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Volatility Analysis of Financial Time Series Using the Multifractal Conditional Diffusion Entropy Method

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Abstract: In this article, we introduce the multifractal conditional diffusion entropy method for analyzing the volatility of financial time series. This method utilizes a q-order diffusion entropy based on a q-weighted time lag scale. The technique of conditional diffusion entropy proves valuable for examining bull and bear behaviors in stock markets across various time scales. Empirical findings from analyzing the Dow Jones Industrial Average (DJI) indicate that employing multi-time lag scales offers greater insight into the complex dynamics of highly fluctuating time series, often characterized by multifractal behavior. A smaller time scale like $t = 2$ to $t = 256$ coincides more with the state of the DJI index than larger time scales like $t = 256$ to $t = 1024$. We observe extreme fluctuations in the conditional diffusion entropy for DJI for a short time lag, while smoother or averaged fluctuations occur over larger time lags.

Keywords: multifractal entropy analysis; diffusion entropy analysis; conditional diffusion entropy; financial time series

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1. Introduction

Practitioners, researchers, and regulators across the domains of economics, mathematics, and physics increasingly prioritize understanding the stability of financial systems in response to the subprime crisis [1]. This heightened importance extends to asset and derivative pricing, asset allocation, and risk management. Concepts and techniques derived from complex systems and econophysics drive investigations into anomalous, chaotic, and non-stationary behavior within economic systems, with entropy playing a crucial role. The literature abounds with examples showcasing abrupt transitions from stable states to radically different ones, building upon prior studies that frame the financial crisis as a complex dynamical system.

In [2], the authors focus on examining scaling and memory phenomena within the context of return intervals associated with stock and currency data. Their findings reveal that a singular scaling function can effectively approximate the distribution function of return intervals. Furthermore, the study underscores robust memory effects, indicating that shorter return intervals are more likely to succeed other short intervals. In contrast, longer intervals tend to be followed by similarly extended periods.

Siokis (2012) analyzes the distribution of the magnitude of significant stock market shocks [3]. The study models the behavior of market index returns before and after significant crashes, aiming to identify statistical patterns. The analysis reveals that, for a considerable number of market crashes, the distribution of market volatility before and after the crash follows the Gutenberg-Richter law, signifying the presence of scale-invariance and self-similarity in the underlying dynamics through a robust power-law relationship.

In [4], Wang et al. examine volatility return intervals for the most heavily traded stocks in the United States markets. The study reveals that the scaling exponent exhibits dependence on the threshold value q, indicating the presence of a multiscaling nature in the distribution of return intervals. They delve into the multiscaling exponent, which characterizes the multiscaling behavior of individual stocks.

The author in [5] investigates the impact of an entropy disturbance in the United States on the entropy levels of other financial markets by employing singular value decomposition on the constituents of stock market indices from different economies.

In [6], the author discusses whether uncertainty and disorder in the stock market reflect entropy. In [7], the authors investigate the volatility of seven stock market indices based on Tsallis and Shannon entropy. Compared to other methods, such as convexity, variance, and vector autoregression (VaR), the authors in [8] find that the information entropy method is a better way to quantify the risk associated with bonds. Therefore, identifying potentially significant factors to reduce the negative consequences on economic systems has received attention recently, even though global financial crises usually result from events generated in the financial industry sectors.

This study aims to extend the Conditional Diffusion Entropy Analysis developed in [9,10] to a Multi-scale Conditional Diffusion Entropy (MS-CDE) as another pertinent method. Our proposed model serves as the multifractal extension of the CDE. It emphasizes the role of Shannon entropy (SE) and Rényi entropy (RE). The monofractal approach utilizes the SE of Scafetta et al. [11] while the multifractal approach employs a combination of SE and RE for different q weights to estimate scaling exponents [12,13]. The study proposes the examination of bull and bear markets using multi-time lag scales to determine MS-CDE. It leverages the study on optimal bin-width in empirical histograms by Jizba et al. [14] to evaluate the underlying probability density function (PDF).

This paper presents the following structure: Section 2 briefly reviews the foundations of the diffusion entropy analysis (DEA) required for determining CDE and MS-CDE. We also provide a brief overview of the q-order DEA, which facilitates the calculation of MS-CDE. Section 3 introduces the time series from the DJI market index sampled daily from April 2013 to April 2021, which we use to generate empirical results. Section 4 showcases the results from the experiment, and corresponding discussions occur in Section 5. Finally, Section 6 concludes the paper.

2. Methodology

2.1. Diffusion Entropy Analysis (DEA)

Let N be the length of a financial time series $\{Y_i\}_{i=1}^N$. The process is as follows [11].

1. Transform the series $\{Y_i\}_{i=1}^N$ into a diffusion process. Consider the series such that it can be written as:
$$\{Y_i, Y_{i+1}, Y_{i+2}, \cdots, Y_{i+t-1}\}$$
where $i = 1, 2, \ldots, N - t + 1$ and $t \in [1, N]$ is the time scale. The matrix ψ_i^j defined as
$$\psi_i^j = Y_{i+j} - Y_{i+j-1},\ j = 0, 1, \cdots, N - t \tag{1}$$
can be regarded as sub-sequences for any given diffusion time t with initial state $\psi_i^0 = 0$. Next, construct a diffusion trajectory for each of these sub-sequences using the stochastic process
$$\eta_t^j = \sum_{i=1}^{t} \psi_i^j, \tag{2}$$
where η_t^j is the new position of the jth particle in the diffusion process.
2. Compute the diffusion entropy. First, partition the x-axis into bin size $B(t)$ and assume that $N_i(t)$ represents the number of particles falling in each bin at time t where $i = 1, 2, \cdots, B(t)$.
3. Determine the optimal bin size B. We note that there is no "best" number of bins, and different bin sizes B reveal different data features. Wider bins are utilized when

the density of the underlying data points is low, reducing sampling-related noise. Conversely, narrower bins are employed when the density is high, enhancing the precision of density estimation. Hence, it proves advantageous to adjust the bin size within a histogram. For our calculations, we utilize the Freedman-Diaconis' rule [15] to determine the bin size B, which is defined as:

$$B = 2\frac{\text{Interquartile range}}{\sqrt[3]{n}}. \tag{3}$$

Freedman-Diaconis' rule is less sensitive to outliers in data compared to the standard deviation, rendering it more robust. Another approach is the Scott's rule [16]. It is defined as $B = \frac{3.5\hat{\sigma}}{\sqrt[3]{n}}$, where $\hat{\sigma}$ denotes the sample standard deviation. Scott's rule works best with data that follows a Gaussian distribution.

4. We approximate the probability density function (PDF) of a particle falling into a bin at time t using the relative frequency as:

$$p(i,t) = \frac{N_i(t)}{N - t + 1}. \tag{4}$$

At each time t, we calculate the diffusion (Shannon) entropy as follows:

$$S(t) = -\sum_{i=1}^{B(t)} p(i,t)\, ln[p(i,t)]. \tag{5}$$

Normalizing the diffusion entropy at time t results in:

$$\bar{S}(t) = \frac{S(t)}{t}. \tag{6}$$

We obtain the linear-log relationship between entropy $S(t)$ and time t [17,18] as:

$$S(t) = A + \delta ln(t). \tag{7}$$

2.2. Conditional Diffusion Entropy (CDE)

Conditional Diffusion Entropy, $C(t)$, provides a framework for distinguishing between bear and bull markets. In a bull market, the market expands, and economic conditions are typically favorable. In contrast, a bear market develops when the economy contracts and most stocks and equities lose value. The CDE, $C(t)$, as shown in [9,10], is defined as:

$$C(t) = 1 + \alpha\, [1 - \bar{S}(t)]. \tag{8}$$

Here, $\bar{S}(t)$ is the normalized entropy as shown in Equation (6). The coefficient α is defined by:

$$\alpha = \begin{cases} -1, & p < q \\ 0, & p = q \\ 1, & p > q \end{cases}. \tag{9}$$

In Equation (9), p and q denote the number of negative and positive values in the series, respectively. Thus, during a bear market, $\alpha > 0$, while during a bull market, $\alpha < 0$. When the market is indifferent or random, $\alpha = 0$, indicating neither a bull nor a bear market.

2.3. q-Order Diffusion Entropy Analysis (Q-DEA)

As multifractality presents a continuous property of time series, techniques that address problems concerning discretization and finite size of histograms employ interpolation approaches, rendering them susceptible to bias. When determining the optimal bin size B for different values of q, the Rényi entropy incorporates both the probabilities p_i and their q-th powers p_i^q for different q. The q-order Scott's rule for determining bin size B_q is given as:

$$B_q = (24\sqrt{\pi})^{1/3} \frac{\sqrt{q}}{\sqrt[6]{2q-1}} \left(\frac{\sum_{k=1}^{m} \sigma_{s_k}^{2(1-q)} / N_{s_k}}{\sum_{k=1}^{m} \sigma_{s_k}^{-(1+2q)}} \right)^{1/3},$$

where $q = 0, 1, 2, 3, 4$ and scale $s_k = 2^k, k = 1, 2, 3, \cdots, m = floor(\log N)$.

We substitute the theoritical standard deviation σ with the empirical standard deviation $\hat{\sigma}$ to get

$$\begin{aligned} \hat{B}_q &= (24\sqrt{\pi})^{1/3} \frac{\sqrt{q}}{\sqrt[6]{2q-1}} \left(\frac{\sum_{k=1}^{m} \hat{\sigma}_{s_k}^{2(1-q)} / N_{s_k}}{\sum_{k=1}^{m} \hat{\sigma}_{s_k}^{-(1+2q)}} \right)^{1/3}. \qquad (10) \\ &\equiv (24\sqrt{\pi})^{1/3} \frac{\sqrt{q}}{\sqrt[6]{2q-1}} N_{q,m}^{\hat{\sigma}}, \end{aligned}$$

where

$$N_{q,m}^{\hat{\sigma}} = \left(\frac{\sum_{k=1}^{m} \hat{\sigma}_{s_k}^{2(1-q)} / N_{s_k}}{\sum_{k=1}^{m} \hat{\sigma}_{s_k}^{-(1+2q)}} \right)^{1/3}.$$

For Freedman-Diaconis' rule, replace the estimated standard deviation with the IQR to get

$$\hat{B}_q = 2.6 \frac{\sqrt{q}}{\sqrt[6]{2q-1}} N_{q,m}^{\hat{IQR}}. \qquad (11)$$

Please see the reference [14] for more details.

To analyze multifractal scaling properties of a time series, we use the q-order entropy, which is a family of Shannon and Rényi entropies defined as:

$$S_q(t) = \begin{cases} -\sum_{i=1}^{B_q(t)} p(i,t)\, ln[p(i,t)], & q = 1 \\ \frac{1}{1-q}\, ln \sum_{i=1}^{B_q(t)} p^q(i,t), & q \neq 1,\ q \in \mathbb{R}^+ \end{cases}. \qquad (12)$$

Here, $q \in \mathbb{R}^+$ denotes the weight assigned to different probabilities of a particle falling in a bin. We constrain $q \geq 0$ because information extraction is compromised for $q < 0$ [19]. This method is called the multifractal diffusion entropy analysis (MFDEA). The DEA corresponds to the q-order DEA where $q = 1$. We express the linear-log relationship between the q-order diffusion entropy $S_q(t)$ and time t as:

$$S_q(t) = A + \delta_q\, ln(t). \qquad (13)$$

2.4. Multi-Scale Conditional Diffusion Entropy (MS-CDE)

The q-order Conditional Diffusion Entropy (MS-CDE) serves as the multifractal extension of the CDE, offering a means to distinguish between bear and bull situations in the market at various q-weights, representing different probabilities of a particle falling in a bin. MS-CDE is defined as:

$$C_q(t) = I + \alpha\, [I - \bar{S}_q(t)], \qquad (14)$$

where I is a vector of ones and $\bar{S}_q(t)$ is the q-order vector of normalized entropy from Equation (12) and

$$\alpha = \begin{cases} -1, & p < q \\ 0, & p = q \\ 1, & p > q \end{cases}. \qquad (15)$$

3. Data

Table 1 displays the start-date and end-date of data used in this paper. We adopt the daily close prices of the Dow Jones Industrial Average (DJI) from April 2013 to April 2021, amounting to 2028 data points. As illustrated in Figure 1, market prices tend to fall during a financial crisis, while they rise during the market recovery period following a crisis. The highlighted portion of the graph in Figure 1 indicates the crash.

Table 1. Close Prices from DJI index.

Data	Start Date	End Date	Median	Mean	Standard Deviation
DJI	5 April 2013	23 April 2021	20,812	21,555	4853.5

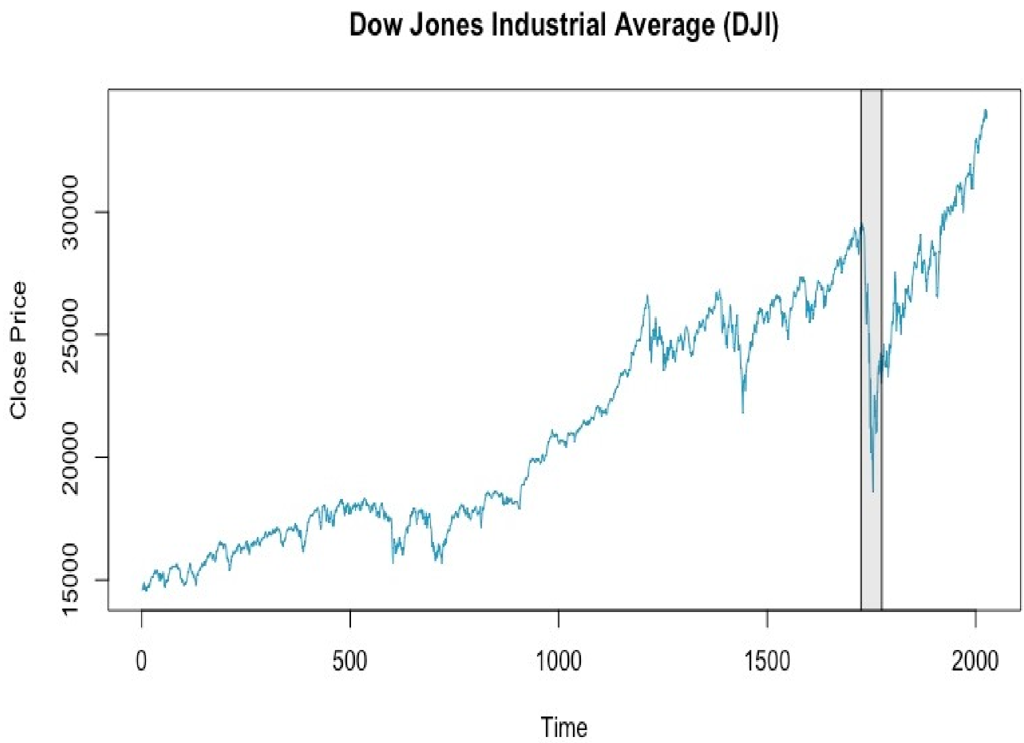

Figure 1. Daily Close Prices Plot of DJI. The grey area corresponds to a period of a financial crash.

4. Results

In this section, we present results by examining the stock market's stability during the COVID-19 pandemic stock market crash using the multi-scale conditional diffusion entropy. The analyzed crash period is highlighted in Figure 1.

4.1. q-Order DEA for Dow Jones Industrial Average

Figure 2 below shows a plot of diffusion entropy versus scale for q weighted in the range 0 to 4 for DJI from April 2013 to April 2021. At certain diffusion times t, entropy decreases with the increase in q weights, whereas entropy increases with the rise in q weights at other diffusion times. This phenomenon occurs because the Rényi entropy changes more rapidly at large q than small q weights.

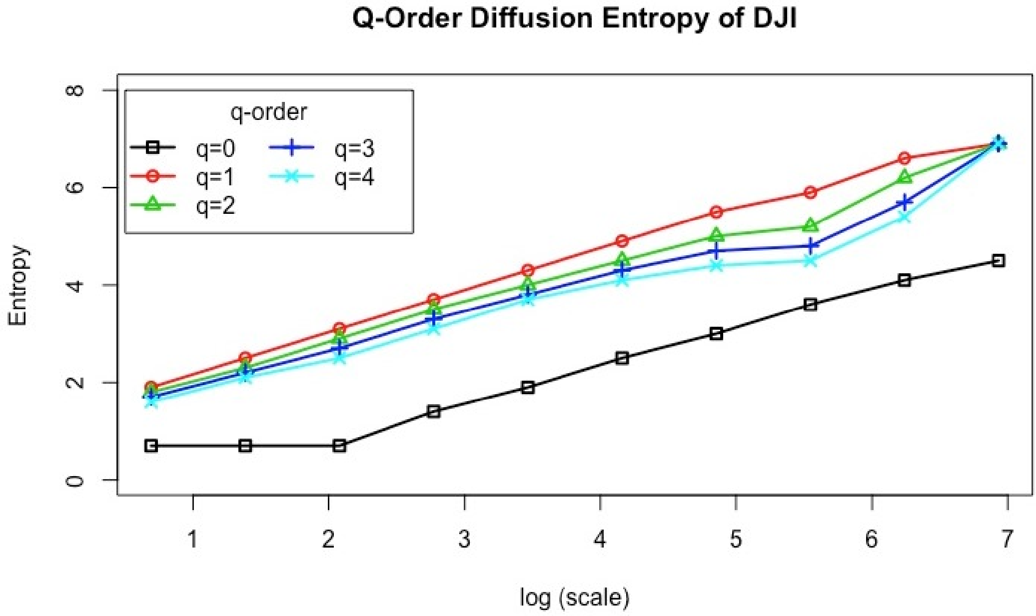

Figure 2. q-order Diffusion Entropy as a function of diffusion time constructed based on different q values of Rényi family of entropies using DJI.

4.2. Multi-Scale Conditional Entropy of DJI from April 2013 to April 2021

Figures 3–12 show the monthly conditional diffusion entropy $C_q(t)$ at different time scales from April 2013 to April 2021. In this representation, $C_q(t) = 1$ signifies random behavior in the financial market, $C_q(t) > 1$ indicates a bull market, and values less than 1 denote a bear market.

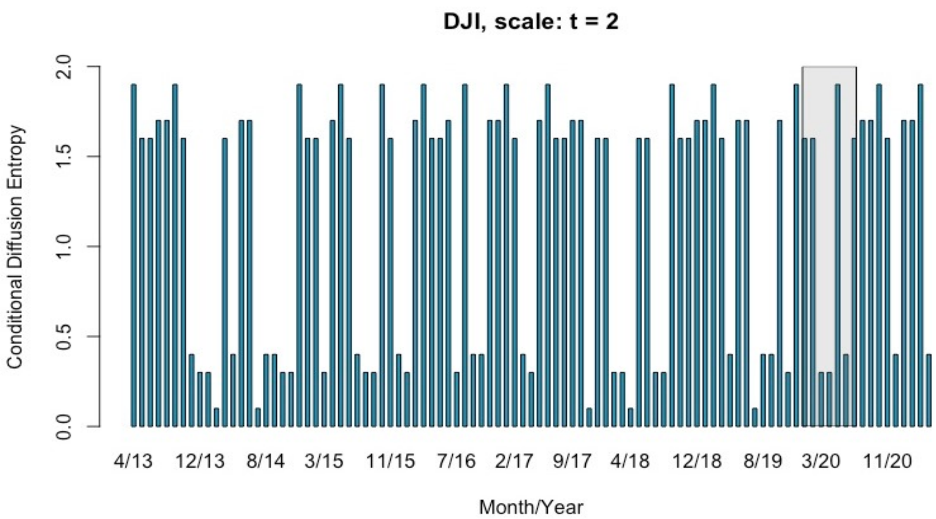

Figure 3. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

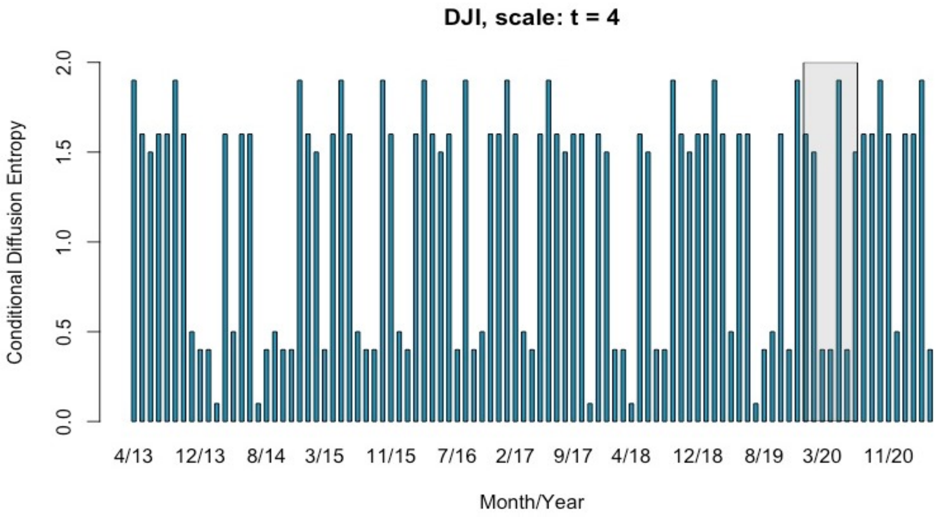

Figure 4. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

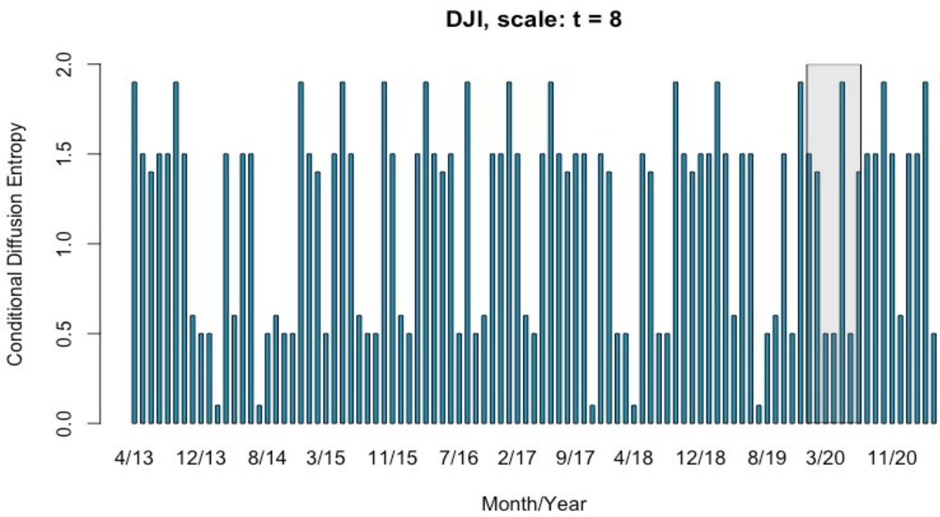

Figure 5. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

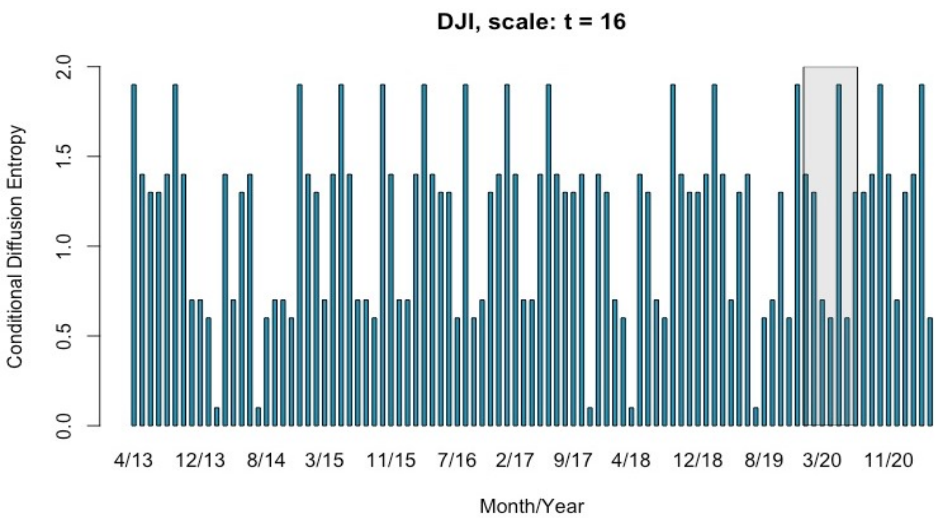

Figure 6. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

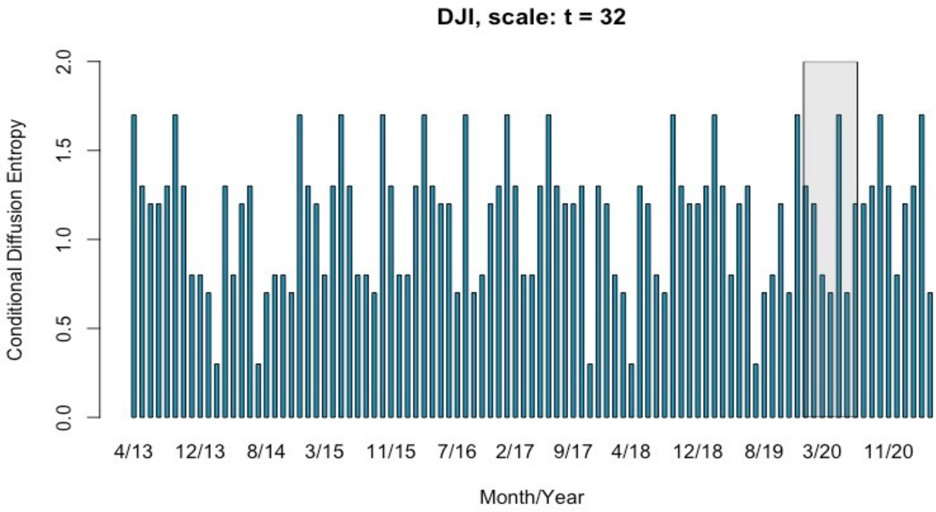

Figure 7. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

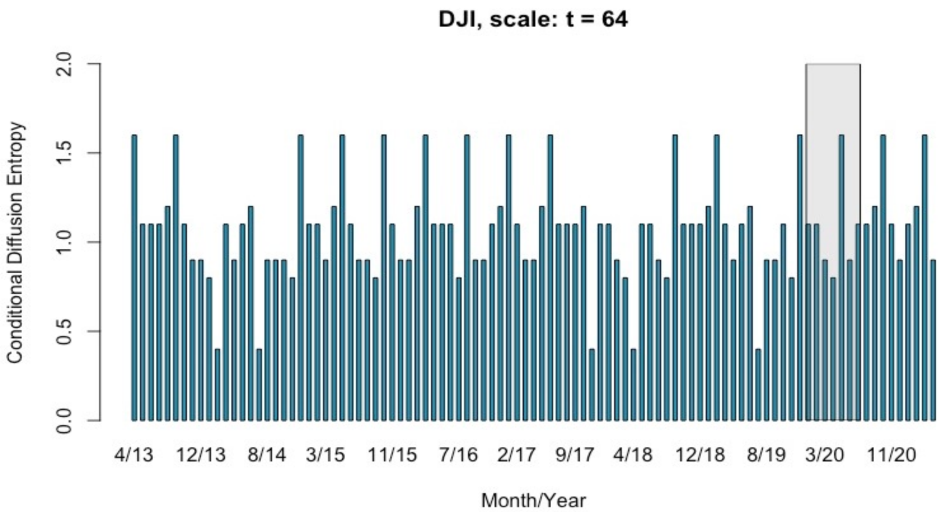

Figure 8. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

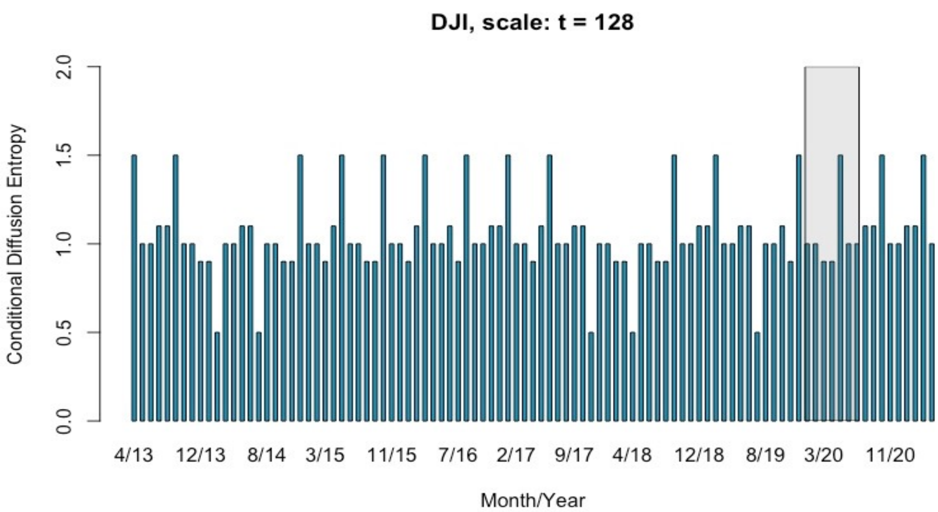

Figure 9. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

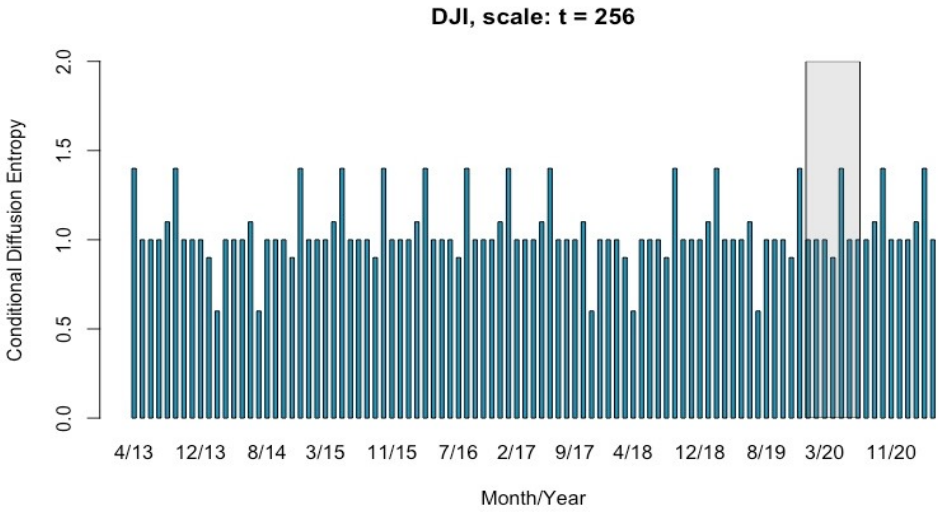

Figure 10. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

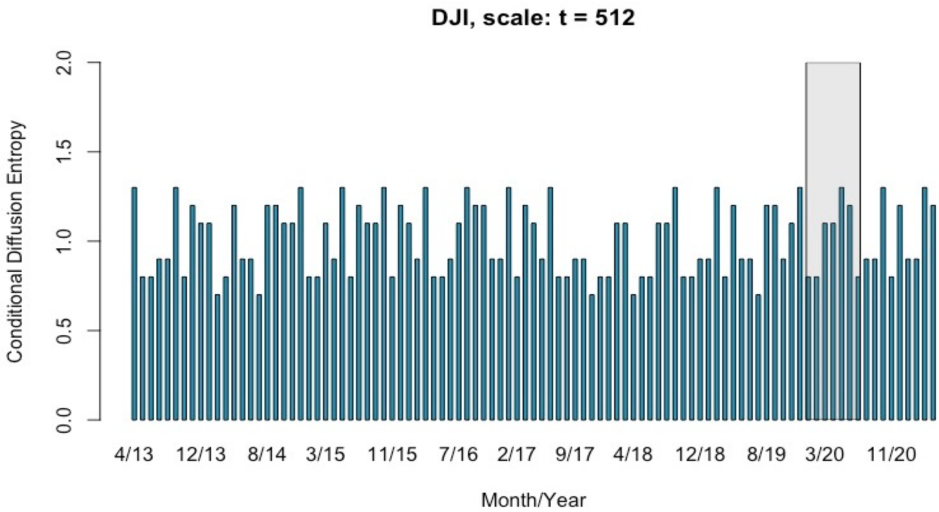

Figure 11. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

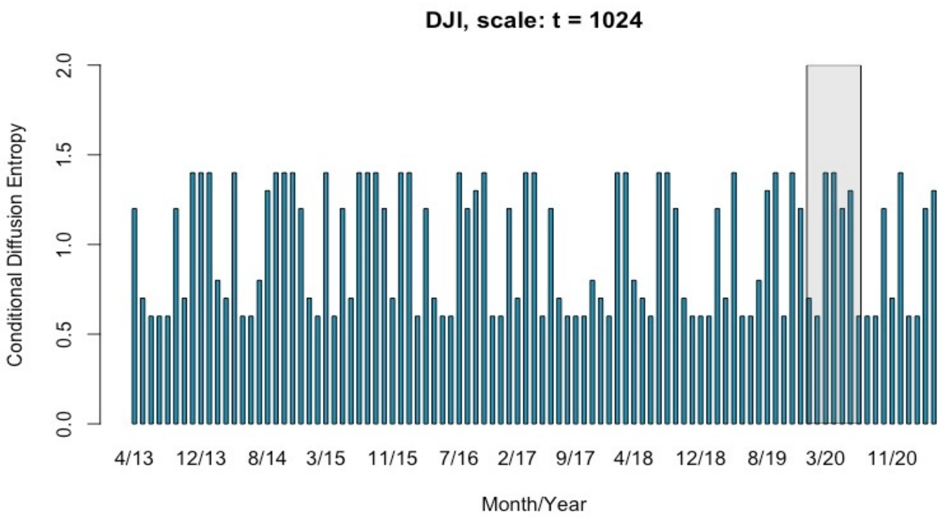

Figure 12. Monthly Conditional Entropy of DJI at Different Time Scales. The grey area are periods of the financial crisis (or bear market).

5. Discussion

Figure 1 displays instances of bull and bear markets in the DJI index. The highlighted portion represents the period from late February 2020 to early April 2020, depicting the 2020 coronavirus stock market crash. During this period, the COVID-19 pandemic spread globally from February 24 to 28, causing a significant decline in global stock markets. The DJI drops 11–12%, marking the most significant weekly decline since the 2007/2008 financial crisis. On March 12, a day after the announcement of a travel ban from Europe, the DJI fell sharply again by 10%. After it became clear that a recession was inevitable, the DJI dropped another 12.93% on March 16. Stock market indices briefly recover to their levels at the end of February 2020 by early June 2020.

Figure 1 shows the fluctuations in the series over time. Many values of the conditional diffusion entropy at time scale $t = 2$ are larger than 1 during bull market periods. In contrast, they are less than 1 during periods of the financial crisis (or bear market), as shown by the grey highlighted part of Figure 3. Observe that, as the time scale increases from $t = 2$ to $t = 256$, the conditional diffusion entropy approach is a random behavior where most of the $C_q(t) \approx 1$ as shown in Figures 4–10. During this period, the market is indifferent. Beyond the time scale of $t = 256$, conditional diffusion entropy is less than 1, as shown in Figures 11 and 12, indicating that the bear market is dominant. Hence, the conditional diffusion entropy value coincides with the state of the stock market but at different time lags, and the conditional diffusion entropy depicts the various states of the stock market. A smaller time scale like $t = 2$ to $t = 256$ coincides more with the state of the stock market than larger time scales like $t = 256$ to $t = 1024$. This is because we observe extreme fluctuations in the conditional diffusion entropy for a short time lag. In contrast, smoother (or averaged-out) fluctuations are observed in larger time scales, resulting in lower values of the conditional diffusion entropy. Comparing the Conditional Diffusion Entropy method developed in [9,10] to our model's results, we observe that Multiscale Conditional Diffusion Entropy Analysis is less prone to incorrectly classifying a bullish market as neutral or bearish. This is because MS-CDE assesses market volatility across multiple time scales, unlike CDE. Thus, MS-CDE conditional diffusion offers a more insightful analysis of stock market volatility. However, one limitation of using MS-CDE is its computational intensity, which is due to the use of multiple time scales in our analysis.

6. Conclusions

This paper investigates the stability of the Dow Jones Industrial Average (DJI) in the US stock markets utilizing multi-scale normalized and conditional diffusion entropy. We discover that conditional diffusion entropy is valuable for analyzing market fluctuations and discerning bear and bull markets. However, careful interpretation of its results is necessary. We illustrate that conditional diffusion entropy offers diverse insights into the market's state at different time lag scales. Hence, employing a multi-time lag scale enhances the analysis of financial market time series, especially considering their often multifractal nature. As a future direction, we plan to extend our inquiry to volatility in other financial securities, such as bonds and cryptocurrency markets, using our innovative approach. Moreover, implementing the method in parallel will mitigate the computational intensity of the model. Additionally, we aim to compare our technique with other multiscale techniques like the wavelet approach or scale-space filtering.

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Abbreviations

List of Acronyms.

Acronyms	Definition
DJI	Dow Jones Industrial Average
SE	Shannon Entropy
RE	Rényi Entropy
VaR	Vector Autoregression
CDE	Conditional Diffusion Entropy
MS-CDE	Multi-Scale Conditional Diffusion Entropy
PDF	Probability Density Function
DEA	Diffusion Entropy Analysis
FD	Freedman-Diaconis Rule
Q-DEA	q-Order Diffusion Entropy Analysis
IQR	Inter quartile Range
MFDEA	Multifractal Diffusion Entropy Analysis

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