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Novel Estimations of Hadamard-Type Integral Inequalities for Raina's Fractional Operators

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Abstract: In the present paper, utilizing a wide class of fractional integral operators (namely the Raina fractional operator), we develop novel fractional integral inequalities of the Hermite–Hadamard type. With the help of the well-known Riemann–Liouville fractional operators, s-type convex functions are derived using the important results. We also note that some of the conclusions of this study are more reasonable than those found under certain specific conditions, e.g., $s = 1$, $\lambda = \alpha$, $\sigma(0) = 1$, and $w = 0$. In conclusion, the methodology described in this article is expected to stimulate further research in this area.

Keywords: Raina fractional operator; s-type convex function; Jensen inequality; Hermite–Hadamard inequality; Hölder inequality; young inequality

MSC: 26D15; 26D10; 26A51; 26A33; 33B20

1. Introduction

Mathematical inequalities are widely recognized as one of the most valuable topics in mathematics due to their practical applications. They have made significant contributions to numerous scientific and engineering domains, such as information theory, coding theory, economics, finance and engineering. Their work has been influential in the study of numerical and partial differential equations. Fractional analysis is a powerful concept that may be used to investigate hidden information that traditional analysis is unable to investigate. To determine if a given fractional differential equation has a solution and whether it is unique, fractional integral inequalities are essential (the Riemann–Liouville, k-Riemann–Liouville, conformable, and Caputo–Fabrizio fractional integral operators are among the most significant operators) [1–4]. The topic of fractional analysis has been extensively studied by mathematicians, who have used various techniques and approaches to examine several fractional derivatives and integrals [5–9].

Convex functions have become increasingly popular in recent years. They are commonly used in various analytical fields across different mathematical specializations. Their usefulness in optimization theory is particularly noteworthy due to the numerous useful properties they exhibit in this area.

The theory of inequalities of convex functions is strongly intertwined. Many famous and useful inequalities, which are useful in applied sciences, have been discovered as a result of convex functions. The classical definition of a convex function on convex sets, which is based on line segments, is the most remarkable. The term can be defined as follows:

Definition 1. *A set $\mathcal{I} \subset \mathbb{R}$ is said to be a convex function if*

$$\Bbbk\delta_1 + (1 - \Bbbk)\delta_2 \in \mathcal{I},$$

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for each $\delta_1, \delta_2 \in \mathcal{I}$ *and* $\Bbbk \in [0,1]$.

Definition 2. *The mapping* $\chi : \mathcal{I} \to \mathbb{R}$ *is said to be a convex function if the following inequality holds:*

$$\chi(\Bbbk\delta_1 + (1-\Bbbk)\delta_2) \leq \Bbbk\chi(\delta_1) + (1-\Bbbk)\chi(\delta_2),$$

for all $\delta_1, \delta_2 \in \mathcal{I}$ *and* $\Bbbk \in [0,1]$. *If* $(-\chi)$ *is convex, then* χ *is said to be concave.*

The definition of a convex function has been used to derive many new and different convex functions. By assigning specific values to these definitions, the basic convex function can be obtained. In this regard, we will discuss the relations between s-convex functions, which is our main topic in this study, and the class of convex functions in [10].

Definition 3. *Let* $s \in [0,1]$. *Then, the real-valued function* $\chi : \mathcal{I} \to R$ *is said to be s-type convex on* $\mathcal{I}$ *if the inequality*

$$\chi(\Bbbk\delta_1 + (1-\Bbbk)\delta_2) \leq [1 - s(1-\Bbbk)]\chi(\delta_1) + (1-s\Bbbk)\chi(\delta_2),$$

holds for all $\delta_1, \delta_2 \in \mathcal{I}$ *and* $\Bbbk \in [0,1]$.

Remark 1. *From Definition 3, we clearly obtain the following:*

1. *If we select* $s = 1$, *then we obtain the traditional convex function.*
2. *If we select* $s = 0$, *then we obtain the definition of a P-function.*
3. *If* χ *is s-type convex on I, then the range of the function* χ *is* $[0, \infty)$.

The literature has several aesthetic inequalities on convex functions, with Jensen's inequality holding a particular place among them. This inequality is widely utilized by academics in information theory and inequality theory, and it can be demonstrated under relatively simple conditions. Jensen's inequality is presented as follows:

Theorem 1 ([11,12]). *If* χ *is a convex function, then Jensen's inequality,*

$$\chi\left(\frac{1}{\eta}\sum_{i=1}^{\eta}\vartheta_i\right) \leq \frac{1}{\eta}\sum_{i=1}^{\eta}\chi(\vartheta_i),$$

and weighted Jensen's inequality,

$$\chi\left(\frac{1}{\mathcal{T}_\eta}\sum_{i=1}^{\eta}\mu_i\vartheta_i\right) \leq \frac{1}{\mathcal{T}_\eta}\sum_{i=1}^{\eta}\mu_i\chi(\vartheta_i),$$

are valid for $\vartheta_i \in \mathcal{I}$ *and* $\mu_i \geq 0$ *with* $i \in \mathbb{N}$ *and* $\mathcal{T}_\eta = \sum_{i=1}^{\eta}\vartheta_i > 0$.

In [13], a more general form of Jensen's inequality depending on the function h is given:

Theorem 2. *Let* $\chi : [\delta_1, \delta_2] \subset \mathcal{I} \to \mathbb{R}$ *be a convex function, and let* $\hbar : \mathcal{I} \to (0, \infty)$ *and* $\zeta : \mathcal{I} \to \mathbb{R}_+ = [0, \infty)$ *be integrable functions. Then,*

$$\chi\left(\frac{\int_{\delta_1}^{\delta_2}\hbar(\Bbbk)\zeta(\Bbbk)d\Bbbk}{\int_{\delta_1}^{\delta_2}\hbar(\Bbbk)d\Bbbk}\right) \leq \frac{\int_{\delta_1}^{\delta_2}\hbar(\Bbbk)\chi(\zeta(\Bbbk))d\Bbbk}{\int_{\delta_1}^{\delta_2}\hbar(\Bbbk)d\Bbbk}. \tag{1}$$

Fundamental mathematics has paid much attention to the Hermite–Hadamard inequality ($\mathcal{H} - \mathcal{H}$ inequality), mainly as a consequence of the extensive use of convex functions and their excellent geometric interpretation. The theory of the inequality has made sig-

nificant progress in recent years [14–16]. One of the main causes of this development is significant inequalities such as the $\mathcal{H}-\mathcal{H}$ inequality. It is interesting to consider how closely the theories of convexity and inequality are related [17–19]. Novel convexity has received a number of new definitions, extensions and generalizations in recent years. At the same time, advances in the theory of convexity inequalities, in particular the theory of integral inequalities, have also received considerable attention. Formally, the $\mathcal{H}-\mathcal{H}$ inequality is written as follows:

Let $\chi : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $\mathcal{I}$ of a real number and $\delta_1, \delta_2 \in \mathcal{I}$ with $\delta_1 < \delta_2$.

$$\chi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\chi(\Bbbk)d\Bbbk \leq \frac{\chi(\delta_1)+\chi(\delta_2)}{2}. \tag{2}$$

If χ is a concave function, then the inequality in (2) will hold in the reverse direction. Based on geometry, the $\mathcal{H}-\mathcal{H}$ inequality provides an upper and lower estimate for the integral mean of any convex function defined on a closed and bounded domain containing the endpoints and midpoints of the function's domain. A number of modifications of the $\mathcal{H}-\mathcal{H}$ inequality (such as Ostrowski, Simpson, Fejer, Hadamard–Jensen–Mercer and Fractional-type inequalities) have been examined in the literature for different classes of convexity, such as (α, m)-convex, quasiconvex, s-convex and h-convex, due to the significance of this inequality.

Fractional calculus is a useful tool for explaining everyday problems and physical occurrences. It is a field of applied mathematics that provides answers more directly connected to real-world issues, enhancing the connections between mathematics and other specialties. Regarding their application fields and spaces, fractional integral and derivative operators have provided novel ideas to fractional analysis. An extension of the idea of the derivative operator which may be extended from the integer order to any rational order is one method to define fractional calculus. Many problems that develop in the disciplines of science and engineering can be resolved with the use of fractional integrals. The Riemann–Liouville fractional integral operators are considered the most helpful and are introduced to advance the theory and provide a foundation for fractional calculus in a more general sense. The operator is defined as follows:

Definition 4 ([20]). *Let $\chi \in L[\delta_1, \delta_2]$. The Riemann–Liouville integrals $J^{\alpha}_{\delta_1^+}\chi$ and $J^{\alpha}_{b^-}\chi$ of order $\alpha > 0$ with $\delta_1 \geq 0$ are defined by*

$$J^{\alpha}_{\delta_1^+}\chi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{\xi}(\xi-\Bbbk)^{\alpha-1}\chi(\Bbbk)d\Bbbk, \quad \xi > \delta_1,$$

and

$$J^{\alpha}_{\delta_2^-}\chi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{\delta_2}(\Bbbk-\xi)^{\alpha-1}\chi(\Bbbk)d\Bbbk, \quad \xi < \delta_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function, and its definition is $\Gamma(\Bbbk) = \int_0^{\infty} e^{-\xi}\xi^{\Bbbk-1}d\xi$. It is to be noted that $J^{\alpha}_{\delta_1^+}\chi(\xi) = J^{\alpha}_{\delta_2^-}\chi(\xi) = \chi(\xi)$; the fractional integral simplifies to the classical integral when $\alpha = 1$.

2. Preliminaries

In 2013, Sarıkaya et al. derived the classical $\mathcal{H}-\mathcal{H}$ inequality as a fractional operator for the first time in [21]. Since then, researchers have obtained new identities using this inequality and special methods. Important studies about this inequality can be found in references [22–25].

Theorem 3. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a positive function with $0 \leq \delta_1 < \delta_2$ and $\delta_1, \delta_2 \in \mathcal{I}$. $\chi \in L[\delta_1, \delta_2]$. If χ is a convex function on $[\delta_1, \delta_2]$, then the following inequality for fractional integrals holds:*

$$\chi\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^{\alpha}}\left[J^{\alpha}_{\delta_1^+}\chi(\delta_2) + J^{\alpha}_{\delta_2^-}\chi(\delta_1)\right] \leq \frac{\chi(\delta_1)+\chi(\delta_2)}{2}.$$

When $\alpha = 1$ is used in the inequality mentioned above, it clearly becomes the classical $\mathcal{H}-\mathcal{H}$ inequality.

The following inequalities provide the basis and methodology for many of the results found in the literature.

The celebrated Young inequality is defined as follows:

Theorem 4 ([26])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\delta_1\delta_2 \leq \frac{1}{p}\delta_1^p + \frac{1}{q}\delta_2^q,$$

where δ_1 and δ_2 are non-negative numbers.

Theorem 5 (Hölder Inequality)**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If χ and g are real functions defined on $[\delta_1, \delta_2]$ such that $|\chi|^p$ and $|\Psi|^q$ are integrable functions on $[\delta_1, \delta_2]$, then*

$$\int_{\delta_1}^{\delta_2} |\chi(\Bbbk)\Psi(\Bbbk)|d\Bbbk \leq \left(\int_{\delta_1}^{\delta_2} |\chi(\Bbbk)|^p d\Bbbk\right)^{\frac{1}{p}} \left(\int_{\delta_1}^{\delta_2} |\Psi(\Bbbk)|^q d\Bbbk\right)^{\frac{1}{q}}.$$

In [27], İşcan obtained the Hölder–İşcan inequality and demonstrated that it provides better boundaries than the Hölder inequality. The literature has revealed numerous novel results through the utilization of the Hölder–İşcan inequality ([27–29]).

Theorem 6 (Hölder–İşcan inequality)**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If χ and Ψ are real functions defined on $[\delta_1, \delta_2]$ such that $|\chi|^p$ and $|\Psi|^q$ are integrable functions on $[\delta_1, \delta_2]$, then*

$$\begin{aligned} & \int_{\delta_1}^{\delta_2} |\chi(\Bbbk)\Psi(\Bbbk)|d\Bbbk \\ \leq\ & \frac{1}{b-\delta_1}\left\{\left(\int_{\delta_1}^{\delta_2}(\delta_2-\Bbbk)|\chi(\Bbbk)|^p d\Bbbk\right)^{\frac{1}{p}}\left(\int_{\delta_1}^{\delta_2}(\delta_2-\Bbbk)|\Psi(\Bbbk)|^q d\Bbbk\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\int_{\delta_1}^{\delta_2}(\Bbbk-\delta_1)|\chi(\Bbbk)|^p d\Bbbk\right)^{\frac{1}{p}}\left(\int_{\delta_1}^{\delta_2}(\Bbbk-\delta_1)|\Psi(\Bbbk)|^q d\Bbbk\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

New generalizations, identities, and results have been obtained in the theory of inequalities through the use of various functions, including the Mittag–Leffler functions and the Hypergeometric functions. The Raina function, which is defined below and related to fractional operators, is one of those functions.

In [30], Raina defined a class of clearly specified functions known as Raina functions by

$$F^{\sigma}_{\varsigma,\lambda}(\xi) = F^{\sigma(0),\sigma(1),\cdots}_{\varsigma,\lambda}(\xi) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\varsigma k+\lambda)}\xi^k, \quad \varsigma, \lambda > 0, \xi \in \mathbb{C} \text{ with } |\xi| < \mathbb{R}^+, \tag{3}$$

where the coefficient $\sigma(k), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is a bounded sequence of positive real numbers and $\mathbb{R}^+$ is a real positive number. The following left-sided and right-sided fractional integral operators, respectively, were defined by Agarwal et al. in [31] with the use of (3):

$$J^{\sigma}_{\varsigma,\lambda.\delta_1^+;w}\chi(\xi) = \int_{\delta_1}^{\xi} (\xi - \Bbbk)^{\lambda-1} F^{\sigma}_{\varsigma,\lambda}[w(\xi - \Bbbk)^{\varsigma}]\chi(\Bbbk)d\Bbbk, \quad \xi > \delta_1, \tag{4}$$

and

$$J^{\sigma}_{\varsigma,\lambda.\delta_2^-;w}\chi(\xi) = \int_{\xi}^{\delta_2} (\Bbbk - \xi)^{\lambda-1} F^{\sigma}_{\varsigma,\lambda}[w(\Bbbk - \xi)^{\varsigma}]\chi(\Bbbk)d\Bbbk, \quad \xi < \delta_2, \tag{5}$$

where $\varsigma, \lambda > 0, w \in \mathbb{R}$ and $\chi(\Bbbk)$ is such that the integrals on the right side exist.

In [32], Yaldız and Sarıkaya presented new fractional integral inequalities of the $\mathcal{H} - \mathcal{H}$ type utilizing a wide class of fractional integral operators. The following Theorem and Lemma present their findings:

Theorem 7. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a convex function on $[\delta_1, \delta_2]$ with $\delta_1 < \delta_2$; then, for fractional integral operators, the following inequalities are valid:*

$$\begin{aligned} & \chi\left(\frac{\delta_1 + \delta_2}{2}\right) \\ \leq\ & \frac{1}{2(\delta_2 - \delta_1)^{\lambda} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \left[(J^{\sigma}_{\varsigma,\lambda.\delta_1^+;w}\chi)(\delta_2) + (J^{\sigma}_{\varsigma,\lambda.\delta_2^-;w}\chi)(\delta_1)\right] \\ \leq\ & \frac{\chi(\delta_1) + \chi(\delta_2)}{2}, \end{aligned}$$

with $\lambda > 0$.

Lemma 1. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable mapping on (δ_1, δ_2) with $\delta_1 < \delta_2$ and $\lambda > 0$. If $\chi' \in L[\delta_1, \delta_2]$, then for fractional integral operators, the following equalities are valid:*

$$\begin{aligned} & \frac{\chi(\delta_1) + \chi(\delta_2)}{2} - \frac{1}{2(\delta_2 - \delta_1)^{\lambda} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \left[(J^{\sigma}_{\varsigma,\lambda.\delta_1^+;w}\chi)(\delta_2) + (J^{\sigma}_{\varsigma,\lambda.\delta_2^-;w}\chi)(\delta_1)\right] \\ =\ & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ & \times \left[\int_0^1 (1 - \Bbbk)^{\lambda} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}(1 - \Bbbk)^{\varsigma}]\chi'(\Bbbk\delta_1 + (1 - \Bbbk)\delta_2)d\Bbbk \right. \\ & \left. - \int_0^1 \Bbbk^{\lambda} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}\Bbbk^{\varsigma}]\chi'(\Bbbk\delta_1 + (1 - \Bbbk)\delta_2)d\Bbbk\right]. \end{aligned}$$

There are many studies by researchers in the literature on the Raina function. Vivas-Cortes et al. established $\mathcal{H} - \mathcal{H}$ inequalities for the Raina fractional integral operator in [33] using the new definition (known as the generalized ϕ-convex function). They also investigated integral inequalities related to the right-hand-side of the $\mathcal{H} - \mathcal{H}$-type inequalities for Raina fractional integrals. A similar study was also attempted in reference [34]. This paper introduces and investigates a generalized s-type convex function of the Raina type. It discusses some algebraic properties and establishes a new version of the $\mathcal{H} - \mathcal{H}$ inequality. In [35], $\mathcal{H} - \mathcal{H}$-type inequalities for Raina fractional operators are derived using a new Lemma and s-convex functions in the second sense. In addition, further results are obtained for certain specific values. And finally, the authors provide a novel general integral identity, including generalized fractional integral operators, in [36]. They established new $\mathcal{H} - \mathcal{H}$-type inequalities for functions whose absolute values of derivatives are convex by using this identity together with a new Lemma. As the Raina function is expressed in terms of fractional operators, it has attracted the interest of many researchers. It is expected that many new generalizations, Theorems and results will be discovered in the future.

The main goal of researchers in mathematical analysis is to obtain new definitions and to find new generalizations, Theorems and results related to these definitions. In this regard, in this paper we aim to obtain new $\mathcal{H}-\mathcal{H}$-type identities for a different type of convex function (called the s-type convex function) and fractional operators using the Raina function. In addition, for selected special values, important results existing in the literature are obtained. This is the main source of motivation for the future work of the researchers.

3. Main Results

Throughout this paper, we use the following notation:

$$\begin{aligned}&\Im(\sigma,\chi,\lambda,\delta_1,\delta_2)\\ =\;&\left|\frac{\chi(\delta_1)+\chi(b)}{2}-\frac{1}{2(\delta_2-\delta_1)^{\lambda}F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\left[(J^{\sigma}_{\varsigma,\lambda,\delta_1^{+};w}\chi)(\delta_2)+(J^{\sigma}_{\varsigma,\lambda,\delta_2^{-};w}\chi)(\delta_1)\right]\right|.\end{aligned}$$

Theorem 8. *Let $\chi:[\delta_1,\delta_2]\to\mathbb{R}$ be a differentiable mapping on (δ_1,δ_2) with $\delta_1<\delta_2$ and $\lambda>0$. If $|\chi'|$is s-type convex on $[\delta_1,\delta_2]$, then for fractional integral operators, the following inequality is valid:*

$$\Im(\sigma,\chi,\lambda,\delta_1,\delta_2)\le(2-s)(\delta_2-\delta_1)\frac{F^{\sigma_1}_{\varsigma,\lambda+2}[w(\delta_2-\delta_1)^{\varsigma}]}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\frac{|\chi'(\delta_1)|+|\chi'(\delta_2)|}{2},\tag{6}$$

where

$$\sigma_1=\sigma(k)\left(1-\frac{1}{2^{\varsigma k+\lambda}}\right).$$

Proof. Using Lemma 1 and when $|\chi'|$ is s-type convex, we find that

$$\begin{aligned}&\Im(\sigma,\chi,\lambda,\delta_1,\delta_2)\\ \le\;&\frac{(\delta_2-\delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(\delta_2-\delta_1)^{\varsigma k}}{\Gamma(\varsigma k+\lambda+1)}\\ &\times\left\{\int_0^{\frac{1}{2}}\left|(1-\mathbb{k})^{\varsigma k+\lambda}-\mathbb{k}^{\varsigma k+\lambda}\right|\left([1-s(1-\mathbb{k})]|\chi'(\delta_1)|+(1-s\mathbb{k})|\chi'(\delta_2)|\right)d\mathbb{k}\right.\\ &\left.+\int_{\frac{1}{2}}^1\left|\mathbb{k}^{\varsigma k+\lambda}-(1-\mathbb{k})^{\varsigma k+\lambda}\right|\left([1-s(1-\mathbb{k})]|\chi'(\delta_1)|+(1-s\mathbb{k})|\chi'(\delta_2)|\right)d\mathbb{k}\right\}\\ =\;&\frac{(\delta_2-\delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(\delta_2-\delta_1)^{\varsigma k}}{\Gamma(\varsigma k+\lambda+1)}\\ &\times\left\{\frac{2-s}{\varsigma k+\lambda+1}\left(1-\frac{1}{2^{\varsigma k+\lambda}}\right)\left[|\chi'(\delta_1)|+|\chi'(\delta_2)|\right]\right\}\\ =\;&\frac{(2-s)(\delta_2-\delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)w^k(\delta_2-\delta_1)^{\varsigma k}}{\Gamma(\varsigma k+\lambda+2)}\left[|\chi'(\delta_1)|+|\chi'(\delta_2)|\right]\\ =\;&(2-s)(\delta_2-\delta_1)\frac{F^{\sigma_1}_{\varsigma,\lambda+2}[w(\delta_2-\delta_1)^{\varsigma}]}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\frac{[|\chi'(\delta_1)|+|\chi'(\delta_2)|]}{2}.\end{aligned}$$

This completes the proof. □

Now, we will talk about a few specific cases of Theorem 8.

Remark 2. *If, in Theorem 8, we take $s=1$, then inequality (6) becomes inequality (2.8) of Theorem 2.2 in [32].*

Remark 3. *If, in Theorem 8, we take $s=1$, $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$, then inequality (6) becomes inequality (3.5) (the Riemann–Liouville fractional integral type) of Theorem 3 in [21].*

Remark 4. *If, in Theorem 8, we take $s = 1$, $\lambda = 1$, $\sigma(0) = 1$ and $w = 0$, then inequality (6) becomes inequality (2.3) of Theorem 2 in [37].*

Theorem 9. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a positive differentiable mapping on (δ_1, δ_2). If $|\chi'|^q$ is an s-type convex function on $[\delta_1, \delta_2]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) &\leq \frac{(\delta_2 - \delta_1)}{2} \frac{F^{\sigma_2}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ &\times \left[\left(\frac{(4-s)}{8} |\chi'(\delta_1)|^q + \frac{(4-3s)}{8} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} + \left(\frac{(4-3s)}{8} |\chi'(\delta_1)|^q + \frac{(4-s)}{8} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} \right],\end{aligned} \tag{7}$$

where

$$\sigma_2 = \sigma(k) \left(\frac{1}{(\varsigma k + \lambda)p + 1} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{(\varsigma k + \lambda)p}} \right)^{\frac{1}{p}},$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda > 0$.

Proof. From Lemma 1 and Hölder's inequality, we obtain

$$\begin{aligned}&\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ &\times \left\{ \left(\int_0^{\frac{1}{2}} \left| (1-\mathbb{k})^{\varsigma k + \lambda} - \mathbb{k}^{\varsigma k + \lambda} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_{\frac{1}{2}}^1 \left| \mathbb{k}^{\varsigma k + \lambda} - (1-\mathbb{k})^{\varsigma k + \lambda} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \right\}.\end{aligned}$$

Using the s-type convex function of $|\chi'|^q$, we have

$$\begin{aligned}&\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ &\times \left\{ \left(\int_0^{\frac{1}{2}} \left[(1-\mathbb{k})^{\varsigma k + \lambda} - \mathbb{k}^{\varsigma k + \lambda} \right]^p d\mathbb{k} \right)^{\frac{1}{p}} \right. \\ &\qquad \left. \times \left(\int_0^{\frac{1}{2}} \left([1 - s(1-\mathbb{k})] |\chi'(\delta_1)|^q + (1 - s\mathbb{k}) |\chi'(\delta_2)|^q \right) d\mathbb{k} \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_{\frac{1}{2}}^1 \left[\mathbb{k}^{\varsigma k + \lambda} - (1-\mathbb{k})^{\varsigma k + \lambda} \right]^p d\mathbb{k} \right)^{\frac{1}{p}} \right. \\ &\qquad \left. \times \left(\int_{\frac{1}{2}}^1 \left([1 - s(1-\mathbb{k})] |\chi'(\delta_1)|^q + (1 - s\mathbb{k}) |\chi'(\delta_2)|^q \right) d\mathbb{k} \right)^{\frac{1}{q}} \right\}\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{(\delta_2-\delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(\delta_2-\delta_1)^{\varsigma k}}{\Gamma(\varsigma k+\lambda+1)}\\
&\times\Bigg\{\left(\int_0^{\frac{1}{2}}\left[(1-\Bbbk)^{(\varsigma k+\lambda)p}-\Bbbk^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_0^{\frac{1}{2}}\left([1-s(1-\Bbbk)]\left|\chi'(\delta_1)\right|^q+(1-s\Bbbk)\left|\chi'(\delta_2)\right|^q\right)d\Bbbk\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left[\Bbbk^{(\varsigma k+\lambda)p}-(1-\Bbbk)^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\qquad\times\left(\int_{\frac{1}{2}}^1\left([1-s(1-\Bbbk)]\left|\chi'(\delta_1)\right|^q+(1-s\Bbbk)\left|\chi'(\delta_2)\right|^q\right)d\Bbbk\right)^{\frac{1}{q}}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{(\delta_2-\delta_1)}{2}\frac{F^{\sigma_2}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\\
&\times\left[\left(\frac{(4-s)}{8}\left|\chi'(\delta_1)\right|^q+\frac{(4-3s)}{8}\left|\chi'(\delta_2)\right|^q\right)^{\frac{1}{q}}+\left(\frac{(4-3s)}{8}\left|\chi'(\delta_1)\right|^q+\frac{(4-s)}{8}\left|\chi'(\delta_2)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Here, we use $[\mathcal{A}-\mathcal{B}]^{\theta}\leq\mathcal{A}^{\theta}-\mathcal{B}^{\theta}$ for any $\mathcal{A}>\mathcal{B}\geq 0$ and $\theta\geq 1$. Furthermore, we can write the following identity:

$$
\begin{aligned}
\left(\int_0^{\frac{1}{2}}\left|(1-\Bbbk)^{\varsigma k+\lambda}-\Bbbk^{\varsigma k+\lambda}\right|^p d\Bbbk\right)^{\frac{1}{p}} &= \left(\int_{\frac{1}{2}}^1\left|\Bbbk^{\varsigma k+\lambda}-(1-\Bbbk)^{\varsigma k+\lambda}\right|^p d\Bbbk\right)^{\frac{1}{p}}\\
&= \left(\frac{-2\cdot\left(\frac{1}{2}\right)^{(\varsigma kp+\lambda p+1)}}{\varsigma kp+\lambda p+1}+\frac{1^{(\varsigma kp+\lambda p+1)}}{\varsigma kp+\lambda p+1}\right)^{\frac{1}{p}}\\
&= \left(\frac{1}{(\varsigma k+\lambda)p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{(\varsigma k+\lambda)p}}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, the proof is completed. □

Remark 5. *If, in Theorem 9, we set $s=1$, then inequality (7) becomes the inequality of Theorem 2.3 in [32].*

Remark 6. *If, in Theorem 9, we choose $s=1$, $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$, then we obtain the inequality of Corollary 2.1 (the Riemann–Liouville fractional integral type) in [32].*

Remark 7. *If we choose $s=1$, $\lambda=\alpha=1$, $\sigma(0)=1$ and $w=0$ in Theorem 9, we obtain the inequality of Corollary 2.2 in [32].*

Theorem 10. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable mapping on (δ_1, δ_2) with $\delta_1 < \delta_2$. If $|\chi'|^q$ is an s-type convex on $[\delta_1, \delta_2]$ for $p > 1$, then the following inequality for Raina-type fractional integrals holds:*

$$\begin{aligned}\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \leq{}& \frac{(\delta_2 - \delta_1)}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ &\times \Bigg[F^{\sigma_3}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \left(\frac{6-5s}{48} |\chi'(\delta_1)|^q + \frac{6-s}{48} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} \\ &+ F^{\sigma_4}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \left(\frac{3-2s}{24} |\chi'(\delta_1)|^q + \frac{3-s}{24} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} \\ &+ F^{\sigma_5}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \left(\frac{3-s}{24} |\chi'(\delta_1)|^q + \frac{3-2s}{24} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} \\ &+ F^{\sigma_6}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \left(\frac{6-s}{48} |\chi'(\delta_1)|^q + \frac{6-5s}{48} |\chi'(\delta_2)|^q \right)^{\frac{1}{q}} \Bigg],\end{aligned}$$

where

$$\begin{aligned}\sigma_3 &= \sigma(k) \left(\frac{1}{(c+2)} - \frac{1}{2(c+1)} \right)^{\frac{1}{p}} \\ \sigma_4 &= \sigma(k) \left(\frac{1 - \left(\frac{1}{2}\right)^{c+1}}{(c+1)} - \frac{1}{(c+2)} \right)^{\frac{1}{p}} \\ \sigma_5 &= \sigma(k) \left(\frac{\left(\frac{1}{2}\right)^{c+1} - 1}{(c+1)} + \frac{1}{(c+2)} \right)^{\frac{1}{p}} \\ \sigma_6 &= \sigma(k) \left(\frac{1}{2(c+1)} - \frac{1}{(c+2)} \right)^{\frac{1}{p}} \\ c &= (\varsigma k + \lambda) p.\end{aligned}$$

Proof. Similarly to in Theorem 9, utilizing Lemma 1 and the Hölder-İşcan inequality, we obtain

$$\begin{aligned}&\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq{}& \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ &\times \Bigg[\left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \mathbb{k} \right) \left| (1-\mathbb{k})^{(\varsigma k+\lambda)} - \mathbb{k}^{(\varsigma k+\lambda)} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \mathbb{k} \right) |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ &+ \left(\int_0^{\frac{1}{2}} \mathbb{k} \left| (1-\mathbb{k})^{(\varsigma k+\lambda)} - \mathbb{k}^{(\varsigma k+\lambda)} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \mathbb{k} |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{1}{2}}^1 (1-\mathbb{k}) \left| \mathbb{k}^{(\varsigma k+\lambda)} - (1-\mathbb{k})^{(\varsigma k+\lambda)} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 (1-\mathbb{k}) |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \\ &+ \left(\int_{\frac{1}{2}}^1 \left(\mathbb{k} - \frac{1}{2} \right) \left| \mathbb{k}^{(\varsigma k+\lambda)} - (1-\mathbb{k})^{(\varsigma k+\lambda)} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left(\mathbb{k} - \frac{1}{2} \right) |\chi'(\mathbb{k}\delta_1 + (1-\mathbb{k})\delta_2)|^q d\mathbb{k} \right)^{\frac{1}{q}} \Bigg].\end{aligned}$$

By utilizing the s-type convexity of $|\chi'|^q$ and $[\mathcal{A}-\mathcal{B}]^\theta \leq \mathcal{A}^\theta - \mathcal{B}^\theta$ for any $\mathcal{A} > \mathcal{B} \geq 0$ and $\theta \geq 1$, we have

$$
\begin{aligned}
&\Im(\sigma,\chi,\lambda,\delta_1,\delta_2)\\
\leq\ & \frac{(\delta_2-\delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\sum_{k=0}^{\infty}\frac{\sigma(k)w^k(\delta_2-\delta_1)^{\varsigma k}}{\Gamma(\varsigma k+\lambda+1)}\\
&\times\Bigg[\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\Bbbk\right)\left[(1-\Bbbk)^{(\varsigma k+\lambda)p}-\Bbbk^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\Bbbk\right)\left([1-s(1-\Bbbk)]|\chi'(\delta_1)|^q+(1-s\Bbbk)|\chi'(\delta_2)|^q\right)d\Bbbk\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}\Bbbk\left[(1-\Bbbk)^{(\varsigma k+\lambda)p}-\Bbbk^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\Bbbk\left([1-s(1-\Bbbk)]|\chi'(\delta_1)|^q+(1-s\Bbbk)|\chi'(\delta_2)|^q\right)d\Bbbk\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}(1-\Bbbk)\left[\Bbbk^{(\varsigma k+\lambda)p}-(1-\Bbbk)^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}(1-\Bbbk)\left([1-s(1-\Bbbk)]|\chi'(\delta_1)|^q+(1-s\Bbbk)|\chi'(\delta_2)|^q\right)d\Bbbk\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left(\Bbbk-\frac{1}{2}\right)\left[\Bbbk^{(\varsigma k+\lambda)p}-(1-\Bbbk)^{(\varsigma k+\lambda)p}\right]d\Bbbk\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left(\Bbbk-\frac{1}{2}\right)\left([1-s(1-\Bbbk)]|\chi'(\delta_1)|^q+(1-s\Bbbk)|\chi'(\delta_2)|^q\right)d\Bbbk\right)^{\frac{1}{q}}\Bigg]\\
\leq\ & \frac{(\delta_2-\delta_1)}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\\
&\times\Bigg[F^{\sigma_3}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{6-5s}{48}|\chi'(\delta_1)|^q+\frac{6-s}{48}|\chi'(\delta_2)|^q\right)^{\frac{1}{q}}\\
&+F^{\sigma_4}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{3-2s}{24}|\chi'(\delta_1)|^q+\frac{3-s}{24}|\chi'(\delta_2)|^q\right)^{\frac{1}{q}}\\
&+F^{\sigma_5}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{3-s}{24}|\chi'(\delta_1)|^q+\frac{3-2s}{24}|\chi'(\delta_2)|^q\right)^{\frac{1}{q}}\\
&+F^{\sigma_6}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{6-s}{48}|\chi'(\delta_1)|^q+\frac{6-5s}{48}|\chi'(\delta_2)|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Hence, the proof is completed. □

Corollary 1. *If we choose $s = 1$ in Theorem 10, we have the Raina fractional integral operator-type inequality:*

$$\begin{aligned}
&\Im(\sigma,\chi,\lambda,\delta_1,\delta_2)\\
\leq\ & \frac{(\delta_2-\delta_1)}{F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]}\Bigg[F^{\sigma_3}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{|\chi'(\delta_1)|^q+5|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\\
&+F^{\sigma_4}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{|\chi'(\delta_1)|^q+2|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+F^{\sigma_5}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{2|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+F^{\sigma_6}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]\left(\frac{5|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Corollary 2. *If we choose $s = 1$, $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 10, we obtain the Riemann–Liouville fractional integral operator-type inequality as follows:*

$$\begin{aligned}
&\left|\frac{\chi(\delta_1)+\chi(\delta_2)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^{\alpha}}\left[(J^{\alpha}_{\delta_1^+}\chi(\delta_2)+(J^{\alpha}_{\delta_2^-}\chi(\delta_1)\right]\right|\\
\leq\ & (\delta_2-\delta_1)\Bigg[\left(\frac{1}{(\alpha p+2)}-\frac{1}{2(\alpha p+1)}\right)^{\frac{1}{p}}\left(\frac{|\chi'(\delta_1)|^q+5|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\\
&+\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p+1}}{(\alpha p+1)}-\frac{1}{(\alpha p+2)}\right)^{\frac{1}{p}}\left(\frac{|\chi'(\delta_1)|^q+2|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+\left(\frac{\left(\frac{1}{2}\right)^{\alpha p+1}-1}{(\alpha p+1)}+\frac{1}{(\alpha p+2)}\right)^{\frac{1}{p}}\left(\frac{2|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{2(\alpha p+1)}-\frac{1}{(\alpha p+2)}\right)^{\frac{1}{p}}\left(\frac{5|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Corollary 3. *If we take $\alpha = 1$ in Corollary 2, we obtain the classical integral inequality as follows:*

$$\begin{aligned}
&\left|\frac{\chi(\delta_1)+\chi(\delta_2)}{2}-\frac{1}{(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}\chi(\mathbb{k})d\mathbb{k}\right|\\
\leq\ & (\delta_2-\delta_1)\Bigg[\left(\frac{1}{(p+2)}-\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\chi'(\delta_1)|^q+5|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\\
&+\left(\frac{1-\left(\frac{1}{2}\right)^{p+1}}{(p+1)}-\frac{1}{(p+2)}\right)^{\frac{1}{p}}\left(\frac{|\chi'(\delta_1)|^q+2|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+\left(\frac{\left(\frac{1}{2}\right)^{p+1}-1}{(p+1)}+\frac{1}{(p+2)}\right)^{\frac{1}{p}}\left(\frac{2|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{24}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{2(p+1)}-\frac{1}{(p+2)}\right)^{\frac{1}{p}}\left(\frac{5|\chi'(\delta_1)|^q+|\chi'(\delta_2)|^q}{48}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Theorem 11. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable function on (δ_1, δ_2) and $\delta_1 < \delta_2$. If $|\chi'|^q$ is s-type convex on $[\delta_1, \delta_2]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ & \times \left[\frac{2}{p} F^{\sigma_7}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] + F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \frac{(2-s)}{q} \frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right], \end{aligned}$$

where

$$\begin{aligned} \sigma_7 &= \sigma(k)\left(\frac{1}{c+1}\right)\left(1 - \frac{1}{2^c}\right) \\ c &= (\varsigma k + \lambda)p. \end{aligned}$$

Proof. Using the well known Young's inequality, we obtain

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ & \times \left[\left(\frac{1}{p} \int_0^{\frac{1}{2}} \left| (1-\Bbbk)^{\varsigma k + \lambda} - \Bbbk^{\varsigma k + \lambda} \right|^p d\Bbbk + \frac{1}{q} \int_0^{\frac{1}{2}} |\chi'(\Bbbk\delta_1 + (1-\Bbbk)\delta_2)|^q d\Bbbk \right) \right. \\ & \left. + \left(\frac{1}{p} \int_{\frac{1}{2}}^1 \left| \Bbbk^{\varsigma k + \lambda} - (1-\Bbbk)^{\varsigma k + \lambda} \right|^p d\Bbbk + \frac{1}{q} \int_{\frac{1}{2}}^1 |\chi'(\Bbbk\delta_1 + (1-\Bbbk)\delta_2)|^q d\Bbbk \right) \right]. \end{aligned}$$

By utilizing the s-type convexity of $|\chi'|^q$ and $[\mathcal{A} - \mathcal{B}]^{\theta} \leq \mathcal{A}^{\theta} - \mathcal{B}^{\theta}$ for any $\mathcal{A} > \mathcal{B} \geq 0$ and $\theta \geq 1$, we obtain

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ & \times \left[\left(\frac{1}{p} \int_0^{\frac{1}{2}} \left| (1-\Bbbk)^{\varsigma k + \lambda} - \Bbbk^{\varsigma k + \lambda} \right|^p d\Bbbk + \frac{1}{q} \int_0^{\frac{1}{2}} \Big([1 - s(1-\Bbbk)] |\chi'(\delta_1)|^q + (1 - s\Bbbk) |\chi'(\delta_2)|^q \Big) d\Bbbk \right) \right. \\ & \left. + \left(\frac{1}{p} \int_{\frac{1}{2}}^1 \left| \Bbbk^{\varsigma k + \lambda} - (1-\Bbbk)^{\varsigma k + \lambda} \right|^p d\Bbbk + \frac{1}{q} \int_{\frac{1}{2}}^1 \Big([1 - s(1-\Bbbk)] |\chi'(\delta_1)|^q + (1 - s\Bbbk) |\chi'(\delta_2)|^q \Big) d\Bbbk \right) \right] \\ \leq & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ & \times \left[\frac{1}{p} \left(\frac{1}{c+1} \right) \left(1 - \frac{1}{2^c} \right) + \frac{1}{q} \left(\frac{(4-s)}{8} |\chi'(\delta_1)|^q + \frac{(4-3s)}{8} |\chi'(\delta_2)|^q \right) \right. \\ & \left. + \frac{1}{p} \left(\frac{1}{c+1} \right) \left(1 - \frac{1}{2^c} \right) + \left(\frac{(4-3s)}{8} |\chi'(\delta_1)|^q + \frac{(4-s)}{8} |\chi'(\delta_2)|^q \right) \right] \\ = & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ & \times \left[\frac{2}{p} \left(\frac{1}{c+1} \right) \left(1 - \frac{1}{2^c} \right) + \frac{2-s}{q} \left(\frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right) \right] \\ = & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ & \times \left[\frac{2}{p} F^{\sigma_7}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] + F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \frac{(2-s)}{q} \frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right]. \end{aligned}$$

This completes the proof. □

Corollary 4. *If we choose $s = 1$ in Theorem 11 , we have*

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \\ & \times \left[\frac{2}{p} F^{\sigma_7}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] + \frac{1}{q} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right]. \end{aligned}$$

σ_7 is as in Theorem 11.

Corollary 5. *If we choose $s = 1$, $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 11, we obtain the following result:*

$$\begin{aligned} & \left| \frac{\chi(\delta_1) + \chi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2 - \delta_1)^{\alpha}} \left[(J^{\alpha}_{\delta_1^+} \chi(\delta_2)) + (J^{\alpha}_{\delta_2^-} \chi(\delta_1)) \right] \right| \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2} \left[\frac{2}{p} \left(\frac{1}{\alpha p + 1} \right) \left(1 - \frac{1}{2^{\alpha p}} \right) + \frac{1}{q} \frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right]. \end{aligned}$$

Corollary 6. *If we choose $\alpha = 1$ in Corollary 5, we have the new result as follows:*

$$\begin{aligned} & \left| \frac{\chi(\delta_1) + \chi(\delta_2)}{2} - \frac{1}{(\delta_2 - \delta_1)} \int_{\delta_1}^{\delta_2} \chi(\mathbb{k}) d\mathbb{k} \right| \\ \leq \frac{(\delta_2 - \delta_1)}{2} & \left[\frac{2}{p} \left(\frac{1}{p+1} \right) \left(1 - \frac{1}{2^p} \right) + \frac{1}{q} \frac{|\chi'(\delta_1)|^q + |\chi'(\delta_2)|^q}{2} \right]. \end{aligned}$$

Theorem 12. *Let $\chi : [\delta_1, \delta_2] \to \mathbb{R}$ be a differentiable function on (δ_1, δ_2) and $\delta_1 < \delta_2$. If $|\chi'|^q$ is concave on $[\delta_1, \delta_2]$, $q > 1$, then for fractional integral operators, the following inequality is valid:*

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2^{1+\frac{1}{q}} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} F^{\sigma_2}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}] \\ & \times \left[\left| \chi' \left(\frac{\delta_1 + 3\delta_2}{4} \right) \right| + \left| \chi' \left(\frac{3\delta_1 + \delta_2}{4} \right) \right| \right]. \end{aligned}$$

σ_2 is as in Theorem 9.

Proof. By using Hölder's inequality, we obtain

$$\begin{aligned} & \Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\ \leq\ & \frac{(\delta_2 - \delta_1)}{2F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2 - \delta_1)^{\varsigma}]} \sum_{k=0}^{\infty} \frac{\sigma(k) w^k (\delta_2 - \delta_1)^{\varsigma k}}{\Gamma(\varsigma k + \lambda + 1)} \\ & \times \left[\left(\int_0^{\frac{1}{2}} \left| (1 - \mathbb{k})^{\varsigma k + \lambda} - \mathbb{k}^{\varsigma k + \lambda} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \chi'(\mathbb{k}\delta_1 + (1 - \mathbb{k})\delta_2 \right|^q d\mathbb{k} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^{1} \left| \mathbb{k}^{\varsigma k + \lambda} - (1 - \mathbb{k})^{\varsigma k + \lambda} \right|^p d\mathbb{k} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \chi'(\mathbb{k}\delta_1 + (1 - \mathbb{k})\delta_2 \right|^q d\mathbb{k} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Since $|\chi'|^q$ is concave on $[\delta_1, \delta_2]$, we can use the Jensen integral inequality

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|\chi'(\Bbbk\delta_1 + (1-\Bbbk)\delta_2\right|^q d\Bbbk \\
\leq\ & \left(\int_0^{\frac{1}{2}} \Bbbk^0 d\Bbbk\right).\left|\chi'\left(\frac{1}{\int_0^{\frac{1}{2}} \Bbbk^0 d\Bbbk}\int_0^{\frac{1}{2}} (\Bbbk\delta_1 + (1-\Bbbk)\delta_2)d\Bbbk\right)\right|^q \\
\leq\ & \frac{1}{2}\left|\chi'\left(\frac{\delta_1+3\delta_2}{4}\right)\right|^q,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left|\chi'(\Bbbk\delta_1 + (1-\Bbbk)\delta_2)\right|^q d\Bbbk \\
\leq\ & \left(\int_{\frac{1}{2}}^1 \Bbbk^0 d\Bbbk\right)\left|\chi'\left(\frac{1}{\int_{\frac{1}{2}}^1 \Bbbk^0 d\Bbbk}\int_{\frac{1}{2}}^1 (\Bbbk\delta_1 + (1-\Bbbk)\delta_2)d\Bbbk\right)\right|^q \\
\leq\ & \frac{1}{2}\left|\chi'\left(\frac{3\delta_1+\delta_2}{4}\right)\right|^q.
\end{aligned}$$

By combining all of the obtained inequalities, the following result is derived:

$$\begin{aligned}
&\Im(\sigma, \chi, \lambda, \delta_1, \delta_2) \\
\leq\ & \frac{(\delta_2-\delta_1)}{2^{1+\frac{1}{q}} F^{\sigma}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}]} F^{\sigma_2}_{\varsigma,\lambda+1}[w(\delta_2-\delta_1)^{\varsigma}] \\
&\times\left[\left|\chi'\left(\frac{3\delta_1+\delta_2}{4}\right)\right| + \left|\chi'\left(\frac{\delta_1+3\delta_2}{4}\right)\right|\right].
\end{aligned}$$

This completes the proof. □

Corollary 7. *If we choose $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 12, we obtain the following result:*

$$\begin{aligned}
&\left|\frac{\chi(\delta_1)+\chi(\delta_2)}{2} - \frac{\Gamma(\alpha+1)}{2(\delta_2-\delta_1)^{\alpha}}\left[(J^{\alpha}_{\delta_1^+}\chi(\delta_2)) + (J^{\alpha}_{\delta_2^-}\chi(\delta_1))\right]\right| \\
\leq\ & \frac{(\delta_2-\delta_1)}{2^{1+\frac{1}{q}}}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}} \\
&\times\left[\left|\chi'\left(\frac{3\delta_1+\delta_2}{4}\right)\right| + \left|\chi'\left(\frac{\delta_1+3\delta_2}{4}\right)\right|\right].
\end{aligned}$$

Corollary 8. *If we choose $\alpha = 1$ in Corollary 7, we have*

$$\begin{aligned}
&\left|\frac{\chi(\delta_1)+\chi(\delta_2)}{2} - \frac{1}{(\delta_2-\delta_1)}\int_{\delta_1}^{\delta_2}\chi(\Bbbk)d\Bbbk\right| \\
\leq\ & \frac{(\delta_2-\delta_1)}{2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^p}\right)^{\frac{1}{p}} \\
&\times\left[\left|\chi'\left(\frac{3\delta_1+\delta_2}{4}\right)\right| + \left|\chi'\left(\frac{\delta_1+3\delta_2}{4}\right)\right|\right].
\end{aligned}$$

4. Conclusions

The Raina function has been extensively studied and developed by researchers since its relation with the fractional integral operator was established. Fractional operators (4) and (5) are among the most significant operators. In this regard, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance, the classical Riemann–Liouville fractional integrals $J^{\alpha}_{\delta_1^+}$ and $J^{\alpha}_{\delta_2^-}$ of order α can be obtained by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in operators (4) and (5).

This paper presents new generalizations and results for the s-type convex function. The motivation for this work is the existence of similar results in the literature. Specifically, we derive $\mathcal{H} - \mathcal{H}$-type inequalities for the Raina function by using well-known inequalities such as Hölder and Young inequalities. Furthermore, by modifying the values of s, λ, α and σ, we obtain new and previously unpublished findings. This method also allows for obtaining new and distinct identities for concave functions. The presented consequences and methods in this work may encourage further investigation in this field by researchers.

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