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Positive Solutions and Their Existence of a Nonlinear Hadamard Fractional-Order Differential Equation with a Singular Source Item Using Spectral Analysis

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Abstract: Based on the spectral analysis method, Gelfand's formula, and the cones fixed point theorem, some positive solutions with their existence of a nonlinear infinite-point Hadamard fractional-order differential equation is achieved on the interval $[a, b]$ under some conditions, and particularly the nonlinear term allows singularities for time and spatial parameters in the present study. Finally, an analysis case is carried out to reveal the principal results.

Keywords: hadamard; fractional-order differential; positive solution; infinite-point; spectral analysis

MSC: 70K50; 34B18



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1. Introduction

The fractional-order differential model can exactly depict the physical and mechanical process containing historical memories and thus establish spatial global relationship briefly. Accordingly, the fractional-order differential model is concise in form and the parameter meaning is obvious, so it becomes one of the effective approaches for modeling complex mechanical and physical behaviors. Boulham et al. [1] proposed an adaptive monitor for projective chaotic synchronization of general class fractional-order systems with chaotic uncertainty affected by unknown input nonlinear factors, and the closed-loop stability was strictly verified by two simulation examples and related comparative studies. Wu et al. [2] used a new nonlinear fractional-order damaged pattern with viscosity, elasticity and plasticity for rock and soil materials to characterize three-stage creep behaviors, and their pattern fully calculated and predicted the deformation with hysteresis derived from the rapid creep in tunnel engineering. Ma et al. [3] developed a fractional-order model termed as SEIR type and subsequently the beingness, uniqueness, boundness of such a fractional-order system were determined. The research conclusions indicated that it is difficult for humans to avoid coexistence with the spread of the COVID-19 epidemic. Yang et al. [4] designed a fractional controller of EV hybrid energy storage system that can significantly improve the control performance with good robustness. In addition, fractional-order models have been found to be extensively applied in photovoltaic panels [5], electromagnetic waves [6], electric circuits [7], viscoelastic materials [8], chemical reactions [9] and biological systems [10,11].

Owing to its so many advantages and fine results while simulating the system, the fractional-order differential model has received increasingly research attentions during the past several years. How to get the existence result of positive solution for the fractional-order differential model as well as its related nonlinear dynamics becomes a hot research direction in the field of fractional-order differential models, and many meaningful results are achieved in recent years [12–17]. Xu et al. [18] depicted the positive solutions of a kind of fractional-order differential models as follows

$${}^H D_{a+}^{\alpha} x(t) + h(t, x(t)) = 0, t \in (a, b),$$

with the boundary value condition

$$x(a) = x'(a) = 0, x(b) = \int_a^b \chi(t) x(t) \frac{dt}{t},$$

where α, a, b are real positive numbers, $2 < \alpha < 3, a < b < +\infty$, ${}^H D_{a+}^{\alpha}$ is the Hadamard fractional derivative of order α , $\chi : [a, b] \rightarrow \mathbb{R}^+$ with $\chi(t) \not\equiv 0, t \in [a, b]$, and $\int_a^b \chi(t) (\ln \frac{t}{a})^{\alpha-1} \frac{dt}{t} \in [0, (\ln \frac{b}{a})^{\alpha-1}]$, $h \in C([a, b] \times \mathbb{R}^+, \mathbb{R}^+)$. Arul and Karthikeyan [19] discussed the existence and uniqueness of solutions with some integral boundary conditions of implicit Hadamard differential equation

$$-{}^H D_{a+}^{\kappa} v(t) = g(t, v(t), {}^H D_{a+}^{\kappa} v(t)), a < t < b,$$

with boundary value condition

$$v(a) = 0, v(b) = \varsigma \int_0^{\sigma} v(s) ds, a < \sigma < b, \varsigma \in \mathbb{R},$$

where ${}^H D_{a+}^{\kappa} v$ is the standard Hadamard fractional derivative of order $1 < \kappa \leq 2$, Arul and Karthikeyan [19] investigated the existence and uniqueness based on Banach and Schauder's fixed point theorem. Wang et al. [20] studied a fractional-order iterative functional differential equation containing parameter on the interval $[a, b]$, showed as

$${}^c D_{a+}^{\alpha} u(t) = f(t, u(t), u(u^{\nu}(t))) + m, t \in [a, b], \nu \in \mathbb{R} \setminus \{0\}, q \in (0, 1), \lambda \in \mathbb{R},$$

with boundary conditions

$$u(t) = \varphi(t), t \in [a_1, a], u(t) = \psi(t), t \in [b, b_1],$$

where ${}^c D_a^q$ is the Caputo fractional derivative of order q with the lower limit a , $a_1 \leq a < b \leq b_1, a_1 \leq a_1^v, b_1^v \leq b_1, f \in C([a, b] \times [a_1, b_1]^2, \mathbb{R}), \varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$. The existence theorem is established by the method of Schauder's fixed point theorems [20], and Wang et al. [20] obtained data dependence of solutions and related parameters.

In this article, an infinite-point Hadamard fractional-order differential equation is taken into account, shown as

$$-{}^H D_{a+}^b \mathcal{V}(t) + \varrho(t) \ell(t, \mathcal{V}(t), {}^H D_{a+}^{\mu} \mathcal{V}(t)) = 0, a < t < b, \quad (1.1)$$

with boundary value conditions

$${}^H D_{a+}^{\mu} \mathcal{V}(a) = {}^H D_{a+}^{\mu+1} \mathcal{V}(a) = 0, {}^H D_{a+}^{\mu} \mathcal{V}(b) = \sum_{j=1}^{\infty} \kappa_j {}^H D_{a+}^{\mu} \mathcal{V}(\varsigma_j), \quad (1.2)$$

where $2.5 < b \leq 3, \kappa_j \geq 0, 0 < \mu < \frac{1}{2}, 1 < \varsigma_1 < \varsigma_2 < \dots < \varsigma_{j-1} < \varsigma_j < \dots < \ell(j = 1, 2, \dots), \sum_{j=1}^{\infty} \kappa_j \left(\frac{\ln \varsigma_j - \ln s}{\ln b - \ln s} \right)^{b-1} < 1$ and $\sum_{j=1}^{\infty} \kappa_j (\ln \frac{\varsigma_j}{a})^{b-1} < (\ln \frac{b}{a})^{b-1}$, $\varrho(t)$ is singular at $t = a$ or/and $t = b$, $\ell : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function and $\ell(t, x, y)$ is continuous, and ${}^H D_{1+}^p$ is the standard Hadamard derivative.

Compared with [18,21], the present study contains the derivative term in the nonlinear term of the equation and we will deal with this difficulty, moreover, and involved infinite points in the boundary conditions. Compared with [19], the nonlinear term is singular in this article, and the method we used is spectral analysis and infinite-points are contained in the boundary conditions. Compared with [22,23], the interval we discussed

is arbitrarily closed interval $[a, b]$, the interval $[1, T]$ or $[0, 1]$ in [22,23] are special cases of interval $[a, b]$, considering the solution of the equation on this interval will bring about a series of problems, and we overcome this difficulty.

2. Preliminaries And Lemmas

Some important definitions and lemmas which may be adopted during the certification in the present study, could be reviewed in the newly published papers, e.g., see [24,25], we introduce some of them herein.

Definition 1 ([24,25]). Let $a > 0$, then the Hadamard-type fractional left integral of order $\beta > 0$ of a function $h : [a, \infty) \rightarrow \mathbb{R}$ is defined by

$${}^H I_{a^+}^\beta h(t) = \frac{1}{\Gamma(\beta)} \int_a^t \left(\ln \frac{t}{\varsigma}\right)^{\beta-1} h(\varsigma) \frac{d\varsigma}{\varsigma}, t \geq a.$$

Definition 2 ([24,25]). Let $a > 0$, $h : [a, \infty) \rightarrow \mathbb{R}$, $t^{n-1}h^{(n-1)}(t) \in AC[a, \infty)$, $n \in \mathbb{N}$, $\beta \in (n-1, n)$, then the Hadamard fractional left derivative of from $[a, +\infty)$ is defined by

$${}^H D_{a^+}^\beta h(t) = \frac{1}{\Gamma(n-\beta)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{\varsigma}\right)^{n-\beta-1} h(\varsigma) \frac{d\varsigma}{\varsigma}, t > a.$$

Lemma 1 ([24,25]). For $\beta \in (n-1, n)$, $n \in \mathbb{N}$, $h \in L[a, \infty)$, $a > 0$, the fractional equation ${}^H D_{a^+}^\beta h(t) + \omega(t) = 0$, $t > a$ has expression of solution

$$x(t) = \sum_{i=1}^n \chi_i \left(\ln \frac{t}{a}\right)^{\beta-i} - \frac{1}{\Gamma(\beta)} \int_a^t \left(\ln \frac{t}{\varsigma}\right)^{\beta-1} \omega(\varsigma) \frac{d\varsigma}{\varsigma}, t \geq a,$$

where $\chi_i \in \mathbb{R}$, $k = 1, 2, \dots, n$.

Let $\mathcal{U}(t) = {}^H D_{a^+}^\mu \mathcal{V}(t)$, $\mathcal{V}(t) \in C[a, b]$, then the BVP (1.1,1.2) can be reduced to a modified model

$$-{}^H D_{a^+}^{b-\mu} \mathcal{U}(t) + \varrho(t) \ell\left(t, {}^H I_{a^+}^\mu \mathcal{U}(t), \mathcal{U}(t)\right) = 0, a < t < b, \quad (2.1)$$

with boundary value condition

$$\mathcal{U}(a) = \mathcal{U}'(a) = 0, \mathcal{U}(b) = \sum_{j=1}^{\infty} \kappa_j \mathcal{U}(\varsigma_j). \quad (2.2)$$

Lemma 2. Given $\mathcal{A} \in L^1(a, b)$, then the model

$${}^H D_{1^+}^{b-\mu} \mathcal{U}(t) + \mathcal{A}(t) = 0, a < t < b, \quad (2.3)$$

with boundary value condition (2.2) can be expressed by

$$\mathcal{U}(t) = \int_a^b \mathcal{H}(t, s) \mathcal{A}(s) \frac{ds}{s}, t \in [a, b], \quad (2.4)$$

where

$$\mathcal{H}(t, s) = \mathcal{G}(t, s) + \frac{\left(\ln \frac{t}{a}\right)^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \mathcal{G}(\varsigma_j, s),$$

$$\mathcal{G}(t, s) = \frac{1}{\left(\ln \frac{b}{a}\right)^{b-\mu-1} \Gamma(b-\mu)} \begin{cases} (\ln t - \ln a)^{b-\mu-1} (\ln b - \ln s)^{b-\mu-1} \\ - (\ln t - \ln s)^{b-\mu-1} \left(\ln \frac{b}{a}\right)^{b-\mu-1}, a \leq s \leq t \leq b, \\ (\ln t - \ln a)^{b-\mu-1} (\ln b - \ln s)^{b-\mu-1}, a \leq t \leq s \leq b, \end{cases} \quad (2.5)$$

in which $\Delta = (\ln \frac{b}{a})^{b-\mu-1} - \sum_{j=1}^{\infty} \kappa_j (\ln \frac{\zeta_j}{a})^{b-\mu-1}$.

Proof. By Lemma 1, Formula (2.3) is changed into an equivalent integral equation

$$\mathcal{U}(t) = -{}^H I_{a+}^{b-\mu} \mathcal{A}(t) + d_1 (\ln \frac{t}{a})^{b-\mu-1} + d_2 (\ln \frac{t}{a})^{b-\mu-2} + d_3 (\ln \frac{t}{a})^{b-\mu-3}.$$

From $u(a) = 0$, we have $d_3 = 0$, then

$$\mathcal{U}'(t) = -{}^H I_{a+}^{b-\mu-1} \mathcal{A}(t) + d_1 (b-\mu-1) (\ln \frac{t}{a})^{b-\mu-2} \cdot \frac{a}{t} + d_2 (b-\mu-2) (\ln \frac{t}{a})^{b-\mu-3} \frac{a}{t},$$

by $\mathcal{U}'(a) = 0$, we have $d_2 = 0$. Hence, we get

$$\begin{aligned} \mathcal{U}(t) &= d_1 (\ln \frac{t}{a})^{b-\mu-1} - {}^H I_{a+}^{b-\mu} \mathcal{A}(t) \\ &= -\frac{1}{\Gamma(b-\mu)} \int_a^t (\ln t - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} + d_1 (\ln \frac{t}{a})^{b-\mu-1}, \end{aligned}$$

therefore, $\mathcal{U}(\zeta_j) = -\frac{1}{\Gamma(b-\mu)} \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} + d_1 (\ln \frac{\zeta_j}{a})^{b-\mu-1}$. Moreover, by $\mathcal{U}(b) = \sum_{j=1}^{\infty} \kappa_j \mathcal{U}(\zeta_j)$, we have

$$\begin{aligned} \mathcal{U}(b) &= -\frac{1}{\Gamma(b-\mu)} \int_a^b (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} + d_1 (\ln \frac{b}{a})^{b-\mu-1} \\ &= \sum_{j=1}^{\infty} \kappa_j \left(-\frac{1}{\Gamma(b-\mu)} \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} + d_1 (\ln \frac{\zeta_j}{a})^{b-\mu-1} \right). \end{aligned}$$

Then, we get

$$\begin{aligned} d_1 \left((\ln \frac{b}{a})^{b-\mu-1} - \sum_{j=1}^{\infty} \kappa_j (\ln \frac{\zeta_j}{a})^{b-\mu-1} \right) &= \frac{1}{\Gamma(b-\mu)} \int_a^b (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(b-\mu)} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s}, \end{aligned}$$

thus, we have

$$d_1 = \frac{1}{\Gamma(b-\mu)\Delta} \int_a^b (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} - \frac{1}{\Gamma(b-\mu)\Delta} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s},$$

hence,

$$\begin{aligned}
\mathcal{W}(t) &= \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \int_a^b (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} - \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - {}^H I_{a+}^{b-\mu} \mathcal{A}(t) \\
&= \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \int_a^b (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} - \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(b-\mu)} \int_a^t (\ln t - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \int_a^b (\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \int_a^b (\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&= \int_a^b \mathcal{G}(t,s) \mathcal{A}(s) \frac{ds}{s} + \frac{1}{\Gamma(b-\mu)\Delta} \int_a^b (\ln \frac{t}{a})^{b-\mu-1} (\ln b - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \int_a^b (\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&= \int_a^b \mathcal{G}(t,s) \mathcal{A}(s) \frac{ds}{s} + \frac{(\ln \frac{b}{a})^{b-\mu-1} - \Delta}{\Delta \Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \int_a^b (\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&\quad - \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Gamma(b-\mu)\Delta} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \\
&= \int_a^b \mathcal{G}(t,s) \mathcal{A}(s) \frac{ds}{s} \\
&\quad + \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Delta} \left(\frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \sum_{j=1}^{\infty} \kappa_j \int_a^b (\ln \frac{\zeta_j}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \right. \\
&\quad \left. - \frac{1}{\Gamma(b-\mu)} \sum_{j=1}^{\infty} \kappa_j \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \right) \\
&= \int_a^b \mathcal{G}(t,s) \mathcal{A}(s) \frac{ds}{s} \\
&\quad + \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \left(\frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu-1}} \int_a^b (\ln \frac{\zeta_j}{a})^{b-\mu-1} (\ln \frac{b}{s})^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \right. \\
&\quad \left. - \frac{1}{\Gamma(b-\mu)} \int_a^{\zeta_j} (\ln \zeta_j - \ln s)^{b-\mu-1} \mathcal{A}(s) \frac{ds}{s} \right) \\
&= \int_a^b \mathcal{H}(t,s) \mathcal{A}(s) \frac{ds}{s}.
\end{aligned}$$

Therefore, the expression (2.4) is right. $\square$

Lemma 3. The properties of the Green function (2.4) are as follows:

- (i) $\mathcal{H}(t,s) \leq \frac{\omega(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})} w$;
- (ii) $\mathcal{H}(t,s) \geq \frac{\vartheta(t)\omega(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu+1}}$,

where

$$\vartheta(t) = (\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{t}), \varpi(t) = (\ln \frac{t}{a}) (\ln \frac{b}{t})^{b-\mu-1}, w = 1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j. \quad (2.6)$$

Proof. From Lemma 5 of [26], we get

$$\mathcal{G}(t, s) \leq \frac{\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})}, \mathcal{G}(t, s) \geq \frac{\vartheta(t)\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu+1}}, \quad (2.7)$$

where $\vartheta(t), \varpi(t)$ is as Lemma 3, $\mathcal{G}(t, s)$ is from (2.5). By (2.7), we get

$$\begin{aligned} \mathcal{H}(t, s) &= \mathcal{G}(t, s) + \frac{(\ln \frac{t}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \mathcal{G}(\xi_j, s) \\ &\leq \frac{\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})} + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \frac{\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})} \\ &\leq \frac{\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})} \left(1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \right) \\ &= \frac{\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})} w. \end{aligned}$$

On the other hand, we can calculate that

$$\mathcal{H}(t, s) \geq \mathcal{G}(t, s) \geq \frac{\vartheta(t)\varpi(s)}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu+1}},$$

then we have (ii) of Lemma 3. As a consequent, we complete the the proof of Lemma 3. $\square$

Let $E = C[a, b]$, $\|\mathcal{U}\| = \max_{a \leq t \leq b} |\mathcal{U}(t)|$, then $(\mathcal{E}, \|\cdot\|)$ is a Banach space. In this paper,

$$\mathcal{P} = \{u \in E : \mathcal{U}(t) \geq 0, t \in [a, b]\},$$

$$\mathcal{K} = \left\{ \mathcal{U} \in P : \mathcal{U}(t) \geq \frac{\vartheta(t)}{(\ln \frac{b}{a})^{b-\mu} w} \|\mathcal{U}\|, t \in [a, b] \right\},$$

where $w = 1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j$. Apparently, $\mathcal{K}$ is a subcone of P , and $(\mathcal{E}, \mathcal{K})$ has an ordering relation. Let $\mathcal{K}_r = \{\mathcal{U} \in \mathcal{K} : \|\mathcal{U}\| < r\}$, $\partial \mathcal{K}_r = \{\mathcal{U} \in \mathcal{K} : \|\mathcal{U}\| = r\}$ and $\overline{\mathcal{K}}_r = \{\mathcal{U} \in \mathcal{K} : \|\mathcal{U}\| \leq r\}$.

Now the following conditions is listed which will be used later.

(A₁) $q : (a, b) \rightarrow \mathbb{R}_+^1$ is nonnegative, $q(t) \not\equiv 0$ and $q(t)$ may be singular at $t = a$, $t = b$, and

$$\int_a^b \varpi(s) q(s) \frac{ds}{s} < +\infty.$$

(A₂) $\ell : [a, b] \times (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}_+^1$ is continuous, and for any $0 < \iota < j < +\infty$,

$$\begin{aligned} \limsup_{m \rightarrow +\infty} \left\{ \sup_{e(m)} \int_{e(m)} \varpi(s) q(s) \ell(s, x_1(s), x_2(s)) \frac{ds}{s} \right\}, \\ x_1 \in \overline{\mathcal{K}}_{\bar{j}} \setminus \mathcal{K}_{\iota}, x_2 \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_{\iota} \} = 0, \end{aligned}$$

where $e(m) = [0, \frac{1}{m}] \cup [\frac{m-1}{m}, 1]$, $\bar{j} = \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^{\mu} j$.

Nonlinear operator $A : \mathcal{K} \setminus \{0\} \rightarrow \mathcal{P}$ and linear operator $\mathcal{T} : \mathcal{E} \rightarrow \mathcal{E}$ are defined, shown as

$$(A\mathcal{U})(t) = \int_a^b \mathcal{H}(t,s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s}, \quad t \in [a, b], \quad (2.8)$$

$$(\mathcal{T}\mathcal{U})(t) = \int_a^b \mathcal{H}(t,s) \varrho(s) \mathcal{U}(s) \frac{ds}{s}, \quad t \in [a, b]. \quad (2.9)$$

Lemma 4 ([27]). Assume $\mathcal{T} : \mathcal{E} \rightarrow \mathcal{E}$, and $\mathcal{T}$ be a linear operator and is continuous, $\mathcal{T}(\mathcal{K}) \subset \mathcal{K}$, where $\mathcal{K}$ is a total cone. If there exist a positive constant d and $\zeta \in \mathcal{E} \setminus (-\mathcal{K})$ that makes $d\mathcal{T}(\zeta) \geq \zeta$, then the spectral radius of $\mathcal{T}$ be greater than 0, and which has a positive eigenfunction in regard to its the first eigenvalue $\lambda = r(\mathcal{T})^{-1}$. This formula is often called Krein-Rutmann's theorem.

Lemma 5 ([27]). The spectral radius of $\mathcal{T}$ meets

$$r(\mathcal{T}) = \lim_{n \rightarrow +\infty} \|\mathcal{T}^n\|^{\frac{1}{n}},$$

where $\mathcal{T}$ is a linear bounded operator, and $\|\cdot\|$ is the norm of operator. This formula is often called Gelfand's formula.

Lemma 6. $\mathcal{T} : \mathcal{K} \rightarrow \mathcal{K}$ defined by (2.9) is a linear operator with complete continuity under the condition (A_1) , and the spectral radius $r(\mathcal{T})$ of $\mathcal{T}$ is unequal zero, furthermore, $\mathcal{T}$ exists a positive eigenfunction ζ in regard to its first eigenvalue $\lambda_1 = (r(\mathcal{T}))^{-1}$.

Proof. For $\forall \mathcal{U} \in \mathcal{K}$, from Lemma 3, one arrives

$$\begin{aligned} \|\mathcal{T}\mathcal{U}\| &= \max_{t \in [a,b]} \int_a^b \mathcal{H}(t,s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \\ &\leq \frac{1}{\Gamma(b-\mu)(\ln \frac{b}{a})} \left(1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \right) \int_a^b \omega(s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \\ &\leq \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \int_a^b \omega(s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \end{aligned} \quad (2.10)$$

Moreover, by Lemma 3, the following result can be derived

$$\mathcal{T}\mathcal{U}(t) \geq \frac{\vartheta(t)}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu+1}} \int_a^b \omega(s) \varrho(s) \mathcal{U}(s) \frac{ds}{s}, \quad t \in [a, b]. \quad (2.11)$$

Formula (2.10) and Formula (2.11) mean $\mathcal{T}$ from $\mathcal{K}$ to $\mathcal{K}$. One finds that $\mathcal{T}$ has completely continuous property from $\mathcal{K}$ to $\mathcal{K}$ by combing (A_2) and the uniform continuous property of $\mathcal{H}(t,s)$ on $[a,b] \times [a,b]$.

Next, based on Krein-Rutmann's theorem, we prove that $\mathcal{T}$ has the first eigenvalue λ_1 and $\lambda_1 > 0$. In reality, by process of proof Lemma 3, there exists $t_0 \in (a,b)$ that makes $\mathcal{H}(t_0, t_0) > 0$. Therefore, there exists $[m,n] \subset (a,b)$ that makes $t_0 \in (m,n)$ and $\mathcal{H}(t,s) > 0$ for $\forall t,s \in [m,n]$. Select $\mathcal{U} \in K$ that makes $\mathcal{U}(t_0) > 0$ and $\mathcal{U}(t) = 0$ for all $t \notin [m,n]$. After that, for $\forall t \in [m,n]$, we have

$$(\mathcal{T}\mathcal{U})(t) = \int_a^b \mathcal{H}(t,s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \geq \int_m^n \mathcal{H}(t,s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} > 0.$$

Hence, there is $d > 0$ such that $d(\mathcal{T}\mathcal{U})(t) \geq \mathcal{U}(t)$ for $t \in [a, b]$, then the spectral radius $r(\mathcal{T}) \neq 0$ according to Lemma 4, furthermore, by Lemma 6, for the given first eigenvalue $\lambda_1 = (r(\mathcal{T}))^{-1}$, there exists a positive eigenfunction φ^* for A that makes

$$\lambda_1 \mathcal{T}\varphi^* = \varphi^*.$$

The proof is over. $\square$

Lemma 7. In case of $(A_1), (A_2)$ hold, then A acts as an operator with complete continuity from $\overline{\mathcal{K}}_J \setminus \mathcal{K}_I$ to $\mathcal{K}$.

Proof. First of all, we show $A(\overline{\mathcal{K}}_J \setminus \mathcal{K}_I) \subset \mathcal{K}$. Actually, for $\forall u \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I, t \in [a, b]$, and combining Lemma 3, one has

$$\begin{aligned} (A\mathcal{U})(t) &= \int_a^b \mathcal{H}(t, s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \\ &\leq \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \int_a^b \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s}, \end{aligned}$$

Then

$$\|(A\mathcal{U})\| \leq \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \int_a^b \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu u(s), u(s)\right) \frac{ds}{s}.$$

On the other side, the following result is obtained from Lemma 3

$$\begin{aligned} (A\mathcal{U})(t) &= \int_a^b \mathcal{H}(t, s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \\ &\geq \frac{\vartheta(t)}{\Gamma(b-\mu)(\ln \frac{b}{a})^{b-\mu+1}} \int_a^b \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \\ &\geq \frac{\vartheta(t)}{(\ln \frac{b}{a})^{b-\mu} w} \|A\mathcal{U}\|, t \in [a, b]. \end{aligned}$$

Consequently, $A(\overline{\mathcal{K}}_J \setminus \mathcal{K}_I) \subset \mathcal{K}$. Next, for $\forall \iota > 0$, we prove

$$\sup_{\mathcal{U} \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I} \int_a^b \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} < +\infty, \quad (2.12)$$

that is to say, $A : \overline{\mathcal{K}}_J \setminus \mathcal{K}_I \rightarrow \mathcal{K}$ is well defined. As a matter of fact, for $\forall \mathcal{U} \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I$, we have

$$\begin{aligned} {}^H I_{a+}^\mu \mathcal{U}(t) &= \frac{1}{\Gamma(\mu)} \int_a^t (\ln t - \ln s)^{\mu-1} \mathcal{U}(s) \frac{ds}{s} \\ &\leq -\frac{1}{\Gamma(\mu)} \|\mathcal{U}\| \int_a^t (\ln t - \ln s)^{\mu-1} d(\ln t - \ln s) \\ &= \frac{1}{\Gamma(\mu+1)} \|\mathcal{U}\| (\ln t - \ln s)^\mu \Big|_a^t = \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^\mu \|\mathcal{U}\| \\ &\leq \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^\mu J, t \in [a, b], \end{aligned} \quad (2.13)$$

Then $\| {}^H I_{a^+}^\mu \mathcal{U} \| \leq \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^\mu J$, hence, by (A_2) , there must be a non-negative integer $v_0 > 1$ such that

$$\sup_{\mathcal{U} \in \overline{K_J} \setminus K_i} \int_{e(v_0)} w(s) \varrho(s) \ell(s, {}^H I_{a^+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} < \frac{\Gamma(b-\mu)(\ln \frac{b}{a})}{w}. \quad (2.14)$$

where

$$w = 1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j.$$

Choosing

$$\varrho = \min \left\{ \frac{1}{(\ln \frac{b}{a})^{b-\mu-1} w}, \frac{(\ln a)^{b-\mu} (-1)^{b-\mu-1}}{\Gamma(\mu) \gamma (\ln \frac{b}{a})^{b-\mu} w}, 1 \right\},$$

$$\bar{\varrho} = \max \left\{ 1, \frac{(\ln \frac{b}{a})^\mu}{\Gamma(\mu+1)} \right\}.$$

Hence, for $\forall \mathcal{U} \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_i$, we get

$$u(t) \leq \|\mathcal{U}\| \leq J \leq \bar{\varrho} J, \quad (2.15)$$

$$\begin{aligned} \mathcal{U}(t) &\geq \frac{\vartheta(t) \|\mathcal{U}\|}{(\ln \frac{b}{a})^{b-\mu} w} = \frac{(\ln \frac{t}{a})^{b-\mu-1} (\ln \frac{b}{t}) \|\mathcal{U}\|}{(\ln \frac{b}{a})^{b-\mu} w} \\ &\geq \frac{(\ln t)^{b-\mu-1}}{(\ln \frac{b}{a})^{b-\mu-1} w} \|\mathcal{U}\| \geq (\ln t)^{b-\mu-1} \bar{\varrho} \|\mathcal{U}\|, \end{aligned} \quad (2.16)$$

by (2.13), we have

$${}^H I_{a^+}^\mu \mathcal{U}(t) \leq \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^\mu \|\mathcal{U}\| \leq \bar{\varrho} \|\mathcal{U}\|, t \in [a, b], \quad (2.17)$$

$$\begin{aligned} {}^H I_{a^+}^\mu \mathcal{U}(t) &= \frac{1}{\Gamma(\mu)} \int_a^t (\ln t - \ln s)^{\mu-1} \mathcal{U}(s) \frac{ds}{s} \\ &\geq \frac{1}{\Gamma(\mu)} \int_a^t (\ln t - \ln s)^{\mu-1} \frac{(\ln \frac{s}{a})^{b-\mu-1} (\ln \frac{b}{s})}{(\ln \frac{b}{a})^{b-\mu} w} \|\mathcal{U}\| \frac{ds}{s} \\ &= \frac{\|\mathcal{U}\|}{\Gamma(\mu) (\ln \frac{b}{a})^{b-\mu} w} \int_a^t (\ln t - \ln s)^{\mu-1} (\ln s - \ln a)^{b-\mu-1} (\ln b - \ln s) d(\ln s), \\ &\quad t \in [a, b]. \end{aligned} \quad (2.18)$$

If $\ln s = \Lambda \ln t$, one gets

$$\begin{aligned}
 & \frac{\|\mathcal{U}\|}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} \int_a^t (\ln t - \ln s)^{\mu-1} (\ln s - \ln a)^{b-\mu-1} (\ln b - \ln s) d(\ln s) \\
 &= \frac{\|\mathcal{U}\|}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} \int_{\frac{\ln a}{\ln t}}^1 (\ln t - \Lambda \ln t)^{\mu-1} (\iota \ln t - \ln a)^{b-\mu-1} (\ln b - \iota \ln t) d\tau \\
 &= \frac{\|\mathcal{U}\|}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} \int_{\frac{\ln a}{\ln t}}^1 (1 - \Lambda)^{\mu-1} (\iota \ln t - \ln a)^{b-\mu-1} (\ln b - \iota \ln t) d\tau \\
 &\geq \frac{\|\mathcal{U}\|}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} \int_0^1 (1 - \Lambda)^{\mu-1} (\Lambda \ln b - \ln a)^{b-\mu-1} (\ln b - \Lambda \ln a) d\tau \\
 &\geq \frac{\|\mathcal{U}\|(\ln a)^{b-\mu-1}}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} \int_0^1 (1 - \Lambda)^{\mu-1} (\Lambda - 1)^{b-\mu-1} \ln a (1 - \Lambda) d\tau \quad (2.19) \\
 &= \frac{\|\mathcal{U}\|(\ln a)^{b-\mu}}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} \int_0^1 (1 - \Lambda)^{\mu-1} (\Lambda - 1)^{b-\mu-1} d\tau \\
 &= \frac{\|\mathcal{U}\|(\ln a)^{b-\mu}}{\Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} (-1)^{b-\mu-1} \int_0^1 (1 - \Lambda)^{b-1} \\
 &= \frac{\|\mathcal{U}\|(\ln a)^{b-\mu}}{\gamma \Gamma(\mu)(\ln \frac{b}{a})^{b-\mu}w} (\ln t)^{\mu-1} (-1)^{b-\mu-1} \\
 &\geq \|\mathcal{U}\|(\ln t)^{\mu-1} \varrho \geq \|u\|(\ln t)^{b-\mu-1} \varrho, t \in [a, b].
 \end{aligned}$$

Thus, for $\forall a + \frac{1}{v_0} \leq t \leq b - \frac{1}{v_0}$, by (2.15)–(2.19), we obtain

$$r\varrho(\ln(a + \frac{1}{v_0}))^{b-\mu-1} \leq \mathcal{U}(t), {}^H I_{a+}^\mu \mathcal{U}(t) \leq \bar{\varrho}j. \quad (2.20)$$

Combing (2.14) and (2.20), we get

$$\begin{aligned}
 & \sup_{\mathcal{U} \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \int_a^b \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \\
 &\leq \sup_{\mathcal{U} \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \int_{e(v_0)} \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \\
 &+ \sup_{\mathcal{U} \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \int_{a+\frac{1}{v_0}}^{b-\frac{1}{v_0}} \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)\right) \frac{ds}{s} \quad (2.21) \\
 &\leq 1 + D_1 \int_a^b \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \frac{ds}{s} < +\infty,
 \end{aligned}$$

where

$$w = 1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \kappa_j,$$

$$D_1 = \max \left\{ \ell(t, \varsigma_1, \varsigma_2) : (t, \varsigma_1, \varsigma_2) \in \left(\frac{1}{v_0}, 1 - \frac{1}{v_0} \right) \times \left(\iota \varrho(\ln(a + \frac{1}{v_0}))^{b-\mu-1}, \bar{\varrho}j \right)^2 \right\}.$$

Thus, (2.12) is valid and indicating that A has the uniformly bounded property on any bounded set.

For $\forall \epsilon > 0$, by (A_2) , there must be a non-negative integer $v_0 > 1$ that makes

$$\sup_{x_1 \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i, x_2 \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \int_{e(v_0)} w(s) \varrho(s) \ell(s, x_1(s), x_2(s)) \frac{ds}{s} < \frac{\epsilon \Gamma(b-\mu)(\ln \frac{b}{a})}{4w}. \quad (2.22)$$

Then, we show that A is continuous from $\overline{\mathcal{K}}_J \setminus \mathcal{K}_I$ to $\mathcal{K}$. Let $\mathcal{U}_k, \mathcal{U}_0 \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I$ and $\|\mathcal{U}_k - \mathcal{U}_0\| \rightarrow 0$ ($k \rightarrow \infty$). On account of $\ell(t, x_1, x_2)$ has uniformly continuous property on

$$\left(a + \frac{1}{v_0}, b - \frac{1}{v_0}\right) \times \left(r\varrho(\ln(a + \frac{1}{v_0}))^{b-\mu-1}, \overline{\varrho}R\right)^2,$$

we have

$$\lim_{k \rightarrow +\infty} \left| \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) - \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_0(s), \mathcal{U}_0(s)\right) \right| = 0$$

Uniformly holds for $t \in \left[a + \frac{1}{v_0}, b - \frac{1}{v_0}\right]$. Then, by Lebesgue control convergence theorem and (A_2) , we have

$$\int_{a+\frac{1}{v_0}}^{b-\frac{1}{v_0}} w(s)\varrho(s) \left| \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) - \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_0(s), \mathcal{U}_0(s)\right) \right| \frac{ds}{s} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Thus, for the ϵ shown as in (2.18), there is a natural number N , for $k > N$, that makes

$$\begin{aligned} & \int_{a+\frac{1}{v_0}}^{b-\frac{1}{v_0}} w(s)\varrho(s) \left| \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) - \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_0(s), \mathcal{U}_0(s)\right) \right| \frac{ds}{s} \\ & < \frac{\epsilon \Gamma(b-\mu)(\ln \frac{b}{a})}{2w}. \end{aligned} \quad (2.23)$$

For $k > N$, by (2.22) and (2.23), one has

$$\begin{aligned} & \|A\mathcal{U}_k - A\mathcal{U}_0\| \\ & \leq \sup_{\mathcal{U}_k \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I} \int_{e(v_0)} \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \varrho(s) \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) \frac{ds}{s} \\ & \quad + \sup_{\mathcal{U}_k \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I} \int_{e(v_0)} \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \varrho(s) \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) \frac{ds}{s} \\ & \quad + \int_{a+\frac{1}{v_0}}^{b-\frac{1}{v_0}} \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} w(s)\varrho(s) \left| \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_k(s), \mathcal{U}_k(s)\right) \right. \\ & \quad \left. - \ell\left(s, {}^H I_{a+}^{\mu} \mathcal{U}_0(s), \mathcal{U}_0(s)\right) \right| \frac{ds}{s} \\ & < 2 \left(\frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \right) \times \frac{\epsilon}{4 \left(\frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \right)} + \left(\frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \right) \frac{\epsilon}{2 \left(\frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \right)} = \epsilon. \end{aligned}$$

Thus, A is continuous from $\overline{\mathcal{K}}_J \setminus \mathcal{K}_I$ to K .

For $\forall$ bounded set Ω and $\Omega \subset \overline{\mathcal{K}}_J \setminus \mathcal{K}_I$, we show that $A(\Omega)$ is equicontinuous. As a matter of fact, by (A_2) , for $\forall \epsilon > 0$, there is a natural number $\kappa_0 > 1$ so that

$$\sup_{x_2 \in \overline{\mathcal{K}}_J \setminus \mathcal{K}_I} \int_{e(\kappa_0)} w(s)\varrho(s) \ell(s, x_1(s), x_2(s)) \frac{ds}{s} < \frac{\epsilon}{4 \left(\frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \right)}.$$

Taking

$$D_2 = \max \left\{ \ell(t, x_1, x_2) : (t, x_1, x_2) \in \left(a + \frac{1}{\kappa_0}, b - \frac{1}{\kappa_0}\right) \times \left(r\varrho(\ln(a + \frac{1}{n_0}))^{b-\mu-1}, \overline{\varrho}J\right)^2 \right\}.$$

On account of $\mathcal{H}(t, s)$ is uniformly continuous on $[a, b] \times [a, b]$, for the ϵ which is defined by above, there is $\delta > 0$, for $\forall s \in [a + \frac{1}{\kappa_0}, b - \frac{1}{\kappa_0}]$, we have

$$|\mathcal{H}(t, s) - \mathcal{H}(t', s)| \leq \frac{\epsilon}{2} \left(D_2 \int_{a+\frac{1}{\kappa_0}}^{b-\frac{1}{\kappa_0}} \varrho(s) \frac{ds}{s} \right)^{-1},$$

for $|t - t'| < \delta, t, t' \in [a, b]$. Thus, for $\forall |t - t'| < \delta, t, t' \in [a, b]$ and $\mathcal{U} \in \Omega$, we obtain

$$\begin{aligned} & \|A\mathcal{U}(t) - A\mathcal{U}(t')\| \\ & \leq 2 \sup_{\mathcal{U} \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \left(\frac{w}{\Gamma(b - \mu)(\ln \frac{b}{a})} \right) \int_{e(\kappa_0)} w(s) \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\ & + \sup_{\mathcal{U} \in \overline{\mathcal{K}}_j \setminus \mathcal{K}_i} \int_{a+\frac{1}{\kappa_0}}^{b-\frac{1}{\kappa_0}} |\mathcal{H}(t, s) - \mathcal{H}(t', s)| \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\ & < 2 \left(\frac{w}{\Gamma(b - \mu)(\ln \frac{b}{a})} \right) \times \frac{\epsilon}{4(\frac{w}{\Gamma(b - \mu)(\ln \frac{b}{a})})} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which demonstrates that $A(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, A is fully continuous from $\overline{\mathcal{K}}_j \setminus \mathcal{K}_i$ to $\mathcal{K}$, then the verification is finished. $\square$

3. Principal Results

In this section, the main results will be provided according to these preliminaries and the following lemmas.

Lemma 8 ([27]). Suppose $\mathcal{K}$ is a cone within Banach space $\mathcal{E}$. Let $A : \overline{\mathcal{K}}_r \rightarrow \mathcal{K}$ is a fully continuous operator. In case of $\mathcal{U}_0 \in \mathcal{K} \setminus \{\theta\}$ so that $\mathcal{U} - A\mathcal{U} \neq \mu \mathcal{U}_0$ for an arbitrary $\mathcal{U} \in \partial \mathcal{K}_r$ and $\mu \geq 0$, thus $i(A, \mathcal{K}_r, \mathcal{K}) = 0$.

Lemma 9 ([27]). Suppose $\mathcal{K}$ is a cone within Banach space E . Let $A : \overline{\mathcal{K}}_r \rightarrow \mathcal{K}$ is a fully continuous operator. In case of $A\mathcal{U} \neq \mu \mathcal{U}$ for an arbitrary $\mathcal{U} \in \partial \mathcal{K}_r$ and $\mu \geq 1$, thus $i(A, \mathcal{K}_r, \mathcal{K}) = 1$.

Theorem 1. Suppose the conditions $(A_1 - A_2)$ are established, and

$$\liminf_{\substack{x_i \rightarrow 0^+ \\ i=1,2}} \min_{t \in [a,b]} \frac{\ell(t, x_1, x_2)}{x_1 + x_2} > \lambda_1, \quad (3.1)$$

$$\limsup_{x_2 \rightarrow +\infty} \max_{t \in [a,b]} \frac{\ell(t, x_1, x_2)}{x_2} < \lambda_1 \quad (3.2)$$

Hold uniformly for $x_i \in [0, +\infty)$ ($i = 1, 2$), and λ_1 is the first eigenvalue of $\mathcal{T}$ which is defined by (2.9), then there's at least one positive solution of the BVP (1.1, 1.2).

Proof. By (3.1), there exists $\iota > 0$, that makes

$$\ell(t, x_1, x_2) \geq \lambda_1(x_1 + x_2), 0 < x_i \leq \iota, i = 1, 2, t \in [a, b]. \quad (3.3)$$

Choosing $\iota_0 = \min\{\iota, \frac{\iota}{\bar{\varrho}}\}$ and for $\forall \mathcal{U} \in \partial \mathcal{K}_{\iota_0}$, one has

$$0 < {}^H I_{a+}^\mu \mathcal{U}(s) \leq \bar{\varrho} \iota_0 \leq \iota, 0 < \mathcal{U}(s) \leq \iota, s \in [a, b]. \quad (3.4)$$

Accordingly, one can arrive at according to (3.3) and (3.4)

$$\begin{aligned}
(A\mathcal{U})(t) &= \int_a^b \mathcal{H}(t,s) \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\
&\geq \lambda_1 \int_a^b \mathcal{H}(t,s) \varrho(s) \left({}^H I_{a+}^\mu \mathcal{U}(s) + \mathcal{U}(s) \right) \frac{ds}{s} \\
&\geq \lambda_1 (\mathcal{T}\mathcal{U})(t), t \in [a, b].
\end{aligned} \tag{3.5}$$

By Lemma 6, $\mathcal{T}$ has a positive eigenfunction ζ in regard to λ_1 , i.e., $\zeta = \lambda_1 \mathcal{T}\zeta$. Then we will show

$$\mathcal{U} - A\mathcal{U} \neq d\zeta, \mathcal{U} \in \partial\mathcal{K}_0, d \geq 0. \tag{3.6}$$

Otherwise, there is $\mathcal{U}_0 \in \partial\mathcal{K}_0$ and $d_0 \geq 0$ that makes $\mathcal{U}_0 - A\mathcal{U}_0 = d_0\zeta$, then $d_0 > 0$ and $\mathcal{U}_0 = A\mathcal{U}_0 + d_0\zeta \geq d_0\zeta$. Assume $\bar{d} = \sup\{d | \mathcal{U}_0 \geq d\zeta\}$, then $\bar{d} \geq d_0$, $\mathcal{U}_0 \geq \bar{d}\zeta$, $\lambda_1 \mathcal{T}\mathcal{U}_0 \geq \lambda_1 \bar{d} \mathcal{T}\zeta = \bar{d}\zeta$. Thus, from (3.5), we get

$$\mathcal{U}_0 = A\mathcal{U}_0 + d_0\zeta \geq \lambda_1 \mathcal{T}\mathcal{U}_0 + d_0\zeta \geq \bar{d}\zeta + d_0\zeta = (\bar{d} + d_0)\zeta,$$

which is a contradiction with the definition of $\bar{d}$. So (3.6) holds and combining Lemma 8, we have

$$i(A, \mathcal{K}_0, \mathcal{K}) = 0. \tag{3.7}$$

Now taking a constant $0 < \sigma < 1$ makes

$$\limsup_{x_2 \rightarrow +\infty} \max_{t \in [a,b]} \frac{\ell(t, x_1, x_2)}{x_2} < \omega \lambda_1, \tag{3.8}$$

and a linear operator $\widetilde{\mathcal{T}}\mathcal{U} = \omega \lambda_1 \mathcal{T}\mathcal{U}$ is defined, then subsequently $\widetilde{\mathcal{T}}$ is a bounded linear operator from $\mathcal{E}$ to $\mathcal{E}$, and $\widetilde{\mathcal{T}}(\mathcal{K}) \subset \mathcal{K}$. Additionally, $\widetilde{\mathcal{T}}\zeta = \omega \lambda_1 \mathcal{T}\zeta = \omega\zeta$, and so the spectral radius of $\widetilde{\mathcal{T}}$ is $r(\widetilde{\mathcal{T}}) = \omega$ and $\widetilde{\mathcal{T}}$ also has the first eigenvalue $r^{-1}(\widetilde{\mathcal{T}}) = \omega^{-1} > 1$. Adopting Gelfand's formula, one can obtain

$$\omega = \lim_{n \rightarrow +\infty} \|\widetilde{\mathcal{T}}^n\|^{\frac{1}{n}}. \tag{3.9}$$

Let $\varepsilon_0 = \frac{1}{2}(1 - \omega)$ and by (3.9), there is a large enough natural number N_0 , and when $n \geq N$, we have $\|\widetilde{\mathcal{T}}^n\| \leq [\omega + \varepsilon_0]^n$. For $\forall \mathcal{U} \in \mathcal{E}$, we define

$$\|\mathcal{U}\|^* = \sum_{i=1}^N [\omega + \varepsilon_0]^{N-i} \|\widetilde{\mathcal{T}}^{i-1}\mathcal{U}\|, \tag{3.10}$$

where $\widetilde{\mathcal{T}}^0 = I$ is the unit operator. Apparently, $\|\cdot\|^*$ is another norm of $\mathcal{E}$.

In addition, according to (3.2), there is $j_1 > \iota$, we have

$$\ell(t, x_1, x_2) \leq \omega \lambda_1 x_2, \text{ for } x_2 \geq j_1, x_i \geq 0, i = 1, 2, t \in [a, b]. \tag{3.11}$$

Choosing

$$J > \max \left\{ j_1, \frac{2(\omega + \varepsilon_0^{N-1})^{-1}}{\varepsilon_0} C^* \right\},$$

where $C^* = \|C\|^*$ and

$$\begin{aligned}
C &= \sup_{\mathcal{U} \in \mathcal{K}_{j_1}} \int_a^b \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\
&< +\infty \text{ (by (2.11))}.
\end{aligned} \tag{3.12}$$

Then we prove that

$$A\mathcal{U} \neq \mu \mathcal{U}, \mathcal{U} \in \partial\mathcal{K}_J, \mu \geq 1. \tag{3.13}$$

If not, then there is $\mathcal{U}_1 \in \partial\mathcal{K}_j$ and $d_1 \geq 1$ that makes $A\mathcal{U}_1 = d_1\mathcal{U}_1$. Assume $\widetilde{\mathcal{U}}(t) = \min\{\mathcal{U}_1(t), j_1\}$ and $D(\mathcal{U}_1) = \{t \in [a, b] : \mathcal{U}_1(t) > j_1\}$. Assume

$$\xi(t) = \frac{\Gamma(b-\mu)(\ln \frac{b}{a})}{w} \vartheta(t).$$

According to $\widetilde{\mathcal{U}} \in C([a, b], [0, +\infty))$, $\xi(t)j \leq \mathcal{U}_1(t) \leq \|\mathcal{U}_1\| = j$ and $A\mathcal{U}_1 = \mu_1\mathcal{U}_1$, we have $\mathcal{U}_1$ meets boundary conditions. Thus, we get $u_1(0) = 0$, hence, there is $0 < t_0 \leq 1$ that makes $u_1(t_0) = R$. Hence, $\widetilde{\mathcal{U}}(t) = \min\{\mathcal{U}_1(t), j_1\} \leq \min\{j, j_1\} = j_1$ for $t \in [a, b]$, and $\widetilde{\mathcal{U}}(t_0) = \min\{\mathcal{U}_1(t_0), j_1\} = \min\{j, j_1\} = j_1$, then we get $\|\widetilde{\mathcal{U}}\| = j_1$. Because

$$\widetilde{\mathcal{U}}(t) = \min\{\mathcal{U}_1(t), j_1\} \geq \min\{\xi(t)j, j_1\} \geq j_1\xi(t), \quad t \in [a, b],$$

Thus, $\widetilde{u} \in \partial\mathcal{K}_{j_1}$. For $t \in D(\mathcal{U}_1)$, $\mathcal{U}_1(t) \geq j_1$, ${}^H I_{a+}^\mu \mathcal{U}_1(t) \geq 0$, by (3.11) and Lemma 2, one gets

$$\begin{aligned} (A\mathcal{U}_1)(t) &= \int_a^b \mathcal{H}(t, s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}_1(s), \mathcal{U}_1(s)\right) \frac{ds}{s} \\ &\leq \int_{D(\mathcal{U}_1)} \mathcal{H}(t, s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}_1(s), \mathcal{U}_1(s)\right) \frac{ds}{s} \\ &\quad + \int_{[a, b] \setminus D(\mathcal{U}_1)} \mathcal{H}(t, s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \mathcal{U}_1(s), \mathcal{U}_1(s)\right) \frac{ds}{s} \\ &\leq \sigma \lambda_1 \int_a^b \mathcal{H}(t, s) \varrho(s) \mathcal{U}_1(s) \frac{ds}{s} \\ &\quad + \int_a^b \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \ell\left(s, {}^H I_{a+}^\mu \widetilde{\mathcal{U}}(s), \widetilde{\mathcal{U}}(s)\right) \frac{ds}{s} \\ &\leq (\widetilde{\mathcal{T}}\mathcal{U}_1)(t) + C, \quad t \in [a, b]. \end{aligned} \quad (3.14)$$

It is noted that $\widetilde{\mathcal{T}}$ is a bounded linear operator and from $\mathcal{H}$ to $\mathcal{K}$, from (3.14) one arrives

$$0 \leq (\widetilde{\mathcal{T}}^j(A\mathcal{U}_1))(t) \leq (\widetilde{\mathcal{T}}^j(\widetilde{\mathcal{T}}\mathcal{U}_1 + C))(t), \quad j = 0, 1, 2, \dots, n-1, \quad t \in [a, b]. \quad (3.15)$$

Then, by (3.15), we have

$$\|(\widetilde{\mathcal{T}}^j(A\mathcal{U}_1))\| \leq \|(\widetilde{\mathcal{T}}^j(\widetilde{\mathcal{T}}\mathcal{U}_1 + C))\|, \quad j = 0, 1, 2, \dots, n-1,$$

that gives rise to

$$\begin{aligned} \|A\mathcal{U}_1\|^\star &= \sum_{i=1}^n [\omega + \varepsilon_0]^{n-i} \|\widetilde{\mathcal{T}}^{i-1}(A\mathcal{U}_1)\| \\ &\leq \sum_{i=1}^n [\omega + \varepsilon_0]^{n-i} \|\widetilde{\mathcal{T}}^{i-1}(\widetilde{\mathcal{T}}\mathcal{U}_1 + C)\| = \|\widetilde{\mathcal{T}}\mathcal{U}_1 + C\|^\star. \end{aligned} \quad (3.16)$$

By $\mathcal{U}_1 \in \partial\mathcal{K}_j$, $\|\mathcal{U}_1\| = j$, and (3.10), the following formula can be developed

$$\|\mathcal{U}_1\|^\star > [\omega + \varepsilon_0]^{n-1} \|\mathcal{U}_1\| = [\omega + \varepsilon_0]^{n-1} R > \frac{2}{\varepsilon_0} C^\star,$$

which results in

$$C^\star < \frac{\varepsilon_0}{2} \|\mathcal{U}_1\|^\star. \quad (3.17)$$

According to (3.10), (3.16) and (3.17), one obtains

$$\begin{aligned}
 \mu_1 \|\mathcal{U}_1\|^* &= \|A\mathcal{U}_1\|^* \leq \|\widetilde{\mathcal{T}}\mathcal{U}_1\|^* + C^* = \sum_{i=1}^n [\sigma + \epsilon_0]^{n-i} \|\widetilde{\mathcal{T}}^i \mathcal{U}_1\| + C^* \\
 &= [\sigma + \epsilon_0] \sum_{i=1}^{n-1} [\sigma + \epsilon_0]^{n-i-1} \|\widetilde{\mathcal{T}}^i \mathcal{U}_1\| + \|\widetilde{\mathcal{T}}^N \mathcal{U}_1\| + C^* \\
 &\leq [\sigma + \epsilon_0] \sum_{i=1}^{N-1} [\sigma + \epsilon_0]^{n-i-1} \|\widetilde{\mathcal{T}}^i \mathcal{U}_1\| + [\sigma + \epsilon_0]^N \|\mathcal{U}_1\| + C^* \\
 &= [\sigma + \epsilon_0] \sum_{i=1}^N [\sigma + \epsilon_0]^{n-i} \|\widetilde{\mathcal{T}}^{i-1} \mathcal{U}_1\| + C^* \\
 &= [\sigma + \epsilon_0] \|\mathcal{U}_1\|^* + C^* \leq [\sigma + \epsilon_0] \|\mathcal{U}_1\|^* + \frac{\epsilon_0}{2} \|\mathcal{U}_1\|^* \\
 &= \left[\frac{1}{4}\sigma + \frac{3}{4} \right] \|\mathcal{U}_1\|^*.
 \end{aligned}$$

Taking care of $d_1 \geq 1$, we have $\frac{1}{4}\omega + \frac{3}{4} \geq 1$, and thus $\omega \geq 1$, which contradict with $0 < \omega < 1$. Thus (3.13) is right, and by Lemma 9, we have

$$i(A, \mathcal{K}_j, \mathcal{K}) = 1. \quad (3.18)$$

Combing (3.7) with (3.18), we have

$$i(A, \mathcal{K}_j \setminus \overline{\mathcal{K}_{i_0, j}}) = i(A, \mathcal{K}_j, \mathcal{K}) - i(A, \mathcal{K}_{i_0}, \mathcal{K}) = 1.$$

Hence, A has at least one fixed point in $K_R \setminus \overline{K}_{r_0}$, that is to say, the BVP (2.1, 2.2) has at least one positive solution, which implies that BVP (1.1, 1.2) also has at least one positive solution. Another case for Equations (1.1) and (1.2) will be discussed herein. For this purpose, for $\forall$ small enough $0 < \epsilon < 1$, we define a linear operator $\mathcal{T}_\epsilon$

$$(\mathcal{T}_\epsilon \mathcal{U})(t) = \int_{a+\epsilon}^{b-\epsilon} \mathcal{H}(t, s) q(s) \mathcal{U}(s) \frac{ds}{s}, \quad t \in [a, b].$$

According to Lemma 7, we have $\mathcal{T}_\epsilon$ from $\mathcal{K}$ to $\mathcal{K}$ denotes a linear operator with complete continuity, too, and the spectral radius $r(\mathcal{T}_\epsilon)$ of $\mathcal{T}_\epsilon$ unequal to 0, and furthermore, $\mathcal{T}_\epsilon$ has a positive eigenfunction ζ_ϵ in regard to its first eigenvalue $\lambda_\epsilon = (r(\mathcal{T}_\epsilon))^{-1}$. $\square$

Lemma 10. Assume that (A_1) holds, then $\mathcal{T}$ has an eigenvalue $\tilde{\lambda}_1$ that makes

$$\lim_{\epsilon \rightarrow 0^+} \lambda_\epsilon = \tilde{\lambda}_1.$$

Proof. Choose $\dots \leq \epsilon_n \leq \dots \leq \epsilon_2 \leq \epsilon_1$ and ϵ_n tend to 0 as $n \rightarrow +\infty$. Then for $\forall n < m$ and $\zeta \in \mathcal{E}$, we get

$$(\mathcal{T}_{\epsilon_n} \zeta)(t) \leq (\mathcal{T}_{\epsilon_m} \zeta)(t) \leq (\mathcal{T} \zeta)(t), \quad t \in [a, b],$$

and

$$(\mathcal{T}_{\epsilon_n}^k \zeta)(t) \leq (\mathcal{T}_{\epsilon_m}^k \zeta)(t) \leq (\mathcal{T}^k \zeta)(t), \quad t \in [a, b], \quad k = 2, 3, \dots,$$

where $\mathcal{T}_{\epsilon_n}^k = \mathcal{T}(\mathcal{T}_{\epsilon_n}^{k-1})$ ($k = 2, 3, \dots$). Thus, $\|\mathcal{T}_{\epsilon_n}^k\| \leq \|\mathcal{T}_{\epsilon_m}^k\| \leq \|\mathcal{T}^k\|$ ($k = 1, 2, \dots$). By Gelfand's formula, we obtain $\lambda_1 \leq \lambda_{\epsilon_m} \leq \lambda_{\epsilon_n}$, and λ_1 is the first eigenvalue of $\mathcal{T}$. According to $\{\lambda_{\epsilon_n}\}$ monotonous property with lower boundedness λ_1 , assume

$$\lambda_{\epsilon_n} \rightarrow \tilde{\lambda}_1 (n \rightarrow +\infty).$$

Now we prove that $\tilde{\lambda}_1$ is one of eigenvalue of $\mathcal{T}$. Assume ζ_{ε_n} is one of positive eigenfunction of $\mathcal{T}_{\varepsilon_n}$ in regard to λ_{ε_n} with $\|\zeta_{\varepsilon_n}\| = 1$ ($n = 1, 2, \dots$), that is,

$$\varphi_{\varepsilon_n}(t) = \lambda_{\varepsilon_n} \int_{a+\varepsilon_n}^{b-\varepsilon_n} \mathcal{H}(t, s) \varrho(s) \zeta_{\varepsilon_n}(s) \frac{ds}{s} = \lambda_{\varepsilon_n} \mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}(t), \quad t \in [a, b]. \quad (3.19)$$

Take notice that

$$\|\mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}\| = \max_{a \leq t \leq b} \int_{a+\varepsilon_n}^{b-\varepsilon_n} \mathcal{H}(t, s) \varrho(s) \zeta_{\varepsilon_n}(s) \frac{ds}{s} \leq \int_a^b \frac{w}{\Gamma(b-\mu)(\ln \frac{b}{a})} \omega(s) \varrho(s) \frac{ds}{s}, \quad n = 1, 2, \dots,$$

and hence $\{\mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}\} \subset \mathcal{E}$ is uniform boundedness. In addition, for $\forall n \in \mathbb{N}$ and $t_1, t_2 \in [a, b]$, one arrives

$$|\mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}(t_1) - \mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}(t_2)| \leq \int_{a+\varepsilon_n}^{b-\varepsilon_n} |\mathcal{H}(t_1, s) - \mathcal{H}(t_2, s)| \varrho(s) \zeta_{\varepsilon_n}(s) \frac{ds}{s}.$$

Since $\mathcal{H}(t, s)$ has uniform continuity property on $[a, b] \times [a, b]$, we have $\{\mathcal{T}_{\varepsilon_n} \zeta_{\varepsilon_n}\} \subset \mathcal{E}$ is equicontinuous. Combining the Arzela-Ascoli theorem, Lemma 3.3 and $\lambda_{\varepsilon_n} \rightarrow \tilde{\lambda}_1$ ($n \rightarrow +\infty$), we have $\zeta_{\varepsilon_n} \rightarrow \zeta_0$ ($n \rightarrow +\infty$). This implies $\|\zeta_0\| = 1$, then by using (3.19), we get

$$\zeta_0(t) = \tilde{\lambda}_1 \int_a^b \mathcal{H}(t, s) \varrho(s) \zeta_0(s) \frac{ds}{s}, \quad t \in [a, b],$$

in short, $\zeta_0 = \tilde{\lambda}_1 \mathcal{T} \zeta_0$. $\square$

Theorem 2. Suppose (A_1-A_2) hold, and

$$\limsup_{\substack{x_i \rightarrow 0^+ \\ i=1,2}} \max_{t \in [a,b]} \frac{\ell(t, x_1, x_2)}{x_2} < \lambda_1, \quad (3.20)$$

$$\liminf_{x_1+x_2 \rightarrow +\infty} \min_{t \in [a,b]} \frac{\ell(t, x_1, x_2)}{x_1 + x_2} > \tilde{\lambda}_1, \quad (3.21)$$

where λ_1 represents the first appeared eigenvalue of $\mathcal{T}$ and $\tilde{\lambda}_1$ is another eigenvalue of $\mathcal{T}$, then the BVP (1.1,1.2) has at least one positive solution.

Proof. First of all, from (3.20), for $\forall t \in [a, b]$, there is $\iota_0 > 0$ such that

$$\ell(t, x_1, x_2) \leq \lambda_1 x_2, 0 < x_1 \leq \frac{(\ln b - \ln a)^\mu}{\Gamma(\mu+1)} \iota_0, 0 < x_2 \leq \iota_0, \quad t \in [a, b]. \quad (3.22)$$

Hence, for any $\mathcal{U} \in \partial K_{\iota_0}$, noticing ${}^H I_{a^+}^\mu \mathcal{U}(s) \leq \frac{(\ln b - \ln a)^\mu}{\Gamma(\mu+1)} \iota_0$, $|\mathcal{U}(s)| \leq \|\mathcal{U}\| = \iota_0$. By (3.22), we get

$$\begin{aligned} (A\mathcal{U})(t) &= \int_a^b \mathcal{H}(t, s) \varrho(s) \ell(s, {}^H I_{a^+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\ &\leq \lambda_1 \int_a^b \mathcal{H}(t, s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \\ &= \lambda_1 (\mathcal{T}\mathcal{U})(t), \quad t \in [a, b]. \end{aligned} \quad (3.23)$$

As a matter of fact, we assume there are no fixed points for on A on $\partial \mathcal{K}_{\iota_0}$. It is necessary to prove

$$A\mathcal{U} \neq d\mathcal{U}, \quad \text{for any } \mathcal{U} \in \partial \mathcal{K}_{\iota_0}, \quad d \geq 1. \quad (3.24)$$

If not, there is $\mathcal{U}_0 \in \partial \mathcal{K}_{\iota_0}$ and $d_0 \geq 1$ such that $A\mathcal{U}_0 = d_0 \mathcal{U}_0$. Then $d_0 > 1$ and from (3.23), we obtain

$$d_0 \mathcal{U}_0 = A\mathcal{U}_0 \leq \lambda_1 \mathcal{T} \mathcal{U}_0. \quad (3.25)$$

According to induction from (3.25), we achieve

$$d_0^n \mathcal{U}_0 \leq \lambda_1^n \mathcal{T}^n \mathcal{U}_0, \quad n = 1, 2, \dots,$$

we easily get

$$\|\mathcal{T}^n\| \geq \frac{\|\mathcal{T}^n \mathcal{U}_0\|}{\|\mathcal{U}_0\|} \geq \frac{\mu_0^n \|\mathcal{U}_0\|}{\lambda_1^n \|\mathcal{U}_0\|} = \frac{\mu_0^n}{\lambda_1^n}.$$

Additionally, one derives the following result through the Gelfand formula

$$r(\mathcal{T}) = \lim_{n \rightarrow \infty} \sqrt[n]{\|\mathcal{T}^n\|} \geq \frac{\mu_0}{\lambda_1} > \frac{1}{\lambda_1},$$

but it contradicts with $r(\mathcal{T}) = \lambda_1^{-1}$. So (3.23) holds. Applying Lemma 3.2, one demonstrates

$$i(A, \mathcal{K}_{l_0}, \mathcal{K}) = 1. \quad (3.26)$$

According to (3.21) and $\lambda_\varepsilon \rightarrow \tilde{\lambda}_1 (\varepsilon \rightarrow 0^+)$, it exists a sufficiently tiny $\varepsilon \in (0, \frac{1}{2})$ and $j > \iota$, we have

$$\ell(t, x_1, x_2) \geq \lambda_\varepsilon (x_1 + x_2) \geq \rho_\varepsilon j, \quad t \in [a, b], \quad (3.27)$$

where λ_ε represents the first appeared eigenvalue $\mathcal{T}_\varepsilon$, $\rho_\varepsilon = 2(\ln(a + \varepsilon))^{b-\mu-1}$. Assume ζ_ε be the positive eigenfunction of $\mathcal{T}_\varepsilon$ with respect to λ_ε , then $\zeta_\varepsilon = \lambda_\varepsilon \mathcal{T}_\varepsilon \zeta_\varepsilon$.

For an any $\mathcal{U} \in \partial \mathcal{K}_j$, $s \in [a + \varepsilon, b - \varepsilon]$, by (2.13)–(2.15), we achieve

$$\begin{aligned} {}^H I_{a+}^\mu \mathcal{U}(s) + \mathcal{U}(s) &= \frac{1}{\Gamma(\mu)} \int_a^t (\ln t - \ln s)^{\mu-1} \mathcal{U}(s) \frac{ds}{s} + \mathcal{U}(s) \\ &\geq \|\mathcal{U}\| (\ln t)^{b-\mu-1} \varrho + \|\mathcal{U}\| (\ln t)^{b-\mu-1} \varrho \\ &\geq 2\|\mathcal{U}\| \varrho (\ln(a + \varepsilon))^{b-\mu-1} = \rho_\varepsilon j. \end{aligned} \quad (3.28)$$

Combing (3.27) with (3.28), one has

$$\begin{aligned} (A\mathcal{U})(t) &= \int_a^b \mathcal{H}(t, s) \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\ &\geq \int_{a+\varepsilon}^{b-\varepsilon} \mathcal{H}(t, s) \varrho(s) \ell(s, {}^H I_{a+}^\mu \mathcal{U}(s), \mathcal{U}(s)) \frac{ds}{s} \\ &\geq \lambda_\varepsilon \int_{a+\varepsilon}^{b-\varepsilon} \mathcal{H}(t, s) \varrho(s) \left({}^H I_{a+}^\mu \mathcal{U}(s) + \mathcal{U}(s) \right) \frac{ds}{s} \\ &\geq \lambda_\varepsilon \int_{a+\varepsilon}^{b-\varepsilon} \mathcal{H}(t, s) \varrho(s) \mathcal{U}(s) \frac{ds}{s} \\ &= \lambda_\varepsilon (\mathcal{T}_\varepsilon \mathcal{U})(t), \quad t \in [a, b]. \end{aligned}$$

By the similar method with the proof of Theorem 1, one gets

$$\mathcal{U} - A\mathcal{U} \neq d\zeta_\varepsilon, \quad \mathcal{U} \in \partial \mathcal{K}_j, \quad d \geq 0,$$

by using Lemma 8, one arrives at

$$i(A, \mathcal{K}_j, \mathcal{K}) = 0. \quad (3.29)$$

Combing (3.26) with (3.29), one obtains

$$i(A, \mathcal{K}_j \setminus \overline{\mathcal{K}_{l_0}}, \mathcal{K}) = i(A, \mathcal{K}_j, \mathcal{K}) - i(A, \mathcal{K}_{l_0}, \mathcal{K}) = -1.$$

Hence, A at least has one fixed point in $K_R \setminus \overline{K_{r_0}}$, which means BVP (2.1,2.2) at least has a positive solution, i.e., the BVP (1.1,1.2) has at least one positive solution. The proof is completed. $\square$

4. Example

Example 1. For a boundary value problem show as follows

$$\begin{cases} -{}^H D_{a+}^{\frac{5}{2}} v(t) = g(t) \ell \left(t, v(t), {}^H D_{a+}^{\frac{1}{4}} v(t) \right), & a < t < b, \\ {}^H D_{a+}^{\frac{1}{4}} v(a) = {}^H D_{a+}^{\frac{5}{4}} v(a) = 0, \\ {}^H D_{a+}^{\frac{1}{4}} v(b) = \sum_{j=1}^{\infty} \eta_j {}^H D_{a+}^{\frac{1}{4}} v(\zeta_j), \end{cases} \quad (4.1)$$

where $b = \frac{5}{2}$, $\mu = \frac{1}{4}$, $a = 1$, $b = 3$, $\kappa_j = \frac{1}{16} \frac{1}{j^4}$, $\zeta_j = e^{\frac{1}{j^4}}$, $\varrho(t) = \frac{1}{3(1-t)(t-3)}$, $\ell(t, x, y) = (x + y)^{-\frac{1}{3}} + |\ln y|$. Clearly, $\varrho(t)$ is singular at $t = a$ or/and $t = b$, $\ell(t, x, y)$ is singular at $x = y = 0$. Let $\mathcal{U}(t) = {}^H D_{a+}^{\frac{1}{4}} \mathcal{V}(t)$, the BVP (4.1) can be reduced to an amended boundary value problem shown as follows

$$\begin{cases} -{}^H D_{a+}^{\frac{9}{4}} \mathcal{U}(t) = \varrho(t) \ell(t, {}^H I_{a+}^{\frac{1}{4}} \mathcal{U}(t), \mathcal{U}(t)), & a < t < b, \\ \mathcal{U}(a) = \mathcal{U}'(a) = 0, \mathcal{U}(b) = \sum_{j=1}^{\infty} \eta_j \mathcal{U}(\zeta_j), \end{cases} \quad (4.2)$$

by simple calculations, we have

$$\Delta = \left(\ln \frac{b}{a} \right)^{b-\mu-1} - \sum_{j=1}^{\infty} \kappa_j \left(\ln \frac{\zeta_j}{a} \right)^{b-\mu-1} = 0.1554,$$

clearly, we have

$$\mathcal{G}(t, s) = \frac{1}{(\ln 3)^{\frac{5}{4}} \Gamma(\frac{9}{4})} \begin{cases} (\ln t)^{\frac{5}{4}} (\ln 3 - \ln s)^{\frac{5}{4}} - (\ln t - \ln s)^{\frac{5}{4}} (\ln 3)^{\frac{5}{4}}, & 1 \leq s \leq t \leq 3, \\ (\ln t)^{\frac{5}{4}} (\ln 3 - \ln s)^{\frac{5}{4}}, & 1 \leq t \leq s \leq 3, \end{cases} \quad (4.3)$$

$$\mathcal{H}(t, s) = \mathcal{G}(t, s) + \frac{(\ln t)^{\frac{5}{4}}}{\Delta} \sum_{j=1}^{\infty} \kappa_j \mathcal{G}(\zeta_j, s),$$

$$w = 1 + \frac{(\ln \frac{b}{a})^{b-\mu-1}}{\Delta} \sum_{j=1}^{\infty} \eta_j = 1 + \frac{(\ln 3)^{\frac{5}{4}}}{0.1554} \sum_{j=1}^{\infty} \frac{1}{16} \frac{1}{j^4} \approx 1.4895,$$

then the cone

$$\mathcal{K} = \{ \mathcal{U} \in C[1, e] : \mathcal{U}(t) \geq \frac{\vartheta(t)}{(\ln 3)w} \|\mathcal{U}\| \geq 0.6150 \vartheta(t) \|\mathcal{U}\| \}.$$

For any $0 < r < R < +\infty$ and $u \in \overline{\mathcal{K}}_R \setminus \mathcal{K}_r$, we have

$$(\ln t)^{\frac{5}{4}} \varrho r \leq \mathcal{U}(t) \leq R \bar{\varrho},$$

$$(\ln t)^{\frac{5}{4}} \varrho r \leq \|u\| (\ln t)^{b-\mu-1} \varrho \leq v(t) = {}^H I_{1+}^{\frac{1}{4}} \mathcal{U}(t) \leq \frac{1}{\Gamma(\frac{5}{4})} (\ln 3)^{\frac{1}{4}} \|\mathcal{U}\| \leq \bar{\varrho} \|\mathcal{U}\|, \quad t \in [1, 3],$$

$$\begin{aligned} \varrho &= \min \left\{ \frac{1}{(\ln \frac{b}{a})^{\gamma-\mu-1} w}, \frac{(\ln a)^{b-\mu} (-1)^{b-\mu-1}}{\Gamma(\mu) \gamma (\ln \frac{b}{a})^{b-\mu} w}, 1 \right\} = \min \left\{ \frac{1}{(\ln 3)^{\frac{5}{4}} w}, \frac{(-1)^{\frac{5}{4}}}{\Gamma(\frac{1}{4}) \gamma (\ln 3)^{\frac{9}{4}} w}, 1 \right\}, \\ \bar{\varrho} &= \max \left\{ 1, \frac{(\ln \frac{b}{a})^{\mu}}{\Gamma(\mu+1)} \right\} = \max \left\{ 1, \frac{(\ln 3)^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \right\}. \end{aligned} \quad (4.4)$$

Since $|\ln u|$ decreases on $(0, 1)$, and increases on $(1, +\infty)$, one must have

$$|\ln u(t)| \leq 2|\ln R| + |(\ln t)^4|, t \in [1, e],$$

$$[x(t) + y(t)]^{-\frac{1}{3}} \leq \left[(\ln t)^4 r + \frac{B(5, \frac{1}{2})}{\Gamma(\mu)} r (\ln t)^{-\frac{1}{2}} \right]^{-\frac{1}{3}}, t \in [1, e].$$

where $B(5, \frac{1}{2})$ is a Beta function. By considering the absolute continuity of the obtained integral, one can derive that

$$\lim_{m \rightarrow \infty} \int_{e(m)} \omega(s) \varrho(t) \left((\ln t)^4 r + \left[(\ln t)^4 r + B(5, \frac{1}{2}) r (\ln t)^{-\frac{1}{2}} \right]^{-\frac{1}{3}} \right) \frac{ds}{s} = 0.$$

Hence,

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \sup_{\substack{x_1 \in \bar{K}_{\bar{R} \setminus K_r} \\ x_2 \in \bar{K}_R \setminus K_r}} \int_{e(m)} \omega(s) \varrho(s) f(s, x_1(s), x_2(s)) \frac{ds}{s} \\ & \leq \limsup_{m \rightarrow \infty} \sup_{\substack{x_1 \in \bar{K}_{\bar{R} \setminus K_r} \\ x_2 \in \bar{K}_R \setminus K_r}} \int_{e(m)} \left(\ln \frac{s}{a} \right) \left(\ln \frac{b}{s} \right)^{b-\mu-1} \frac{1}{3(1-s)(s-3)} \left[(x_1 + x_2)^{-\frac{1}{3}} + |\ln x_2| \right] \frac{ds}{s} \\ & \leq \limsup_{m \rightarrow \infty} \int_{e(m)} \left(\ln \frac{s}{a} \right) \left(\ln \frac{b}{a} \right)^{\frac{5}{4}} \frac{1}{3(1-s)(s-3)} \left((\ln t)^4 r + \left[(\ln t)^4 r + B(5, \frac{1}{2}) r (\ln t)^{-\frac{1}{2}} \right]^{-\frac{1}{3}} \right) \frac{ds}{s} = 0, \end{aligned}$$

where $\bar{R} = \frac{1}{\Gamma(\mu+1)} (\ln \frac{b}{a})^\mu R = \frac{1}{\Gamma(\frac{5}{4})} (\ln 3)^{\frac{1}{4}} R$. So (H_1) holds. On the other hand, it is obvious that

$$\begin{aligned} & \liminf_{\substack{x_1 \rightarrow 0^+ \\ x_2 \rightarrow 0^+}} \inf_{t \in [1, 3]} \frac{f(t, x_1, x_2)}{x_1 + x_2} = \liminf_{\substack{x_1 \rightarrow 0^+ \\ x_2 \rightarrow 0^+}} \frac{(x_1 + x_2)^{-\frac{1}{3}} + |\ln x_2|}{x_1 + x_2} = +\infty, \\ & \limsup_{\substack{x_1 + x_2 \rightarrow +\infty \\ x_2 \rightarrow +\infty}} \sup_{t \in [1, 3]} \frac{f(t, x_1, x_2)}{x_2} = \limsup_{\substack{x_1 + x_2 \rightarrow +\infty \\ x_2 \rightarrow +\infty}} \sup_{t \in [1, 3]} \frac{((x_1 + x_2)^{-\frac{1}{3}} + |\ln x_2|)}{x_2} = 0, \end{aligned}$$

which implies that

$$\liminf_{\substack{x_1 \rightarrow 0^+ \\ x_2 \rightarrow 0^+}} \inf_{t \in [1, 3]} \frac{f(t, x_1, x_2)}{x_1 + x_2} > \lambda_1 > \limsup_{\substack{x_1 + x_2 \rightarrow +\infty \\ x_2 \rightarrow +\infty}} \sup_{t \in [1, 3]} \frac{f(t, x_1, x_2)}{x_2}.$$

So far, all the condition proposed in Theorem 1 have been meet. Consequently, Theorem 1 ensures that there must be one or more positive solutions to Equation (4.2). Equivalently, one or more positive solutions can be determined for Equation (4.2). It is worth mentioning that the calculation of fractional derivatives generally involves the beta function. In this example, because a specific explicit expression is provided, the beta function appears. However, in the general setting, if the function is abstract, then such the beta function will not appear.

5. Conclusions

The present study was concerned with the positive solutions and their existence for a nonlinear infinite-point Hadamard fractional differential equation. Using the fixed point theorem in cones, the nonlinear singularities on time and spatial variables are achieved through the spectral analysis of the relevant linear operator and Gelfand's formula. Compared with some existing literature discussing the same interval we derived herein, we overcome a significant difficulty that the consideration of solutions on this interval brings

about a series of problems. Compared with some other existing literature, the nonlinear term of the equation contains the derivative term, and the infinite point is involved in the boundary conditions. These are the main contributions and innovations of the present research.

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