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Study of a Coupled Ψ –Liouville–Riemann Fractional Differential System Characterized by Mixed Boundary Conditions

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Abstract: This research is concerned with the existence and uniqueness of solutions for a coupled system of Ψ –Riemann–Liouville fractional differential equations. To achieve this objective, we establish a set of necessary conditions by formulating the problem as an integral equation and utilizing well-known fixed-point theorems. By employing these mathematical tools, we demonstrate the existence and uniqueness of solutions for the proposed system. Additionally, to illustrate the practical implications of our findings, we provide several examples that showcase the main results obtained in this study.

Keywords: fractional differential equation; iterative methods; Banach space; Ψ –Liouville–Riemann fractional operators



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1. Introduction

Numerous authors have tackled boundary value problems (BVPs) encompassing various types of fractional derivative operators, including, but not limited to, Caputo–Liouville, Riemann–Liouville, Ψ –Riemann–Liouville [1], Hilfer [2], κ –Riemann–Liouville [3], and ψ –Hilfer [4]. Recent research papers [5–7] delve into multivalued BVPs that incorporate Hilfer and Caputo–Hadamard-type fractional derivative operators. To grasp the fundamental concepts of fractional calculus, consult references such as the book [8]. Notably, the Hilfer fractional derivative offers a unified definition for both Riemann–Liouville and Caputo fractional derivatives. For practical applications involving the Hilfer fractional derivative operator, readers are referred to [9–12]. Traditionally, derivatives are taken with respect to variables. However, fractional derivatives can also be taken with respect to other functions. This type of derivative is a generalization of the traditional derivative, and it is commonly used in fractional calculus to describe complex systems with non-integer-order dynamics. By taking the fractional derivative with respect to another function, researchers can better model the behavior of these systems and gain a deeper understanding of their underlying dynamics; for more details, see [13–20]. In recent years, fixed-point theory has emerged as a crucial tool in the analysis of differential equations, including fractional differential systems. The Banach Fixed-Point Theorem provides a foundational result, establishing conditions for the existence and uniqueness of fixed points for contraction mappings in complete metric spaces [21]. Further extending these ideas, Krasnoselskii’s fixed-point theorem handles mappings composed of a compact continuous operator and a contraction, broadening the scope of applicable systems [22]. The Nonlinear Alternative of the Leray–Schauder type provides another powerful tool, offering alternatives for the existence of fixed points in Banach spaces under compactness assumptions [23]. These theoretical advancements have found significant applications in the study of fractional differential equations. For instance, recent works have successfully applied fixed-point theorems to establish existence results for fractional pantograph equations and advection–diffusion equations under various

initial and boundary conditions [24]. These contributions underscore the versatility of fixed-point theory in addressing complex problems in fractional calculus. The study of fractals and their applications in various fields has garnered significant attention in recent years. Fractals, which are complex structures characterized by self-similarity and fractional dimensions, have been found to be particularly useful in modeling and analyzing natural phenomena. For instance, recent research has demonstrated the effectiveness of fractal analysis in understanding complex systems and improving algorithmic approaches in mathematics and computer science (see [25,26]). In particular, advancements in mathematical methods have shown promise in enhancing our understanding of fractal dimensions and their applications in signal processing and image analysis [27]. These methods not only provide insights into the intrinsic properties of fractals but also open up new avenues for practical applications in various scientific and engineering disciplines. In recent studies [15–17], significant attention was directed towards establishing findings pertaining to a specific category of coupled system of fractional integral problems. These problems were associated with fractional differential equations (FDEs) and featured movable boundary conditions. In [28], the researchers focused their efforts on investigating and analyzing the following form of problem structure:

$$\begin{cases} {}^c D_{0+}^{\ell_1} u(\varsigma) = \gamma_1(\varsigma, \omega(\varsigma)), & \varsigma \in [0, 1], \\ {}^c D_{0+}^{\ell_2} \omega(\varsigma) = \gamma_2(\varsigma, u(\varsigma)), & \varsigma \in [0, 1], \\ u(0) = 0, \quad \omega(0) = 0, \\ u(1) = \int_0^a u(s)ds, \quad \omega(1) = \int_0^b \omega(s)ds, \end{cases} \quad (1)$$

with $\gamma_1, \gamma_2 \in C([0, 1], \mathbb{R}_+)$, $1 < \ell_1, \ell_2 < 2$, and $0 < a, b < 1$. Recently, Rezapour [29] constructed a new analysis studying the problem

$$\begin{cases} {}^C D^{\xi_1} ({}^C D^{\xi_2} \xi)(\varsigma) = K(\varsigma, \xi(\varsigma), {}^C D^{\xi_2} \xi(\varsigma)), & \varsigma \in [0, 1], \\ {}^C D^{\xi_1^*} ({}^C D^{\xi_2^*} \xi)(\varsigma) = M(\varsigma, \xi(\varsigma), {}^C D^{\xi_2^*} \xi(\varsigma)), & \varsigma \in [0, 1], \\ \gamma \xi(0) = \delta \xi(1) = \alpha {}^C D^{\xi_2} \xi(0) = \beta {}^C D^{\xi_2} \xi(1) = 0, \\ \gamma^* \xi(0) = \delta^* \xi(1) = \alpha^* {}^C D^{\xi_2^*} \xi(0) = \beta^* {}^C D^{\xi_2^*} \xi(1) = 0, \end{cases} \quad (2)$$

such that $\xi_1, \xi_1^* \in (1, 2]$, $\xi_2, \xi_2^* \in (1, 2]$ and $\gamma, \delta, \alpha, \beta, \gamma^*, \delta^*, \alpha^*, \beta^* \in \mathbb{R}_+$. In addition, the authors used the operator ${}^C D^{(\cdot)}$ to deal with fraction derivatives according to Caputo's definition. Furthermore, the continuous single-valued functions K and M , defined on $[0, 1] \times \mathbb{R}^2$ are considered, taking values in $\mathbb{R}$. In [30], the authors proposed a new formulation, marking the first instance of a coupled system of three-point integral pantograph FDEs using the corresponding Caputo's operators in

$$\begin{cases} {}^{CC} D_{\varsigma_0}^{\mu, \xi_1} \varphi(\varsigma) = \omega_1(\varsigma, \Psi(\varsigma), \Psi(\lambda \varsigma)), & \varsigma \in \mathbb{O} = [\varsigma_0, Z], \\ {}^{CC} D_{\varsigma_0}^{\mu, \xi_2} \Psi(\varsigma) = \omega_2(\varsigma, \varphi(\varsigma), \varphi(\lambda \varsigma)), \\ \varphi(\varsigma_0) = 0, \quad a_1 \varphi(Z) + a_2 {}^{RC} I_{\varsigma_0}^{\mu, \theta} \varphi(c) = m_1, & c \in \mathbb{O}, \\ \Psi(\varsigma_0) = 0, \quad b_1 \Psi(Z) + b_2 {}^{RC} I_{\varsigma_0}^{\mu, \theta} \Psi(d) = m_2, & d \in \mathbb{O}, \end{cases} \quad (3)$$

where ${}^{CC} D_{\varsigma_0}^{\mu, \xi_i}$ stands for the corresponding Caputo's derivatives of order $\xi_i \in (1, 2)$ with $\mu \in (0, 1)$ for $i = 1, 2$, and ${}^{RC} I_{\varsigma_0}^{\mu, \theta}$ denotes the RL-corresponding integral of order $\theta > 0$. Furthermore, $a_1, a_2, b_1, b_2, m_1, m_2 \in \mathbb{R}$, $\lambda \in (0, 1)$, with

$$\omega_i : [\varsigma_0, Z] \times \mathbb{R}^2 \longrightarrow \mathbb{R},$$

are continuous for $i = 1, 2$.

In this work, we aim to extend the results of the aforementioned articles within the context of Ψ -Riemann–Liouville fractional derivatives. Specifically, we focus on the

existence and uniqueness of solutions by employing Banach's fixed-point theorem and Krasnoselskii's fixed-point theorem, similar to the previous studies. Additionally, we introduce the use of another fixed-point theorem, the nonlinear alternative of the Leray–Schauder type. Furthermore, we explore the Ulam–Hyers stability of the solutions. The importance of Ψ –Riemann–Liouville fractional-order derivatives lies in their ability to generalize and extend traditional derivatives. For example, if we make $\Psi(\varsigma) = \varsigma$, we obtain the classical Riemann–Liouville fractional operator, and if we make $\Psi(\varsigma) = \ln(\varsigma)$, we obtain the Hadamard fractional operator. Conversely, Ψ –Riemann–Liouville fractional-order derivatives enhance the ability to model and analyze systems with complex, non-local, or memory-dependent behaviors, making them a valuable tool in theoretical and applied contexts. In particular, we delve into the coupled fractional BVP formulation, which can be described as follows:

$$\begin{cases} D_{0+}^{\varrho, \Psi} \vartheta(\varsigma) = \hat{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)), & \varsigma \in \mathcal{O} = [0, 1] \\ D_{0+}^{\omega, \Psi} \chi(\varsigma) = \check{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)), & \varsigma \in \mathcal{O} = [0, 1], \end{cases} \quad (4)$$

subject to the following mixed boundary conditions:

$$\begin{cases} \vartheta(0) = 0, & \vartheta(1) = P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2), & P_1, P_2 \in \mathbb{R} \\ \chi(0) = 0, & \chi(1) = Q_1 I_{0+}^{\omega_1, \Psi} \vartheta(\eta_1) + Q_2 I_{0+}^{\omega_2, \Psi} \vartheta(\eta_2), & Q_1, Q_2 \in \mathbb{R}, \end{cases} \quad (5)$$

where $D_{0+}^{\theta, \Psi}$ denotes the Ψ –RL fractional derivative of order $\theta \in \{\varrho, \omega, \sigma_1, \sigma_2\}$, $1 < \varrho, \omega < 2$, $0 < \sigma_1, \sigma_2 < 1$, $\hat{\ell}, \check{\ell} \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$, $I_{0+}^{\omega_1, \Psi}, I_{0+}^{\omega_2, \Psi}$ are the Ψ –RL fractional integrals with order $\omega_1 > 0$, $\omega_2 > 0$, respectively, and $0 < \xi_1, \xi_2, \eta_1, \eta_2 < 1$.

The rest of the article is structured as follows: In Section 2, we recall the main tools, preliminaries, and basic definitions with useful auxiliary propositions to be used later. In Section 3, we state and prove the existence and uniqueness of the results for our main problem. To illustrate our results, a series of examples are presented and discussed in Section 4. Finally, a concluding section is given to summarize the main steps of our work.

2. Preliminary Findings and Essential Tools

We will commence this section by introducing the definitions associated with the coupled Ψ –Riemann–Liouville system (4) and (5).

Definition 1 ([1]). For $\varphi : (0, +\infty) \rightarrow \mathbb{R}$, the integral

$$(I_{0+}^{\varrho} \varphi)(\varsigma) = \frac{1}{\Gamma(\varrho)} \int_0^{\varsigma} (\varsigma - s)^{\varrho-1} \varphi(s) ds,$$

is referred to as the Riemann–Liouville fractional integral of order $\varrho > 0$, where $\Gamma(\varrho)$ is the gamma function

Definition 2 ([1]). For a continuous function $\varphi : (0, +\infty) \rightarrow \mathbb{R}$, the equation

$$\mathcal{D}_{0+}^{\varrho} \varphi(\varsigma) = \frac{1}{\Gamma(n - \varrho)} \left(\frac{d}{d\varsigma} \right)^n \int_0^{\varsigma} (\varsigma - s)^{n-\varrho-1} \varphi(s) ds = \left(\frac{d}{d\varsigma} \right)^n I_{0+}^{n-\varrho} \varphi(\varsigma),$$

is called the Riemann–Liouville derivative of order $n - 1 < \varrho \leq n$.

Definition 3 ([31]). Consider a function $\varphi : [a, b] \rightarrow \mathbb{R}$ that is integrable, and let $\Psi \in C^n[a, b]$ be an increasing mapping satisfying the condition $\Psi'(\varsigma) \neq 0$ for all $\varsigma \in [a, b]$. The Ψ –Riemann–Liouville integral of order $\varrho > 0$ for the function φ is defined as:

$$I_{a+}^{\varrho, \Psi} \varphi(\varsigma) = \frac{1}{\Gamma(\varrho)} \int_a^{\varsigma} \Psi'(s) (\Psi(\varsigma) - \Psi(s))^{\varrho-1} \varphi(s) ds.$$

Similarly, the Ψ -Riemann–Liouville derivative of order $q > 0$ for the same function is given by:

$$\begin{aligned} D_{a^+}^{q;\Psi} \varphi(\varsigma) &= \left(\frac{1}{\Psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n I_{a^+}^{n-q;\Psi} \varphi(\varsigma) \\ &= \frac{1}{\Gamma(n-q)} \left(\frac{1}{\Psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n \int_a^\varsigma \Psi'(s) (\Psi(\varsigma) - \Psi(s))^{n-q-1} \varphi(s) ds, \end{aligned}$$

where $n-1 < q \leq n$.

For any positive values of μ and a , the following semigroup specification holds true

$$I_{a^+}^{\mu;\Psi} I_{a^+}^{a;\Psi} u(\varsigma) = I_{a^+}^{\mu+a;\Psi} u(\varsigma).$$

In the case where $\Psi(\varsigma) = \varsigma$, the Ψ -operators given in Definition 3 coincide with the classical Riemann–Liouville integral and derivative, as presented in Definitions 1 and 2, respectively. This establishes a direct correspondence between the Ψ -operators and the well-known Riemann–Liouville fractional operators. Conversely, when $\Psi(\varsigma) = \ln(\varsigma)$, the Ψ -operators return to the Hadamard fractional integral and derivative, as previously defined in references [1,31,32]. This relationship highlights the versatility of Ψ -operators, allowing them to encompass both the classical and Hadamard fractional operators depending on the choice of the weight function $\Psi(\varsigma)$.

Lemma 1. For $1 < q < 2$, and $\vartheta : [a, b] \rightarrow \mathbb{R}$, the following relation holds

$$I_{0^+}^{q;\Psi} D_{0^+}^{q;\Psi} \vartheta(\varsigma) = \vartheta(\varsigma) - a_1(\Psi(\varsigma) - \Psi(0))^{q-1} - a_2(\Psi(\varsigma) - \Psi(0))^{q-2}.$$

Definition 4 ([31]). Let $\Psi \in C^m[a, b]$ with $\Psi'(\varsigma) \neq 0$ for all $\varsigma \in [a, b]$. Then, we present the spaces

$$\mathcal{AC}[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R}, \quad f(x) = f(a) + \int_a^x g(t) dt, \quad g \in L^1([a, b]) \right\}.$$

$$\mathcal{AC}^{m;\Psi}[a, b] = \left\{ g : [a, b] \rightarrow \mathbb{R}, \quad g^{[m-1]} = \left(\frac{1}{\Psi'(\varsigma)} \frac{d}{d\varsigma} \right)^{m-1} g \in \mathcal{AC}[a, b] \right\}.$$

Theorem 1 (Banach's fixed-point theorem [21,31]). Let $(Y, \|\cdot\|)$ be a non-empty Banach space, and let $F : Y \rightarrow Y$ be a contraction mapping. That is, there exists a constant $k \in (0, 1)$ such that for all $x, y \in Y$,

$$\|F(x) - F(y)\| \leq k\|x - y\|.$$

Then, F has a unique fixed point $x^* \in Y$ (i.e., $T(x^*) = x^*$).

Theorem 2 (The Nonlinear Alternative of the Leray–Schauder Type [31]). Let X be a convex closed subset of a Banach space Y . Let B be an open subset of X with $0 \in B$, and let $G : \bar{B} \rightarrow X$ be a continuous and compact mapping.

Then, either:

- (1) The mapping G has a fixed point in $\bar{B}$, or
- (2) There exists $x \in \partial B$ and $0 < \kappa < 1$ such that $x = \kappa Gx$.

Theorem 3 ((Krasnoselskii's fixed-point theorem [22,31])). Let $\mathcal{N}$ be a closed, bounded, convex, and nonempty subset of a Banach space Y . Moreover, let the operators G_1 and G_2 have the properties:

- (a) $G_1x + G_2y \in \mathcal{N}$ for all $x, y \in \mathcal{N}$,
- (b) G_1 is compact and continuous,
- (c) G_2 is a contraction.

Then, $\exists z \in \mathcal{N}$ so that $z = G_1z + G_2z$.

To transform our problem into an equivalent integral equation, we require the following Proposition

Proposition 1. Let $\omega_i > \sigma_i$, for $i = 1, 2$. Then, (ϑ, χ) is a solution of the Problem

$$\begin{cases} D_{0+}^{\varrho, \Psi} \vartheta(\varsigma) = \hat{h}(\varsigma), & \varsigma \in \mathcal{O} = [0, 1], \\ D_{0+}^{\omega, \Psi} \chi(\varsigma) = \check{h}(\varsigma), & \varsigma \in \mathcal{O} = [0, 1], \\ \vartheta(0) = 0, \quad \vartheta(1) = P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2), & P_1, P_2 \in \mathbb{R}, \\ \chi(0) = 0, \quad \chi(1) = Q_1 I_{0+}^{\omega_1, \Psi} \vartheta(\eta_1) + Q_2 I_{0+}^{\omega_2, \Psi} \vartheta(\eta_2), & Q_1, Q_2 \in \mathbb{R}, \end{cases} \quad (6)$$

and it is equivalent to

$$\begin{aligned} \vartheta(\varsigma) = & I_{0+}^{\varrho, \Psi} \hat{h}(\varsigma) + \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{\Lambda} \\ & \times \left[k_1 (\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - I_{0+}^{\varrho, \Psi} \hat{h}(1) \right) \right. \\ & \left. + \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \Delta_1 I_{0+}^{\omega, \Psi} \check{h}(1) \right], \end{aligned} \quad (7)$$

and

$$\begin{aligned} \chi(\varsigma) = & I_{0+}^{\omega, \Psi} \check{h}(\varsigma) + \frac{(\Psi(\varsigma) - \Psi(0))^{\omega-1}}{\Lambda} \left[\frac{Q_1}{k_1} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{Q_2}{k_1} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \frac{1}{k_1} I_{0+}^{\omega, \Psi} \check{h}(1) \right. \\ & \left. + \Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1) \right], \end{aligned} \quad (8)$$

where

$$\Delta_1 = \frac{P_1 \Gamma(\omega)}{\Gamma(\omega - \sigma_1)} (\Psi(\xi_1) - \Psi(0))^{\omega-\sigma_1-1} + \frac{P_2 \Gamma(\omega)}{\Gamma(\omega - \sigma_2)} (\Psi(\xi_2) - \Psi(0))^{\omega-\sigma_2-1}, \quad (9)$$

$$\Delta_2 = \frac{Q_1 \Gamma(\varrho)}{\Gamma(\varrho + \omega_1)} (\Psi(\eta_1) - \Psi(0))^{\varrho+\omega_1-1} + \frac{Q_2 \Gamma(\varrho)}{\Gamma(\varrho + \omega_2)} (\Psi(\eta_2) - \Psi(0))^{\varrho+\omega_2-1}, \quad (10)$$

$$\Lambda = (\Psi(1) - \Psi(0))^{\varrho-1} (\Psi(1) - \Psi(0))^{\omega-1} - \Delta_2 \Delta_1, \quad (11)$$

$$k_1 = (\Psi(1) - \Psi(0))^{1-\varrho}, \quad (12)$$

with $\Lambda \neq 0$ and $k_1 \neq 0$.

Proof. The first implication: (6) implies (7) and (8).

Suppose that (ϑ, χ) is the solution (6). Applying $I_{0+}^{\varrho, \Psi}$ and $I_{0+}^{\omega, \Psi}$ on the first and second D_{0+}^{Ψ} FDEs in (6), respectively, and by Lemma 1, we find

$$\vartheta(\varsigma) = I_{0+}^{\varrho, \Psi} \hat{h}(\varsigma) + a_1 (\Psi(\varsigma) - \Psi(0))^{\varrho-1} + a_2 (\Psi(\varsigma) - \Psi(0))^{\varrho-2}, \quad (13)$$

$$\chi(\varsigma) = I_{0+}^{\omega, \Psi} \check{h}(\varsigma) + b_1 (\Psi(\varsigma) - \Psi(0))^{\omega-1} + b_2 (\Psi(\varsigma) - \Psi(0))^{\omega-2}, \quad (14)$$

with constants a_1 , a_2 , b_1 , and b_2 . By exploiting the boundary conditions $\vartheta(0) = 0$ and $\chi(0) = 0$ in (13) and (14), we obtain $a_2 = 0$ and $b_2 = 0$. By using the Ψ -RL fractional

integral and derivative of order δ_1 and δ_2 , respectively, with $\delta_1 \in \{\omega_1, \omega_2\}$, $\delta_2 \in \{\sigma_1, \sigma_2\}$, we obtain

$$I_{0+}^{\delta_1, \Psi} \vartheta(\eta_1) = I_{0+}^{\varrho+\delta_1, \Psi} \hat{h}(\eta_1) + a_1 \frac{\Gamma(\varrho)}{\Gamma(\varrho+\delta_1)} (\Psi(\eta_1) - \Psi(0))^{\varrho+\delta_1-1},$$

and

$$D_{0+}^{\delta_2, \Psi} \chi(\xi_1) = I_{0+}^{\omega-\delta_2, \Psi} \check{h}(\xi_1) + b_1 \frac{\Gamma(\omega)}{\Gamma(\omega-\delta_2)} (\Psi(\xi_1) - \Psi(0))^{\omega-\delta_2-1}.$$

$$\vartheta(1) = P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2), \quad P_1, P_2 \in \mathbb{R}.$$

By replacing the values $\delta_1 = \omega_1$, $\delta_1 = \omega_2$, $\delta_2 = \sigma_1$, and $\delta_2 = \sigma_2$ and using the second condition of (6), we obtain

$$I_{0+}^{\varrho, \Psi} \hat{h}(1) + a_1 (\Psi(1) - \Psi(0))^{\varrho-1} = P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) + b_1 \Delta_1, \quad (15)$$

and

$$I_{0+}^{\omega, \Psi} \check{h}(1) + b_1 (\Psi(1) - \Psi(0))^{\omega-1} = Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) + a_1 \Delta_2, \quad (16)$$

where Δ_1 and Δ_2 are defined in (9) and (10), respectively, which leads to

$$a_1 = (\Delta_1 \Delta_2 + \Lambda) \frac{P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - I_{0+}^{\varrho, \Psi} \hat{h}(1)}{(\Psi(1) - \Psi(0))^{\varrho-1} \Lambda} \\ + \frac{\Delta_1 Q_2}{\Lambda} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{\Delta_1 Q_2}{\Lambda} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \frac{\Delta_1 I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda},$$

and

$$b_1 = \frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + (\Psi(1) - \Psi(0))^{\varrho-1} Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2)}{\Lambda} \\ - \frac{(\Psi(1) - \Psi(0))^{\varrho-1} I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda} + \frac{\Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1)}{\Lambda}.$$

Upon substituting the prescribed values for the constants a_1 and b_1 into Equations (13) and (14), we derive Equations (7) and (8).

The reverse implication: (7) and (8) implies (6)

We have

$$\vartheta(\varsigma) = I_{0+}^{\varrho, \Psi} \hat{h}(\varsigma) + \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{\Lambda} \\ \times \left[k_1 (\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - I_{0+}^{\varrho, \Psi} \hat{h}(1) \right) \right. \\ \left. + \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \Delta_1 I_{0+}^{\omega, \Psi} \check{h}(1) \right], \quad (17)$$

and

$$\chi(\varsigma) = I_{0+}^{\omega, \Psi} \check{h}(\varsigma) + \frac{(\Psi(\varsigma) - \Psi(0))^{\omega-1}}{\Lambda} \left[\frac{Q_1}{k_1} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{Q_2}{k_1} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) \right. \\ \left. - \frac{1}{k_1} I_{0+}^{\omega, \Psi} \check{h}(1) + \Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1) \right], \quad (18)$$

By using the Ψ -RL fractional derivative of order ϱ and ω on Equations (17) and (18), respectively, we obtain

$$\begin{cases} D_{0+}^{\varrho, \Psi} \vartheta(\varsigma) = \hat{h}(\varsigma), & \varsigma \in \mathcal{O} = [0, 1], \\ D_{0+}^{\omega, \Psi} \chi(\varsigma) = \check{h}(\varsigma), & \varsigma \in \mathcal{O} = [0, 1]. \end{cases} \quad (19)$$

We now check that the integral equations satisfy all three conditions.

1. From (17), we obtain $\vartheta_1(0) = 0$.
2. For the second condition

$$\vartheta(1) = P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2). \quad (20)$$

From Equation (17), we have

$$\begin{aligned} \vartheta(1) &= I_{0+}^{\varrho, \Psi} \hat{h}(1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{\Lambda} \\ &\times \left[k_1(\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - I_{0+}^{\varrho, \Psi} \hat{h}(1) \right) \right. \\ &\left. + \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \Delta_1 I_{0+}^{\omega, \Psi} \check{h}(1) \right]. \end{aligned} \quad (21)$$

Conversely, by using the Ψ -RL fractional derivative of order σ_1 and σ_2 on Equation (18), we obtain

$$\begin{aligned} D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) &= I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \frac{\Gamma(\omega)}{\Gamma(\omega - \sigma_1)} (\Psi(\xi_1) - \Psi(0))^{\omega-\sigma_1-1} \\ &\times \left[\frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_1}{\Lambda} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_2}{\Lambda} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) \right. \\ &\left. - \frac{(\Psi(1) - \Psi(0))^{\varrho-1} I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda} + \frac{\Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1)}{\Lambda} \right], \end{aligned} \quad (22)$$

and

$$\begin{aligned} D_{0+}^{\sigma_2, \Psi} \chi(\xi_2) &= I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) + \frac{\Gamma(\omega)}{\Gamma(\omega - \sigma_2)} (\Psi(\xi_2) - \Psi(0))^{\omega-\sigma_2-1} \\ &\times \left[\frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_1}{\Lambda} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_2}{\Lambda} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) \right. \\ &\left. - \frac{(\Psi(1) - \Psi(0))^{\varrho-1} I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda} + \frac{\Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1)}{\Lambda} \right]. \end{aligned} \quad (23)$$

Substituting the value of $D_{0+}^{\sigma_1, \Psi} \chi(\xi_1)$ and $D_{0+}^{\sigma_2, \Psi} \chi(\xi_2)$ into the following equation

$$\begin{aligned}
P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2) &= P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \frac{P_1 \Gamma(\omega)}{\Gamma(\omega-\sigma_1)} (\Psi(\xi_1) - \Psi(0))^{\omega-\sigma_1-1} \\
&\times \left[\frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_1}{\Lambda} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_2}{\Lambda} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) \right. \\
&\left. - \frac{(\Psi(1) - \Psi(0))^{\varrho-1} I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda} + \frac{\Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1)}{\Lambda} \right] \\
&+ P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) + \frac{P_2 \Gamma(\omega)}{\Gamma(\omega-\sigma_2)} (\Psi(\xi_2) - \Psi(0))^{\omega-\sigma_2-1} \\
&\times \left[\frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_1}{\Lambda} I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1} Q_2}{\Lambda} I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) \right. \\
&\left. - \frac{(\Psi(1) - \Psi(0))^{\varrho-1} I_{0+}^{\omega, \Psi} \check{h}(1)}{\Lambda} + \frac{\Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{h}(1)}{\Lambda} \right].
\end{aligned} \tag{24}$$

Simplifying Equation (24), we obtain

$$\begin{aligned}
P_1 D_{0+}^{\sigma_1, \Psi} \chi(\xi_1) + P_2 D_{0+}^{\sigma_2, \Psi} \chi(\xi_2) &= I_{0+}^{\varrho, \Psi} \hat{h}(1) + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{\Lambda} \\
&\times \left[k_1 (\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{h}(\xi_1) + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{h}(\xi_2) - I_{0+}^{\varrho, \Psi} \hat{h}(1) \right) \right. \\
&\left. + \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{h}(\eta_1) + \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{h}(\eta_2) - \Delta_1 I_{0+}^{\omega, \Psi} \check{h}(1) \right] \\
&= \vartheta(1).
\end{aligned}$$

3. Similarly, we obtain

$$\chi(0) = 0, \chi(1) = Q_1 I_{0+}^{\omega_1, \Psi} \vartheta(\eta_1) + Q_2 I_{0+}^{\omega_2, \Psi} \vartheta(\eta_2).$$

Hence, the proof is completed.

A series of computations demonstrate that the converse aspect of the argument is valid. Hence, the proof is finalized. $\square$

Let us consider $\mathcal{E}$ as the space of continuous functions gifted with the norm $\|\vartheta\| = \max_{\varsigma \in [0,1]} |\vartheta(\varsigma)|$. This makes $\mathcal{E}$ a Banach space. Now, if we take the Cartesian product $\mathcal{E} \times \mathcal{E}$, and define the norm on this space as $\|(\vartheta, \chi)\| = \|\vartheta\| + \|\chi\|$, we obtain another Banach space.

Using Proposition 1, we define the operator $F : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E} \times \mathcal{E}$ as

$$F(\vartheta, \chi)(\varsigma) = (F_1(\vartheta, \chi)(\varsigma), F_2(\vartheta, \chi)(\varsigma)),$$

where

$$\begin{aligned}
F_1(\vartheta, \chi)(\varsigma) &= I_{0+}^{\varrho, \Psi} \hat{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)) + \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{\Lambda} \\
&\times \left[k_1(\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{\ell}(\xi_1, \vartheta(\xi_1), \chi(\xi_1)) \right. \right. \\
&+ P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2)) - I_{0+}^{\varrho, \Psi} \hat{\ell}(1, \vartheta(1), \chi(1)) \Big) \\
&+ \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1)) \\
&+ \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2)) - \Delta_1 I_{0+}^{\varrho, \Psi} \check{\ell}(1, \vartheta(1), \chi(1)) \Big], \\
F_2(\vartheta, \chi)(\varsigma) &= I_{0+}^{\omega, \Psi} \check{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)) + \frac{(\Psi(\varsigma) - \Psi(0))^{\omega-1}}{\Lambda} \left[\frac{Q_1}{k_1} I_{0+}^{\varrho+\omega_1, \Psi} \hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1)) \right. \\
&+ \frac{Q_2}{k_1} I_{0+}^{\varrho+\omega_2, \Psi} \hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2)) - \frac{1}{k_1} I_{0+}^{\omega, \Psi} \check{\ell}(1, \vartheta(1), \chi(1)) \\
&+ \Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{\ell}(\xi_1, \vartheta(\xi_1), \chi(\xi_1)) \\
&+ \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2)) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{\ell}(1, \vartheta(1), \chi(1)) \Big].
\end{aligned}$$

It is worth noting that the problem of finding a fixed point (ϑ, χ) satisfying $F(\vartheta, \chi) = (\vartheta, \chi)$ is equivalent to the nonlinear problem represented by Equations (4) and (5). To simplify the computational process, we introduce the following notation for the sake of convenience:

$$\begin{aligned}
R_1 &= \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} \right. \\
&+ |\Delta_1| |Q_1| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} + \Delta_1 |Q_2| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} \Big], \\
R_2 &= k_1(|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} + |P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \right) \\
&+ |\Delta_1| \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega + 1)}, \\
R_3 &= \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega + 1)} + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \left(\frac{1}{k_1} \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega + 1)} \right. \\
&+ |\Delta_2 P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} + |\Delta_2 P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \Big), \\
R_4 &= \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \left[\frac{|Q_1|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} + \frac{|Q_2|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} \right. \\
&+ |\Delta_2| \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} \Big]. \tag{25}
\end{aligned}$$

3. Results of Existence and Uniqueness

Here, we utilize Banach and Krasnoselskii fixed-point theorems and the Leray–Schauder nonlinear alternative (referenced as [33]) to establish our main results of the existence and uniqueness $(\mathbb{E}, \mathbb{U})$. Together, these two powerful fixed-point theorems provide a solid theoretical foundation for analyzing and characterizing the solutions of (4) and (5).

Theorem 4. Let $\hat{\ell}, \check{\ell} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy the Lipschitz condition, that is, for all $\varsigma \in [0, 1]$ and $\wp_i, \wp_i^* \in \mathbb{R}$, $i = 1, 2$,

$$\begin{aligned} |\hat{\ell}(z, \wp_1, \wp_2) - \hat{\ell}(z, \wp_1^*, \wp_2^*)| &\leq m_1 |\wp_1 - \wp_1^*| + m_2 |\wp_2 - \wp_2^*|, \\ |\check{\ell}(z, \wp_1, \wp_2) - \check{\ell}(z, \wp_1^*, \wp_2^*)| &\leq n_1 |\wp_1 - \wp_1^*| + n_2 |\wp_2 - \wp_2^*|, \end{aligned} \quad (26)$$

with real constants $m_i, n_i, i = 1, 2$. In addition, we assume that

$$1 > (R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2), \quad (27)$$

where $R_i, i \in \{1, 2, 3, 4\}$, are expressed by (25). Thus, the fractional boundary value problem (FBVP) (4) and (5) admits only one solution on $[0, 1]$.

Proof. We define the set $\mathcal{B}_r = \{(\vartheta, \chi) \in \mathcal{E} \times \mathcal{E} : \|(\vartheta, \chi)\| < r\}$, such that

$$\begin{aligned} \frac{(R_1 + R_4)D_1 + (R_2 + R_3)D_2}{1 - (R_2 + R_3)(n_1 + n_2) - (R_1 + R_4)(m_1 + m_2)} &\leq r, \\ \sup_{\varsigma \in [0, 1]} |\hat{\ell}(\varsigma, 0, 0)| = D_1 < \infty \quad \text{and} \quad \sup_{\varsigma \in [0, 1]} |\check{\ell}(\varsigma, 0, 0)| = D_2 < \infty. \end{aligned} \quad (28)$$

Then, under the assumptions (26) and (28), taking (ϑ, χ) , we obtain

$$\begin{aligned} &|F_1(\vartheta, \chi)(\varsigma)| \\ &\leq I_{0+}^{\varrho, \Psi} [|\hat{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)) - \hat{\ell}(\varsigma, 0, 0)| + |\hat{\ell}(\varsigma, 0, 0)|] \\ &+ \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1 (|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| I_{0+}^{\omega-\sigma_1, \Psi} [|\check{\ell}(\xi_1, \vartheta(\xi_1), \chi(\xi_1)) - \right. \right. \\ &\quad \left. \left. - (\xi_1, 0, 0)| + |(\xi_1, 0, 0)|] \right. \right. \\ &+ |P_2| I_{0+}^{\omega-\sigma_2, \Psi} [|\check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2)) - (\xi_2, 0, 0)| + |(\xi_2, 0, 0)|] \\ &+ I_{0+}^{\varrho, \Psi} [\hat{\ell}(1, \vartheta(1), \chi(1)) - \hat{\ell}(1, 0, 0) + \hat{\ell}(1, 0, 0)] \\ &+ |\Delta_1 Q_1| I_{0+}^{\varrho+\omega_1, \Psi} [|\hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1)) - (\eta_1, 0, 0)| + |(\eta_1, 0, 0)|] \\ &+ |\Delta_1 Q_2| I_{0+}^{\varrho+\omega_2, \Psi} [|\hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2)) - (\eta_2, 0, 0)| + |(\eta_2, 0, 0)|] \\ &\left. + |\Delta_1| I_{0+}^{\omega, \Psi} [|\check{\ell}(1, \vartheta(1), \chi(1)) - \check{\ell}(1, 0, 0)| + |\check{\ell}(1, 0, 0)|] \right] \\ &\leq \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\ &+ \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1 (|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \right. \right. \\ &+ |P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) + \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} \\ &\times (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \left. \right) + |\Delta_1 Q_1| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\ &+ |\Delta_1 Q_2| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \end{aligned}$$

$$\begin{aligned}
& + |\Delta_1| \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \Big] \\
& = \left[\frac{(\Psi(1) - \Psi(0))^\varrho}{\Gamma(\varrho + 1)} + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1 (|\Delta_1 \Delta_2| + |\Lambda|) \frac{(\Psi(1) - \Psi(0))^\varrho}{\Gamma(\varrho + 1)} \right. \right. \\
& + |\Delta_1 Q_1| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} + |\Delta_1 Q_2| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} \Big] (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\
& + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1 (|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} \right. \right. \\
& + |P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \Big) + |\Delta_1| \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} \Big] (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& = R_1(m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) + R_2(n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& \leq (R_1 m_1 + R_2 n_1 + R_1 m_2 + R_2 n_2) r + R_1 D_1 + R_2 D_2.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
& |F_2(\vartheta, \chi)(\varsigma)| \\
& \leq \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \\
& \left[\frac{|Q_1|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \right. \\
& + \frac{|Q_2|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\
& + \frac{1}{k_1} \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& + |\Delta_2 P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& + |\Delta_2 P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& + |\Delta_2| \frac{(\Psi(1) - \Psi(0))^\varrho}{\Gamma(\varrho + 1)} (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \Big] \\
& = \left[\frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \left(\frac{1}{k_1} \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} \right. \right. \\
& + |\Delta_2 P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} \\
& + |\Delta_2 P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \Big) \Big] (n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) \\
& + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \left[\frac{|Q_1|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} \right. \\
& + \frac{|Q_2|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} + |\Delta_2| \frac{(\Psi(1) - \Psi(0))^\varrho}{\Gamma(\varrho + 1)} \Big] (m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\
& = R_3(n_1 \|\vartheta\| + n_2 \|\chi\| + D_2) + R_4(m_1 \|\vartheta\| + m_2 \|\chi\| + D_1) \\
& \leq (R_3 n_1 + R_4 m_1 + R_3 n_2 + R_4 m_2) r + R_3 D_2 + R_4 D_1.
\end{aligned}$$

We obtain

$$\begin{aligned}\|F(\vartheta, \chi)\| &= \|(F_1(\vartheta, \chi), F_2(\vartheta, \chi))\| = \|F_1(\vartheta, \chi)\| + \|F_2(\vartheta, \chi)\| \\ &\leq \left[(R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2) \right] r \\ &\quad + (R_1 + R_4)D_1 + (R_2 + R_3)D_2 \leq r,\end{aligned}$$

which yields that $F(\mathcal{B}_r) \subseteq \mathcal{B}_r$, and since $(\vartheta, \chi) \in \mathcal{B}_r$ is an arbitrary element. Conversely, for $(\vartheta_1, \chi_1), (\vartheta_2, \chi_2) \in \mathcal{E} \times \mathcal{E}$ and $\varsigma \in [0, 1]$, we obtain

$$\begin{aligned}&|F_1(\vartheta_2, \chi_2)(\varsigma) - F_1(\vartheta_1, \chi_1)(\varsigma)| \\ &\leq I_{0+}^{\varrho, \Psi} (|\hat{\ell}(\varsigma, \vartheta_2(\varsigma), \chi_2(\varsigma)) - \hat{\ell}(\varsigma, \vartheta_1(\varsigma), \chi_1(\varsigma))|) \\ &+ \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| I_{0+}^{\omega-\sigma_1, \Psi} (|\check{\ell}(\xi_1, \vartheta_2(\xi_1), \chi_2(\xi_1)) \right. \right. \\ &- \check{\ell}(\xi_1, \vartheta_1(\xi_1), \chi_1(\xi_1))|) + |P_2| I_{0+}^{\omega-\sigma_2, \Psi} (|\check{\ell}(\xi_2, \vartheta_2(\xi_2), \chi_2(\xi_2)) - \check{\ell}(\xi_2, \vartheta_1(\xi_2), \chi_1(\xi_2))|) \\ &+ I_{0+}^{\varrho, \Psi} (|\hat{\ell}(1, \vartheta_2(1), \chi_2(1)) - \hat{\ell}(1, \vartheta_1(1), \chi_1(1))|) \\ &+ |\Delta_1 Q_1| I_{0+}^{\varrho+\omega_1, \Psi} (|\hat{\ell}(\eta_1, \vartheta_2(\eta_1), \chi_2(\eta_1)) - \hat{\ell}(\eta_1, \vartheta_1(\eta_1), \chi_1(\eta_1))|) \\ &+ |\Delta_1 Q_2| I_{0+}^{\varrho+\omega_2, \Psi} (|\hat{\ell}(\eta_2, \vartheta_2(\eta_2), \chi_2(\eta_2)) - \hat{\ell}(\eta_2, \vartheta_1(\eta_2), \chi_1(\eta_2))|) \\ &\left. + |\Delta_1| I_{0+}^{\omega, \Psi} (|\check{\ell}(1, \vartheta_2(1), \chi_2(1)) - \check{\ell}(1, \vartheta_1(1), \chi_1(1))|) \right] \\ &\leq (R_1 m_1 + R_2 n_1) \|\vartheta_2 - \vartheta_1\| + (R_1 m_2 + R_2 n_2) \|\chi_2 - \chi_1\|.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|F_1(\vartheta_2, \chi_2)(\varsigma) - F_1(\vartheta_1, \chi_1)(\varsigma)\| & \\ &\leq (R_1 m_1 + R_2 n_1 + R_1 m_2 + R_2 n_2) [\|\vartheta_2 - \vartheta_1\| + \|\chi_2 - \chi_1\|].\end{aligned}\tag{29}$$

Similarly, one can find that

$$\begin{aligned}\|F_2(\vartheta_2, \chi_2)(\varsigma) - F_2(\vartheta_1, \chi_1)(\varsigma)\| & \\ &\leq (R_3 n_1 + R_4 m_1 + R_3 n_2 + R_4 m_2) [\|\vartheta_2 - \vartheta_1\| + \|\chi_2 - \chi_1\|].\end{aligned}\tag{30}$$

Thus, in view of (29) and (30) it follows that,

$$\begin{aligned}\|F(\vartheta_2, \chi_2)(\varsigma) - F(\vartheta_1, \chi_1)(\varsigma)\| & \\ &\leq \left[(R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2) \right] [\|\vartheta_2 - \vartheta_1\| + \|\chi_2 - \chi_1\|].\end{aligned}$$

Considering the condition (27), it can be observed that the mapping F satisfies the criteria for being a contraction. Consequently, from Banach's contraction, we conclude that F possesses a single, distinct fixed point. As a result, the problem (4) and (5) unequivocally possesses one solution within the interval $[0, 1]$. $\square$

Theorem 5. Let $\hat{\ell}, \check{\ell} \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ such that, for all $\varsigma \in [0, 1]$ and $\vartheta_i, \chi_i \in \mathbb{R}, i \in \{1, 2\}$,

$$\begin{aligned}|\hat{\ell}(\varsigma, \vartheta_1, \chi_1)| &\leq \kappa_0 + \kappa_1 |\vartheta_1| + \kappa_2 |\chi_1| \\ |\check{\ell}(\varsigma, \vartheta_2, \chi_2)| &\leq \nu_0 + \nu_1 |\vartheta_2| + \nu_2 |\chi_2|\end{aligned}\tag{31}$$

with real constants $\kappa_i, v_i, i \in \{0, 1, 2\}$, with $\kappa_i, v_i > 0$. Then, the System (4) and (5) has at least one solution on $[0, 1]$ provided that

$$(R_1 + R_4)\kappa_1 + (R_2 + R_3)v_1 < 1 \quad \text{and} \quad (R_1 + R_4)\kappa_2 + (R_2 + R_3)v_2 < 1,$$

such that $R_i, i \in \{1, 2, 3, 4\}$, are given by (25).

Proof. Note that, since $\hat{\ell}$ and $\check{\ell}$ are continuous, it follows that the mapping F is continuous as well. Moreover, we can demonstrate that F is also completely continuous.

Let us consider a bounded set S in $\mathcal{E} \times \mathcal{E}$. Thus, within this set, there exist positive constants L_1 and L_2 such that

$$|\hat{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma))| \leq L_1, \quad |\check{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma))| \leq L_2, \quad (\vartheta, \chi) \in S.$$

Therefore, for all $(\vartheta, \chi) \in S$, we obtain

$$\begin{aligned} |F_1(\vartheta, \chi)(\varsigma)| &\leq \left[\frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho + 1)} \right. \right. \\ &+ |\Delta_1 Q_1| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} + |\Delta_1 Q_2| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} \left. \left. \right] \right] L_1 \\ &+ \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} \right. \right. \\ &+ |P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \left. \left. \right) + |\Delta_1| \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega + 1)} \right] L_2 \\ &= R_1 L_1 + R_2 L_2, \end{aligned}$$

which yields

$$\|F_1(\vartheta, \chi)(\varsigma)\| \leq R_1 L_1 + R_2 L_2.$$

Analogously, one can obtain

$$\|F_2(\vartheta, \chi)(\varsigma)\| \leq R_3 L_1 + R_4 L_2.$$

Hence,

$$\|F(\vartheta, \chi)(\varsigma)\| \leq (R_1 + R_3)L_1 + (R_2 + R_4)L_2.$$

As a result, the operator F exhibits uniform boundedness. To demonstrate the equicontinuity of F , consider ς_1 and ς_2 in the interval $[0, 1]$ such that $\varsigma_1 < \varsigma_2$. Consequently, we obtain

$$\begin{aligned} &|F_1(\vartheta, \chi)(\varsigma_2) - F_1(\vartheta, \chi)(\varsigma_1)| \\ &\leq \frac{1}{\Gamma(\varrho)} \left| \int_0^{\varsigma_1} [\Psi(s)'(\Psi(\varsigma_2) - \Psi(s))^{\varrho-1} - \Psi(s)'(\Psi(\varsigma_1) - \Psi(s))^{\varrho-1}] \hat{\ell}(s, \vartheta(s), \chi(s)) ds \right. \\ &+ \frac{1}{\Gamma(\varrho)} \int_{\varsigma_1}^{\varsigma_2} [\Psi(s)(\Psi(\varsigma_2) - \Psi(s))^{\varrho-1}] \hat{\ell}(s, \vartheta(s), \chi(s)) ds \left. \right| \\ &+ \frac{(\Psi(\varsigma_2) - \Psi(0))^{\varrho-1} - (\Psi(\varsigma_1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \\ &\times \left[|k_1(\Delta_1 \Delta_2 + \Lambda)| \left(|P_1| I_{0+}^{\omega-\sigma_1, \Psi} |\check{\ell}(\tilde{\varsigma}_1, \vartheta(\tilde{\varsigma}_1), \chi(\tilde{\varsigma}_1))| \right) \right] \end{aligned}$$

$$\begin{aligned}
& + |P_2| I_{0+}^{\omega-\sigma_2, \Psi} |\check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2))| + I_{0+}^{\varrho, \Psi} |\hat{\ell}(1, \vartheta(1), \chi(1))| \\
& + |\Delta_1 Q_1| I_{0+}^{\varrho+\omega_1, \Psi} |\hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1))| \\
& + |\Delta_1 Q_2| I_{0+}^{\varrho+\omega_2, \Psi} |\hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2))| + |\Delta_1| I_{0+}^{\omega, \Psi} |\check{\ell}(1, \vartheta(1), \chi(1))| \Big] \\
& \leq \frac{L_1}{\Gamma(\varrho+1)} \left[2|(\Psi(\xi_2) - \Psi(\xi_1))^{\varrho-1}| + |(\Psi(\xi_2) - \Psi(0))^{\varrho-1} - (\Psi(\xi_1) - \Psi(0))^{\varrho-1}| \right] \\
& + \frac{(\Psi(\xi_2) - \Psi(0))^{\varrho-1} - (\Psi(\xi_1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(\Delta_1 \Delta_2 + \Lambda) \left(L_2 |P_1| \frac{(\Psi(\xi_1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega-\sigma_1+1)} \right. \right. \\
& + L_2 |P_2| \frac{(\Psi(\xi_2) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega-\sigma_2+1)} + L_1 \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho+1)} \Big) + L_1 |\Delta_1 Q_1| \frac{(\Psi(\xi_1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho+\omega_1+1)} \\
& + L_1 |\Delta_1 Q_2| \frac{(\Psi(\xi_1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho+\omega_2+1)} + L_2 |\Delta_1| \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega+1)} \Big] \rightarrow 0 \text{ as } \xi_2 - \xi_1 \rightarrow 0,
\end{aligned}$$

independently of $(\vartheta, \chi) \in S$. Hence, $F_1(\vartheta, \chi)$ is equicontinuous. Similarly, we can establish the equicontinuity of $F_2(\vartheta, \chi)$. Hence, based on the preceding arguments, we can conclude that the operator $F(\vartheta, \chi)$ is completely continuous.

Finally, we will demonstrate that the set

$$\mathcal{D} = \{(\vartheta, \chi) \in \mathcal{E} \times \mathcal{E} : (\vartheta, \chi) = wF(\vartheta, \chi), 0 \leq w \leq 1\},$$

is bounded. Let $(\vartheta, \chi) \in \mathcal{D}$, then $(\vartheta, \chi) = wF(\vartheta, \chi)$ for all $\varsigma \in [0, 1]$ and that

$$\vartheta(\varsigma) = wF_1(\vartheta, \chi)(\varsigma), \quad \chi(\varsigma) = wF_2(\vartheta, \chi)(\varsigma).$$

Thus, we have

$$\begin{aligned}
|\hat{\ell}(\varsigma, \vartheta_1, \chi_1)| & \leq \kappa_0 + \kappa_1 |\vartheta_1| + \kappa_2 |\chi_1|, \\
|\check{\ell}(\varsigma, \vartheta_2, \chi_2)| & \leq \nu_0 + \nu_1 |\vartheta_1| + \nu_2 |\chi_2|,
\end{aligned}$$

$$\begin{aligned}
|\vartheta| & \leq \left[\frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \frac{(\Psi(1) - \Psi(0))^{\varrho}}{\Gamma(\varrho+1)} \right. \right. \\
& + |\Delta_1 Q_1| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho+\omega_1+1)} + |\Delta_1 Q_2| \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho+\omega_2+1)} \Big] \Big] (\kappa_0 + \kappa_1 |\vartheta_1| + \kappa_2 |\chi|) \\
& + \frac{(\Psi(1) - \Psi(0))^{\varrho-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega-\sigma_1+1)} \right. \right. \\
& + |P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega-\sigma_2+1)} \Big) + |\Delta_1| \frac{(\Psi(1) - \Psi(0))^{\omega}}{\Gamma(\omega+1)} \Big] (\nu_0 + \nu_1 |\vartheta_1| + \nu_2 |\chi|),
\end{aligned}$$

$$\begin{aligned}
|\chi| \leq & \left[\frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{\Lambda} \left(\frac{1}{k_1} \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega + 1)} \right. \right. \\
& + |\Delta_2 P_1| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_1}}{\Gamma(\omega - \sigma_1 + 1)} \\
& \left. \left. + |\Delta_2 P_2| \frac{(\Psi(1) - \Psi(0))^{\omega-\sigma_2}}{\Gamma(\omega - \sigma_2 + 1)} \right) \right] (\nu_0 + \nu_1 |\vartheta_1| + \nu_2 |\chi|) + \frac{(\Psi(1) - \Psi(0))^{\omega-1}}{|\Lambda|} \\
& \times \left[\frac{|Q_1|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_1}}{\Gamma(\varrho + \omega_1 + 1)} + \frac{|Q_2|}{k_1} \frac{(\Psi(1) - \Psi(0))^{\varrho+\omega_2}}{\Gamma(\varrho + \omega_2 + 1)} + |\Delta_2| \frac{(\Psi(1) - \Psi(0))^\varrho}{\Gamma(\varrho + 1)} \right] \\
& \times (\kappa_0 + \kappa_1 |\vartheta_1| + \kappa_2 |\chi|).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|\vartheta\| + \|\chi\| \leq & (R_1 + R_4)\kappa_0 + (R_2 + R_3)\nu_0 + [(R_1 + R_4)\kappa_1 + (R_2 + R_3)\nu_1] \|\vartheta\| \\
& + [(R_1 + R_4)\kappa_2 + (R_2 + R_3)\nu_2] \|\chi\|,
\end{aligned}$$

which can be expressed as

$$\|(\vartheta, \chi)\| \leq \frac{(R_1 + R_4)\kappa_0 + (R_2 + R_3)\nu_0}{\mathbb{M}_0},$$

where

$$\mathbb{M}_0 = \min\{1 - (R_1 + R_4)\kappa_1 - (R_2 + R_3)\nu_1, 1 - (R_1 + R_4)\kappa_2 - (R_2 + R_3)\nu_2\}.$$

Consequently, the Leray–Schauder alternative (Theorem 2) can be invoked, leading to the conclusion that the operator F possesses at least one fixed point. As a result, System (4) and (5) exhibits at least one solution on the interval $[0, 1]$. $\square$

Theorem 6. Consider the assumptions that $\hat{\ell}$ and $\check{\ell}$ are continuous functions defined on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, satisfying Condition (31) of Theorem 5. Additionally, we assume that:
(Con₁) There exist P and $Q \in C([0, 1], \mathbb{R}_+)$ such that

$$|\hat{\ell}(\varsigma, \vartheta, \chi)| \leq P(\varsigma), \quad \check{\ell}(\varsigma, \vartheta, \chi) \leq Q(\varsigma), \quad \text{for each } (\varsigma, \vartheta, \chi) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

Then, the problem (4) and (5) possesses at least one solution on $[0, 1]$, provided that

$$(R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2) < 1. \quad (32)$$

Proof. To start the decomposition process, we will express the operator F as a combination of four individual operators: $F_{1,1}$, $F_{1,2}$, $F_{2,1}$, and $F_{2,2}$, as

$$\begin{aligned}
F_{1,1}(\vartheta, \chi)(\varsigma) &= I_{0+}^{\varrho, \Psi} \hat{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)), \quad \varsigma \in [0, 1], \\
F_{1,2}(\vartheta, \chi)(\varsigma) &= \frac{(\Psi(\varsigma) - \Psi(0))^{\varrho-1}}{\Lambda} \left[k_1(\Delta_1 \Delta_2 + \Lambda) \left(P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{\ell}(\xi_1, \vartheta(\xi_1), \chi(\xi_1)) \right. \right. \\
&\quad + P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2)) - I_{0+}^{\varrho, \Psi} \hat{\ell}(1, \vartheta(1), \chi(1)) \Big) \\
&\quad + \Delta_1 Q_1 I_{0+}^{\varrho+\omega_1, \Psi} \hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1)) \\
&\quad + \Delta_1 Q_2 I_{0+}^{\varrho+\omega_2, \Psi} \hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2)) - \Delta_1 I_{0+}^{\omega, \Psi} \check{\ell}(1, \vartheta(1), \chi(1)) \Big], \quad \varsigma \in [0, 1], \\
F_{2,1}(\vartheta, \chi)(\varsigma) &= I_{0+}^{\omega, \Psi} \check{\ell}(\varsigma, \vartheta(\varsigma), \chi(\varsigma)), \quad \varsigma \in [0, 1], \\
F_{2,2}(\vartheta, \chi)(\varsigma) &= \frac{(\Psi(\varsigma) - \Psi(0))^{\omega-1}}{\Lambda} \left[\frac{Q_1}{k_1} I_{0+}^{\varrho+\omega_1, \Psi} \hat{\ell}(\eta_1, \vartheta(\eta_1), \chi(\eta_1)) \right. \\
&\quad + \frac{Q_2}{k_1} I_{0+}^{\varrho+\omega_2, \Psi} \hat{\ell}(\eta_2, \vartheta(\eta_2), \chi(\eta_2)) - \frac{1}{k_1} I_{0+}^{\omega, \Psi} \check{\ell}(1, \vartheta(1), \chi(1)) \\
&\quad + \Delta_2 P_1 I_{0+}^{\omega-\sigma_1, \Psi} \check{\ell}(\xi_1, \vartheta(\xi_1), \chi(\xi_1)) \\
&\quad + \Delta_2 P_2 I_{0+}^{\omega-\sigma_2, \Psi} \check{\ell}(\xi_2, \vartheta(\xi_2), \chi(\xi_2)) - \Delta_2 I_{0+}^{\varrho, \Psi} \hat{\ell}(1, \vartheta(1), \chi(1)) \Big], \quad \varsigma \in [0, 1].
\end{aligned}$$

Observe that $F_1 = F_{1,1} + F_{1,2}$ and $F_2 = F_{2,1} + F_{2,2}$. Consider the set

$$B_\mu = \{(\vartheta, \chi) \in \mathcal{E} \times \mathcal{E} : \|(\vartheta, \chi)\| \leq \mu\},$$

with $(R_1 + R_4)\|P\| + (R_2 + R_3)\|Q\| \leq \mu$. Similarly to the proof of Theorem 5, one can derive:

$$|F_{1,1}(\vartheta, \chi)(\varsigma) + F_{1,2}(\vartheta, \chi)(\varsigma)| \leq R_1\|P\| + R_2\|Q\|,$$

and

$$|F_{2,1}(\vartheta, \chi)(\varsigma) + F_{2,2}(\vartheta, \chi)(\varsigma)| \leq R_4\|P\| + R_3\|Q\|.$$

Therefore, we obtain

$$\|F_1(\vartheta, \chi) + F_2(\vartheta, \chi)\| \leq \|P\|(R_1 + R_4) + \|Q\|(R_2 + R_3) < \mu.$$

Consequently, $F_1(\vartheta, \chi) + F_2(\vartheta, \chi) \in B_\mu$. Next, it will be established that the operator $(F_{1,2}, F_{2,2})$ is a contraction. As demonstrated during the proof of Theorem 4, for $(\vartheta_1, \chi_1), (\vartheta_2, \chi_2) \in B_\mu$, one can find that

$$\begin{aligned}
& |F_{1,2}(\vartheta_2, \chi_2)(\varsigma) - F_{1,2}(\vartheta_1, \chi_1)(\varsigma)| \\
& \leq \frac{(\Psi(\varsigma) - \Psi(0))^{q-1}}{|\Lambda|} \left[k_1(|\Delta_1 \Delta_2| + |\Lambda|) \left(|P_1| I_{0+}^{\omega-\sigma_1, \Psi} (|\check{\ell}(\xi_1, \vartheta_2(\xi_1), \chi_2(\xi_1)) \right. \right. \\
& - \check{\ell}(\xi_1, \vartheta_1(\xi_1), \chi_1(\xi_1))|) \\
& + |P_2| I_{0+}^{\omega-\sigma_2, \Psi} (|\check{\ell}(\xi_2, \vartheta_2(\xi_1), \chi_2(\xi_2)) - \check{\ell}(\xi_2, \vartheta_1(\xi_1), \chi_1(\xi_2))|) \\
& + I_{0+}^{q, \Psi} (|\hat{\ell}(1, \vartheta_2(1), \chi_2(1)) - \hat{\ell}(1, \vartheta_1(1), \chi_1(1))|) \\
& + |\Delta_1 Q_1| I_{0+}^{q+\omega_1, \Psi} (|\hat{\ell}(\eta_1, \vartheta_2(\eta_1), \chi_2(\eta_1)) - \hat{\ell}(\eta_1, \vartheta_1(\eta_1), \chi_1(\eta_1))|) \\
& + |\Delta_1 Q_2| I_{0+}^{q+\omega_2, \Psi} (|\hat{\ell}(\eta_2, \vartheta_2(\eta_2), \chi_2(\eta_2)) - \hat{\ell}(\eta_2, \vartheta_1(\eta_2), \chi_1(\eta_2))|) \\
& + |\Delta_1| I_{0+}^{\omega, \Psi} (|\check{\ell}(1, \vartheta_2(1), \chi_2(1)) - \check{\ell}(1, \vartheta_1(1), \chi_1(1))|) \Big] \\
& \leq (R_1 m_1 + R_2 n_1) \|\vartheta_2 - \vartheta_1\| + (R_1 m_2 + R_2 n_2) \|\chi_2 - \chi_1\|, \tag{33}
\end{aligned}$$

and

$$\begin{aligned}
& |F_{2,2}(\vartheta_2, \chi_2)(\varsigma) - F_{2,2}(\vartheta_1, \chi_1)(\varsigma)| \\
& \leq (R_3 n_1 + R_4 m_1) \|\vartheta_2 - \vartheta_1\| + (R_3 n_2 + R_4 m_2) \|\chi_2 - \chi_1\| \tag{34}
\end{aligned}$$

From Equations (33) and (34), we obtain

$$\begin{aligned}
& \|(F_{1,2}, F_{2,2})(\vartheta_2, \chi_2) - (F_{1,2}, F_{2,2})(\vartheta_1, \chi_1)(\varsigma)\| \\
& \leq ((R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2)) (\|\vartheta_2 - \vartheta_1\| + \|\chi_2 - \chi_1\|).
\end{aligned}$$

This indicates that the operator $(F_{1,2}, F_{2,2})$ can be regarded as a contraction, thanks to the satisfaction of Condition (32). Moreover, considering the continuity properties of $\hat{\ell}$ and $\check{\ell}$, we can conclude that the operator $(F_{1,1}, F_{2,1})$ is continuous. also, we have:

$$\|(F_{1,1}, F_{2,1})(\vartheta, \chi)\| \leq \frac{(\Psi(1) - \Psi(0))^q}{\Gamma(q+1)} \|P\| + \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega+1)} \|Q\|,$$

as

$$\|F_{1,1}(\vartheta, \chi)\| \leq \frac{(\Psi(1) - \Psi(0))^q}{\Gamma(q+1)} \|P\|,$$

and

$$\|F_{2,1}(\vartheta, \chi)\| \leq \frac{(\Psi(1) - \Psi(0))^\omega}{\Gamma(\omega+1)} \|Q\|.$$

Thus, $(F_{1,1}, F_{2,1})B_\mu$ is uniformly bounded.

In the next step, we establish that the set $(F_{1,1}, F_{2,1})B_\mu$ is equicontinuous. For $\varsigma_1, \varsigma_2 \in [0, 1]$, $\varsigma_1 < \varsigma_2$ and for all $(\vartheta, \chi) \in B_\mu$, we have

$$\begin{aligned}
& |F_{1,1}(\vartheta(\varsigma_2), \chi(\varsigma_2)) - F_{1,1}(\vartheta(\varsigma_1), \chi(\varsigma_1))| \\
& \leq \frac{1}{\Gamma(\varrho)} \left| \int_0^{\varsigma_1} [\Psi(s)(\Psi(\varsigma_2) - \Psi(s))^{\varrho-1} - \Psi(s)(\Psi(\varsigma_1) - \Psi(s))^{\varrho-1}] \hat{\ell}(s, \vartheta(s), \chi(s)) ds \right. \\
& \quad \left. + \frac{1}{\Gamma(\varrho)} \int_{\varsigma_1}^{\varsigma_2} [\Psi(s)(\Psi(\varsigma_2) - \Psi(s))^{\varrho-1}] \hat{\ell}(s, \vartheta(s), \chi(s)) ds \right| \\
& \leq \frac{\|P\|}{\Gamma(\varrho+1)} \left| [2|(\Psi(\varsigma_2) - \Psi(\varsigma_1))^{\varrho-1}| + |(\Psi(\varsigma_2) - \Psi(0))^{\varrho-1} - (\Psi(\varsigma_1) - \Psi(0))^{\varrho-1}|] \right| \rightarrow 0 \\
& \text{as } \varsigma_2 - \varsigma_1 \rightarrow 0,
\end{aligned}$$

independently of $(\vartheta, \chi) \in B_\mu$. Analogously, one can obtain that

$$F_{2,1}(\vartheta(\varsigma_2), \chi(\varsigma_2)) - F_{2,1}(\vartheta(\varsigma_1), \chi(\varsigma_1)) \rightarrow 0, \text{ as } \varsigma_1 \rightarrow \varsigma_2.$$

Consequently, as ς_1 approaches ς_2 , then $|(F_{1,1}, F_{2,1})(\vartheta, \chi)(\varsigma_2) - (F_{1,1}, F_{2,1})(\vartheta, \chi)(\varsigma_1)| \rightarrow 0$. This implies that $(F_{1,1}, F_{2,1})$ exhibits equicontinuity. By applying the Arzelà–Ascoli theorem, we can deduce that the operator $(F_{1,1}, F_{2,1})$ is compact on B_μ . Consequently, the conditions required by the Krasnoselskii fixed-point theorem (Theorem 3) are satisfied. Thus, we can conclude that the System (4) and (5) possesses at least one solution on the interval $[0, 1]$. $\square$

4. Illustrative Examples

Example 1. Consider the FBVP:

$$\begin{cases} D_{0+}^{1.84, 1+\varsigma^3} \vartheta(\varsigma) = \frac{1}{\varsigma+2023} \frac{|\vartheta+\chi|}{|\vartheta+\chi|+1} + \frac{1}{2024} \varsigma + 0.2025, & \varsigma \in [0, 1], \\ D_{0+}^{1.22, 1+\varsigma^3} \chi(\varsigma) = \frac{1}{5\varsigma^2+50} \sin\left(\frac{|\vartheta+\chi|}{|\vartheta+\chi|+1}\right) + \exp(\varsigma+2000) + 0.2222, & \varsigma \in [0, 1], \end{cases} \quad (35)$$

supplemented with the mixed boundary conditions

$$\begin{cases} \vartheta(0) = 0, & \vartheta(1) = 5D_{0+}^{0.18, 1+\varsigma^3} \chi(0.3) + 20.5D_{0+}^{0.27, 1+\varsigma^3} \chi(0.4), \\ \chi(0) = 0, & \chi(1) = Q_1 I_{0+}^{5.33, 1+\varsigma^3} \vartheta(0.2) + Q_2 I_{0+}^{10.22, 1+\varsigma^3} \vartheta(0.2), \end{cases} \quad (36)$$

Here, $\rho = 1.84$, $\omega = 1.22$, $\omega_1 = 5.33$, $\omega_2 = 10.22$, $\sigma_1 = 0.18$, $\sigma_2 = 0.27$, $Q_1 = 20$, $Q_2 = 10$, $P_1 = 5$, $P_2 = 20.5$, $\eta_1 = 0.2$, $\eta_2 = 0.2$, $\xi_1 = 0.3$, $\xi_2 = 0.4$, $\Psi = 1 + \varsigma^3$.

A simple computation gives:

$$\mathbb{K} = 1, R_1 \approx 1.1716, R_2 \approx 13.3695, R_3 \approx 1.7953, R_4 \approx 0.0028.$$

Consider the nonlinear Lipschitzian unbounded functions $\hat{\ell}$ and $\check{\ell}$ defined on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ as follows:

$$\hat{\ell}(\varsigma, \vartheta, \chi) = \frac{1}{\varsigma+2023} \frac{|\vartheta+\chi|}{|\vartheta+\chi|+1} + \frac{1}{2024} \varsigma + 0.2025, \quad (37)$$

$$\check{\ell}(\varsigma, \vartheta, \chi) = \frac{1}{5\varsigma^2+50} \sin\left(\frac{|\vartheta+\chi|}{|\vartheta+\chi|+1}\right) + \exp(\varsigma+2000) + 0.2222, \quad (38)$$

which satisfy the Lipschitz condition:

$$|\hat{\ell}(\varsigma, \vartheta_1, \chi_1) - \hat{\ell}(\varsigma, \vartheta_2, \chi_2)| \leq \frac{1}{2023} |\vartheta_1 - \vartheta_2| + \frac{1}{2023} |\chi_1 - \chi_2|,$$

$$|\check{\ell}(\varsigma, \vartheta_1, \chi_1) - \check{\ell}(\varsigma, \vartheta_2, \chi_2)| \leq \frac{1}{50} |\vartheta_1 - \vartheta_2| + \frac{1}{50} |\chi_1 - \chi_2|,$$

with Lipschitz constants $m_1 = m_2 = 1/2023$, $n_1 = n_2 = 1/50$. Furthermore,

$$(R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2) \approx 0.6077 < 1.$$

Hence, the conditions of Theorem 4 are verified and it implies that Problem (35) and (36) possesses one solution on $[0, 1]$.

Example 2. Let us consider the system:

$$\begin{cases} D_{0+}^{1.91, 1+\zeta^2} \vartheta(\zeta) = \frac{\pi}{\zeta+2\pi^2} + \frac{1}{5\pi^2} \vartheta + \frac{3}{(1+\pi)^4} \sin(|\chi|), & \zeta \in [0, 1], \\ D_{0+}^{1.44, 1+\zeta^2} \chi(\zeta) = \frac{1+\cos^2(\vartheta\chi)}{\zeta+2\pi^2} + \frac{\vartheta^2 \exp(-|\chi|)}{5(1+\vartheta)} + \frac{\chi(1+\cos^8(\vartheta))}{(\zeta^5+20)}, & \zeta \in [0, 1], \end{cases} \quad (39)$$

subject with the mixed boundary conditions

$$\begin{cases} \vartheta(0) = 0, & \vartheta(1) = 0.77D_{0+}^{0.28, 1+\zeta^2} \chi(0.39) + 0.88D_{0+}^{0.37, 1+\zeta^2} \chi(0.47), \\ \chi(0) = 0, & \chi(1) = 0.44I_{0+}^{51.33, 1+\zeta^2} \vartheta(0.19) + 0.99I_{0+}^{15.22, 1+\zeta^2} \vartheta(0.29). \end{cases} \quad (40)$$

Here, $\rho = 1.91$, $\omega = 1.44$, $\omega_1 = 51.33$, $\omega_2 = 15.22$, $\sigma_1 = 0.28$, $\sigma_2 = 0.37$, $Q_1 = 0.44$, $Q_2 = 0.99$, $P_1 = 0.77$, $P_2 = 0.88$, $\eta_1 = 0.19$, $\eta_2 = 0.29$, $\xi_1 = 0.39$, $\xi_2 = 0.47$, $\Psi = 1 + \zeta^2$.

A simple computation leads to:

$$\mathbb{K} = 1, R_1 \approx 1.0848, R_2 \approx 2.5626, R_3 \approx 1.5679, R_4 \approx 1.9175 \times 10^{-15}$$

Consider the functions $\hat{\ell}$ and $\check{\ell}$ defined as nonlinear Lipschitzian unbounded functions on the domain $[0, 1] \times \mathbb{R} \times \mathbb{R}$, given by:

$$\hat{\ell}(\zeta, \vartheta, \chi) = \frac{\pi}{\zeta+2\pi^2} + \frac{1}{5\pi^2} \vartheta + \frac{3}{(1+\pi)^4} \sin(|\chi|), \quad (41)$$

$$\check{\ell}(\zeta, \vartheta, \chi) = \frac{1+\cos^2(\vartheta\chi)}{\zeta+2\pi^2} + \frac{\vartheta^2 \exp(-|\chi|)}{5(1+\vartheta)} + \frac{\chi(1+\cos^8(\vartheta))}{(\zeta^5+20)}. \quad (42)$$

Clearly, $|\hat{\ell}(\zeta, \vartheta, \chi)| \leq \frac{1}{2\pi} + \frac{1}{5\pi^2} |\vartheta| + \frac{3}{(1+\pi)^4} |\chi|$ and $|\check{\ell}(\zeta, \vartheta, \chi)| = \frac{1}{\pi} + \frac{1}{5} |\vartheta| + \frac{1}{10} |\chi|$ with $\kappa_0 = \frac{1}{2\pi}$, $\kappa_1 = \frac{1}{5\pi^2}$, $\kappa_2 = \frac{3}{(1+\pi)^4}$, $\nu_0 = \frac{1}{2\pi^2}$, $\nu_1 = \frac{1}{5}$, $\nu_2 = \frac{1}{10}$. Moreover,

$$(R_1 + R_4)\kappa_1 + (R_2 + R_3)\nu_1 \approx 0.8532 < 1 \quad \text{and} \quad (R_1 + R_4)\kappa_2 + (R_2 + R_3)\nu_2 \approx 0.4267 < 1.$$

Consequently, based on the result obtained from Theorem 5, we can conclude that Problem (39) and (40) possesses at least one solution on the interval $[0, 1]$.

Example 3. Consider the following FBVP:

$$\begin{cases} D_{0+}^{1.33, 1+\zeta^4} \vartheta(\zeta) = \frac{\exp(-\zeta^2-1) \arctan(\vartheta)}{45} + \frac{1}{\pi^5} \sin(\chi) + \frac{1}{22} \zeta + \frac{1}{\pi^5}, & \zeta \in [0, 1], \\ D_{0+}^{1.04, 1+\zeta^4} \chi(\zeta) = \frac{|\vartheta|}{20(|\vartheta|+1)} + \frac{|\chi|}{50|\chi|+50} + \exp(\zeta^2+2) + \frac{3}{100}, & \zeta \in [0, 1], \end{cases} \quad (43)$$

subject with the mixed boundary conditions

$$\begin{cases} \vartheta(0) = 0, & \vartheta(1) = 1.77D_{0+}^{0.47, 1+\zeta^4} \chi(0.89) + 1.88D_{0+}^{0.77, 1+\zeta^4} \chi(0.77), \\ \chi(0) = 0, & \chi(1) = 0.77I_{0+}^{33.33, 1+\zeta^4} \vartheta(0.69) + 0.88I_{0+}^{22.22, 1+\zeta^4} \vartheta(0.69). \end{cases} \quad (44)$$

Here, $\rho = 1.33$, $\omega = 1.04$, $\omega_1 = 33.33$, $\omega_2 = 22.22$, $\sigma_1 = 0.47$, $\sigma_2 = 0.87$, $Q_1 = 0.77$, $Q_2 = 0.88$, $P_1 = 1.77$, $P_2 = 1.88$, $\eta_1 = 0.69$, $\eta_2 = 0.79$, $\xi_1 = 0.89$, $\xi_2 = 0.77$, $\Psi = 1 + \zeta^4$.

By a simple computation, we obtain

$$\mathbb{K} = 1, R_1 \approx 1.6832, R_2 \approx 6.1372, R_3 \approx 1.9654, R_4 \approx 5.9581 \times 10^{-24}.$$

Consider the functions $\hat{\ell}$ and $\check{\ell}$ defined as nonlinear Lipschitzian unbounded functions on the domain $[0, 1] \times \mathbb{R} \times \mathbb{R}$, given by:

$$\begin{aligned}\hat{\ell}(\varsigma, \vartheta, \chi) &= \frac{\exp(-\varsigma^2 - 1) \arctan(\vartheta)}{45} + \frac{1}{\pi^5} \sin(\chi) + \frac{1}{22} \varsigma + \frac{1}{\pi^5}, \\ \check{\ell}(\varsigma, \vartheta, \chi) &= \frac{|\vartheta|}{20(|\vartheta| + 1)} + \frac{|\chi|}{50|\chi| + 50} + \exp(\varsigma^2 + 2) + \frac{3}{100}.\end{aligned}$$

Then, we have

$$\begin{aligned}|\hat{\ell}(\varsigma, \vartheta, \chi)| &= \frac{\exp(-\varsigma^2 - 1)}{45} + \frac{2}{\pi^5} + \frac{1}{22} \varsigma, \\ |\check{\ell}(\varsigma, \vartheta, \chi)| &= \frac{1}{10} + \exp(\varsigma^2 + 2),\end{aligned}$$

and

$$\begin{aligned}|\hat{\ell}(\varsigma, \vartheta_1, \chi_1) - \check{\ell}(\varsigma, \vartheta_2, \chi_2)| &\leq \frac{1}{45} |\vartheta_1 - \vartheta_2| + \frac{1}{\pi^5} |\chi_1 - \chi_2|, \\ |\check{\ell}(\varsigma, \vartheta_1, \chi_1) - \check{\ell}(\varsigma, \vartheta_2, \chi_2)| &\leq \frac{1}{20} |\vartheta_1 - \vartheta_2| + \frac{1}{50} |\chi_1 - \chi_2|,\end{aligned}$$

with Lipschitz constants $m_1 = 1/45, m_2 = 1/\pi^5, n_1 = 1/20, n_2 = 1/50$.

Furthermore, $(R_2 + R_3)(n_1 + n_2) + (R_1 + R_4)(m_1 + m_2) \approx 0.6218 < 1$. Consequently, according to the statement in Theorem 6, we can deduce that Problem (43) and (44) possesses at least one solution within the interval $[0, 1]$.

5. Conclusions and Perspectives

In this research, we have proven the existence, uniqueness, and stability of solutions for a coupled fractional differential system characterized by mixed boundary conditions and incorporating fractional ψ -Riemann–Liouville operators. To solve this problem, we used a fixed-point approach by reformulating the original problem into an equivalent fixed-point problem, which allowed us to exploit specific fixed-point theorems due to Banach, Krasnoselskii, and Leray–Schauder, which are particularly appropriate for our analysis. It is important to note that the problem we investigated introduces a unique setup that extends the scope of traditional studies. The results of this work have broad applicability, offering insights that go beyond the limitations of previously investigated cases.

The techniques and results presented in this paper can be potentially generalized to other types of fractional differential equations and boundary conditions. Future research could explore these generalizations to broaden the applicability of the methods, while the paper focuses on analytical results, developing robust numerical methods to approximate solutions for the discussed fractional boundary values problems would be an important next step. This would help in validating the theoretical results and in solving real-world problems in which analytical solutions are difficult to obtain. The fractional differential equations studied in this paper have applications in various fields, including physics, engineering, and finance. Investigating specific applications in which these types of equations model real phenomena could lead to new insights and practical advancements.

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