck's constant and m signifies the mass of the particle. To find U, we combine this equation with the Poisson equation:

$$-\Delta_x U = \beta|\Phi|^2, \quad \lim_{|x|\to\infty} U = 0,$$

where $|\Phi(x,t)|^2$ represents the anticipated particle density for a pure quantum state in the spatial domain $\mathbb{R}^3_x$ at time t. The value of β is +1 when the Coulomb force is repulsive and +1 when it is attractive. Our primary focus in this paper is the repulsive case, and the Poisson equation represents the repulsive character of the Coulomb force.

Over the past three decades, the Schrödinger-Poisson system

$$\begin{cases} -\Delta u + M(x)\phi u + U(x)u = h(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = M(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

has been the subject of extensive research because of its wide physical applications. The model like (2) proposed by Benci [3] has been used to describe the relationship between the nonlinear steady-state Schrödinger equation and the electrostatic field, and it is widely used in quantum mechanical models and semiconductor theory. Under the specific hypothesis of U and M, Liu and Guo [4] proved that, by utilizing variational methods, system (2) has a minimum of one positive ground state solution. In [5], Zhang et al. demonstrated the existence of high-energy solutions for system (2) by employing the linking theorem with $h(x,u) = u^5$. In [6], Zhong and Tang explored system (2) where $U \equiv 1, h = \mu f(x)u + |u|^4 u$, and established the problem has at least one ground state sign-changing solution by employing the constraint variational method.

In the framework of fractional Laplacian systems, there are numerous results related to the fractional Schrödinger-Poisson system. Here we list some results related to our paper. Zhang et al. [7] investigated the fractional Schrödinger-Poisson system with subcritical and critical nonlinear terms:

$$\begin{cases} (-\Delta)^s u + M(x)\phi u + U(x)u = h(x,u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^t\phi = M(x)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{3}$$

Through a perturbation method, they obtained the existence of positive solutions and detailed the asymptotic of solutions. Employing Pohozaev-Nehari manifold, the monotonicity trick and global compactness Lemma, Teng in [8] obtained the existence of ground state solutions for (3) with $h(x,u) = Q(x)|u|^{2_s^*}u + K(x)f(u)$. With the help of the Ljusternik-Schnirelmann theory and penalization techniques, Ambrosio in [9] proved the concentration and multiplicity of positive solutions for system (3) with $M \equiv 1, h(x,u) = g(u) + |u|^{2_s^*-2}u$.

Lately, the logarithmic Schrödinger equation expressed as

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi + V(x)\Psi - \Psi\log\Psi^2, \;\; N \geq 3 \tag{4}$$

with $\Psi : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{C}$, has garnered significant attention because of its profound impact in various fields, including effective quantum, quantum mechanics, and Bose-Einstein condensation. Finding the standing waves of (4), which are represented by $\Psi(x,t) = \exp(-i\lambda t)u(x)$ where $\lambda \in \mathbb{R}$, is essential. This substitution transforms the equation into

$$-\Delta u + V(x)u = u\log u^2, \text{ in } \mathbb{R}^N. \tag{5}$$

The associated energy functional can be expressed as

$$\hat{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left((V(x)+1)u^2 + |\nabla u|^2\right)dx - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 dx.$$

Nonetheless, $\hat{I}$ might not be well-defined in $H^1(\mathbb{R}^3)$ as there is a $u \in H^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} u^2 \log u^2 dx = -\infty$. More precisely, we consider the case where $N = 1$ and u is a smooth function defined as $u(x) = (\sqrt{x \log x})^{-1}$ for $x \geq 2$ and $u(x) = 0$ for $x \leq 0$. In this scenario, u belongs to $H^1(\mathbb{R}^3)$ and $I(u) = +\infty$, assuming V grows slowly enough, such as $V(x) = (\log x)^{1/2}$ when $x \geq 2$. To resolve this problem, various techniques have been developed by researchers. Next we review some established results about logarithmic Schrödinger equations. In [10], the authors applied genus theory and the minimax principles for lower semicontinuous functionals as detailed in [11] to find multiple solutions for the problem (5) with periodic potential. Later, inspired by the ideas presented in [10], Ji and Szulkin in [12] proved the existence of multiple solutions for the Equation (5) where V meets

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x|\to\infty} V(x) = \infty$.

(V_2) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\lim_{|x|\to\infty} V(x) = \sup_{x\in\mathbb{R}^N} V(x) =: \mathbb{V}_\infty \in (-1, \infty)$ and spectrum $\sigma((-\Delta) + V + 1) \subset (0, \infty)$.

When the potential meets (V_1), they acquire the existence of infinitely many solutions for (5) and there exists a ground state solution for (5) when the potential meets (V_2). By employing variational methods, Alves and Ji in [13] established the existence of multi-bump positive solutions for the equation similar to (5). Another subject that has gained growing attention lately is the logarithmic Schrödinger-Poisson system. Recently, Peng [14] considered existence and concentration of positive solutions for the logarithmic Schrödinger-Poisson system

$$\begin{cases} -\epsilon^2 \Delta u - \phi u + V(x)u = u \log u^2, & \text{in } \mathbb{R}^3, \\ -\epsilon^2 \Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

via variational method and penalization scheme under local assumption that potential meets.

Inspired by the above studies, this paper explores the existence of multiple solutions for the logarithmic fractional Schrödinger-Poisson system. To the best of our knowledge, in the fractional scenario, literature on the Schrödinger-Poisson system with logarithmic features is relatively scarce. In the following, we present the main results.

Theorem 1. *Assume that V satisfies (V_1) with $N = 3$, problem (1) possesses infinitely many solutions $\pm u_n$ such that $\lim_{n\to\infty} I(\pm u_n) = \infty$.*

Theorem 2. *Assume that V satisfies (V_2) with $N = 3$, problem (1) possesses a ground state solution.*

Let us outline the main challenges we faced in this paper: Because of the logarithmic terms, $u \in X$ may exist such that $\int_{\mathbb{R}^3} u^2 \log u^2 = -\infty$, which can result in the corresponding functional attaining $+\infty$. Therefore, the functional is not well-defined in H, which makes traditional variational methods inapplicable in this situation. To find solutions for (1), similar to [10], we decompose the functional into the sum of a C^1 functional and a lower semicontinuous convex functional. As far as we know, there are few available results about multiplicity of solutions for fractional Schrödinger-Poisson system, even in the Laplacian setting.

Remark 1. *Our results outline two key differences compared to those of [10]: (i) Our equation includes not only logarithmic term but also the nonlocal term $\phi(x)u$; (ii) We extend the Equation (5) to the fractional Laplacian setting.*

This paper is divided into the following sections. The second section provides a review of several lemmas that are utilized throughout the paper. In the third section, we give the proof of Theorem 1. The fourth section is dedicated to demonstrating Theorem 2.

Throughout this article, we note the following:

- C and C_i are different positive constants.
- The norm $\|u\|_p$ is defined as $\left(\int_{\mathbb{R}^3}|u|^p\right)^{\frac{1}{p}}$.
- Define $B_l(u)$ as an open ball with radius $l > 0$ centered at u, and let $B_l := B_l(0)$.
- For a functional I on H, denote by A the critical point set of I, $I_e := \{u \in X : I(u) \geq e\}$, $I^f := \{u \in X : I(u) \leq f\}$, $I_e^f := I_e \cap I^f$ and $A_d = A \cap I_d^d$.

2. Preliminaries

Let us first define the homogeneous fractional Sobolev space $D^{t,2}(\mathbb{R}^3)$ as

$$D^{t,2}(\mathbb{R}^3) = \{u \in L^{2_t^*}(\mathbb{R}^3) \,|\, |\zeta|^t \hat{u}(\zeta) \in L^2(\mathbb{R}^3)\}$$

which represents the closure of $C_0^\infty(\mathbb{R}^3)$ in relation to the norm

$$\|u\|_{D^{t,2}} = \left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{t}{2}}u|^2dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3}|\zeta|^{2t}|\hat{u}(\zeta)|^2d\zeta\right)^{\frac{1}{2}}$$

for $t \in (0,1)$; see [1] for more details.

Through the Fourier transform [1], the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined as

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3}(|\zeta|^{2s}|\hat{u}(\zeta)|^2 + |\hat{u}(\zeta)|^2)d\zeta < \infty\right\}$$

equipped with the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3}|\zeta|^{2s}|\hat{u}(\zeta)|^2 + |\hat{u}(\zeta)|^2\right)^{\frac{1}{2}}.$$

Based on the Plancherel theorem, it follows that $\||\zeta|^s u\|_2 = \|(-\Delta)^{\frac{s}{2}}u\|_2$ and $\|u\|_2 = \|\hat{u}\|_2$.

Therefore,

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2 + |u(x)|^2\right)^{\frac{1}{2}}.$$

Alternatively, the Sobolev space $H^s(\mathbb{R}^3)$ is described by

$$H^s(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \iint_{\mathbb{R}^6}\frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}dxdy < \infty\right\}.$$

This space is endowed with a norm determined by

$$\|u\| = \left(\iint_{\mathbb{R}^6}\frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}dxdy + \int_{\mathbb{R}^3}u^2dx\right)^{\frac{1}{2}}.$$

Based on Propositions 3.4 and 3.6 in [1], it can be established that

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \frac{1}{C(s)}\iint_{\mathbb{R}^6}\frac{(u(x)-u(y))^2}{|x-y|^{3+2s}}dxdy.$$

A widely recognized fact is that $H^s(\mathbb{R}^3)$ is continuously embedded into $L^r(\mathbb{R}^3)$ for every $r \in [2, 2_s^*]$ and compactly embedded into $L^r_{loc}(\mathbb{R}^3)$ for every $r \in [1, 2_s^*)$, where $2_s^* = 6/(3-2s)$; see [1] for more details.

In Theorem 1, let

$$H := \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 + (V(x)+1)^+ u^2 dx < \infty \right\} \tag{6}$$

with $V^{\pm} := \max\{\pm V, 0\}$. Evidently, H is a Hilbert space endowed with the inner product:

$$\langle u, \varphi \rangle := \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} (V(x)+1)^+ u\varphi dx.$$

It is standard to prove that the space H can be continuously embedded into $L^r(\mathbb{R}^3)$ for all $r \in [2, 2_s^*]$, and locally compact embedded into $L^r_{loc}(\mathbb{R}^3)$ for any $r \in [1, 2_s^*)$, we refer to [15] for more details.

In Theorem 2, our working space will be $H^s(\mathbb{R}^3)$. When V meets the conditions stated in Theorem 2, $H := H^s(\mathbb{R}^3)$ is endowed with the inner product:

$$\langle u, \varphi \rangle := \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}} dxdy + \int_{\mathbb{R}^3} (V(x)+1) u\varphi dx.$$

Note that we do not assume the global positivity of $V+1$. In fact, we require that $\sigma((-\Delta)^s + V + 1) \subset (0, \infty)$, which implies that the quadratic form

$$u \mapsto \int_{\mathbb{R}^3} |(-\triangle)^{\frac{s}{2}} u|^2 + (V(x)+1)u^2 dx$$

is positive definite on $H^s(\mathbb{R}^3)$.

It is well known that if $4s + 2t \geq 3$, there is a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ for every $u \in H^s(\mathbb{R}^3)$, which is guaranteed through the Lax-Milgram theorem, see for example [16]. This unique function satisfies the equation

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v \, dx = \int_{\mathbb{R}^3} u^2 v \, dx, \quad \forall v \in D^{t,2}(\mathbb{R}^3)$$

which indicates that ϕ_u^t is a weak solution to

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3.$$

Additionally, the expression for ϕ_u^t is given by

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2s}} dy, \quad x \in \mathbb{R}^3$$

with

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma\left(\frac{3-2t}{2}\right)}{\Gamma(t)}.$$

This function is referred to as the t-Riesz potential.

By substituting $\phi^t = \phi_u^t$ into system (1), we see that system (1) can be reformulated as a single equation

$$(-\Delta)^s u + V(x)u - \phi_u^t u = u \log u^2, \qquad x \in \mathbb{R}^3. \tag{7}$$

It is standard to show that the energy functional I related to problem (7) is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (V(x)+1) u^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} u \log u^2 dx.$$

Definition 1. *A solution to Equation (7) means a function $u \in H$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^3)$ (i.e., $I(u) < \infty$) and*

$$\int_{\mathbb{R}^3} \Big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) uv \Big) dx - \int_{\mathbb{R}^3} \phi_u^t uv dx = \int_{\mathbb{R}^3} uv \log u^2 dx, \ \ \forall v \in C_0^\infty(\mathbb{R}^3).$$

First of all, we outline several properties of ϕ_u^t.

Lemma 1. *If $u \in H$ and $4s + 2t \geq 3$, then the following properties hold:*

(i) *$\phi_u^t \geq 0$ in $\mathbb{R}^3$ and $\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C\|u\|^4_{\frac{12}{3+2t}} \leq C\|u\|^4$, $\|\phi_u^t\|_{D^{t,2}} \leq C\|u\|^2_{\frac{12}{3+2t}} \leq C\|u\|^2$.*

(ii) *If $z \in \mathbb{R}^3$ with $u_z(x) = u(x+z)$, it follows that $\phi_{u_z}^t(x) = \phi_u^t(x+z)$ and $\int_{\mathbb{R}^3} \phi_{u_z}^t u_z^2 dx = \int_{\mathbb{R}^3} \phi_u^t u^2 dx$.*

(iii) *If $u_n \rightharpoonup u$ in H, it follows that*

$$\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx$$

and $\phi_{u_n}^t \rightharpoonup \phi_u^t$ in $D^{t,2}(\mathbb{R}^3)$.

(iv) *If u_n, v_n are bounded in H with $\|u_n - v_n\|_{\frac{12}{3+2t}} \to 0$, then*

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n z dx \to \int_{\mathbb{R}^3} \phi_{v_n}^t v_n z dx \ for\ any\ z \in H.$$

(v) *If $u_n \rightharpoonup u$ in H, then it follows that*

$$\int_K \phi_{u_n}^t u_n v dx \to \int_k \phi_u^t uv dx$$

for any $v \in C_0^\infty(\mathbb{R}^3)$ and the compact support K of v.

Proof. We just need to verify (iv) and (v) since the verifications for (i), (ii) and (iii) are available in Lemma 2.1 of [14]. Following the ideas from Lemma 2.2 in [16], we can prove (iv) and (v).

- Verification of (iv): Applying Hölder's inequality along with the condition $4s + 2t > 3$ which implies $H \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^3} \phi_{u_n}^t u_n v dx - \int_{\mathbb{R}^3} \phi_{v_n}^t v_n v dx \right| \\
&= \left| \int_{\mathbb{R}^3} \phi_{u_n}^t (u_n - v_n) v dx + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u_n^2(x) - v_n^2(x)) \frac{v_n(y) v(y)}{|x-y|^{3-2t}} dx dy \right| \\
&\leq \int_{\mathbb{R}^3} |u_n^2(x) - v_n^2(x)| (\phi_{v_n}^t)^{\frac{1}{2}} (\phi_v^t)^{\frac{1}{2}} dx + \|u_n - v_n\|_{\frac{12}{3+2t}} \|v\|_{\frac{12}{3+2t}} \|\phi_{u_n}^t\|_{2_t^*} \\
&\leq \|\phi_{v_n}^t\|_{2_t^*}^{\frac{1}{2}} \|\phi_v^t\|_{2_t^*}^{\frac{1}{2}} (\|u_n\|_{\frac{12}{3+2t}} + \|v_n\|_{\frac{12}{3+2t}}) \|u_n - v_n\|_{\frac{12}{3+2t}} \\
&+ \|\phi_{u_n}^t\|_{2_t^*} \|u_n - v_n\|_{\frac{12}{3+2t}} \|v\|_{\frac{12}{3+2t}} \to 0
\end{aligned}$$

  for any $v \in H$.
- Verification of (v): Using Hölder's inequality, $4s + 2t > 3$ and (iii) which implies

$$\left(\int_K |\phi_{u_n}^t - \phi_u^t|^{2_t^*} dx \right)^{\frac{1}{2_t^*}} \to 0.$$

Then we can conclude that

$$\begin{aligned}\left|\int_K \phi_{u_n}^t u_n v dx - \int_K \phi_u^t u v dx\right| &= \left|\int_K \phi_{u_n}^t (u_n - u) v dx + \int_K (\phi_{u_n}^t - \phi_u^t) u v dx\right| \\ &\leq \left(\int_K |u_n - u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{12}} \|v\|_{\frac{12}{3+2t}} \|\phi_{u_n}^t\|_{2_t^*} \\ &+ \left(\int_K |\phi_{u_n}^t - \phi_u^t|^{2_t^*} dx\right)^{\frac{1}{2_t^*}} \|u\|_{\frac{12}{3+2t}} \|v\|_{\frac{12}{3+2t}} \\ &\leq C_1 \|u_n\|^2 \left(\int_K |u_n - u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{12}} \|v\|_{\frac{12}{3+2t}} \\ &+ \left(\int_K |\phi_{u_n}^t - \phi_u^t|^{2_t^*} dx\right)^{\frac{1}{2_t^*}} \|u\|_{\frac{12}{3+2t}} \|v\|_{\frac{12}{3+2t}} \to 0.\end{aligned}$$

As desired.

□

As in [10], we define

$$A_1 := \begin{cases} 0 & m = 0, \\ -\frac{1}{2} m^2 \log m^2 & 0 < |m| \leq \vartheta, \\ \frac{1}{2} m^2 (\log \vartheta^2 + 3) + 2\vartheta |m| - \frac{1}{2}\vartheta^2 & |m| \geq \vartheta, \end{cases}$$

and

$$A_2(m) := \frac{1}{2} m^2 \log m^2 + A_1(m).$$

Consequently,

$$A_2(m) - A_1(m) = \frac{1}{2} m^2 \log m^2.$$

By choosing a sufficiently small $\vartheta > 0$, we know that A_1 is convex, $A_1, A_2 \in C^1(\mathbb{R}, \mathbb{R})$ and $|A_2'| \leq C_p |m|^{p-1}$, where $p \in (2, 2_s^*)$.

Denote

$$\Psi(u) := \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) + 1) u^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} A_2(u) dx,$$

$$G(u) := \int_{\mathbb{R}^3} A_1(u) dx.$$

Hence, $I(u) = \Psi(u) + G(u)$, $\Psi \in C^1(H, \mathbb{R})$, $G \geq 0$. Obviously, G is a convex function. Besides, by Fatou's lemma, we may conclude that G is lower semicontinuous (see [17] Lemma 2.9). Therefore, the critical point theory described in [13] is applicable to the functional I.

Definition 2 (see [11]). *Let H be a Banach space and $I = \Psi + G$, where $\Psi \in C^1(H, \mathbb{R})$ and $G : H \to (-\infty, \infty]$. Moreover, G is lower semicontinuous, convex and $G \not\equiv +\infty$.*

(i) *$D(I) := \{u \in H : I(u) < +\infty\}$ is named the effective domain of I.*

(ii) *For any $u \in D(I)$. We define*

$$\partial I(u) := \{\xi \in H^* : G(z) - G(u) + \langle \Psi'(u), z - u\rangle \geq \langle \xi, z - u\rangle\}$$

as the subdifferential of I at u, with $z \in H$.

(iii) *For all $z \in H$, supposing that $u \in D(I)$ and $0 \in \partial I(u)$, i.e.*

$$G(z) - G(u) + \langle \Psi'(u), z - u\rangle \geq 0,$$

then $u \in H$ is a critical point of I.

(iv) *Assume $I(u_n)$ is bounded and there exists $\varepsilon_n \to 0^+$ such that*

$$G(z) - G(u_n) + \langle \Psi'(u_n), z - u_n \rangle \geq -\varepsilon_n \|z - u_n\| \text{ for all } z \in H, \tag{8}$$

then (u_n) is a Palais-Smale sequence for I.

(v) *I fulfills the Palais-Smale condition if Palais-Smale sequence has a convergent subsequence.*

Lemma 2 (see Proposition 2.3 of [12])**.** *If $u \in D(I)$, then there is a unique $\xi \in H^*$ such that $\partial I(u) = \{\xi\}$, i.e.,*

$$G(z) - G(u) + \langle \Psi'(u), z - u \rangle \geq \langle \xi, z - u \rangle \text{ for this } \xi \text{ and all } z \in H.$$

Furthermore,

$$\int_{\mathbb{R}^N} A_1'(u) v dx + \langle \Psi'(u), v \rangle = \langle \xi, v \rangle \text{ for any } v \in H \text{ such that } A_1'(u)v \in L^1(\mathbb{R}^3).$$

This unique ξ is defined as $I'(u)$.

Lemma 3. (i) *If $u \in D(I)$, then $I'(u) = 0$ if and only if (u, ϕ_u^t) is a solution of (1).*

(ii) *If $I(u_n)$ is bounded, then (u_n) is a Palais-Smale sequence if and only if $I'(u_n) = 0$.*

(iii) *If $I(u_n)$ is bounded above, $u_n \rightharpoonup u$ and $I'(u_n) \to 0$, it follows that u is a critical point.*

Proof. The proof of (i) and (ii) is similar to Lemma 2.4 of [12]. Now we prove (iii). In fact, we can deduce that G is weakly lower semicontinuous due to the lower semicontinuity and convexity of G. Therefore, $G(u) < \infty$ and $u \in D(I)$. By (v) of Lemma 1 and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^3)$ for each $r \in [1, 2_s^*)$,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle I'(u_n), v \rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} v + V(x) u_n v\right) dx - \int_{\mathbb{R}^3} u_n v \log u_n^2 dx - \int_{\mathbb{R}^3} \phi_{u_n}^t u_n v dx \\
&= \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + V(x) u v\right) dx - \int_{\mathbb{R}^3} u v \log u^2 dx - \int_{\mathbb{R}^3} \phi_u^t u v dx \\
&= \langle I'(u), v \rangle
\end{aligned}$$

for all $v \in C_0^\infty(\mathbb{R}^3)$. □

For all $u \in H$, we define by A the critical point set of I, for which $A = \{u \in D(I) : I'(u) = 0\}$. The subsequent pseudo-gradient vector field will be significant in the upcoming sections:

Proposition 1 (see Lemma 2.7 of [10])**.** *If there is a set of points $(u_i) \subset D(I) \backslash A$, a locally finite countable covering (M_i) of $D(I) \backslash A$ and a locally Lipschitz continuous vector field $F : D(I) \backslash A \to H$, then the following conclusions hold:*

(i) *$\| F(u) \| \leqslant 1$ and $\langle I'(u), F(u) \rangle > g(u)$, where $g(u) := \min \frac{1}{2} \| I'(u_i) \|$ for all i such that $u \in M_i$.*

(ii) *F is odd in u.*

(iii) *F possesses locally compact support. That is, for each $u_0 \in D(I) \setminus A$ there is a neighbourhood U_0 of u_0 in $D(I) \setminus A$ and $K > 0$ such that $\operatorname{supp} F(u) \subset B_K(0)$ for any $u \in U_0$.*

Corollary 1. *For any $b \in \mathbb{R}$, we can construct $(M_i), (u_i)$, and F on $\{u \in H : b < I(u) < \infty\} \backslash A$, where $(M_i), (u_i)$, and F satisfy all properties in Proposition 1. (i.e., $D(I) \backslash A$ can be substituted with $\{u \in H : b < I(u) < \infty\} \backslash A$ all the time).*

In addition, we will require a logarithmic Sobolev inequality in [18] applicable to all $u \in H^s(\mathbb{R}^3)$, stated as follows:

$$\int_{\mathbb{R}^3} u^2 \log u^2 dx \leq \|u\|_2^2 \log \|u\|_2^2 + \frac{\alpha^2}{\pi^s}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - \left(3 + \frac{3}{s}\log \alpha + \log \frac{s\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2s})}\right)\|u\|_2^2 \quad (9)$$

for any $\alpha > 0$.

3. Proof of Theorem 1

This section introduces several lemmas that will be utilized later. Firstly, we will demonstrate that the functional I fulfills the Palais-Smale condition.

Lemma 4. *The functional I fulfills the Palais-Smale condition.*

Proof. First, let us demonstrate the boundedness of the sequence (u_n). Select $h \in \mathbb{R}$ such as $I(u_n) \leq h$ for all n. As $n \to \infty$, we have

$$2h + o(1)\|u_n\| \geq 2I(u_n) - \langle I'(u_n), u_n\rangle = \|u_n\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \geq \|u_n\|_2^2. \quad (10)$$

Using (9), we conclude that

$$\int_{\mathbb{R}^3} u^2 \log u^2 dx \leq \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + C_1(\log \|u\|_2^2 + 1)\|u\|_2^2, \quad (11)$$

by choosing a sufficiently small $\alpha > 0$. Consequently, by employing (10) and (11), we have

$$\begin{aligned} 4h + o(1)\|u_n\| &\geq 4I(u_n) - \langle I'(u_n), u_n\rangle \\ &= \|u_n\|^2 + \|u_n\|_2^2 - \int_{\mathbb{R}^3}(V(x)+1)^- u_n^2 dx - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 dx \\ &\geq \frac{1}{2}\|u_n\|^2 - C_2(\|u_n\|+1)^{1+\theta}, \end{aligned}$$

where we take $0 < \theta < 1$. Therefore, the sequence (u_n) is bounded and for some u, $u_n \rightharpoonup u$ in H, after passing to a subsequence. Due to the compactness of the embedding $H \hookrightarrow L^r(\mathbb{R}^3)$ for $r \in [2, 2_s^*)$, as shown in [19], $u_n \to u$ in $L^r(\mathbb{R}^3)$. Substituting $v = u$ into (8), we deduce that

$$\begin{aligned} &\langle u_n, u - u_n\rangle - \int_{\mathbb{R}^3} \phi_{u_n}^t (u - u_n) u_n dx - \int_{\mathbb{R}^3} V(x)^- u_n (u - u_n) dx \\ &+ G(u) - G(u_n) - \int_{\mathbb{R}^3} A_2'(u_n)(u - u_n) dx \geq -\varepsilon_n \|u - u_n\|, \end{aligned}$$

where

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n}^t (u - u_n) u_n dx &\leq \|u_n\|_{\frac{12}{3+2t}} \|\phi_{u_n}^t\|_{2_t^*} \|u_n - u\|_{\frac{12}{3+2t}} \\ &\leq C\|u_n\|^3 \|u_n - u\|_{\frac{12}{3+2t}} \to 0 \end{aligned}$$

as $n \to \infty$. Thus,

$$G(u) - G(u_n) + \|u\|^2 - \|u_n\|^2 + o(1) \geq o(1).$$

Since $\liminf_{n\to\infty} G(u_n) \geq G(u)$ and $\liminf_{n\to\infty} \|u_n\|^2 \geq \|u\|^2$, the above inequalities lead to $\|u_n\| \to \|u\|$. Consequently, $u_n \to u$ in H. □

Lemma 5 (see Lemma 3.3 of [12])**.** *Suppose $A_d = \emptyset$, there is $\varepsilon_0 > 0$ such that no Palais-Smale sequences exists in $I_{d-2\varepsilon_0}^{d+2\varepsilon_0}$.*

Assume $A_d = \emptyset$ and consider ε_0 as defined in Lemma 5. Define $\kappa : H \to [0,1]$ as an even function is locally Lipschitz continuous, with $\kappa = 0$ on $I^{d-\varepsilon_0}$ and $\kappa > 0$ elsewhere. Let the flow η be denoted by

$$\begin{cases} \frac{d}{dt}\eta(t,u) = -\kappa(\eta(t,u))F(\eta(t,u)), \\ \eta(0,u) = u, \ u \in I^{d+\varepsilon_0}. \end{cases} \tag{12}$$

The vector field F which is defined on $\{u \in F : d - 2\varepsilon_0 < I(u) < \infty\} \backslash A \to H$ is described according to Corollary 1. It should be noted that $\kappa(u)F(u) = 0$ if $I(u) < d - \varepsilon_0$. In [10], it has been proved $t \mapsto I(\eta(t,u))$ is differentiable and

$$\frac{\partial I(\eta(t,u))}{\partial t} = \langle I'(\eta(t,u), \frac{\partial}{\partial t}\eta(t,u)\rangle.$$

Therefore, according to (i) of Propositon 1, we see that

$$\frac{\partial I(\eta(t,u))}{\partial t} \leq 0, \ t \mapsto I(\eta(t,u))$$

is non-increasing. Taking into account $\|F(u)\| \leq 1$ and $A \cap I_{d-\varepsilon_0}^{d+\varepsilon_0} = \emptyset$ for any $t \geq 0$, there exists $\eta(t,u)$.

Lemma 6 (see Proposition 3.4 of [12])**.** *Assume $A_d = \emptyset$ and let ε_0 be as defined in Lemma 5. If $\varepsilon \in (0, \varepsilon_0)$, then for each compact set $W \subset I^{d+\varepsilon} \cap C_0^\infty(\mathbb{R}^3)$, there is $T > 0$ such that $\eta(T, W) \subset I^{d-\varepsilon}$.*

Given that H is separable and $C_0^\infty(\mathbb{R}^3)$ is dense in H, it is possible to find a sequence of subspaces, denoted as (H_k), each within $C_0^\infty(\mathbb{R}^3)$ and of dimension k, such that $H = \overline{\cup_{k=1}^\infty H_k}$. Define Z_k as the orthogonal complement of H_k in H, denoted by $Z_k := H_k^\perp$. Let

$$P_k := \{u \in H_k : \|u\| \leq \tau_k\}, \ Q_k := \{u \in Z_{k-1} : \|u\| = \sigma_k\},$$

where $\tau_k > \sigma_k > 0$.

Lemma 7 (see Lemma 3.4 of [20])**.** *If $\alpha \in C(P_k, H)$ is odd and $\alpha|_{\partial P_k} = id$, then $\alpha(P_k) \cap Q_k \neq \emptyset$.*

Lemma 8. *There exists $\tau_k > \sigma_k > 0$ such that*

$$g_k := \max_{u \in H_k, \|u\| = \tau_k} I(u) \leq 0 \ \textit{for all } k \ \textit{ and } \ f_k := \inf_{u \in Q_k} I(u) \to \infty \textit{ as } k \to \infty.$$

Proof. Set $u = sv$ with $u \in H_k$ and $\|v\|_2 = 1$. Subsequently,

$$I(sv) \leq \frac{1}{2}s^2\left(\|v\|^2 - \int_{\mathbb{R}^3}(V(x)+1)^- v^2 dx - \int_{\mathbb{R}^3} v^2 \log v^2 dx - \log s^2\right).$$

Considering that all norms in H_k are equivalent, coupled with $H_k \subset C_0^\infty(\mathbb{R}^3)$, we can conclude that both integrals above are uniformly bounded. Therefore, as $s \to \infty$, $I(sv) \to -\infty$ uniformly for all v. This implies that there exists τ_k such that $g_k \leq 0$. Additionally, τ_k can be selected to be arbitrarily large as needed.

Set

$$\omega_k := \max_{u \in Z_{k-1}, \|u\| = 1} \|u\|_{\frac{12}{3+2t}}.$$

In order to prove $\omega_k \to 0$, we refer to Lemma 3.8 in [20]. Specifically, we give the following proof. The sequence ω_k is both positive and decreasing, leading to the conclusion that $\omega_k \to \omega \geq 0$. Additionally, there exists a sequence $u_k \in Z_{k-1}$ with $\|u_k\| = 1$ such that $\|u_k\|_{\frac{12}{3+2t}} \geq \omega_k/2$. Considering that $u_k \rightharpoonup 0$ in H, it follows $u_k \to 0$ in $L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. Consequently, we can conclude that $\omega = 0$.

Employing (11) as demonstrated in Lemma 4, one has

$$\begin{aligned} I(u) &\geq \frac{1}{4}\|u\|^2 - C_1\|u\|_2^2 - C_2(1+\|u\|_2^{2\theta}) - C_3\|u\|^4_{\frac{12}{3+2t}} \\ &\geq \frac{1}{4}\|u\|^2 - C_4\|u\|^2_{\frac{12}{3+2t}} - C_5\|u\|^{2\theta}_{\frac{12}{3+2t}} - C_4\|u\|^4_{\frac{12}{3+2t}} - C_2, \end{aligned}$$

where $\theta \in (1,2)$. Set $\sigma_k = \frac{1}{\omega_k}$ and $\|u\| = \sigma_k$. Thus, as $k \to \infty$, it can be concluded that

$$I(u) \geq \frac{1}{4}\sigma_k^2 - C_5 - C_4 - C_2 \to \infty,$$

which implies that $f_k \to \infty$. Given that τ_k can be selected such that $\tau_k > \sigma_k$, this proof is thus completed. □

Proof of Theorem 1. Set

$$\Gamma_k := \left\{\alpha \in C(P_k, H) : \alpha|_{\partial P_k} = id, \alpha(P_k) \textit{ has compact support and } \alpha \textit{ is odd}\right\}$$

and

$$d_k = \inf_{\alpha \in \Gamma_k} \max_{u \in P_k} I(\alpha(u)).$$

According to Lemma 7, we find that $\alpha(P_k) \cap Q_k \neq \emptyset$, which leads to the conclusion that $d_k \geq f_k \to \infty$. What remains to be shown is that $A_{d_k} \neq \emptyset$ for sufficiently large k. Assuming the opposite, select $\varepsilon_0, \varepsilon$ and T according to Lemma 6. Consider $\alpha \in \Gamma_k$ such that $\alpha(P_k) \subset I^{d_k+\varepsilon}$. Set $\gamma(u) := \eta(T, \alpha(u))$, with η representing the flow given in (12). By (ii) of Proposition 1, α is odd. Given that $\eta(T,u) = u$ for any $u \in I^{d_k-\varepsilon_0}$, it follows $\gamma|_{\partial P_k} = id$, and therefore $\gamma \in \Gamma_k$. According to Lemma 6, we have $\gamma(P_k) \subset I^{d_k-\varepsilon}$, which contradicts the definition of d_k.

4. Proof of Theorem 2

In this section, our work is conducted in the space $H = H^s(\mathbb{R}^3)$ where the functional is defined as

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 dx. \tag{13}$$

Lemma 9. *If $I(u_n)$ is bounded above and $I'(u_n) \to 0$, then (u_n) is bounded.*

Proof. Selecting $h \in \mathbb{R}$ such as $I(u_n) \leq h$ for any n, we derive as $n \to \infty$

$$\begin{aligned} 4h + o(1)\|u_n\| &\geq 4I(u_n) - \langle I'(u_n), u_n\rangle \\ &\geq \frac{1}{2}\|u_n\|^2 - C_2(\|u_n\|+1)^{1+\theta}, \end{aligned}$$

where $0 < \theta < 1$. Here, we have used (10) once more, by choosing α in (9) to be sufficiently small. Consequently, $1/2$ in (11) is replaced by a constant a, ensuring that $a\|(-\Delta)^{\frac{s}{2}}u\|_2^2 \leq \frac{1}{2}\|u\|^2$. □

We next consider a limiting problem

$$(-\Delta)^s u + \mathbb{V}_\infty u - \phi_u^t u = u \log u^2 \quad in\ \mathbb{R}^3. \tag{14}$$

The associated energy functional is given by

$$I_\infty(u) = \frac{1}{2}\left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 + (\mathbb{V}_\infty + 1)u^2 dx\right) - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 dx. \tag{15}$$

Consider the Nehari manifold for I, denoted as

$$\mathbb{X} := \{u \in D(I) \setminus \{0\} : \langle I'(u), u\rangle = 0\}.$$

In a similar way, the Nehari manifold for I_∞ is denoted by $\mathbb{X}_\infty$. Following the ideas from [10], we can prove problem (14) exists a nontrivial solution u^* and $\inf\limits_{u \in X_\infty} I = I(u^*) > 0$. It is obvious that u^* is a ground state solution to (14). We first elaborate on the differences from the Section 2.1 of [10]. It is worth noting that Lemma 10, Lemma 12, and Lemma 13 in this paper correspond to Lemma 2.10, Lemma 2.13, and Lemma 2.14 in [10], respectively. For the reader's convenience, we restate these lemmas below.

Lemma 10 (see Lemma 2.10 of [10])**.** $\rho := \inf\{\|j-k\| : j, k \in A, j \neq k\} > 0$.

Lemma 11 (see Lemma 2.11 of [10])**.** *If $(u_n), (v_n) \subset H$ are two Palais-Smale sequence, then one of the following holds:* $\limsup_{n\to\infty} \|u_n - v_n\| \geq \rho$ *or* $\|u_n - v_n\| \to 0$ *as* $n \to \infty$.

Proof. Choose $q = 12/(3+2t)$, where q is in $(2, 2_s^*)$. Hence there exists a constant $C > 0$ to ensure $|A_2'(s)| < C|s|^{q-1}$. Let us first assume $\|u_n - v_n\|_q \to 0$ as $n \to \infty$. By Lemma 9, it follows that $(u_n), (v_n)$ are bounded in H. By (8) and (iv) of Lemma 1, we conclude that

$$\begin{aligned}
o(1) + \|u_n - v_n\|^2 &= \|u_n - v_n\|^2 - \int_{\mathbb{R}^3} (u_n - v_n)(\phi_{u_n}^t u_n - \phi_{v_n}^t v_n)dx \\
&\quad - \int_{\mathbb{R}^3} (u_n - v_n)(u_n \log u_n^2 - v_n \log v_n^2)dx + o(1) \\
&= \langle I'(u_n), u_n - v_n\rangle - \langle I'(v_n), u_n - v_n\rangle \\
&= \int_{\mathbb{R}^3} (u_n - v_n)(A_2'(u_n) - A_2'(v_n))dx + \langle \Psi'(u_n), u_n - v_n\rangle - \langle \Psi'(v_n), u_n - v_n\rangle \\
&\leq C \int_{\mathbb{R}^3} |u_n - v_n|(|v_n|^{q-1} + |u_n|^{q-1})dx + 2\varepsilon_n \|u_n - v_n\| \\
&\leq C_1 \|u_n - v_n\|_q + 2\varepsilon_n \|u_n - v_n\|.
\end{aligned}$$

So $\|u_n - v_n\| \to 0$ as $n \to \infty$.

Suppose now that $\|u_n - v_n\|_q \nrightarrow 0$. Using Lions' lemma (see Lemma 1.21 of [20] or Lemma I.1 of [21]), it is easy to find a sequence $(a_n) \subset \mathbb{R}^3$ and $\vartheta > 0$ such that, for sufficiently large n,

$$\int_{B_1(a_n)} (u_n - v_n)^2 dx \geq \vartheta.$$

By (ii) of Lemma 1, I is invariant under translations by elements of $\mathbb{R}^3$, the subsequence (a_n) can be assumed to be bounded. Thus, after taking a subsequence, $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ and $u \neq v$. By (iii) of Lemma 3, we have $u, v \in A$. Hence

$$\limsup_{n\to\infty} \|u_n - v_n\| \geq \|u - v\| \geq \rho.$$

This finishes the proof. □

Remark 2. *As in Remark 2.12 of [10], the conclusions of Lemmas 10–13 remain valid on I^h, we just need to show that $u, v \in I^h$ within the argument for Lemma 11. By the lower semicontinuity of $\|u\|_2^2$ and (iii) of Lemma 1,*

$$\begin{aligned}
h &\geq I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o(1) = \frac{1}{2}\|u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx + o(1) \\
&\geq \frac{1}{2}\|u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 dx + o(1) = I(u) - \frac{1}{2}\langle I'(u), u\rangle + o(1) = I(u) + o(1).
\end{aligned}$$

Therefore, $u \in I^h$ and similarly, $v \in I^h$.

Now, we turn our focus to the flow $\tilde{\eta}$ as defined by

$$\begin{cases} \frac{d}{dt}\tilde{\eta}(t,u) = -F(\tilde{\eta}(t,u)), \\ \tilde{\eta}(0,u) = u, \ u \in D(I)\backslash A. \end{cases} \tag{16}$$

Denote the maximal existence time for the trajectory $t \mapsto \tilde{\eta}(t,u)$ as $(Y^-(u), Y^+(u))$.

Lemma 12 (see Lemma 2.13 of [10])**.** *Let $u \in D(I)\backslash A$. Then there are two possible outcomes: either $Y^+(u) = +\infty$, $\lim_{t\to Y^+(u)} I(\tilde{\eta}) = -\infty$ or $\lim_{t\to Y^+(u)} \tilde{\eta}(t,u)$ exists and is a critical point of I.*

Let $h > 0$, select $\varepsilon_1 > 0$ such that $I_{h-2\varepsilon_1}^{h+2\varepsilon_1} \cap A = A_h$.

Lemma 13 (see Lemma 2.14 of [10])**.** *For every $\mu > 0$ there is $\varepsilon \in (0, \varepsilon_1)$ such that*

$$\lim_{t\to Y^+(u)} I(\tilde{\eta}(t,u)) < d - \varepsilon, \quad \textit{whenever } u \in I^{d+\varepsilon} \setminus U_\mu(A_h).$$

Furthermore, for all $t \in [0, Y^+(u))$, we have $\tilde{\eta}(t,u) \notin U_{\mu\backslash 2}(A_h) \cap I_{h-\varepsilon}$.

Lemma 14. *There is $a, b > 0$ such that $I(u) \geq 0$ holds for any $u \in B_a(0)$ and $I(u) \geq b$ for any $u \in S_b(0)$.*

Proof. For the proof we mimic that of Lemma 2.15 in [10]. In view of (i) of Lemma 1, $|A_2'(s)| < C|s|^{p-1}$ and $G \geq 0$, we find that $I(u) \geq \Psi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$. Therefore, the desired result follows. □

Inspired by the idea of [10], we can prove that problem (7) possesses a ground state solution. In fact, for any $u \in D(I)\backslash\{0\}$, we define $\varphi_u := I(lu), l > 0$; then, we obtain

$$\varphi_u'(l) = l\left(\frac{1}{2}\|u\|^2 - l^2\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} u^2 \log u^2 dx - 2\|u\|_2^2 \log l - \|u\|_2^2\right). \tag{17}$$

Rewrite $\varphi_u'(l) = l g_u(l)$, where

$$g_u(l) = \frac{1}{2}\|u\|^2 - l^2\int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} u^2 \log u^2 dx - 2\|u\|_2^2 \log l - \|u\|_2^2. \tag{18}$$

It can be easily infered from (18) that $\lim_{l\to 0^+} g_u(l) > 0$ and $\lim_{l\to+\infty} g_u(l) < 0$. Obviously, for all $l > 0$, $g_u'(l) < 0$, so there exists a unique l_0 such that $g_u(l_0) = 0$. Note that $l_0 > 0$, we confirm that $\varphi_u'(l_0) = 0$ from (17) and l_0 is the only intersection of the $\varphi_u'(l)$ with $\mathbb{X}$. Furthermore, $\varphi_u(l) \to -\infty$ as $l \to \infty$. When $\|u\| = 1$, the mapping $l \mapsto \Psi(lu)$ increases for all $0 < l < l_0$ (where l_0 is independent of u) and $l \mapsto G(lu)$ increases for all $l > 0$ thanks to its convexity. Therefore, $\mathbb{X}$ is bounded away from the origin. Set

$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, I(\gamma(1)) < 0\}$$

and

$$c := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} I(\gamma(s)), \ c_{\mathbb{X}} := \inf_{u\in\mathbb{X}} I(u).$$

Based on Lemma 14, we have that $c \geq b > 0$. Obviously, $c \leq c_{\mathbb{X}}$. Suppose that for some $\varepsilon_1 > 0$, there are no nontrivial solution with energy levels below $c + \varepsilon_1$. By Remark 2, we can apply Lemma 13 with $U_\mu(A_c) = \emptyset$ and a sufficiently small $\varepsilon < \varepsilon_1$. We explore the flow denoted by

$$\begin{cases} \frac{d}{dt}\eta^*(t,u) = -\kappa^*(\eta^*(t,u))F(\eta^*(t,u)), \\ \eta^*(0,u) = u, \ u \in I^{c+\varepsilon}, \end{cases} \tag{19}$$

where $\kappa^* : H \to [0,1]$ is locally Lipschitz continuous such that $\kappa^* = 0$ on $I^{c/2}$, $\kappa^* > 0$ elsewhere. By Lemma 13, we obtain a contradiction and a sequence of nontrivial solutions u_n. Hence we deduce that $c = c_{\mathbb{X}}$ and thus $I(u_n) \to c$. Furthermore, we assume $u_n \rightharpoonup u$ in H as $n \to \infty$. According to (iii) of Lemma 3, u is a solution of (14). If $\|u_n\|_{\frac{12}{3+2t}} \to 0$, then

$$o(1) = \langle I'(u_n), u_n \rangle \geq \|u_n\|^2 - C_1 \int_{\{u_n^2 \geq \frac{1}{e}\}} |u_n|^{\frac{12}{3+2t}} dx - C_2 \|u_n\|^4_{\frac{12}{3+2t}} \tag{20}$$

which implies that $\|u_n\| \to 0$ as $n \to \infty$. This contradicts the assumption that $(u_n) \subset \mathbb{X}$. Therefore, $\|u_n\|_{\frac{12}{3+2t}} \nrightarrow 0$ and then we can find a sequence $(a_n) \subset \mathbb{R}^3$ and $\vartheta > 0$ such that for large n,

$$\int_{B_1(a_n)} u_n^2 dx \geq \vartheta,$$

thanks to Lions' lemma (see Lemma 1.21 of [20] or Lemma I.1 of [21]). Using the method applied in Lemma 11, we can assume that the sequence (a_n) remains bounded after necessary translations. Hence, for the (translated) sequence (u_n), it follows $u_n \rightharpoonup u \neq 0$ as $n \to \infty$. Based on (iii) of Lemma 3, $u \in A$, so $I(u) \geq c$. Following the reasoning in Remark 2, we also conclude that $I(u) \leq c$. Thus, $I(u) = \inf_{u \in \mathbb{X}} I$, indicating that u is a ground state solution.

Remark 3. *It is worth mentioning that if $V(x) = \mathbb{V}_\infty > -1$, then the above results remains valid. In this case, there is a nontrivial solution $u^* \neq 0$ for (14) and satisfying $\inf_{u \in \mathbb{X}_\infty} I_\infty = I_\infty(u^*)$.*

Lemma 15. (i) *If $V \not\equiv \mathbb{V}_\infty$, then $c_{\mathbb{X}} < c^*$, where $c^* := \inf_{u \in \mathbb{X}_\infty} I_\infty(u)$.*

(ii) *If $I(u_n) \to d \in (0, c^*)$ and $I'(u_n) \to 0$ then $u_n \rightharpoonup u \neq 0$; after takeing a subsequence, u is a critical point of I and $I(u) \leq d$.*

Proof. (i) Choose $l_0 > 0$ such that $l_0 u^* \in \mathbb{X}$, where $u^* \neq 0$ is a ground state solution of (14). Considering $V < \mathbb{V}_\infty$ in some open set and $u^* \neq 0$, $l \mapsto I_\infty(lu^*)$ for all $l > 0$ has a unique maximum at $l = 1$,

$$c_{\mathbb{X}} \leq I(l_0 u^*) < I_\infty(l_0 u^*) \leq I_\infty(u^*) = c^*.$$

(ii) According to Lemma 9, we have $u_n \rightharpoonup u$ in H after passing to a subsequence. Furthermore, as stated in (iii) of Lemma 3, u is a critical point of I. Following the same argument as in Remark 2, we get $I(u) \leq d$. Now there is only the task of demonstrating that $u \not\equiv 0$. Indirectly, let us assume $u \equiv 0$. Given that $V(x) \to \mathbb{V}_\infty$ as $|x| \to \infty$ and $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$, we find

$$I_\infty(u_n) - I(u_n) = \frac{1}{2} \int_{\mathbb{R}^3} (\mathbb{V}_\infty - V(x)) u_n^2 dx \to 0.$$

Thus, $I_\infty(u_n) \to d$. By applying the Sobolev inequality and the Hölder inequality, and choosing v such that $\|v\| = 1$, we derive

$$\begin{aligned} |\langle I'_\infty(u_n) - I'(u_n), v \rangle| &\leq \int_{\mathbb{R}^3} |u_n||v|(\mathbb{V}_\infty - V(x)) dx \\ &\leq C \left(\int_{\mathbb{R}^3} (\mathbb{V}_\infty - V(x))^2 u_n^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

The expression on the right converges to 0 uniformly when $\|v\| = 1$, $I'_\infty(u_n) - I'(u_n) \to 0$. Therefore, $I'_\infty(u_n) \to 0$. If $\|u_n\|_{\frac{12}{3+2t}} \to 0$, according to (20), we have $u_n \to 0$ in H. By setting $v = 0$ in (8), we obtain

$$\langle \Psi'(u_n), -u_n \rangle - G(u) \geq -\varepsilon_n \|u_n\|.$$

This implies that $G(u_n) \to 0$. Thus, $I(u_n) \to 0$, which contradicts the assumption that $I(u_n) \to d > 0$. Consequently, $\|u_n\|_{\frac{12}{3+2t}} \nrightarrow 0$. By means of Lions' lemma, we deduce that there exist sequences $(b_n) \subset \mathbb{R}^3$ and $\vartheta > 0$ such that for large n,

$$\int_{B_1(b_n)} u_n^2 dx \geq \vartheta.$$

Set $w_n(x) := u_n(x + b_n)$. By (ii) of Lemma 1, I_∞ is invariant under translations by elements of $\mathbb{R}^3$, thus $I_\infty(w_n) \to d$ and $I'_\infty(w_n) \to 0$. Furthermore,

$$\int_{B_1(0)} w_n^2 dx = \int_{B_1(b_n)} u_n^2 dx \geq \vartheta.$$

and hence $w_n \rightharpoonup w \neq 0$ after taking a subsequence. Thus w is a nontrivial critical point of I_∞ satisfying $I_\infty(w) \leq d < I_\infty(u^*)$. Consequently, this leads to a contradiction. □

Proof of Theorem 2. According to Remark 3, if $V \equiv \mathbb{V}_\infty$, then u^* is the exact solution we seek for. Therefore, suppose $V(x) < \mathbb{V}_\infty$ for some x. Let us assume that there is $\varepsilon_0 \in (0, c/2)$ such that there are no Palais-Smale sequences in $I_{c-2\varepsilon_0}^{c+2\varepsilon_0}$. Let $\varepsilon \in (0, \varepsilon_0)$. Select $\gamma \in \Gamma$ so that $\gamma([0,1]) \subset I^{c+2/\varepsilon}$. It can be assumed $I(\gamma(1)) < -\varepsilon/2$. Let $\tau_K \in C^1(\mathbb{R}^3, [0,1])$ be such that $\tau_K = 1$ if $|x| \leq K$, $\tau_K = 0$ if $|x| \geq 2K$ and $\|(-\Delta)^{\frac{s}{2}} \tau_K\|_2 \leq 1$. Let $u_K(x) := \tau_K u(x)$. We observe that $\|u_K - u\| \to 0$ uniformly in $u \in \gamma([0,1])$ as $K \to +\infty$. Given that $\Psi \in C^1(H, \mathbb{R})$, there is $K > 0$ such that for any $u \in \gamma([0,1])$, we have $\Psi(u_K) \leq \Psi(u) + \varepsilon/2$. Additionally, since A_1 is convex and $|u_K| \leq |u|$, we conclude that $G(u_K) \leq G(u)$. Therefore, $\gamma_K([0,1]) \subset I^{c+\varepsilon}$ and $I(\gamma_K(1)) < 0$, where $\gamma_K(s) := \tau_K \gamma(s)$. By the definition of compact support, we can conclude that γ_K has compact support and since $\gamma_K(0) = 0$, we obatin $\gamma_K \in \Gamma$. According to Lemma 6, let $\alpha_K(s) = \eta(T, \gamma_K(s))$ and we derive $\alpha_K \in \Gamma$, leading to $\alpha_K([0,1]) \subset I^{c-\varepsilon}$, thereby conflicting with the definition of c. Due to the fact that ε_0 may be taken arbitrarily small, there is a sequence u_n such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Through Lemma 15, we acquire a nontrivial critical point u of I, fulfilling $I(u) \leq c$. Therefore, $u \in \mathbb{X}$. Consequently, $c = c_{\mathbb{X}}$ and u is a ground state solution. Thus, the proof is complete.

5. Conclusions

In the article, we investigate the existence of solutions for a logarithmic fractional Schrödinger-Poisson system with a potential which may change sign. Due to the lack of smoothness of the functional I, we employed the approach explored in [10,12] to establish our results. Some main results are as follows:

(i) When the potential was coercive, we applied some arguments of the Fountain theorem to show $d_k \to \infty$. Subsequently, by using the deformation lemma (Lemma 6), we demonstrated that $A_{d_k} \neq \emptyset$ for sufficiently large k. Consequently, we established the existence of infinitely many solutions for the system (1).

(ii) When the potential was bounded, following the ideas from [10], we proved the existence of a ground state solution for problem (14). Additionally, by utilizing the deformation lemma (Lemma 6) and the Nehari method, we confirmed the existence of a ground state solution for the system (1).

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