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Existence of Mild Solutions to Delay Diffusion Equations with Hilfer Fractional Derivative

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Abstract: Because of the prevalent time-delay characteristics in real-world phenomena, this paper investigates the existence of mild solutions for diffusion equations with time delays and the Hilfer fractional derivative. This derivative extends the traditional Caputo and Riemann–Liouville fractional derivatives, offering broader practical applications. Initially, we constructed Banach spaces required to handle the time-delay terms. To address the challenge of the unbounded nature of the solution operator at the initial moment, we developed an equivalent continuous operator. Subsequently, within the contexts of both compact and non-compact analytic semigroups, we explored the existence and uniqueness of mild solutions, considering various growth conditions of nonlinear terms. Finally, we presented an example to illustrate our main conclusions.

Keywords: mild solutions; fractional diffusion equation; fixed-point theorems; delay

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1. Introduction

Differential equations with fractional derivatives play a pivotal role across a wide spectrum of fields such as the natural sciences, engineering, biological sciences, finance, and physics, capturing the interest of numerous scholars [1–7]. The phenomenon of time delay, a ubiquitous occurrence in the fabric of real-life scenarios, has spurred a significant body of research. In recent years, growing interest has been observed in exploring the characteristics of solutions to delay differential equations, particularly focusing on the existence and stability of solutions involving the Caputo fractional derivative, as highlighted in studies [8–10].

The derivative, known as the Hilfer derivative, extends numerous derivatives including the Caputo derivative and the Riemann–Liouville derivative, offering a broad range of application value. Compared with the others, there is relatively little research on the existence and controllability of solutions to delay differential equations with the Hilfer fractional derivative. Kavitha and Vijayakumar [11] studied the existence of mild solutions to the following equation in a Hilbert space H

$$\begin{cases} D_{0+}^{\mu,\nu}u(t) = Au(t) + A_1u(t-\tau) + Bu(t) + f(t,u(t-\tau)), & t \in (0,\infty), \\ I_{0+}^{(1-\mu)(1-\nu)}u(t) = \varphi(t), & t \in [-\tau,0], \end{cases}$$

where $\mu \in [0,1]$ and $\nu \in (0,1)$, B is a bounded operator, K is another Hilbert space, f is continuous, and there exist a positive constant $\nu_1 \in (0,\nu)$and$m \in L^{\frac{1}{\nu_1}}(K,\mathbb{R}^+)$, such that for every $u_1,u_2 \in H$, $\| f(t,u_1) - f(t,u_2)\| \leq mt^{(1-\mu)(1-\nu)}\| u_1 - u_2 \|_H$. The existence and controllability of mild solutions to fractional differential equations with other types of delays and the Hilfer fractional derivative can be found in [12–14].

Motivated by the literature above, we consider the existence of mild solutions of the following Dirichlet-type initial-boundary-value problem

$$\begin{cases} D_{0+}^{\mu,\nu}u(x,t) = Au(x,t) + f(x,t,u(x,t-\tau_0),u(x,t-\tau_1),\cdots, \\ u(x,t-\tau_n)), \qquad x\in\Omega, \quad t\in(0,b], \\ u|_{\partial\Omega}=0, \\ I_{0+}^{(1-\mu)(1-\nu)}u(x,0^+)=\varphi(x,0), \qquad x\in\Omega, \\ u(x,t)=\frac{t^{(\mu-1)(1-\nu)}\varphi(x,t)}{\Gamma(\mu(1-\nu)+\nu)}, \qquad x\in\Omega, \quad t\in[-r,0), \end{cases} \tag{1}$$

where $D_{0+}^{\mu,\nu}$ represents the Hilfer fractional derivative, $\mu,\nu\in(0,1)$, $I_{0+}^{(1-\mu)(1-\nu)}u(x,0^+)$ is the limit of $I_{0+}^{(1-\mu)(1-\nu)}u(x,t)$ as $t\to 0^+$, $b>0$, $\Omega\subset\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. f and φ are given functions, $\tau_0=0$, $\tau_1,\tau_2,\cdots,\tau_n\in[0,r]$, $r>0$.

$$Au(x,t)=\sum_{p=1}^{n}\sum_{q=1}^{n}\partial_{x_p}\left[a_{pq}(x)\partial_{x_q}u(x,t)\right]-b(x)u(x,t), \tag{2}$$

where a_{pq} and b are real valued functions that satisfy

$$a_{pq}\in C^1(\overline{\Omega}), \qquad 1\le p,q\le n,$$

$$\sum_{p,q=1}^{n}a_{pq}(x)\vartheta_p\vartheta_q\ge\varsigma|\vartheta|^2, \qquad \vartheta\in\mathbb{R}^n, \quad x\in\overline{\Omega},$$

$$b\in C(\overline{\Omega}), \qquad b(x)\ge b_0>0, \quad x\in\overline{\Omega},$$

with some constant $\varsigma>0$.

The introduction of n delay terms increases the complexity when studying the existence of solutions. To address this problem, we need to consider the specific continuous function space $\mathcal{C}_{[-r,0]}^{n}=\mathcal{C}_{[-r,0]}\times\mathcal{C}_{[-r,0]}\times\cdots\times\mathcal{C}_{[-r,0]}$. Because of the unboundedness and continuity of solutions to equations containing Hilfer fractional derivatives at zero, we examine the initial value of the Hilfer fractional diffusion equation with delay in the form of $\frac{t^{(\mu-1)(1-\nu)}\varphi(x,t)}{\Gamma(\mu(1-\nu)+\nu)}$ on the interval $[-r,0)$, and introduce a new solution operator to ensure the meaningfulness of the studied equation's solutions at zero.

In the main results, we initially assume the compactness of the analytic semigroup and relax the continuity condition for the function f required by reference [11], demanding instead that f be continuous in other variables for almost all of the time variables. Additionally, we assume that the norm of f is governed by the $L^{\frac{1}{\nu_1}}((0,b],R^+)$ norm of delayed terms with $\nu_1\in[0,\nu)$. Based on these assumptions, we employ the Leray–Schauder fixed-point theorem to demonstrate the existence of mild solutions. On this basis, we do not impose compactness on the analytic semigroup and we stipulate that the measure of f is controlled by the measures of delayed terms. Then, we utilize a non-compact measure approach to further prove the existence of mild solutions. Lastly, we assume that the norms of f satisfy a Lipschitz condition in another space Y. Following this, we apply the Banach contraction mapping principle to prove the existence and uniqueness of mild solutions. However, the interval $\mu\in[0,1]$ in [11] is not applicable in this study because the proof of the strong continuity of $R_{\mu,\nu}(t)=t^{(1-\mu)(1-\nu)}S_{\mu,\nu}(t)$ necessitates the condition that μ is not zero. For simplicity, this paper restricts μ to the interval $(0,1)$. Moreover, Theorem 3.3 in the Gu and Trujillo [15] represents a special case of Theorem 2 in this paper when f has no time delays.

The structure of this manuscript is articulated as follows. Section 2 delineates the requisite space and norm pertinent to this study, alongside a review of some foundational results. Subsequently, it proceeds to articulate the solution operator for Equation (4). Section 3 leverages fixed-point theorems, under specified conditions, to establish the existence of a solution for Equation (4). An illustrative example that underscores the derived outcomes is presented in Section 4. Concluding the discourse, Section 5 offers an all-encompassing recapitulation of the paper's content.

2. Preliminaries

Let $X = L^2(\Omega)$ be a Banach space, where the norm is $\|\cdot\|$. The space of all continuous functions maps J into X is denoted by $\mathcal{C}_J = C(J, X)$, where J is an interval and $J \subset R$. For arbitrary $y \in \mathcal{C}_J$ and closed interval J, we define the norm $\| y \|_\infty = \sup_{t \in J} \| y(t) \|$. The Lebesgue measurable functions $\omega : J \to R$ with $1 \le p \le \infty$ construct a Banach space, which is written as $L^p(J, R)$.

Define

$$Y = \left\{ u \in \mathcal{C}_{(0,b]}, \quad \lim_{t \to 0} t^{(1-\mu)(1-\nu)} u(t) \quad \text{is a finite constant} \right\},$$

where the norm $\|\cdot\|_Y$ is

$$\| u \|_Y = \sup_{t \in (0,b]} \| t^{(1-\mu)(1-\nu)} u(t) \|.$$

Then, with the norm $\|\cdot\|_Y$, Y becomes a Banach space.

Definition 1 ([16]). *The Riemann–Liouville derivative of the real function f is defined by*

$$D^p_{a+} f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{p+1-n}} ds, \qquad t > a, \quad n = [p] + 1,$$

where $p > 0$, and $\Gamma(\cdot)$ denote the Gamma function.

Definition 2 ([16]). *The fractional integral of the real function f is defined by*

$$I^p_{a+} f(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{f(s)}{(t-s)^{1-p}} ds, \qquad t > a,$$

where $p > 0$, and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3 ([17]). *The definition of the generalized fractional Riemann–Liouville derivative with order $\mu, \nu \in (0,1)$ and lower limit a is*

$$D^{\mu,\nu}_{a+} f(t) = I^{\mu(1-\nu)}_{a+} \frac{d}{dt} I^{(1-\mu)(1-\nu)}_{a+} f(t),$$

provided the right-hand side is well-defined.

Definition 4 ([18]). *Let $\Omega_X \subset X$ be bounded. Then, the Kuratowski measure of noncompactness is defined by*

$$\lambda(S) = \inf \left\{ \sup\{|x-y|, \quad x, y \in S_i\} | i = 1, \cdots, n, \text{and } S = \bigcup_{i=1}^{n} S_i \right\},$$

where $S \subset \Omega_X$.

Lemma 1 ([19]). *Let $S_1, S_2 \subset X$, and S_1, S_2 be bounded. Furthermore, let c be a real number. The noncompactness measure possesses the following properties*

(i) $S_1 \subset S_2$ implies that $\lambda(S_1) \le \lambda(S_2)$;
(ii) $\lambda(cS_1) = |c|\lambda(S_1)$;
(iii) $\lambda(\overline{co}S_1) = \lambda(S_1)$, where $\overline{co}S_1$ represents the convex closure of S_1.

Lemma 2 ([20]). *Suppose $B \subset X$ is bounded. Subsequently, there is a countable subset $B_0 \subset B$, satisfying $\lambda(B) \le 2\lambda(B_0)$.*

Lemma 3 ([21]). *Let $W \subset C_J$, where W is both equicontinuous and bounded. Subsequently, $\lambda(W(t))$ is continuous for $t \in J$, and $\lambda(W) = \max_{t\in J} \lambda(W(t))$.*

Lemma 4 ([22]). *Let $W = \{w_n\}_{n=1}^{\infty} \subset C_J$ be countable and bounded. Assuming there exists a function $h \in L^1(J, R^+)$, such that*

$$\| w_n(t) \| \le h(t), \quad \text{for} \quad n \in N \quad \text{and} \quad a.e. \quad t \in J.$$

Then, we have

$$\lambda\left(\left\{\int_J w_n(t)dt : n \in \mathbb{N}\right\}\right) \le 2\int_J \lambda(W(t))dt,$$

where $\lambda(W(t))$ is the Lebesgue integral on J.

Definition 5 ([23]). *Let $S \subset X$ be nonempty. If for any bounded set $B \subset S$ and continuous mapping $T : S \to X$, there exists a constant $k \in [0,1)$ such that*

$$\lambda(T(B)) \le k\lambda(B),$$

then T is called k-set-contractive.

We define $A : D(A) \subset X \to X$ with domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ (see [24]) and $(Au)(t)x = Au(x,t)$. Then, A generates an analytic semigroup $\{Q(t)\}_{t\ge 0}$ on X. Without losing generality, we assume that $\{Q(t)\}_{t\ge 0}$ is a uniformly bounded linear operator. Thus, there exists $M \ge 1$, such that

$$M := \sup_{t\in[0,+\infty)} \| Q(t) \| < \infty. \tag{3}$$

Set $u_{0_t}(\tau_0)(x) = u(t-\tau_0)(x) = u(x,t-\tau_0), u_{1_t}(\tau_1)(x) = u(t-\tau_1)(x) = u(x,t-\tau_1), \cdots, u_{n_t}(\tau_n)(x) = u(t-\tau_n)(x) = u(x,t-\tau_n)$ and $f(t,u(t-\tau_0),u(t-\tau_1),\cdots,u(t-\tau_n))(x) = f(x,t,u(x,t-\tau_0),u(x,t-\tau_1),\cdots,u(x,t-\tau_n))$, then (1) can be formulated in an abstract Cauchy problem form as

$$\begin{cases} D_{0+}^{\mu,\nu}u(t) = Au(t) + f(t,u_{0_t},u_{1_t},\cdots,u_{n_t}), & t\in(0,b], \\ I_{0+}^{(1-\mu)(1-\nu)}u(0^+) = \varphi(0), \\ u(t) = \frac{t^{(\mu-1)(1-\nu)}\varphi(t)}{\Gamma(\mu(1-\nu)+\nu)}, & t\in[-r,0), \end{cases} \tag{4}$$

where $f : [0,b] \times C_{[0,b]} \times C_{[-r,0]}^n \to X$ is the given functions satisfying some assumptions, $\varphi \in C_{[-r,0]}$.

Lemma 5 ([25]). *For a measurable function $G : [0,b] \to X$, if $\| G \|$ is Lebesgue integrable, then G is called the Bochner integrable.*

The equivalent integral equation for Equation (4) is given by

$$u(t) = \begin{cases} \frac{\varphi(0)}{\Gamma(\mu(1-\nu)+\nu)}t^{(\mu-1)(1-\nu)} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}[Au+ \\ f(s,u_{0_s},u_{1_s},\cdots,u_{n_s})]ds, & t\in(0,b], \\ \frac{t^{(\mu-1)(1-\nu)}\varphi(t)}{\Gamma(\mu(1-\nu)+\nu)}, & t\in[-r,0). \end{cases} \tag{5}$$

Similar to [15], we obtain the following result.

Lemma 6. *If integral Equation (5) holds, then we have*

$$u(t) = \begin{cases} S_{\mu,\nu}(t)\varphi(0) + \int_0^t K_\nu(t-s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds, & t \in (0, b], \\ \frac{t^{(\mu-1)(1-\nu)}\varphi(t)}{\Gamma(\mu(1-\nu)+\nu)}, & t \in [-r, 0). \end{cases} \tag{6}$$

where

$$S_{\mu,\nu}(t) = I_{0+}^{\mu(1-\nu)} K_\nu(t), \qquad K_\nu(t) = t^{\nu-1} P_\nu(t)$$

with

$$P_\nu(t) = \int_0^\infty \nu\theta M_\nu(\theta) Q(t^\nu \theta) d\theta.$$

The wright function $M_\nu(\theta)$, where $\nu \in (0,1)$ and $\theta \in \mathbb{C}$, is defined as the infinite series

$$M_\nu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\nu n)}.$$

This function satisfies the integral equality

$$\int_0^\infty \theta^\varrho M_\nu(\theta) d\theta = \frac{\Gamma(1+\varrho)}{\Gamma(1+\nu\varrho)}, \qquad \theta \geq 0.$$

Definition 6. *We define the mild solution of Equation (4) as a function $u \in \mathcal{C}_{(0,b]}$ for which u satisfies*

$$u(t) = \begin{cases} S_{\mu,\nu}(t)\varphi(0) + \int_0^t K_\nu(t-s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds, & t \in (0, b], \\ \frac{t^{(\mu-1)(1-\nu)}\varphi(t)}{\Gamma(\mu(1-\nu)+\nu)}, & t \in [-r, 0). \end{cases}$$

Lemma 7 ([15]). *For any $t > 0$, by the continuity of $Q(t)$, we know that $P_\nu(t)$ is continuous according to the uniform operator topology.*

Lemma 8 ([15]). *For any fixed $t > 0$, $\{K_\nu(t)\}_{t>0}$ and $\{S_{\mu,\nu}(t)\}_{t>0}$ are bounded linear operators, which means that, for any $x \in X$*

$$\| K_\nu(t)x \| \leq \frac{Mt^{\nu-1}}{\Gamma(\nu)} \| x \| \qquad \text{and} \qquad \|S_{\mu,\nu}(t)x \| \leq \frac{Mt^{(\mu-1)(1-\nu)}}{\Gamma(\mu(1-\nu)+\nu)} \| x \|.$$

Lemma 9. *Let $R_{\mu,\nu}(t) = t^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t)$, $t > 0$. Thus $\{R_{\mu,\nu}(t)\}_{t>0}$ is continuous according to the uniform operator topology.*

Proof. Let $\varepsilon > 0$ be fixed. By Lemma 7, we know that for $\forall \ t_0 > 0$, there exists $\delta > 0$, such that

$$\| P_\nu(t_2) - P_\nu(t_1) \| \leq \frac{\Gamma(\nu + \mu(1-\nu))}{\Gamma(\nu)} \varepsilon, \tag{7}$$

for $t_2 > t_1 \geq t_0$, and $|t_2 - t_1| < \delta$.

Subsequently, we have

$$\begin{aligned}
&\| R_{\mu,\nu}(t_2) - R_{\mu,\nu}(t_1) \| \\
&= \frac{1}{\Gamma(\mu(1-\nu))} \| t_2^{(1-\mu)(1-\nu)} \int_0^{t_2} (t_2 - s)^{\mu(1-\nu)-1} s^{\nu-1} P_\nu(s) ds \\
&- t_1^{(1-\mu)(1-\nu)} \int_0^{t_1} (t_1 - s)^{\mu(1-\nu)-1} s^{\nu-1} P_\nu(s) ds \| \\
&= \frac{1}{\Gamma(\mu(1-\nu))} \int_0^1 (1-\gamma)^{\mu(1-\nu)-1} \gamma^{\nu-1} \| P_\nu(t_2\gamma) - P_\nu(t_1\gamma) \| d\gamma,
\end{aligned}$$

and

$$\int_0^1 (1-\gamma)^{\mu(1-\nu)-1} \gamma^{\nu-1} d\gamma = B(\nu, \mu(1-\nu)) = \frac{\Gamma(\nu)\Gamma(\mu(1-\nu))}{\Gamma(\nu + \mu(1-\nu))},$$

where $B(\cdot,\cdot)$ represents the Beta function (see [26]). Therefore, we obtain

$$\| R_{\mu,\nu}(t_2) - R_{\mu,\nu}(t_1) \| \leq \varepsilon.$$

That is, by the arbitrariness of t_0, $R_{\mu,\nu}(t)(t > 0)$ is continuous in the uniform operator topology. □

Remark 1. *$\{R_{\mu,\nu}(t)\}_{t>0}$ is strongly continuous, that is,*

$$\| R_{\mu,\nu}(t_2)x - R_{\mu,\nu}(t_1)x \| \to 0 \quad \text{as} \quad t_1 \to t_2,$$

for any $t_1, t_2 \in (0, b]$ and $\forall \ x \in X$.

Remark 2. *In the context of the Beta function $B(\nu, \mu(1-\nu))$, both μ and ν are required to be strictly positive. Therefore, in this article, order $\mu, \nu \in (0, 1)$.*

Set $u(t) = t^{(\mu-1)(1-\nu)} v(t)$. Due to the limit of $t^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t)\varphi(0)$ being equal to $\frac{\varphi(0)}{\Gamma(\mu(1-\nu)+\nu)}$ as $t \to 0^+$, for any $v \in \mathcal{C}_{[-r,b]}$ we set

$$(Tv)(t) = \begin{cases} t^{(1-\mu)(1-\nu)} \Big[S_{\mu,\nu}(t)\varphi(0) + \int_0^t K_\nu(t-s) f(s, u_{0_s}, u_{1_s}, \cdots, \\ u_{n_s})\, ds \Big], \qquad t \in (0, b], \\ \frac{\varphi(t)}{\Gamma(\mu(1-\nu)+\nu)}, \qquad t \in [-r, 0]. \end{cases}$$

Obviously, u is a mild solution of Equation (4), which is equivalent to T has a fixed point on $\mathcal{C}_{[-r,b]}$.

Lemma 10 ([23])**.** *Let $B \subset X$ be a bounded closed convex set and that the operator $T : B \to B$ is k-set-contractive. Then, T has a fixed point in B.*

Lemma 11 ([23])**.** *For completely continuous mapping $T : D \to D$, where D is a convex subset of X containing 0. One of the following must hold: T has a fixed point, or the set $\{z \in D, \quad z = \xi T(z)\}$ is unbounded, where $\xi \in (0, 1)$.*

3. Main Results

In order to obtain the existence of mild solutions for Equation (4), we provide some assumptions.

(H$_1$) $Q(t)$ is compact for each $t > 0$.

(H$_2$) The function $f(t, \cdot, \cdots, \cdot) : \mathcal{C}_{[0,b]} \times \mathcal{C}^n_{[-r,0]} \to X$ is continuous for almost all $t \in (0, b]$. Additionally, for each $u_{0_t} \in \mathcal{C}_{[0,b]}, u_{k_t} \in \mathcal{C}_{[-r,0]} (k = 1, 2, \cdots, n)$, the function $f(\cdot, u_{0_t}, u_{1_t}, \cdots, u_{n_t}) : (0, b] \to X$ is strongly measurable.

(H$_3$) There exists constants $\nu_1 \in [0, \nu)$ and $m \in L^{\frac{1}{\nu_1}}((0, b], R^+)$, such that

$$I_{0+}^{\nu} m \in C\big((0, b], R^+\big), \qquad \lim_{t \to 0^+} t^{(1-\mu)(1-\nu)} I_{0+}^{\nu} m(t) = 0,$$

and

$$\| f(t, u_{0_t}, u_{1_t}, \cdots, u_{n_t}) \| \leq m(t),$$

for all $u_{0_t} \in \mathcal{C}_{[0,b]}$, $u_{k_t} \in \mathcal{C}_{[-r,0]} (k = 1, 2, \cdots, n)$ and almost all $t \in [0, b]$.

(H$_4$) For any bounded, equicontinuous and countable sets $B_k \subset X (k = 0, 1, \cdots, n)$, some constants $L_k > 0 \quad (k = 0, 1, \cdots, n)$ exist, such that

$$\lambda(f(t, B_0, B_1, \cdots, B_n)) \leq \sum_{k=0}^{n} L_k \lambda(B_k), \qquad t \in (0, b].$$

(H$_5$) Assume that $f : [0, b] \times \mathcal{C}_{[0,b]} \times \mathcal{C}^n_{[-r,0]} \to X$ is a continuous function. There exists a non-negative continuous function $\rho_k(\cdot) (k = 0, 1, \cdots, n)$, satisfying

$$\| f(t, u_{0_t}, u_{1_t}, \cdots, u_{n_t}) - f\big(t, u'_{0_t}, u'_{1_t}, \cdots, u'_{n_t}\big) \| \leq \sum_{k=0}^{n} \rho_k(t) \| u - u' \|_Y, \qquad t \in (0, b],$$

for any $u_{0_t}, u'_{0_t} \in \mathcal{C}_{[0,b]}$, $u_{k_t}, u'_{k_t} \in \mathcal{C}_{[-r,0]} (k = 1, 2, \cdots, n)$.

Let $B_{k_0} = \{v \in \mathcal{C}_{[-r,b]}, \quad \| v \|_\infty \leq k_0\}$. Then, $B_{k_0} \subset \mathcal{C}_{[-r,b]}$ is a bounded, closed, and convex set.

Theorem 1. *Under the assumption that condition (H$_1$)–(H$_3$) holds, Cauchy Equation (4) has a mild solution.*

Proof. In view of Lemma 8, for $t \in (0, b]$, it follows that

$$\| t^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t) \varphi(0) \| \leq \frac{M}{\Gamma(\mu(1-\nu)+\nu)} \| \varphi \|_\infty. \tag{8}$$

For $v \in B_{k_0}$, according to (H$_2$), and $u_{k_t} (k = 0, 1, \cdots, n)$ is continuous in t, and we have $(t-s)^{\nu-1} \in L^{\frac{1}{1-\nu_1}}[0, t]$ for $t \in (0, b]$ and $\nu_1 \in [0, \nu)$. Considering

$$a = \frac{\nu - 1}{1 - \nu_1} \in (-1, 0), \qquad M_1 = \| m \|_{L^{\frac{1}{\nu_1}}[0,b]}.$$

By applying (H$_3$) and Hölder inequality, for $t \in (0, b]$, we derive

$$\begin{aligned} & t^{(1-\mu)(1-\nu)} \int_0^t \| (t-s)^{\nu-1} f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) \| ds \\ & \leq t^{(1-\mu)(1-\nu)} \left(\int_0^t (t-s)^{\frac{\nu-1}{1-\nu_1}} ds \right)^{1-\nu_1} \| m \|_{L^{\frac{1}{\nu_1}}[0,t]} \\ & \leq \frac{M_1}{(1+a)^{1-\nu_1}} b^{2+a-\mu-\nu-\nu_1-a\nu_1+\mu\nu}. \end{aligned} \tag{9}$$

We obtain by Lemma 8 and (9) that

$$\begin{aligned} & t^{(1-\mu)(1-\nu)} \int_0^t \| K_\nu(t-s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) \| ds \\ & \leq \frac{M b^{(1-\mu)(1-\nu)}}{\Gamma(\nu)} \int_0^t \| (t-s)^{\nu-1} f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) \| ds \\ & \leq \frac{M M_1}{\Gamma(\nu)(1+a)^{1-\nu_1}} b^{2+a-\mu-\nu-\nu_1-a\nu_1+\mu\nu}, \qquad \text{for} \quad t \in (0, b]. \end{aligned} \tag{10}$$

Consequently, $\| K_\nu(t-s)f(s,u_{0_s},u_{1_s},\cdots,u_{n_s}) \|$ is a Lebesgue integrable for $s\in[0,t]$ and $t\in(0,b]$. By Lemma 5, we know that $K_\nu(t-s)f(s,u_{0_s},u_{1_s},\cdots,u_{n_s})$ is a Bochner integrable for $s\in[0,t]$ and $t\in(0,b]$. Therefore, operator T is well-defined in $[0,b]$.

Firstly, we demonstrate that T is a completely continuous operator.

Assume $\{v_n\}\subseteq B_{k_0}$ with $v_n\to v$ on B_{k_0}, where $v_n=t^{(1-\mu)(1-\nu)}u_n$, $v=t^{(1-\mu)(1-\nu)}u$. Then, by (H$_2$) and the fact that $u_{n,k_t}\to u_{k_t}(k=0,1,\cdots,n)$ for $t\in(0,b]$, we have

$$f(s,u_{n,0_s},u_{n,1_s},\cdots,u_{n,n_s})\to f(s,u_{0_s},u_{1_s},\cdots,u_{n_s}),\qquad \text{a.e.}\quad t\in(0,b],\quad \text{as}\quad n\to\infty.$$

Considering that

$$\begin{aligned}&(t-s)^{\nu-1}\| f(s,u_{n,0_s},u_{n,1_s},\cdots,u_{n,n_s})-f(s,u_{0_s},u_{1_s},\cdots,u_{n_s}) \|\\ &\le (t-s)^{\nu-1}2m(s),\qquad \text{a.e.in}\quad [0,t),\end{aligned}$$

by the dominated convergence theorem, we have

$$\begin{aligned}&\| (Tv_n)(t)-(Tv)(t) \|\\ &\le \frac{M}{\Gamma(\nu)}t^{(1-\mu)(1-\nu)}\int_0^t(t-s)^{\nu-1}\| f(s,u_{n,0_s},u_{n,1_s},\cdots,u_{n,n_s})\\ &-f(s,u_{0_s},u_{1_s},\cdots,u_{n_s}) \|ds\\ &\to 0,\qquad \text{as}\quad n\to\infty.\end{aligned}$$

Thus, we know that T is continuous.

Subsequently, we prove that $\{Tv,\ v\in B_{k_0}\}$ is relatively compact. It suffices to demonstrate that $\{Tv,\ v\in B_{k_0}\}$ is uniformly bounded and equicontinuous, and $\{(Tv)(t),\ v\in B_{k_0}\}$ is relatively compact in X for any $t\in(0,b]$.

According to (H$_3$), there is a constant $k_0>0$, such that

$$M\left(\frac{\| \varphi \|_\infty}{\Gamma(\mu(1-\nu)+\nu)}+\sup_{t\in[0,b]}\left\{t^{(1-\mu)(1-\nu)}I_{0+}^{\nu}m(t)\right\}\right)\le k_0.$$

For $t\in[0,b]$, by (H$_3$) and Lemma 8, we obtain

$$\begin{aligned}&\| (Tv)(t) \|\\ &\le t^{(1-\mu)(1-\nu)}\left[\| S_{\mu,\nu}(t)\varphi(0) \| +\| \int_0^t K_\nu(t-s)f(s,u_{0_s},u_{1_s},\cdots,u_{n_s})ds \|\right]\\ &\le \frac{M\| \varphi \|_\infty}{\Gamma(\mu(1-\nu)+\nu)}+\frac{Mt^{(1-\mu)(1-\nu)}}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}\| f(s,u_{0_s},u_{1_s},\cdots,u_{n_s}) \|ds\\ &\le M\left(\frac{\| \varphi \|_\infty}{\Gamma(\mu(1-\nu)+\nu)}+\sup_{t\in[0,b]}\left\{t^{(1-\mu)(1-\nu)}I_{0+}^{\nu}m(t)\right\}\right)\le k_0.\end{aligned}$$

Hence, $\| Tv \|\le k_0$, for any $v\in B_{k_0}$, which means $\{Tv,\quad v\in B_{k_0}\}$ is uniformly bounded. For $v\in B_{k_0}$, when $t_1=0,t_2\in(0,b]$, we obtain

$$\begin{aligned}&\| (Tv)(t_2)-(Tv)(0) \|\\ &=t_2^{(1-\mu)(1-\nu)}\| S_{\mu,\nu}(t_2)\varphi(0)+\int_0^{t_2}K_\nu(t_2-s)f(s,u_{0_s},u_{1_s},\cdots,u_{n_s})ds \|\\ &\le t_2^{(1-\mu)(1-\nu)}\left[\| S_{\mu,\nu}(t_2) \| \| \varphi \|_\infty+\frac{M}{\Gamma(\nu)}\int_0^{t_2}(t_2-s)^{\nu-1}m(s)ds\right]\\ &\to 0\qquad \text{as}\quad t_2\to 0.\end{aligned}$$

Take $v \in B_{k_0}$, and $t_1, t_2 \in (0, b]$, we obtain

$$\begin{aligned}
&\| (Tv)(t_2) - (Tv)(t_1) \| \\
&= \| t_2^{(1-\mu)(1-\nu)} \Big[S_{\mu,\nu}(t_2)\varphi(0) + \textstyle\int_0^{t_2} K_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \Big] \\
&- t_1^{(1-\mu)(1-\nu)} \Big[S_{\mu,\nu}(t_1)\varphi(0) + \textstyle\int_0^{t_1} K_\nu(t_1 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \Big] \| \\
&\leq \| t_2^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_2)\varphi(0) - t_1^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_1)\varphi(0) \| \\
&+ \| \textstyle\int_{t_1}^{t_2} t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&+ \| \textstyle\int_0^{t_1} t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \\
&- \textstyle\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&+ \| \textstyle\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \\
&- \textstyle\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_1 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&\leq \| \Big[t_2^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_2) - t_1^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_1) \Big] \varphi(0) \| \\
&+ \tfrac{M}{\Gamma(\nu)} \| \textstyle\int_{t_1}^{t_2} t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} m(s) ds \| \\
&+ \tfrac{M}{\Gamma(\nu)} \textstyle\int_0^{t_1} \Big[t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} - t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} \Big] m(s) ds \\
&+ \| \textstyle\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&- \| \textstyle\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_1 - s)] f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&\leq I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \| t_2^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_2) - t_1^{(1-\mu)(1-\nu)} S_{\mu,\nu}(t_1) \| \| \varphi \|_\infty \\
&= \| R_{\mu,\nu}(t_2) - R_{\mu,\nu}(t_1) \| \| \varphi \|_\infty, \\
I_2 &= \frac{M}{\Gamma(\nu)} \| \int_0^{t_2} t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} m(s) ds \\
&- \int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} m(s) ds \|, \\
I_3 &= \frac{2M}{\Gamma(\nu)} \int_0^{t_1} \Big[t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} - t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} \Big] m(s) ds, \\
I_4 &= \| \int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_2 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \\
&- \| \int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} P_\nu(t_1 - s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \|.
\end{aligned}$$

It is obvious that $I_1 \to 0$ as $t_1 \to t_2$ by Remark 1. By condition (H_3), it can be deduced that $\lim_{t_1 \to t_2} I_2 = 0$. Noting that

$$\begin{aligned}
&\Big[t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} - t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} \Big] m(s) \\
&\leq t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} m(s),
\end{aligned}$$

and $\int_0^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} m(s) ds$ exists, then by the dominated convergence theorem, we obtain

$$\int_0^{t_1} \Big[t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} - t_2^{(1-\mu)(1-\nu)} (t_2 - s)^{\nu-1} \Big] m(s) ds \to 0, \quad \text{as} \quad t_1 \to t_2,$$

then, it can be deduced that $\lim_{t_1 \to t_2} I_3 = 0$. For sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned} I_4 \leq & \int_0^{t_1-\varepsilon} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} \| P_\nu(t_2 - s) - P_\nu(t_1 - s) \| \\ & \| f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) \| ds \\ & + \int_{t_1-\varepsilon}^{t_1} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} \| P_\nu(t_2 - s) - P_\nu(t_1 - s) \| \\ & \| f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) \| ds \\ & \leq t_1^{(1-\mu)(1-\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} m(s) ds \sup_{s \in [0, t_1-\varepsilon]} \| P_\nu(t_2 - s) - P_\nu(t_1 - s) \| \\ & + \frac{2M}{\Gamma(\nu)} t_1^{(1-\mu)(1-\nu)} \int_{t_1-\varepsilon}^{t_1} (t_1 - s)^{\nu-1} m(s) ds \\ & \leq I_{41} + I_{42} + I_{43}, \end{aligned}$$

where

$$\begin{aligned} & I_{41} = \tfrac{k_0 \Gamma(\nu)}{M} \sup_{s \in [0, t_1-\varepsilon]} \| P_\nu(t_2 - s) - P_\nu(t_1 - s) \|, \\ & I_{42} = \tfrac{2M}{\Gamma(\nu)} \| t_1^{(1-\mu)(1-\nu)} \int_0^{t_1} (t_1 - s)^{\nu-1} m(s) ds \| \\ & - \tfrac{2M}{\Gamma(\nu)} \| (t_1 - \varepsilon)^{(1-\mu)(1-\nu)} \int_0^{t_1-\varepsilon} (t_1 - \varepsilon - s)^{\nu-1} m(s) ds \|, \\ & I_{43} = \tfrac{2M}{\Gamma(\nu)} \int_0^{t_1-\varepsilon} (t_1 - \varepsilon)^{(1-\mu)(1-\nu)} (t_1 - \varepsilon - s)^{\nu-1} m(s) ds \\ & - \tfrac{2M}{\Gamma(\nu)} \int_0^{t_1-\varepsilon} t_1^{(1-\mu)(1-\nu)} (t_1 - s)^{\nu-1} m(s) ds. \end{aligned}$$

By Lemma 7, we know that $I_{41} \to 0$ as $t_1 \to t_2$. Similarly, one can establish the proof for $I_2 \to 0$ and $I_3 \to 0$, then, we have $I_{42} \to 0$ and $I_{43} \to 0$ as $\varepsilon \to 0$. Therefore, $I_4 \to 0$ independently of $v \in B_{k_0}$ as $t_1 \to t_2$, $\varepsilon \to 0$. Thus, $\| (Tv)(t_2) - (Tv)(t_1) \| \to 0$ independently of $v \in B_{k_0}$ as $t_1 \to t_2$, which means that $\{Tv, \quad v \in B_{k_0}\}$ is equicontinuous.

Now, we need to establish that for any $t \in [-r, b]$, $\{(Tv)(t), v \in B_{k_0}\}$ is relatively compact in X.

Clearly, for any $t \in [-r, 0]$, $\{(Tv)(t), v \in B_{k_0}\}$ is relatively compact in X. Assume $t \in (0, b]$ is fixed. For $\forall \quad \delta > 0$ and $\forall \quad \varepsilon \in (0, t)$, operator $T_{\varepsilon,\delta}$ is defined on B_{k_0} as follows

$$\begin{aligned} (T_{\varepsilon,\delta} v)(t) = & \frac{t^{(1-\mu)(1-\nu)} \nu}{\Gamma(\mu(1-\nu))} \int_0^t \int_0^\infty (t - s)^{\mu(1-\nu)-1} s^{\nu-1} \theta M_\nu(\theta) Q(s^\nu \theta) \varphi(0) d\theta ds \\ & + t^{(1-\mu)(1-\nu)} \int_0^{t-\varepsilon} \int_\delta^\infty (t - s)^{\nu-1} \nu \theta M_\nu(\theta) Q\big((t - s)^\nu \theta\big) \\ & f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \\ & = \frac{t^{(1-\mu)(1-\nu)} \nu}{\Gamma(\mu(1-\nu))} \int_0^t \int_0^\infty (t - s)^{\mu(1-\nu)-1} s^{\nu-1} \theta M_\nu(\theta) \\ & [Q(\varepsilon^\nu \delta) Q(s^\nu \theta - \varepsilon^\nu \delta)] \varphi(0) d\theta ds \\ & + t^{(1-\mu)(1-\nu)} \int_0^{t-\varepsilon} \int_\delta^\infty (t - s)^{\nu-1} \nu \theta M_\nu(\theta) \big[Q(\varepsilon^\nu \delta) Q\big((t - s)^\nu \theta - \varepsilon^\nu \delta\big)\big] \\ & f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \\ & = Q(\varepsilon^\nu \delta) \frac{t^{(1-\mu)(1-\nu)} \nu}{\Gamma(\mu(1-\nu))} \int_0^t \int_0^\infty (t - s)^{\mu(1-\nu)-1} s^{\nu-1} \theta M_\nu(\theta) \\ & Q(s^\nu \theta - \varepsilon^\nu \delta) \varphi(0) d\theta ds \\ & + Q(\varepsilon^\nu \delta) t^{(1-\mu)(1-\nu)} \int_0^{t-\varepsilon} \int_\delta^\infty (t - s)^{\nu-1} \nu \theta M_\nu(\theta) Q\big((t - s)^\nu \theta - \varepsilon^\nu \delta\big) \\ & f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds, \end{aligned}$$

where $v \in B_{k_0}$. By (H_1), we know that $Q(\varepsilon^\nu \delta)(\varepsilon^\nu \delta > 0)$ is compact, then we obtain that the set $\{(T_{\varepsilon,\delta} v)(t),\ v \in B_{k_0}\}$ is relatively compact in X for $\forall\ \delta > 0$ and $\forall\ \varepsilon \in (0, t)$. Furthermore, for each $v \in B_{k_0}$, we have

$$\begin{aligned}
&\| (Tv)(t) - (T_{\varepsilon,\delta} v)(t) \| \\
&= \nu t^{(1-\mu)(1-\nu)} \\
&\| \int_0^t \int_0^\delta \theta (t-s)^{\nu-1} M_\nu(\theta) Q\big((t-s)^\nu \theta\big) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \\
&+ \int_0^t \int_\delta^\infty \theta (t-s)^{\nu-1} M_\nu(\theta) Q\big((t-s)^\nu \theta\big) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \\
&- \int_0^{t-\varepsilon} \int_\delta^\infty \theta (t-s)^{\nu-1} M_\nu(\theta) Q\big((t-s)^\nu \theta\big) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \| \\
&\le \nu t^{(1-\mu)(1-\nu)} \\
&\Big[\| \int_0^t \int_0^\delta \theta (t-s)^{\nu-1} M_\nu(\theta) Q\big((t-s)^\nu \theta\big) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \| \\
&+ \| \int_{t-\varepsilon}^t \int_\delta^\infty \theta (t-s)^{\nu-1} M_\nu(\theta) Q\big((t-s)^\nu \theta\big) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) d\theta ds \| \Big] \\
&\le \nu M b^{(1-\mu)(1-\nu)} \Bigg[\left(\int_0^t (t-s)^{\frac{\nu-1}{1-\nu_1}} ds \right)^{1-\nu_1} \| m \|_{L^{\frac{1}{\nu_1}}[0,t]} \int_0^\delta \theta M_\nu(\theta) d\theta \\
&+ \left(\int_{t-\varepsilon}^t (t-s)^{\frac{\nu-1}{1-\nu_1}} ds \right)^{1-\nu_1} \| m \|_{L^{\frac{1}{\nu_1}}[t-\varepsilon,t]} \int_0^\infty \theta M_\nu(\theta) d\theta \Bigg] \\
&\le \frac{\nu M M_1 b^{2+a-\mu-\nu-\nu_1-a\nu_1+\mu\nu}}{(1+a)^{1-\nu_1}} \int_0^\delta \theta M_\nu(\theta) d\theta \\
&+ \frac{\nu M M_1 \varepsilon^{(1+a)(1-\nu_1)} b^{(1-\mu)(1-\nu)}}{\Gamma(1+\nu)(1+a)^{1-\nu_1}}.
\end{aligned}$$

Consequently, for $t > 0$, there are relatively compact sets close to $\{(Tv)(t),\ v \in B_{k_0}\}$ arbitrarily. This implies that the set $\{(Tv)(t),\ v \in B_{k_0}\}$ is also relatively compact in X.

The relative compactness of $\{Tv,\ v \in B_{k_0}\}$ follows from the Arzela–Ascoli theorem. This, combined with the continuity of T, leads to the conclusion that $T : B_{k_0} \to B_{k_0}$ is completely continuous.

We set

$$M_2 = \{v \in B_{k_0}, \quad v = \eta T v, \quad \eta \in (0,1)\}.$$

Obviously, $0 \in M_2$. For $v \in M_2$, $t \in (0, b]$, we have

$$\begin{aligned}
\| v(t) \| &\le \eta t^{(1-\mu)(1-\nu)} \left[\| S_{\mu,\nu}(t)\varphi(0) \| + \| \int_0^t K_\nu(t-s) f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) ds \| \right] \\
&\le \eta \left[\frac{M}{\Gamma(\mu(1-\nu)+\nu)} \| \varphi \|_\infty + \frac{M M_1}{\Gamma(\nu)(1+a)^{1-\nu_1}} b^{2+a-\mu-\nu-\nu_1-a\nu_1+\mu\nu} \right] \\
&< \frac{M}{\Gamma(\mu(1-\nu)+\nu)} \| \varphi \|_\infty + \frac{M M_1}{\Gamma(\nu)(1+a)^{1-\nu_1}} b^{2+a-\mu-\nu-\nu_1-a\nu_1+\mu\nu}.
\end{aligned}$$

For $t \in [-r, 0]$, we have

$$\| v(t) \| \le \eta \frac{\| \varphi \|_\infty}{\Gamma(\mu(1-\nu)+\nu)} < \frac{\| \varphi \|_\infty}{\Gamma(\mu(1-\nu)+\nu)}.$$

Therefore, from Lemma 11, T has a fixed point. That is, Equation (4) has a mild solution. □

Theorem 2. *Under the assumptions (H_2), (H_3), (H_4), Equation (4) has a mild solution if*

$$\frac{4Mb^{\mu(\nu-1)+1}}{\nu\Gamma(\nu)}\sum_{k=0}^{n}L_k<1.$$

Proof. Similar to Theorem 1, $T:B_{k_0}\to B_{k_0}$ is continuous, and $\{Tv,\ v\in B_{k_0}\}$ is uniformly bounded and equicontinuous. Let $G=\overline{co}T(B_{k_0})$. Then, it is simple to demonstrate that T maps G into itself and $G\subset B_{k_0}$ is equicontinuous. By Lemma 2, we have that for any $B\subset G$, there exists a countable set $B_0=\{w_n\}\subset B$, such that

$$\lambda(T(B))\leq 2\lambda(T(B_0)).$$

The equicontinuity of G implies that $B_0\subset B$ is also equicontinuous.

By (H_4), for any $t\in(0,b]$, we have

$$\begin{aligned}
&\lambda(T(B_0)(t))\\
&=\lambda\Big(t^{(1-\mu)(1-\nu)}S_{\mu,\nu}(t)\varphi(0)+\int_0^t t^{(1-\mu)(1-\nu)}K_\nu(t-s)\\
&f(s,w_n(s-\tau_0),w_n(s-\tau_1),\cdots,w_n(s-\tau_n))ds)\\
&\leq\frac{2M}{\Gamma(\nu)}b^{(1-\mu)(1-\nu)}\int_0^t(t-s)^{\nu-1}\\
&\lambda(f(s,w_n(s-\tau_0),w_n(s-\tau_1),\cdots,w_n(s-\tau_n)))ds\\
&\leq\frac{2M}{\Gamma(\nu)}\sum_{k=0}^{n}L_k b^{(1-\mu)(1-\nu)}\int_0^t(t-s)^{\nu-1}\lambda(B_0(s))ds\\
&\leq\frac{2Mb^{\mu(\nu-1)+1}}{\nu\Gamma(\nu)}\sum_{k=0}^{n}L_k\lambda(B).
\end{aligned}\tag{11}$$

When $t\in[-r,0]$, (11) is clearly true. Then, as $T(B_0)\subset G$ is bounded and equicontinuous, it follows from Lemma 3 that

$$\lambda(T(B_0))=\max_{t\in[-r,b]}\lambda(T(B_0)(t)).$$

Therefore, we have

$$\lambda(T(B))\leq\frac{4Mb^{\mu(\nu-1)+1}}{\nu\Gamma(\nu)}\sum_{k=0}^{n}L_k\lambda(B).$$

As $\frac{4Mb^{\mu(\nu-1)+1}}{\nu\Gamma(\nu)}\sum_{k=0}^{n}L_k<1$, then $T:B_{k_0}\to B_{k_0}$ is a k-set-contractive operator. Thus, we know from Lemma 10 that the Cauchy Equation (4) has a mild solution. □

Theorem 3. *Under assumptions (H_5), Equation (4) has a unique mild solution provided*

$$\frac{Mb^{(1-\mu)(1-\nu)}}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}\sum_{k=0}^{n}\rho_k(s)ds<1.$$

Proof. It is obvious that $S_{\mu,\nu}(t)\varphi(0)$ exists, $K_\nu(t-s)f(s,u_{0_s},u_{1_s},\cdots,u_{n_s})ds$ is the Bochner integrable for $s\in[0,t]$ and $t\in(0,b]$. Moreover, $T:B_{k_0}\to B_{k_0}$.

For $v, v' \in B_{k_0}$, according to (H5), we have

$$\begin{aligned}
&\| (Tv)(t) - (Tv')(t) \| \\
&= t^{(1-\mu)(1-\nu)} \| \int_0^t K_\nu(t-s)\big[f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s})ds - f\big(s, u'_{0_s}, u'_{1_s}, \cdots, u'_{n_s}\big)\big] ds \| \\
&\le \frac{Mb^{(1-\mu)(1-\nu)}}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \| f(s, u_{0_s}, u_{1_s}, \cdots, u_{n_s}) - f\big(s, u'_{0_s}, u'_{1_s}, \cdots, u'_{n_s}\big) \| ds \\
&\le \frac{Mb^{(1-\mu)(1-\nu)}}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \sum_{k=0}^{n} \rho_k(s) \| u - u' \|_Y ds \\
&\le \frac{Mb^{(1-\mu)(1-\nu)}}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \sum_{k=0}^{n} \rho_k(s) ds \| v - v' \|_\infty.
\end{aligned}$$

As $\frac{Mb^{(1-\mu)(1-\nu)}}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \sum_{k=0}^{n} \rho_k(s) ds < 1$, by Banach contraction principle we concluded that Cauchy Equation (4) has a unique mild solution. □

4. An Example

Let $X = L^2([0,1], R)$, we study the fractional delay diffusion equations

$$\begin{cases}
D_{0+}^{\frac{2}{3},\frac{1}{6}} u(x,t) = \partial_x^2 u(x,t) + f(x,t,u(x,t),u(x,t-\tau)), \\
(x,t) \in [0,1] \times (0,b], \\
u(0,t) = u(1,t) = 0, \quad t \in (0,b], \\
I_{0+}^{\frac{5}{18}} u(x,0^+) = \varphi(x,0), \quad x \in [0,1], \\
u(x,t) = \frac{t^{-\frac{5}{18}} \varphi(x,t)}{\Gamma\left(\frac{13}{18}\right)}, \qquad (x,t) \in [0,1] \times [-r,0),
\end{cases} \tag{12}$$

where $D_{0+}^{\frac{2}{3},\frac{1}{6}}$ represents the Hilfer fractional derivative, $\mu = \frac{2}{3}, \nu = \frac{1}{6}$, $\tau \in [0,r]$, $r > 0$, $\varphi \in \mathcal{C}_{[-r,0]}$.

We let $Au = u''$, and have

$$D(A) = \big\{u(\cdot) \in X, \quad u, u' \text{ are absolutely continuous}, \quad u'' \in X, u(0) = 0 = u(1)\big\}.$$

Subsequently, A generates a strongly continuous semigroup $\{Q(t)\}_{t \ge 0}$, and

$$Q(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, p_n)p_n,$$

where the normalized eigenvector $p_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ corresponds to the eigenvalues $n^2 (n \in \mathbb{N})$ of A. Furthermore, (H1) is established, $Q(t)(t > 0)$ is compact, continuous in the uniform operator topology, and $\| Q(t) \| \le e^{-t}, \quad t \ge 0$. Additionally, we have two operators $S_{\frac{2}{3},\frac{1}{6}}(t)$ and $K_{\frac{1}{6}}(t)$ are defined by

$$S_{\frac{2}{3},\frac{1}{6}}(t) = \frac{1}{6\Gamma\left(\frac{5}{9}\right)} \int_0^t \int_0^\infty \theta (t-s)^{-\frac{4}{9}} s^{-\frac{5}{6}} M_{\frac{1}{6}}(\theta) Q\Big(s^{\frac{1}{6}}\theta\Big) d\theta ds,$$

$$K_{\frac{1}{6}}(t) = \frac{1}{6} t^{-\frac{5}{6}} \int_0^\infty \theta M_{\frac{1}{6}}(\theta) Q\Big(t^{\frac{1}{6}}\theta\Big) d\theta.$$

Clearly,

$$\| K_{\frac{1}{6}}(t) \| \le \frac{t^{-\frac{5}{6}}}{\Gamma\Big(\frac{1}{6}\Big)}, \qquad \| S_{\frac{2}{3},\frac{1}{6}}(t) \| \le \frac{t^{-\frac{5}{18}}}{\Gamma\Big(\frac{13}{18}\Big)}, \qquad t > 0.$$

We let $u(t)x = u(x,t), u(t-\tau)x = u(x,t-\tau)$ and $f(t,u(t),u(t-\tau))x = f(x,t,u(x,t), u(x,t-\tau))$, where $x \in [0,1]$, $t \in (0,b]$. Then, (12) can be converted into the problem in X

$$\begin{cases} D_{0+}^{\frac{2}{3},\frac{1}{6}} u(t) = Au(t) + f(t,u(t),u(t-\tau)), & t \in (0,b], \\ I_{0+}^{\frac{5}{18}} u(0^+) = \varphi(0), \\ u(t) = \frac{t^{-\frac{5}{18}}\varphi(t)}{\Gamma(\frac{13}{18})}, & t \in [-r,0). \end{cases} \quad (13)$$

We take $f(t,u(t),u(t-\tau)) = t^2[\sin u(t) + \sin u(t-\tau)]$; then, (H_1)–(H_3), (H_4), (H_5) are satisfied, where $\rho_0(t) = \rho_1(t) = L_0 = L_1 = b^2$. According to Theorem 1, Theorem 2, or Theorem 3, (13) there is a mild solution provided $b^{\frac{47}{18}} < \frac{1}{12}\Gamma\left(\frac{1}{6}\right)$.

5. Conclusions

Compared with other fractional derivatives, such as the Riemann–Liouville derivative and Caputo derivative, the Hilfer fractional derivative is built on a new theoretical foundation of fractional calculus, which provides a more complete and unified definition to better describe the behavior of complex systems. Therefore, studying the existence of solutions to the Hilfer fractional delay diffusion equation contributes to a deeper understanding of the behavior of such equations and provides accurate mathematical models for solving practical problems. This endeavor holds significant academic significance in optimizing engineering designs, predicting outcomes, and controlling natural systems. The main focus of this article is to investigate the existence of solutions to delay diffusion equations with Hilfer fractional derivatives. Under the assumption that the analytic semigroup is compact or non-compact, the Leray–Schauder fixed-point theorem and non-compactness measure method were employed to prove the existence of mild solutions, while Banach contraction mapping principle was utilized to establish the uniqueness of mild solutions. Moving forward, we aim to further explore the regularity and stability of mild solution to delay diffusion equations with the Hilfer fractional derivative.

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