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Certain Geometric Study Involving the Barnes–Mittag-Leffler Function

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Abstract: The main purpose of this paper is to study certain geometric properties of a class of analytic functions involving the Barnes–Mittag-Leffler function. The main mathematical tools are the monotonicity patterns of some class of functions associated with the gamma and digamma functions. Furthermore, some consequences and examples are presented.

Keywords: convex function; starlike function; univalent function; Barnes–Mittag-Leffler function

MSC: 30H10; 30C45; 30D15

1. Introduction and Motivation

The Geometric Function Theory has become one of the important growing research subjects of complex analysis due to its numerous applications in some areas of science. The present far-reaching development of the geometric properties of analytic functions has attracted a large number of contributions [1–9]. In recent years, due to its numerous applications in various branches of science and engineering, the Mittag-Leffler function is one of the most important special functions whose geometric properties were extensively studied by several researchers (see, for instance, [10–14]). The Barnes–Mittag-Leffler function is connected to the Mittag-Leffler function. In the present investigation, we aim to present some geometric properties of a class of functions associated with the Barnes–Mittag-Leffler function. Let us recall some basic definitions and analytic description related to the context. Let $\mathcal{H}$ denote the class of all analytic functions in the unit disk

$$\mathcal{D} = \Big\{\xi \in \mathbb{C} : \quad |\xi| < 1\Big\}.$$

Let $\mathcal{A}$ be the class of analytic function ϕ, satisfying the normalization conditions $\phi(0) = 0$ and $\phi'(0) = 1$, i.e.,

$$\phi(\xi) = \xi + \sum_{l=2}^{\infty} \rho_l \xi^l \qquad (\forall \xi \in \mathcal{D}). \tag{1}$$

The function $\phi \in \mathcal{A}$ is called a starlike function in $\mathcal{D}$, if ϕ is univalent in $\mathcal{D}$ and $\phi(\mathcal{D})$ is a starlike domain with respect to the origin. It is worth mentioning that the characterization of such a starlike function (see Duren's book [15]) is

$$\Re\left(\frac{\xi\phi'(\xi)}{\phi(\xi)}\right) > 0 \qquad (\forall \xi \in \mathcal{D}).$$

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MacGregor, in [16], proved that if a function $\phi \in \mathcal{A}$ such that

$$\left|\frac{\phi(\xi)-\xi}{\xi}\right| < 1 \quad (\forall \xi \in \mathcal{D}),$$

then ϕ is starlike in

$$\mathcal{D}_{\frac{1}{2}} = \left\{\xi : \xi \in \mathbb{C}, \quad |\xi| < \frac{1}{2}\right\}.$$

If ϕ is a univalent function in $\mathcal{D}$ and $\phi(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then ϕ is said to be a convex function in $\mathcal{D}$. It is worth mentioning that the characterization of such a convex function in $\mathcal{D}$ is

$$\Re\left(1+\frac{\xi\phi''(\xi)}{\phi'(\xi)}\right) > 0 \quad (\forall \xi \in \mathcal{D}).$$

Again, MacGregor, in [17], established that for any analytic function $\phi \in \mathcal{A}$ that satisfies the following inequality:

$$\left|\phi'(\xi)-1\right| < 1,$$

for $\xi \in \mathcal{D}$, then ϕ is convex in $\mathcal{D}_{\frac{1}{2}}$. It is well known that an analytic function ϕ is convex if and only if the function $\xi\phi'$ is starlike.

An analytic function ϕ in $\mathcal{A}$ is called close to convex in the open unit disk $\mathcal{D}$ if there exists a starlike function ϕ in the unit disk such that

$$\Re\left(\frac{\xi\phi'(\xi)}{\phi(\xi)}\right) > 0 \quad (\forall \xi \in \mathcal{D}).$$

It can be easily verified that every starlike (or convex) function is close to convex. Furthermore, it is worth mentioning here that in 1935, Ozaki [18] established that any analytic function $\phi : \mathcal{D} \to \mathbb{C}$ such that

$$\phi(\xi) = \xi + \sum_{l=2}^{\infty} \lambda_{2l-1}\xi^{2l-1},$$

and it satisfies

$$1 \geq 3\lambda_3 \geq \cdots \geq (2l-1)\lambda_{2l-1} \geq \cdots \geq 0,$$

or if

$$1 \leq 3\lambda_3 \leq \cdots \leq (2l-1)\lambda_{2l-1} \leq \cdots \leq 2,$$

then ϕ is close to convex with respect to the function $\frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right)$. In 1941, Ozaki [19] derived that for any function ϕ in the form (1) such that

$$1 \leq 2\lambda_2 \leq \cdots \leq l\lambda_l \leq \cdots \leq 2,$$

or if

$$1 \geq 2\lambda_2 \geq \cdots \geq l\lambda_l \geq \cdots \geq 0,$$

then ϕ is close to convex with respect to the function $-\log(1-\xi)$.

A function ϕ in $\mathcal{A}$ is called uniformly convex in $\mathcal{D}$, if for every circular arc ζ contained in $\mathcal{D}$ with the center $\eta \in \mathcal{D}$, the image arc $\phi(\zeta)$ is convex with respect to $\phi(\eta)$; for more details see [20]. We denote by UCV the class of all functions that are uniformly convex. It is also important to note here that V. Ravichandran [21] proved that if a function $\phi \in \mathcal{A}$ satisfies the following condition:

$$\left|\frac{\xi\phi''(\xi)}{\phi'(\xi)}\right| < \frac{1}{2} \quad (\forall \xi \in \mathcal{D}), \tag{2}$$

then $\phi \in$ UCV.

The two-parameter Mittag-Leffler function $E_{\nu,\eta}(\xi)$ is defined by [22,23] as follows:

$$E_{\nu,\eta}(\xi) = \sum_{l=0}^{\infty} \frac{\xi^l}{\Gamma(\nu l + \eta)} \quad (\xi, \nu, \eta \in \mathbb{C},\ \min(\Re(\nu), \Re(\eta)) > 0). \tag{3}$$

For some properties involving the two-parameter Mittag-Leffler function, we refer to [24,25]. The Barnes–Mittag-Leffler function $E^{(a)}_{\kappa,\nu}(s;\xi)$ is defined by [26] as follows:

$$E^{(s)}_{\kappa,\nu}(a;\xi) := \sum_{l=0}^{\infty} \frac{\xi^l}{(l+a)^s \Gamma(\kappa l + \nu)}. \tag{4}$$

To discuss some geometric properties of the Barnes–Mittag-Leffler function, we consider the following normalized form:

$$\begin{aligned} \mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi) &= a^s \Gamma(\nu) \xi E^{(s)}_{\kappa,\nu}(a;\xi) \\ &= \sum_{l=1}^{\infty} \rho^{(s)}_l(\nu,\kappa,a) \xi^l, \end{aligned} \tag{5}$$

where:

$$\rho^{(s)}_l(\nu,\kappa,a) = \frac{a^s \Gamma(\nu)}{(a+l-1)^s \Gamma(l\kappa + \nu - \kappa)},\ l \geq 1.$$

Here, we use ψ to denote the digamma (psi) function defined by

$$\begin{aligned} \psi(r) &= \frac{\Gamma'(r)}{\Gamma(r)} \quad (r > 0) \\ &= -\gamma - \frac{1}{r} + \sum_{l=1}^{\infty} \frac{r}{l(l+r)}, \end{aligned}$$

where $\gamma = -\psi(1)$ is an Euler–Mascheroni constant.

2. The Main Results

Theorem 1. *Assume that* $\min(a,\kappa,\nu) > 0$ *and* $s \geq 0$. *The following statements hold:*
(a). If the following inequalities

$$\frac{2\Gamma(\nu)}{\Gamma(\kappa+\nu)} \leq \left(\frac{a+1}{a}\right)^s \text{ and } 2\kappa\psi(\kappa+\nu) \geq 1,$$

hold, then the function $\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)$ *is close to convex with respect to the function* $-\log(1-\xi)$.
(b). If the following inequalities

$$\frac{3\Gamma(\nu)}{\Gamma(\kappa+\nu)} \leq \left(\frac{a+1}{a}\right)^s \text{ and } 3\kappa\psi(\kappa+\nu) \geq 2,$$

hold true, then the function $\varphi^{(s)}_{\kappa,\nu}(a;\xi) := a^s \Gamma(\nu) \xi E^{(s)}_{\kappa,\nu}(a;\xi^2)$ *is close to convex with respect to the function* $\frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right)$.

Proof. (a). By using the given condition, we obtain the following:

$$2\rho^{(s)}_2(\nu,\kappa,a) \leq 1.$$

For $n \geq 2$, a simple computation gives the following:

$$l\rho_l^{(s)}(\nu,\kappa,a) - (l+1)\rho_{l+1}^{(s)}(\nu,\kappa,a) \geq \frac{a^s\Gamma(\nu)\left[l\Gamma(l\kappa+\nu) - (l+1)\Gamma(l\kappa+\nu-\kappa)\right]}{(l+a)^s\Gamma(l\kappa+\nu)\Gamma(l\kappa+\nu-\kappa)}. \tag{6}$$

Let:

$$\delta_{\kappa,\nu}(t) = \frac{t}{\Gamma(\kappa t+\nu-\kappa)} \quad (t \geq 2).$$

Then, we obtain the following:

$$\delta'_{\kappa,\nu}(t) = \frac{1-\kappa t\psi(\kappa t+\nu-\kappa)}{\Gamma(\kappa t+\nu-\kappa)} =: \frac{\Xi_{\kappa,\nu}(t)}{\Gamma(\kappa t+\nu-\kappa)}.$$

In view of the fact that the digamma function $t \mapsto \psi(t)$ is increasing in $(0,\infty)$, we obtain that the function $t \mapsto \Xi_{\kappa,\nu}(t)$ is decreasing on $[2,\infty)$. In particular, for $t \geq 2$ we have

$$\Xi_{\kappa,\nu}(t) \leq \Xi_{\kappa,\nu}(2) \leq 0,$$

by our assumption. Consequently, the function $t \mapsto \delta_{\kappa,\nu}(t)$ is also decreasing in $[2,\infty)$. By (6), the sequence

$$\left\{l\rho_l^{(s)}(\nu,\kappa,a)\right\}_{l\geq 2},$$

is decreasing. Hence, the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ is close to convex with respect to the function $-\log(1-\xi)$.

(b). From (4) we obtain the following:

$$\varphi_{\kappa,\nu}^{(s)}(a;\xi) = \xi + \sum_{l=2}^{\infty}\lambda_{2l-1}^{(s)}(\nu,\kappa,a)\xi^{2l-1},$$

where:

$$\lambda_{2l-1}^{(a)}(\nu,\kappa,a) = \rho_l^{(s)}(\nu,\kappa,a) \quad (l \geq 2).$$

Under our assumption, we have

$$3\lambda_3^{(s)}(\nu,\kappa,a) = 3\rho_2^{(s)}(\nu,\kappa,a) \leq 1.$$

Moreover, for $l \geq 2$ we obtain the following:

$$\begin{aligned}(2l-1)\lambda_{2l-1}^{(s)}(\nu,\kappa,a) - (2l+1)\lambda_{2l+1}^{(s)}(\nu,\kappa,a)& \\ \geq \frac{a^s\Gamma(\nu)\left[(2l-1)\Gamma(l\kappa+\nu) - (2l+1)\Gamma(l\kappa+\nu-\kappa)\right]}{(l+a)^s\Gamma(l\kappa+\nu-\kappa)\Gamma(l\kappa+\nu)}&.\end{aligned} \tag{7}$$

By a similar argument as in the proof of part (a), we derive that the function

$$t \mapsto \tilde{\delta}_{\kappa,\nu}(t) = \frac{2t-1}{\Gamma(\kappa t+\nu-\kappa)},$$

is decreasing on $[2,\infty)$ if $3\kappa\psi(\kappa+\nu) - 2 \geq 0$ and consequently, we deduce that the sequence

$$\left\{(2l-1)\lambda_{2l-1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq 2},$$

is decreasing. This completes the proof. □

Taking $(\kappa,\nu) = (\frac{3}{2},1)$ in part (a) (resp. part (b)) of Theorem 1, we derive the following results.

Corollary 1. *Assume that $a > 0$ and $s \geq 0$. The following statements hold:*
(a). If the following inequality

$$8a^s \leq 3\sqrt{\pi}(a+1)^s,$$

is valid, then the function $\mathcal{E}^{(s)}_{\frac{3}{2},1}(a;\xi)$ is close to convex with respect to the function $-\log(1-\xi)$.
(b). If the following inequality

$$a^s \leq \frac{\sqrt{\pi}}{4}(a+1)^s,$$

hold true, then the function $\varphi^{(s)}_{\frac{3}{2},1}(a;\xi)$ is close to convex with respect to the function $\frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right)$.

Remark 1. *Since any close-to-convex function $\phi : \mathcal{D} \to \mathbb{C}$ is univalent in $\mathcal{D}$ (see, for instance, [18,27]), we deduce by Theorem 1 that the functions $\xi \mapsto \mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)$ and $\xi \mapsto \varphi^{(s)}_{\kappa,\nu}(a;\xi)$ are univalent in $\mathcal{D}$ under the given assumptions asserted in Theorem 1.*

Taking $s = 0$ in Theorem 1, we derive the following result:

Corollary 2. *Assume that $\min(a,\kappa,\nu) > 0$. The following statements hold:*
(a). If the following inequalities

$$\frac{\Gamma(\nu)}{\Gamma(\kappa+\nu)} \leq \frac{1}{2} \quad \text{and} \quad 2\kappa\psi(\kappa+\nu) \geq 1,$$

hold true, then the function $\mathbb{E}_{\kappa,\nu}(\xi)$ defined by

$$\mathbb{E}_{\kappa,\nu}(\xi) = \Gamma(\nu)\xi E_{\kappa,\nu}(\xi), \tag{8}$$

is close to convex with respect to the function $-\log(1-\xi)$.
(b). If the following inequalities

$$\frac{\Gamma(\nu)}{\Gamma(\nu+\kappa)} \leq \frac{1}{3} \quad \text{and} \quad 3\kappa\psi(\kappa+\nu) \geq 2,$$

hold true, then the function $\Theta_{\kappa,\nu}(\xi) := \Gamma(\nu)\xi E_{\kappa,\nu}(\xi^2)$ is close to convex with respect to the function $\frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right)$.

Remark 2. *In [11] (Theorem 4), the authors proved that the function $\mathbb{E}_{\kappa,\nu}(\xi)$ is close to convex with respect to the function $-\log(1-\xi)$ for $\min(\kappa,\nu) \geq 1$. However, if we let $\nu = \frac{1}{2}$ and $\kappa = \frac{5}{2}$, we obtain that the function $\mathbb{E}_{\frac{5}{2},\frac{1}{2}}(\xi)$ is close to convex with respect to the function $-\log(1-\xi)$. More precisely, Corollary 2 improves the corresponding result proved in [11].*

Theorem 2. *Let the parameter space be the same as in Theorem 1. Also, we assume that $4\Gamma(\nu) \leq \Gamma(\nu+\kappa)$. Then, we have the following:*

$$\Re\left\{\left(\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)\right)'\right\} > \frac{1}{2} \quad (\forall \xi \in \mathcal{D}).$$

Proof. By using (5) we have the following:

$$\left(\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)\right)' = \sum_{l=1}^{\infty} l\rho^{(s)}_l(\nu,\kappa,a)\xi^{l-1}.$$

In the proof of part (a) of Theorem 1, we have derived that the sequence

$$\left\{l\rho^{(s)}_l(\nu,\kappa,l)\right\}_{l\geq 1},$$

is decreasing. Furthermore, for $l \geq 1$ we have the following:

$$l\rho_l^{(s)}(\nu,\kappa,a) - 2(l+1)\rho_{l+1}^{(s)}(\nu,\kappa,a)) \geq \frac{a^s\Gamma(\nu)}{(l+a)^s}\left[\frac{l}{\Gamma(l\kappa+\nu-\kappa)} - \frac{2(l+1)}{\Gamma(l\kappa+\nu)}\right]. \tag{9}$$

Moreover, due to the monotonicity of the digamma function, we conclude that the function

$$r \mapsto \frac{\Gamma(r+\alpha)}{\Gamma(r)} \quad (\alpha > 0),$$

is increasing on $(0,\infty)$, and consequently the sequence $(\theta_l)_{l\geq 1}$ defined by

$$\theta_l := \theta_l(\kappa,\nu) = \frac{\Gamma(l\kappa+\nu)}{\Gamma(\kappa(l-1)+\nu)} \quad (l \geq 1)$$

is increasing. Therefore, the sequence $(\epsilon_l)_{l\geq 1}$ defined by

$$\epsilon_l := \epsilon_l(\kappa,\nu) = \frac{l\Gamma(l\kappa+\nu)}{(l+1)\Gamma(\kappa(l-1)+\nu)} \quad (l \geq 1),$$

is also increasing. Then, for $l \geq 1$ we obtain the following:

$$\epsilon_l(\kappa,\nu) \geq \epsilon_1(\kappa,\nu) = \frac{\Gamma(\kappa+\nu)}{2\Gamma(\nu)}. \tag{10}$$

But in our condition, we have the following:

$$\frac{\Gamma(\nu+\kappa)}{2\Gamma(\nu)} \geq 2.$$

Hence, according to the above inequality and (10), we obtain the following:

$$\epsilon_l(\kappa,\nu) \geq 2 \quad (l \geq 1). \tag{11}$$

Then, in view of (9) and (11), we find that

$$l\rho_l^{(s)}(\nu,\kappa,a) - 2(l+1)\rho_{l+1}^{(s)}(\nu,\kappa,a) \geq 0,$$

for $l \geq 1$, but $l\rho_l^{(s)}(\nu,\kappa,a) > 0$ for each $l \geq 1$. Then, the sequence

$$\left\{l\rho_l^{(s)}(\nu,\kappa,a)\right\}_{l\geq 1},$$

is convex. On the other hand, according to Fejér [28], if a function $\phi(\xi) = \sum_{l=1}^{\infty} \alpha_l \xi^{l-1}$ where $\alpha_1 = 1$ and $\alpha_l \geq 0$ for $l \geq 2$ is analytic in the unit disc $\mathcal{D}$ and if the sequence $(\alpha_l)_{l\geq 1}$ is decreasing and convex, then

$$\Re(\phi(\xi)) > \frac{1}{2} \quad (\forall \xi \in \mathcal{D}).$$

This proves the result asserted by Theorem 2. □

Theorem 3. *Assume that* $\min(\kappa,\nu) \geq 1, s \geq 0$ *and* $a > 0$ *such that*

$$(e-1)a^s\Gamma(\nu) \leq (a+1)^s\Gamma(\kappa+\nu).$$

Then, the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Proof. Let $\xi \in \mathcal{D}$. Then, we have the following:

$$\begin{aligned}\left|\frac{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}-1\right| &< \sum_{l=2}^{\infty}\rho_{l}^{(s)}(\nu,\kappa,a)\\ &= \sum_{l=1}^{\infty}\frac{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)}{l!},\end{aligned} \tag{12}$$

where:

$$\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a) = l!\rho_{l+1}^{(s)}(\nu,\kappa,a). \tag{13}$$

Let:

$$f_{\kappa,\nu}(t) = \frac{\Gamma(t+1)}{\Gamma(\kappa t+\nu)} \quad (t>0, \min(\kappa,\nu)\geq 1).$$

Differentiating $f_{\kappa,\nu}(t)$ with respect to t, we obtain the following:

$$\frac{f'_{\kappa,\nu}(t)}{f_{\kappa,\nu}(t)} = \psi(t+1)-\kappa\psi(\kappa t+\nu).$$

However, the function $t \mapsto \psi(t)$ is increasing on $(0,\infty)$, and thus the function $t \mapsto f_{\kappa,\nu}(t)$ is decreasing on $(0,\infty)$. We thus rewrite the sequence $\left\{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq 1}$ as

$$\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a) = \left(\frac{a^s\Gamma(\nu)}{(l+a)^s}\right) f_{\kappa,\nu}(l) \quad (l\geq 1).$$

Then, the sequence

$$\left\{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq 1},$$

is decreasing, as is the product of two positive and decreasing sequences. Hence, by this observation and (12), we obtain the following:

$$\begin{aligned}\left|\frac{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}-1\right| &< \frac{\Gamma(\nu)a^s}{\Gamma(\kappa+\nu)(a+1)^s}\sum_{l=1}^{\infty}\frac{1}{l!}\\ &= \frac{\Gamma(\nu)a^s(e-1)}{\Gamma(\kappa+\nu)(a+1)^s}\\ &\leq 1.\end{aligned} \tag{14}$$

Thus, the proof is complete. □

Upon taking $\nu=\kappa=1$ in Theorem 3, we obtain the following result reads as follows:

Corollary 3. *Let $a>0$ and $s\geq 0$, such that the following condition*

$$a^s(e-1) \leq (a+1)^s,$$

is valid. Then, the function $\mathcal{E}_{1,1}^{(s)}(a;\xi)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$.

We set $s=1$ in Corollary 3, and compute the following result:

Corollary 4. *If $0<a\leq\frac{1}{e-2}$, then the function $\mathcal{E}_{1,1}^{(s)}(1;\xi)$ is starlike in $\mathcal{D}_{\frac{1}{2}}$.*

Taking $s=0$ in Theorem 3, we derive the following statement:

Corollary 5. *Let* $\min(\kappa, \nu) \geq 1$, *such that*

$$(e-1)\Gamma(\nu) \leq \Gamma(\kappa+\nu).$$

Then, the function $\mathbb{E}_{\kappa,\nu}(\xi)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Remark 3. *In [13] (Theorem 2.4), the authors derived that the function* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$ *for* $\kappa \geq 1$ *and* $\nu \geq \frac{\sqrt{5}+1}{2} \approx 1.61\cdots$. *Moreover, we can verify that the function* $\mathbb{E}_{2,\nu}(\xi)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$ *for any* $\nu \geq 1$.

Theorem 4. *Let* $\min(\nu, \kappa) \geq 2, a > 0$ *and* $s \geq 0$. *Then, the function* $\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$.

Proof. A straightforward computation gives

$$\begin{aligned}\left|\left(\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)\right)' - 1\right| &< \sum_{l=1}^{\infty}(l+1)\rho^{(s)}_{l+1}(\nu,\kappa,a)\\ &= \sum_{l=1}^{\infty}\frac{\hat{\rho}^{(s)}_{l+1}(\nu,\kappa,a)}{l!},\end{aligned} \tag{15}$$

where:

$$\hat{\rho}^{(s)}_{l+1}(\nu,\kappa,a) = (l+1)!\rho^{(s)}_{l+1}(\nu,\kappa,a) \quad (l \geq 1). \tag{16}$$

Note that the function $t \mapsto g_{\kappa,\nu}(t)$ defined by

$$g_{\kappa,\nu}(t) = \frac{\Gamma(t+2)}{\Gamma(\kappa t+\nu)},$$

is decreasing on $(0,\infty)$ for $\min(\nu,\kappa) \geq 2$. This in turn implies that the sequence

$$\left\{\hat{\rho}^{(s)}_{l+1}(\nu,\kappa,a)\right\}_{l\geq 1},$$

is decreasing. From this fact and with the help of (15), we obtain the following:

$$\begin{aligned}\left|\left(\mathcal{E}^{(s)}_{\kappa,\nu}(a;\xi)\right)' - 1\right| &< \frac{\Gamma(3)a^s\Gamma(\nu)}{(a+1)^s\Gamma(\kappa+\nu)}\sum_{l=1}^{\infty}\frac{1}{l!}\\ &= \frac{a^s\Gamma(3)\Gamma(\nu)(e-1)}{\Gamma(\kappa+\nu)(a+1)^s},\end{aligned} \tag{17}$$

which is less than or equal to 1 by the condition $\min(\kappa,\nu) \geq 2$. □

Taking in Theorem 4 the values $\kappa = \nu = 2$, we compute the following result:

Corollary 6. *Let* $a > 0$ *and* $s \geq 0$. *Then, the function* $\mathcal{E}^{(s)}_{2,1}(a;\xi)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$.

Taking $s = 0$ in Theorem 4, we derive the following result:

Corollary 7. *Let* $\min(\kappa,\nu) \geq 2$ *and* $a > 0$. *Then, the function* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$.

Remark 4. *In [13] (Theorem 2.4), the authors proved that the function* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$ *if* $\kappa \geq 1$ *and* $\nu \geq \frac{\sqrt{17}+3}{2} \approx 3.65\cdots$. *However, in view of Corollary 7, we deduce that the function* $\mathbb{E}_{2,\nu}(\xi)$ *is convex in* $\mathcal{D}_{\frac{1}{2}}$ *for* $\nu \geq 2$.

Theorem 5. *Let* $\min(\nu, \kappa) \geq 1, s \geq 0$ *and* $a > 0$. *If*

$$(2e-1)\Gamma(\nu)a^s \leq (a+1)^s\Gamma(\kappa+\nu),$$

then the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ *is starlike in* $\mathcal{D}$.

Proof. To prove that the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ is starlike in $\mathcal{D}$, it suffices to prove the inequality

$$\Re\left(\frac{\xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'}{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}\right) > 0 \quad (\forall \xi \in \mathcal{D}).$$

For this objective, it suffices to establish that

$$\left|\frac{\xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'}{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)} - 1\right| < 1, \tag{18}$$

for $\xi \in \mathcal{D}$. By (5) and routine algebra, we obtain the following:

$$\begin{aligned}\left|(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))' - \frac{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}\right| &< \left|\sum_{l=1}^{\infty} l\rho_{l+1}^{(s)}(\nu,\kappa,a)\right| \\ &= \sum_{l=1}^{\infty}\frac{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)}{(l-1)!},\end{aligned} \tag{19}$$

where $\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)$ is defined in (13). By our condition, the sequence

$$\left\{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq 1},$$

is decreasing (see the proof of Theorem 3) and consequently we have the following:

$$\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a) \leq \frac{a^s\Gamma(\nu)}{(a+1)^s\Gamma(\nu+\kappa)}. \tag{20}$$

By combining the above inequality with (19), we obtain the following:

$$\left|(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))' - \frac{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}\right| < \frac{a^s e\Gamma(\nu)}{(a+1)^s\Gamma(\kappa+\nu)} \quad (\forall \xi \in \mathcal{D}). \tag{21}$$

Furthermore, we have the following:

$$\begin{aligned}\left|\frac{\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}\right| &> 1 - \sum_{l=1}^{\infty}\frac{\tilde{\rho}_{l+1}^{(s)}(\nu,\kappa,a)}{l!} \\ &\geq 1 - \frac{\Gamma(\nu)a^s(e-1)}{\Gamma(\kappa+\nu)(a+1)^s} > 0.\end{aligned} \tag{22}$$

Then, having (21) and (22) in mind, we deduce that the inequality (18) is valid for $\xi \in \mathcal{D}$. □

Specifying $\kappa = \nu = 1$ in Theorem 5, we obtain the following result:

Corollary 8. *For any* $a > 0$ *and* $s > 0$ *such that*

$$(2e-1)a^s \leq (a+1)^s,$$

the function $\mathcal{E}_{1,1}^{(s)}(a;\xi)$ *is starlike in* $\mathcal{D}$.

Taking $s = 1$ in Corollary 8, we derive the following result reads as follows:

Corollary 9. *If* $0 < a \leq \frac{1}{2(e-1)}$, *then the function* $\mathcal{E}_{1,1}^{(1)}(a;\xi)$ *is starlike in* $\mathcal{D}$.

Example 1. *The function* $\mathcal{E}_{1,1}^{(1)}(1/5;\xi)$ *is starlike in* $\mathcal{D}$.

If we take $s = 0$ in Theorem 5, we compute the following sufficient conditions for the starlikeness property of the function $\mathbb{E}_{\kappa,\nu}(\xi)$ in the unit disk.

Corollary 10. *Let* $\min(\kappa,\nu) \geq 1$. *If*

$$\Gamma(\nu)(2e-1) \leq \Gamma(\kappa+\nu),$$

then the function $\mathbb{E}_{\kappa,\nu}(\xi)$ *is starlike in* $\mathcal{D}$.

Remark 5. *In [13] (Theorem 2.2), the authors proved that the function* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is starlike in* $\mathcal{D}$ *if* $\kappa \geq 1$ *and* $\nu \geq \frac{\sqrt{17}+3}{2} \approx 3.65\cdots$. *Also, in [11] (Theorem 2), Noreen et al. obtained that the function* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is starlike in* $\mathcal{D}$ *if* $\kappa \geq 2.67$ *and* $\nu \geq 1$. *Moreover, in view of the above Corollary, we conclude that the function* $\mathbb{E}_{2,\nu}(\xi)$ *is starlike in* $\mathcal{D}$ *for any* $\nu \geq \frac{-1+\sqrt{8e-3}}{2} \sim 1.66\cdots$. *Moreover, a numerical computation shows that the function* $\mathbb{E}_{\kappa,2}(\xi)$ *is starlike in* $\mathcal{D}$ *for any* $\kappa \geq 1.76$.

Theorem 6. *Let* $\kappa \geq \max\left(1, \sqrt{\frac{\nu}{2}}\right), s \geq 0$ *and* $a > 0$. *If inequalities*

$$\kappa\log(\kappa+\nu) - \frac{\kappa}{\kappa+\nu} > \frac{5}{6} + \log(3) \quad \text{and} \quad 4a^s\Gamma(\nu)(e-1) \leq (a+1)^s\Gamma(\nu+\kappa),$$

are valid, then the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ *is convex in* $\mathcal{D}$.

Proof. According to the analytic characterizations of convex functions, to prove that the function $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)$ is convex in $\mathcal{D}$ it is enough to prove that the function

$$\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi) = \xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'$$

is starlike in $\mathcal{D}$. For this, it suffices to prove the inequality

$$\left|\frac{\xi(\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi))'}{\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi)} - 1\right| < 1 \quad (\forall \xi \in \mathcal{D}). \tag{23}$$

For any $\xi \in \mathcal{D}$, we obtain the following:

$$\begin{aligned}\left|(\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi))' - \frac{\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi)}{\xi}\right| &< \sum_{l=1}^{\infty} l(l+1)\rho_{l+1}^{(s)}(\nu,\kappa,a) \\ &= \sum_{l=1}^{\infty} \frac{l\Gamma(l+2)\rho_{l+1}^{(s)}(\nu,\kappa,a)}{l!}.\end{aligned} \tag{24}$$

We set the following:

$$h_{\kappa,\nu}(r) = \frac{r\Gamma(r+2)}{\Gamma(\kappa r+\nu)} \quad (r \geq 1).$$

Then,

$$\begin{aligned}\frac{h'_{\kappa,\nu}(r)}{h_{\kappa,\nu}(r)} &= \phi_{\kappa,\nu}(r)\\ &=: \frac{1}{r} + \psi(r+2) - \kappa\psi(\kappa r+\nu)\end{aligned}$$

We now make use of inequality [29], using Equation (17):

$$\log(r) - \frac{1}{r} < \psi(r) < \log(r) - \frac{1}{2r}, \tag{25}$$

to obtain the following:

$$\phi_{\kappa,\nu}(r) < \tilde{\phi}_{\kappa,\nu}(r) := \frac{r+4}{2r(r+2)} + \frac{\kappa}{\kappa r+\nu} + \log(r+2) - \kappa\log(\kappa r+\nu). \tag{26}$$

A simple computation gives the following:

$$\tilde{\phi}'_{\kappa,\nu}(r) = \frac{\kappa r(1-\kappa)+\nu-2\kappa^2}{(r+2)(\kappa r+\nu)} - \frac{r^2+8r+8}{2r^2(r+2)^2} - \frac{\kappa^2}{(\kappa r+\nu)^2} < 0, \tag{27}$$

for $\kappa \geq \max\left(1, \sqrt{\frac{\nu}{2}}\right)$. This in turn implies that the function $r \mapsto \tilde{\phi}_{\kappa,\nu}(r)$ is decreasing on $[1,\infty)$. As $\tilde{\phi}_{\kappa,\nu}(1) < 0$, we deduce that the function $r \mapsto h_{\kappa,\nu}(r)$ is decreasing on $[1,\infty)$. This fact implies that the sequence

$$\left\{l\Gamma(l+2)\rho^{(s)}_{l+1}(\nu,\kappa,a)\right\}_{l\geq 1},$$

is also decreasing. Thus, by (24) we obtain the following:

$$\begin{aligned}\left|(\mathcal{F}^{(s)}_{\kappa,\nu}(a;\xi))' - \frac{\mathcal{F}^{(s)}_{\kappa,\nu}(a;\xi)}{\xi}\right| &< \sum_{l=1}^{\infty}\frac{\Gamma(3)\rho^{(a)}_2(\nu,\kappa,s)}{l!}\\ &= \frac{2a^s\Gamma(\nu)(e-1)}{(a+1)^s\Gamma(\nu+\kappa)}.\end{aligned} \tag{28}$$

Furthermore, for any $\xi \in \mathcal{D}$, we find that:

$$\begin{aligned}\left|\frac{\mathcal{F}^{(s)}_{\kappa,\nu}(a;\xi)}{\xi}\right| &> 1 - \sum_{l=1}^{\infty}(l+1)\rho^{(s)}_{l+1}(\nu,\kappa,a)\\ &= 1 - \sum_{l=1}^{\infty}\frac{\Gamma(l+2)\rho^{(s)}_{l+1}(\nu,\kappa,a)}{l!}.\end{aligned} \tag{29}$$

As we showed above, the function $r \mapsto h_{\kappa,\nu}(r)$ is decreasing on $[1,\infty)$ and consequently the function $r \mapsto \frac{h_{\kappa,\nu}(r)}{r}$ is also decreasing on $[1,\infty)$, which in turn implies that the sequence

$$\left\{\Gamma(l+2)\rho^{(s)}_{l+1}(\nu,\kappa,a)\right\}_{l\geq 1}$$

is decreasing. Therefore, we obtain the following:

$$\begin{aligned}\left|\frac{\mathcal{F}^{(s)}_{\kappa,\nu}(a;\xi)}{\xi}\right| &> 1 - \sum_{l=1}^{\infty}\frac{2\rho^{(s)}_2(\nu,\kappa,a)}{l!}\\ &= 1 - \frac{2a^s(e-1)\Gamma(\nu)}{(a+1)^s\Gamma(\nu+\kappa)}.\end{aligned} \tag{30}$$

Hence, by combining (28) and (30) we obtain the following:

$$\left|\frac{\xi(\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi))'}{\mathcal{F}_{\kappa,\nu}^{(s)}(a;\xi)}-1\right|<\frac{2a^s(e-1)\Gamma(\nu)}{\Gamma(\kappa+\nu)(a+1)^s-2a^s(e-1)\Gamma(\nu)}\quad(\forall\xi\in\mathcal{D}),\tag{31}$$

which is less than 1 by our assumption. □

Specifying $s=0$ in Theorem 6, we obtain the following:

Corollary 11. *Let* $\kappa\geq\max\left(1,\sqrt{\frac{\nu}{2}}\right)$ *such that the following inequalities*

$$\kappa\log(\nu+\kappa)-\frac{\kappa}{\nu+\kappa}>\log(3)+\frac{5}{6}\ \text{ and }\ 4(e-1)\Gamma(\nu)\leq\Gamma(\kappa+\nu).$$

hold true, and then the function $\mathbb{E}_{\kappa,\nu}(\xi)$ *is convex in* $\mathcal{D}$.

Remark 6. *In [11] (Theorem 7), it is shown that* $\mathbb{E}_{\kappa,\nu}(\xi)$ *is convex in* $\mathcal{D}$ *if* $\kappa\geq1$ *and* $\nu\geq3.57\cdots$. *By Corollary 11, we deduce that the function* $\mathbb{E}_{3,\nu}(\xi)$ *is convex in* $\mathcal{D}$ *for* $\nu\geq1.09$.

Theorem 7. *Let* $\kappa\geq\max\left(1,\sqrt{\frac{\nu}{2}}\right), a>0$, *and* $s\geq0$. *Assume that the following inequalities*

$$\kappa\log(\kappa+\nu)-\frac{\kappa}{\kappa+\nu}>\frac{5}{6}+\log(3)\ \text{ and }\ 6a^s\Gamma(\nu)(e-1)\leq\Gamma(\kappa+\nu)(a+1)^s,$$

hold true. Then, $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi)\in UCV$.

Proof. Assume that $\xi\in\mathcal{D}$. A routine algebra gives the following:

$$\begin{aligned}\xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))''&=\sum_{l=1}^{\infty}l(l+1)\rho_{l+1}^{(s)}(\nu,\kappa,a)\xi^l\\&=\sum_{l=1}^{\infty}\frac{l\Gamma(l+2)\rho_{l+1}^{(s)}(\nu,\kappa,a)\xi^l}{l!}.\end{aligned}\tag{32}$$

By examining the proof of Theorem 6, we observe that the sequence

$$\left\{l\Gamma(l+2)\rho_{l+1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq1},$$

is decreasing. In view of (32) we have the following:

$$\left|\xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))''\right|\leq\frac{2a^s\Gamma(\nu)(e-1)}{\Gamma(\kappa+\nu)(a+1)^s}.\tag{33}$$

Moreover, for $\xi\in\mathcal{D}$ we obtain the following:

$$\left|(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'\right|\geq1-\sum_{l=1}^{\infty}\frac{\Gamma(l+2)\rho_{l+1}^{(s)}(\nu,\kappa,a)}{l!}.\tag{34}$$

Similarly to the proof of Theorem 6, the sequence

$$\left\{\Gamma(l+2)\rho_{l+1}^{(s)}(\nu,\kappa,a)\right\}_{l\geq1},$$

is decreasing. Then, we have the following:

$$\begin{aligned}\left|(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'\right| &\geq 1-\sum_{l=1}^{\infty}\frac{\Gamma(3)\rho_2^{(s)}(\nu,\kappa,a)}{l!}\\ &= \frac{(a+1)^s\Gamma(\kappa+\nu)-2a^s\Gamma(\nu)(e-1)}{(a+1)^s\Gamma(\kappa+\nu)}.\end{aligned} \tag{35}$$

Hence, by (33) and (35) we have the following:

$$\begin{aligned}\left|\frac{\xi(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))''}{(\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi))'}\right| &< \frac{2a^s\Gamma(\nu)(e-1)}{(a+1)^s\Gamma(\kappa+\nu)-2a^s\Gamma(\nu)(e-1)}\\ &\leq \frac{1}{2}.\end{aligned}$$

Then, in view of (2) we conclude that $\mathcal{E}_{\kappa,\nu}^{(s)}(a;\xi) \in UCV$. □

Finally, we take $s = 0$ in Theorem 7 and we compute the following result involving the normalized Mittag-Leffler function, which reads as follows:

Corollary 12. *Let* $\kappa \geq \max\left(1, \sqrt{\frac{\nu}{2}}\right)$. *If the inequality*

$$\kappa\log(\nu+\kappa)-\frac{\kappa}{\kappa+\nu} > \frac{5}{6}+\log(3) \text{ and } 6\Gamma(\nu)(e-1) \leq \Gamma(\nu+\kappa),$$

holds true, then the function $\mathbb{E}_{\kappa,\nu}(\xi) \in UCV$.

Remark 7. *The authors [10] (Theorem 2.6) proved that the function* $\mathbb{E}_{\kappa,\nu}(\xi) \in UCV$ *if* $\kappa \geq 1$ *and* $\nu \geq 9.112\cdots$. *In view of Corollary 12, we deduce that the function* $\mathbb{E}_{3,\nu}(\xi) \in UCV$ *for* $\nu \geq 1.34$.

3. Conclusions

In this paper, we have presented some new geometric properties of a class of functions associated with the Barnes–Mittag-Leffler function in the unit disk. Some special cases of our main results were also established. More precisely, some new sufficient conditions imposed on the parameters of such a class of functions involving the Mittag-Leffler functions satisfy certain geometric properties. Some results in our present investigation are (presumably) new.

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