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On the Controllability of Coupled Nonlocal Partial Integrodifferential Equations Using Fractional Power Operators

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Abstract: In this research paper, we investigate the controllability in the α-norm of a coupled system of integrodifferential equations with state-dependent nonlocal conditions in generalized Banach spaces. We establish sufficient conditions for the system's controllability using resolvent operator theory introduced by Grimmer, fractional power operators, and fixed-point theorems associated with generalized measures of noncompactness for condensing operators in vector Banach spaces. Finally, we present an application example to validate the proposed methodology in this research.

Keywords: state-dependent nonlocal condition; fixed-point theorem; controllability; integrodifferential system; generalized Banach space; generalized measures of noncompactness; condensing operator

MSC: 93B05; 34G20; 34K30; 34K10

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1. Introduction

In this work, we consider the controllability problem of a system of partial functional integrodifferential equations (PFIDEs) involving nonlocal conditions of the form

$$\begin{cases} \zeta_1'(t) &= A\zeta_1(t) + \int_0^t \Upsilon(t-s)\zeta_1(s)\,ds + f_1(t,\zeta_1(t),\zeta_2(t)) + \mathbf{C_1}u_1(t), \quad t \in J := [0,a], \\ \zeta_2'(t) &= A\zeta_2(t) + \int_0^t \Upsilon(t-s)\zeta_2(s)\,ds + f_2(t,\zeta_1(t),\zeta_2(t)) + \mathbf{C_2}u_2(t), \quad t \in J := [0,a], \\ \zeta_1(0) &= \zeta_{0,1} + H_1(\sigma_1(\zeta_1),\zeta_1), \\ \zeta_2(0) &= \zeta_{0,2} + H_2(\sigma_2(\zeta_2),\zeta_2), \end{cases} \tag{1}$$

where $\mathcal{X}$ represents a Banach space and the states $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$ take values in $\mathcal{X}$. The operator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$. $(\Upsilon(t))_{t\geq 0}$ denotes a family of closed linear operators, with the domain $D(A) \subset D(\Upsilon(t))$. Additionally, $\zeta_{0,1}, \zeta_{0,2} \in \mathcal{X}$. The functions $f_i : J \times \mathcal{X}_\alpha \times \mathcal{X}_\alpha \to \mathcal{X}$, $H_i : J \times C(J;\mathcal{X}_\alpha) \to \mathcal{X}_\alpha$, and $\sigma_i : C(J;\mathcal{X}_\alpha) \to J$, for $i = 1,2$, are given and are determined later. $C_i : U \to \mathcal{X}$, $i = 1,2$, are bounded linear operators. The control inputs $u_i(\cdot)$, $i = 1,2$, are given functions in $L^2(J;U)$, a Banach space of admissible control functions, with U as a Banach space. Here, $\mathcal{X}_\alpha$ denotes the domain where the fractional power operator A^α is defined and is equipped with an appropriate norm described later.

Control theory is an interdisciplinary field within engineering and applied mathematics that focuses on understanding the behavior of dynamic systems. Controllability is one of the vital and important problems in control theory and engineering, enabling the control and improvement of system performance for stability and efficiency in complex dynamic environments. The concept of controllability originated in finite dimensions, with an extension to the infinite-dimensional case first proposed in 1971, followed by further advancements [1,2]. Generally, controllability refers to the ability to steer the control system from any initial state to a desired state by applying an admissible set of control inputs u within a finite time (see [3,4]). Numerous scholars have thoroughly examined the controllability of diverse nonlinear dynamical systems; one can consider the papers [5–8].

In recent years, partial integrodifferential equations (PIDEs) have garnered extensive attention as an active research area with intensive investigation. PIDEs serve as a valuable instrument for modeling and describing complex systems in different phenomena in engineering and physics [9]. A highly effective method for studying these equations involves transforming them into integrodifferential evolution equations within abstract spaces and studying the resulting equation using resolvent operator theory.

Grimmer et al. [10–12] demonstrated the existence of solutions for the integrodifferential evolution equation in the form

$$\begin{cases} \nu'(t) = A\nu(t) + \int_0^t \Upsilon(t-s)\nu(s)\,ds + h(t), \;\; t \geq 0, \\ \nu(0) = \nu_0, \end{cases} \tag{2}$$

in Banach space $\mathcal{X}$, where the function $h : \mathbb{R}^+ \to \mathcal{X}$ is continuous. Using the resolvent operator corresponding to the following homogeneous linear equation, they established the representation as well as the existence and uniqueness of solutions for Equation (2)

$$\begin{cases} \nu'(t) = A\nu(t) + \int_0^t \Upsilon(t-s)\nu(s)\,ds, \;\; t \geq 0, \\ \nu(0) = \nu_0. \end{cases}$$

The resolvent operator is crucial for resolving Equation (2) in both strict and weak senses, effectively substituting the need for the conventional C_0-semigroup. Based on Grimmer's work [10], many authors have applied resolvent operator theory to study different topics related to nonlinear integrodifferential evolution equations (see, for instance, [13,14]).

On the other hand, problems involving nonlocal Cauchy conditions have demonstrated better effects and more significant applicability than conventional problems [15]. In particular, the works proposed by Byszewski et al. [16,17] represent the first attempt to study semilinear evolution equations with a nonlocal condition. So far, many researchers have focused on studying the different topics of different types of evolution equations and integrodifferential evolution equations subject to nonlocal conditions employing a variety of common methods such as fixed point theorems, see [13,18–20].

As is well known, the controllability of PFIDEs is a challenging problem. Several authors have extensively studied, using semigroup methods, the controllability of nonlinear systems represented by PFIDEs particularly when $\Upsilon(\cdot) = 0$ and the nonlinear term $f(\cdot,\cdot)$ includes an integral term, and many interesting findings have been obtained; see for instance, [21,22], and the references cited therein. When $\Upsilon(\cdot) \neq 0$, research on the controllability of various integrodifferential systems has attracted great interest from mathematicians in recent years [23]. For instance, in [23], the authors have investigated the controllability problem for a class of nonlocal PFIDEs in Banach spaces. They achieved the controllability results through the utilization of the resolvent operator and the measure of noncompactness (MNC), without requiring the compactness of the resolvent operator.

In recent years, coupled systems of differential equations have garnered considerable interest from researchers for their practical importance in mathematical modeling, especially when dealing with highly complex systems. In 2009, Precup [24] emphasized the importance of matrices converging to zero and the significance of vector-valued norms in investigating semilinear operator systems. Since then, several researchers have investigated the existence of solutions for systems of differential equations by applying the vector version of fixed-point theorems in generalized Banach spaces (GBSs) [25–28]. For instance, in [27,28], authors combined the approach of a measure of noncompactness with matrices converging to zero to study coupled systems of differential equations. Controllability of a system of differential equations is crucial in various fields of study. Attaining controllability in these systems is advantageous, as it allows for the effective manipulation of the dynamics of interconnected phenomena. Researchers have shown controllability properties through investigations of systems such as coupled wave equations on manifolds [29], coupled Stokes or Navier–Stokes systems [30], and coupled Korteweg–de Vries equations [31]. To the best of our knowledge, the controllability of coupled systems of integrodifferential equations, subject to state-dependent nonlocal conditions in GBSs, has not yet been studied.

Inspired by the works [27,28], we utilize Schaefer's fixed point theorem to investigate the controllability problem for system (1) via resolvent operators in the sense given by Grimmer. We stress here that the nonlocal condition in (1) is more general. The functions $H_i(\cdot,\cdot)$ for $i = 1,2$ in the nonlocal conditions are state-dependent, thus generalizing many nonlocal conditions mentioned in the literature. In addition, we do not require the compactness assumption of the resolvent operator and functions $H_i(\cdot,\cdot)$, $i = 1,2$, in the nonlocal conditions. Moreover, in numerous practical models, the nonlinear term f may incorporate spatial derivatives. In such scenarios, the problem cannot be addressed in the whole space $\mathcal{X}$, thus making the discussion in [23] invalid. To ensure that the results are accurate for this type of system, we mainly investigate the problem by utilizing the α-norm and fractional power operator.

We conclude this section by briefly outlining the structure of this manuscript. In Section 2, we start by presenting some fundamental concepts and definitions necessary for deriving our findings. Section 3 is devoted to establishing the controllability of the nonlocal system (1) in the α-norm. Lastly, in Section 4, an illustrative example is presented to demonstrate the applicability of the proposed theoretical findings.

2. Basic Concepts and Preliminaries

We primarily recall some notations, provide definitions, and present initial results that are necessary to demonstrate our main findings in this section.

We begin this section by presenting some preliminaries on the fractional power operator. In this article, let $\mathcal{Y} = D(A)$ be endowed with the graph norm $\|\cdot\|_1$. The operator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$. The resolvent set of A is denoted by $\varrho(A)$, which includes 0 (i.e., $0 \in \varrho(A)$). Thus, one can define the fractional power operator A^α for $\alpha \in (0,1]$ as a closed linear operator on its dense set $D(A^\alpha)$ with the following norm

$$\|\zeta\|_\alpha = \|A^\alpha \zeta\|, \text{ for } \zeta \in D(A^\alpha).$$

Let $\mathcal{X}_\alpha$ denote the space $(D(A^\alpha), \|\cdot\|_\alpha)$; clearly, $\mathcal{X}_\alpha$ is a Banach space for every $\alpha \in (0,1]$. In addition, $C(J;\mathcal{X}_\alpha)$ denotes the Banach space of all continuous functions from J to $\mathcal{X}_\alpha$ with the norm

$$\|\zeta\|_C = \sup_{t\in J} \|A^\alpha \zeta(t)\|, \ \zeta \in C(J;\mathcal{X}_\alpha).$$

Next , we recall well-known basic definitions and essential facts of GBS.

Definition 1 ([32])**.** *Let $\mathcal{X}$ be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A mapping $\|\cdot\| : \mathcal{X} \to \mathbb{R}^n_+$ is called a vector-valued norm on $\mathcal{X}$ if it satisfies the subsequent conditions:*

(c_1) $\|\zeta\| \geq 0$ for all $\zeta \in \mathcal{X}$; if $\|\zeta\| = 0$, then $\zeta = (0,\ldots,0)$;
(c_2) $\|\lambda\zeta\| = |\lambda|\|\zeta\|$ for all $\zeta \in \mathcal{X}$, and $\lambda \in \mathbb{K}$;
(c_3) $\|\zeta_1 + \zeta_2\| \leq \|\zeta_1\| + \|\zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathcal{X}$.

We point out that a generalized normed space is denoted by the pair $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$. Moreover, when the generalized metric induced by $\|\cdot\|_{\mathcal{X}}$ (i. e., $d(\zeta_1,\zeta_2)= \|\zeta_1 - \zeta_2\|_{\mathcal{X}}$) is complete, the space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is referred to as a GBS, where

$$\|\zeta_1 - \zeta_2\|_{\mathcal{X}} = \begin{pmatrix} \|\zeta_1 - \zeta_2\|_1 \\ \vdots \\ \|\zeta_1 - \zeta_2\|_n \end{pmatrix}.$$

Remark 1. *The notions of Cauchy sequence, convergent sequence, continuity, completeness, open subsets, and closed subsets in a GBS in the sense of Perov are similar to those for standard metric spaces.*

Throughout this article, let $\mathscr{C}_\alpha := C(J;\mathcal{X}_\alpha) \times C(J;\mathcal{X}_\alpha)$ be equipped with the vector norm $\|\cdot\|_{\mathscr{C}_\alpha}$ defined as

$$\|\mathfrak{u}\|_{\mathscr{C}_\alpha} = \Big(\|\mathfrak{u}_1\|_C, \|\mathfrak{u}_2\|_C\Big),$$

for all $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2)$. Thus, clearly, $\Big(\mathscr{C}_\alpha, \|\cdot\|_{\mathscr{C}_\alpha}\Big)$ is a GBS.

Lemma 1 ([33])**.** *A square matrix $\mathcal{M}$ of real numbers is said to be convergent to zero if and only if its spectral radius $\varrho(\mathcal{M})$ is strictly less than 1. In other words, this means that all the eigenvalues of $\mathcal{M}$ are in the open unit disc, i.e., $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $\det(\mathcal{M} - \lambda I) = 0$, where I denotes the unit matrix of $\mathscr{M}_{n\times n}(\mathbb{R})$.*

Lemma 2 ([34]). *Let*

$$\Xi = \begin{pmatrix} \xi_{11} & -\xi_{12} \\ -\xi_{21} & \xi_{22} \end{pmatrix},$$

where $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22} \geq 0$ *and* $\det(\Xi) > 0$. *Thus,* Ξ^{-1} *is order-preserving.*

In the sequel, we introduce some fundamental results and definitions related to the theory of resolvent operators [10,11].

Definition 2 ([10]). *A family* $(R(t))_{t\geq 0}$ *of bounded linear operators on* $\mathcal{X}$ *is called a resolvent operator for*

$$\begin{aligned} \zeta'(t) &= A\zeta(t) + \int_0^t \Upsilon(t-s)\zeta(s)\, ds, \quad \text{for } t \geq 0, \\ \zeta(0) &= \zeta_0 \in \mathcal{X}, \end{aligned} \tag{3}$$

if

(a) $R(0) = I$ *and* $\|R(t)\| \leq Me^{\omega t}$ *for some constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$.

(b) *For each* $\zeta \in \mathcal{X}$ *and* $t \geq 0$, *the function* $R(t)\zeta$ *is continuous.*

(c) $R(t) \in \mathscr{L}(\mathcal{Y})$ *for* $t \geq 0$. *For any* $\zeta \in \mathcal{Y}$, $R(\cdot)\zeta \in C^1(J;\mathcal{X}) \cap C(J;\mathcal{Y})$ *such that for each* $t \geq 0$, *we have*

$$\begin{aligned} R'(t)\zeta &= AR(t)\zeta + \int_0^t \Upsilon(t-s)R(s)\zeta\, ds \\ &= R(t)A\zeta + \int_0^t R(t-s)\Upsilon(s)\zeta\, ds. \end{aligned}$$

Next , we impose the following assumptions on the operators A and $(\Upsilon(t))_{t\geq 0}$ for Equation (3), as introduced in [12]:

(V_1) An analytic semigroup on $\mathcal{X}$ is generated by the operator A. Let $(\Upsilon(t))_{t\geq 0}$ be a closed operator on $\mathcal{X}$, with a domain at least $D(A)$ for almost every $t \geq 0$, with $\Upsilon(t)\zeta$ is strongly measurable for every $\zeta \in D(A)$, and $\|\Upsilon(t)\zeta\| \leq \eta(t)\|\zeta\|$ for $\eta \in L^1_{loc}(0,+\infty)$ with $\eta^*(\lambda)$ absolutely convergent for $Re(\lambda) > 0$.

(V_2) There exists a bounded operator $\varrho(\lambda) := (\lambda I - A - \Upsilon^*(\lambda))^{-1}$ on $\mathcal{X}$, which is analytic for λ in the region Λ defined as

$$\Lambda = \Big\{\lambda \in \mathbb{C} : |\arg(\lambda)| < \frac{\pi}{2} + b\Big\},$$

for $b \in (0, \pi/2)$. In Λ, if $0 < \varepsilon \leq |\lambda|$, there is a constant $M = M(\varepsilon) > 0$ such that $\|\varrho(\lambda)\| \leq M|\lambda|^{-1}$.

(V_3) $A\varrho(\lambda) \in \mathscr{L}(\mathcal{X})$ for $\lambda \in \Lambda$, and is analytic from Λ to $\mathscr{L}(\mathcal{X})$. Furthermore, for $\lambda \in \Lambda$, $\Upsilon^*(\lambda)$ belongs to $\mathscr{L}(\mathcal{Y},\mathcal{X})$, and $\Upsilon^*(\lambda)\varrho(\lambda) \in \mathscr{L}(\mathcal{Y},\mathcal{X})$. Given $\varepsilon > 0$, there is $M = M(\varepsilon) > 0$ so that $\zeta \in D(A)$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$, $\|A\varrho(\lambda)\zeta\| + \|\Upsilon^*(\lambda)\varrho(\lambda)\zeta\| \leq M|\lambda|^{-1}\|\zeta\|_1$ and $\|\Upsilon^*(\lambda)\| \to 0$ as $|\lambda| \to +\infty$ in Λ. Additionally, $\|A\varrho(\lambda)\zeta\| \leq M|\lambda|^{-n}\|\zeta\|$ for some $n > 0$, and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$. Moreover, there is $D \subset D(A^2)$ that is dense in $\mathcal{Y}$ such that $A(D)$ and $\Upsilon^*(\lambda)(D)$ are contained in $\mathcal{Y}$ and $\|\Upsilon^*(\lambda)\zeta\|_1$ is bounded for every $\zeta \in D$ and $\lambda \in \Lambda$ with $|\lambda| \geq \varepsilon$.

Based on [12], it can be deduced that under the mentioned hypotheses (V_1)–(V_3), there is a resolvent operator $(R(t))_{t\geq 0}$ for the system (3) defined by

$$R(t) = \begin{cases} (2\pi i)^{-1} \displaystyle\int_\Gamma e^{\lambda t}\varrho(\lambda)\, d\lambda, & t > 0, \\ I, & t = 0. \end{cases}$$

where $\varrho(\lambda) = (\lambda I - A - \Upsilon^*(\lambda))^{-1}$ and Γ denotes a contour type employed to acquire an analytic semigroup. We can select a contour Γ to be included in the region Λ formed by Γ_i, $i = 1,2,3$, where

$$\begin{aligned} \Gamma_1 &= \Big\{re^{i\phi} : r \geq 1\Big\}, \\ \Gamma_2 &= \Big\{e^{i\theta} : -\phi \leq \theta \leq \phi\Big\}, \\ \Gamma_3 &= \Big\{re^{-i\phi} : r \geq 1\Big\}, \end{aligned}$$

for $\phi \in \left(\frac{\pi}{2}, \frac{\pi}{2}+b\right)$ and $b > 0$. The above curves are oriented so that the imaginary part of λ is increasing on Γ_1 and Γ_2. Furthermore, the operator $R(t)$ is analytic, and there are some constants $M, M_\alpha > 0$ such that

$$\|R(t)\| \leq M \text{ and } \|A^\alpha R(t)\| \leq M_\alpha t^{-\alpha}, \ \ t \in]0,a], \ \ \alpha \in [0,1].$$

We point out that, in general, $R(t)$ and A^α do not necessarily commute. Fortunately, this commutative condition can be achieved in many cases. For instance, consider the case where $\Upsilon(t)$ can be expressed as $\Upsilon(t) = \beta(t)A$, with β representing a scalar function given over the interval $]0, +\infty[$. In such instances, the linear Equation (3) transforms into

$$\begin{cases} \zeta'(t) = A\zeta(t) + \displaystyle\int_0^t \beta(t-s)A\zeta(s)\, ds, & \text{for } t > 0, \\ \zeta(0) = \zeta_0 \in \mathcal{X}. \end{cases} \tag{4}$$

If we impose some conditions $(V_1') - (V_3')$ in [13] on the system (4), then, according to [12], the conditions $(V_1) - (V_3)$ are satisfied, and hence $R(t)$ is analytic. In this scenario, $A^\alpha R(t)\zeta = R(t)A^\alpha\zeta$ for any $\zeta \in D(A^\alpha)$. For the sake of simplicity, this condition is always considered to be valid.

Lemma 3 ([35])**.** *$R(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathscr{L}(\mathcal{X})$.*

Lemma 4 ([19])**.** *$AR(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathscr{L}(\mathcal{X})$.*

Now, we present the notion of a generalized measure of noncompactness (MNC).

Definition 3 ([27])**.** *Let $\mathcal{X}$ be a GBS and $(\mathcal{A}, \leq)$ be a partially ordered set. A mapping $\vartheta : \mathcal{P}(\mathcal{X}) \to \mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}$ is referred to as a generalized MNC on $\mathcal{X}$, if for every $\mathfrak{D} \in \mathcal{P}(\mathcal{X})$, the following condition holds:*

$$\vartheta(\overline{co}\mathfrak{D}) = \vartheta(\mathfrak{D}),$$

where

$$\vartheta(\mathfrak{D}) := \begin{pmatrix} \vartheta_1(\mathfrak{D}) \\ \vdots \\ \vartheta_n(\mathfrak{D}) \end{pmatrix}.$$

An illustration of an MNC is the Kuratowski MNC μ, established for $\Theta \subset \mathcal{X}$ as follows:

$$\begin{aligned} \mu(\Theta) \quad := \quad & \inf\Big\{\varepsilon \in \mathbb{R}^n_+ : \text{there are } n \in \mathbb{N} \text{ such that } \Theta \text{ is included in a finite cover} \\ & \text{of sets, each having diameters less than or equal to } \varepsilon\Big\}. \end{aligned}$$

Lemma 5 ([36])**.** *Let $\Pi \subset C(a,b)$ be bounded and equicontinuous. Then, $\bar{co}(\Pi) \subset C(a,b)$ is also bounded and equicontinuous.*

Definition 4 ([27])**.** *Let $\mathcal{X}, \mathcal{Y}$ be two generalized normed spaces and a map $N : \mathcal{X} \to \mathcal{Y}$. N is said to be an Ξ-contraction (with respect to ϑ) if there exists $\Xi \in \mathscr{M}_{n\times n}(\mathbb{R}^+)$ converging to zero such that, for every $\mathfrak{D} \in \mathcal{P}(\mathcal{X})$, we have*

$$\vartheta(N(\mathfrak{D})) \leq \Xi\vartheta(\mathfrak{D}).$$

Finally, to prove our result, we conclude this section by stating a version of Schaefer's fixed-point theorem for ϑ-condensing operators in a GBS.

Theorem 1 ([27])**.** *Let $\mathcal{X}$ be a GBS, and let $N : \mathcal{X} \to \mathcal{X}$ be a continuous and ϑ-condensing operator. Suppose that the set*

$$\mathscr{A}_\lambda := \Big\{\zeta \in \mathcal{X} : \zeta = \lambda N(\zeta), \lambda \in (0,1)\Big\}$$

is bounded. Then, N possesses a fixed point.

3. Controllability Results

Our focus in this section is on investigating the controllability of the nonlocal system (1). We impose some conditions guaranteeing the controllability of the mild solution. We begin by defining the mild solution for system (1) as follows.

Definition 5. *A mild solution of the coupled system* (1) *is a function* $(\zeta_1, \zeta_2) \in \mathscr{C}_\alpha$ *such that*

$$\begin{cases} \zeta_1(t) = R(t)(\zeta_{0,1} + H_1(\sigma_1(\zeta_1), \zeta_1)) + \int_0^t R(t-s)\,[f_1(s, \zeta_1(s), \zeta_2(s)) + \mathbf{C_1} u_1(s)]\,ds, \quad t \in J. \\ \zeta_2(t) = R(t)(\zeta_{0,2} + H_2(\sigma_2(\zeta_2), \zeta_2)) + \int_0^t R(t-s)\,[f_2(s, \zeta_1(s), \zeta_2(s)) + \mathbf{C_2} u_2(s)]\,ds, \quad t \in J. \end{cases} \tag{5}$$

Definition 6. *The nonlocal system* (1) *is said to be controllable on the interval* J *if for any* $\zeta_{0,1}, \zeta_{0,2}, \zeta_{1,a}, \zeta_{2,a} \in \mathcal{X}$, *there are a pair of controls* $u_1, u_2 \in L^2(J; U)$ *such that the mild solution* $(\zeta_1(\cdot), \zeta_2(\cdot))$ *of* (1) *satisfies the terminal condition* $(\zeta_1(a), \zeta_2(a)) = (\zeta_{1,a}, \zeta_{2,a})$.

We impose the following sufficient assumptions to guarantee the controllability of the coupled system (1):

($\mathbf{H}_1$) The operator resolvent is continuous in the uniform topology.

($\mathbf{H}_2$) The functions $f_i : J \times \mathcal{X}_\alpha \times \mathcal{X}_\alpha \to \mathcal{X}$, for $i = 1, 2$, satisfy the following:

(i) The functions $f_i(t, \cdot, \cdot) : \mathcal{X}_\alpha \times \mathcal{X}_\alpha \to \mathcal{X}$ are continuous for every $t \in J$, and the functions $f_i(\cdot, \zeta_1, \zeta_2) : J \to \mathcal{X}$ are measurable for every $\zeta_1, \zeta_2 \in \mathcal{X}_\alpha$.

(ii) There exist $0 \le q < 1 - \alpha$ and functions $\mathcal{P}_i(\cdot) \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$, $\mathcal{Q}_i(\cdot) \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$ for $i = 1, 2$, such that for every $t \in J$ and $\zeta_1, \zeta_2 \in \mathcal{X}_\alpha$, we have

$$\|f_i(t, \zeta_1, \zeta_2)\| \le \mathcal{P}_i(t)\|\zeta_1\|_\alpha + \mathcal{Q}_i(t)\|\zeta_2\|_\alpha.$$

(iii) For $i = 1, 2$, there are functions $\ell_{f_i} \in L^1(J, \mathbb{R}^+)$ such that, for any bounded set $\mathfrak{D} \times \widetilde{\mathfrak{D}} \subset \mathcal{X}_\alpha \times \mathcal{X}_\alpha$ and every $t \in J$, we have

$$\mu\Big(f_i(t, \mathfrak{D}, \widetilde{\mathfrak{D}})\Big) \le \ell_{f_i}(t)\Big(\mu(\mathfrak{D}) + \mu(\widetilde{\mathfrak{D}})\Big).$$

($\mathbf{H}_3$) The nonlocal functions $H_i(\cdot, \cdot) : J \times C(J; \mathcal{X}_\alpha) \to \mathcal{X}_\alpha$ and the function $\sigma(\cdot) : C(J; \mathcal{X}_\alpha) \to J$ are both continuous and satisfy the following:

(i) For each $i = 1, 2$, there are positive constants $\aleph_i$ and $\hbar_i$ such that

$$\|H_1(\sigma_1(\zeta_1), \zeta_1)\|_\alpha \le \aleph_1 + \hbar_1\|\zeta_1\|_C, \ \ \zeta_1 \in C(J; \mathcal{X}_\alpha),$$

and

$$\|H_2(\sigma_2(\zeta_2), \zeta_2)\|_\alpha \le \aleph_2 + \hbar_2\|\zeta_2\|_C, \ \ \zeta_2 \in C(J; \mathcal{X}_\alpha).$$

(ii) There exist $\ell_{H_i} > 0$ such that for any bounded $\Pi_i \subset C(J; \mathcal{X}_\alpha)$, $i = 1, 2$, we have

$$\mu\Big(H_i(\sigma_i(\Pi_i), \Pi_i)\Big) \le \ell_{H_i} \sup_{t \in J} \Big\{\mu(\Pi_i(t))\Big\}.$$

($\mathbf{H}_4$) (i) The linear operators $\mathcal{W}_i : L^2([0, a]; U) \longrightarrow \mathcal{X}$, $i = 1, 2$, are defined as follows

$$\mathcal{W}_i u_i = \int_0^a R(a-s)\mathbf{C_i} u_i(s)\,ds \tag{6}$$

such that these operators have bounded inverse operators $(\mathcal{W}_i)^{-1}$ taking values in $L^2(J; U)$ $/Ker\mathcal{W}_i$.

(ii) There are positive constants M_{w_i} and M_{c_i}, $i = 1, 2$, satisfying

$$\|\mathcal{W}_i^{-1}\| \le M_{w_i}, \text{ and } \|C_i\| \le M_{c_i}.$$

(iii) There exist $k_{w_i}(\cdot) \in L^1(J; \mathbb{R}^+)$ and $k_{c_i} \ge 0$ such that for any bounded sets $\mathfrak{D}_1 \subset \mathcal{X}$ and $\mathfrak{D}_2 \subset U$,

$$\mu(\mathcal{W}_i^{-1}(\mathfrak{D}_1)(t)) \le k_{w_i}(t)\mu(\mathfrak{D}_1), \ \ \mu(C_i(\mathfrak{D}_2)(t)) \le k_{c_i}\mu(\mathfrak{D}_2(t)).$$

Lemma 6. *If the hypotheses* $(\mathbf{H}_2)(ii)$, $(\mathbf{H}_3)(i)$, *and* $(\mathbf{H}_4)(i)-(ii)$ *are satisfied. Then, the following bounds hold for the control inputs* $u_1(t)$ *and* $u_2(t)$ *of the system (1):*

$$\begin{cases} \mathfrak{M}_{u_1} = M_{w_1}\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\ \qquad\qquad + M\Big(\dfrac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big], \\ \mathfrak{M}_{u_2} = M_{w_2}\Big[\|\zeta_{2,a}\| + M\big[\|\zeta_{0,2}\| + \|A^{-\alpha}\|\big(\aleph_2 + \hbar_2\|\zeta_2\|_C\big)\big] \\ \qquad\qquad + M\Big(\dfrac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_2\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big], \end{cases} \tag{7}$$

where

$$\begin{cases} u_1(t) = \mathcal{W}_1{}^{-1}\Big\{\zeta_{1,a} - R(a)\zeta_1(0) - \displaystyle\int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t) \\ u_2(t) = \mathcal{W}_2{}^{-1}\Big\{\zeta_{2,a} - R(a)\zeta_2(0) - \displaystyle\int_0^a R(a-s)f_2(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t) \end{cases} \tag{8}$$

for $t \in J$.

Proof. Indeed, for $t \in [0,a]$, we have

$$\begin{aligned} \|u_1(t)\| &= \Big\|\mathcal{W}_1^{-1}\Big\{\zeta_{1,a} - R(a)\big[\zeta_{0,1} + H_1(\sigma_1(\zeta_1),\zeta_1)\big] - \int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big\| \\ &= \Big\|\mathcal{W}_1^{-1}\Big\{\zeta_{1,a} - R(a)\big[\zeta_{0,1} + A^{-\alpha}A^{\alpha}H_1(\sigma_1(\zeta_1),\zeta_1)\big] \\ &\quad - \int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big\| \\ &\le M_{w_1}\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\ &\quad + M\int_0^a \Big(\mathcal{P}_1(s)\|\zeta_1\|_\alpha + \mathcal{Q}_1(s)\|\zeta_2\|_\alpha\Big)\,ds\Big] \\ &\le M_{w_1}\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\ &\quad + M\Big(\|\zeta_1\|_C\int_0^a \mathcal{P}_1(s)\,ds + \|\zeta_2\|_\alpha\int_0^a \mathcal{Q}_1(s)\,ds\Big)\Big] \\ &\le M_{w_1}\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\ &\quad + M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big] \\ &= \mathfrak{M}_{u_1}. \end{aligned}$$

Similarly, we can demonstrate that

$$\begin{aligned} \|u_2(t)\| &= \Big\|\mathcal{W}_2^{-1}\Big\{\zeta_{2,a} - R(a)\Big[\zeta_{0,2} + H_2(\sigma_2(\zeta_2),\zeta_2)\Big] - \int_0^a R(a-s)f_2(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big\| \\ &\le M_{w_2}\Big[\|\zeta_{2,a}\| + M\big[\|\zeta_{0,2}\| + \|A^{-\alpha}\|\big(\aleph_2 + \hbar_2\|\zeta_2\|_C\big)\big] \\ &\quad + M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_2\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big] \\ &= \mathfrak{M}_{u_2}. \end{aligned}$$

□

Theorem 2. *Assume that the hypotheses* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$ *are valid. Then, the coupled system* (1) *is controllable on* J *given that*

$$\max\Big\{G_1^{\Delta};\widetilde{G}_2^{\Delta}\Big\} < 1,$$

where

$$G_1^{\Delta} = M_\alpha \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q} \|\mathcal{P}_1\|_{L^{\frac{1}{q}}} + M\hbar_1$$
$$+ M_\alpha M_{c_1} M_{w_1} \hbar_1 \Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) + M_\alpha M_{c_1} M_{w_1} M \Big(\frac{q-1}{q}\Big)^{q-1} \frac{a^{3-\alpha-q}}{1-\alpha} \|\mathcal{P}_1\|_{L^{\frac{1}{q}}},$$
$$\widetilde{G}_2^{\Delta} = M_\alpha \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q} \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M \|A^{-\alpha}\| \hbar_2 \Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)$$
$$+ M_\alpha M_{c_2} M_{w_2} M \Big(\frac{q-1}{q}\Big)^{q-1} \Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big) \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}.$$

Proof. Before proceeding with the proof, using techniques from [27], we consider an example of MNC in $\mathscr{C}_\alpha := C(J;\mathcal{X}_\alpha) \times C(J;\mathcal{X}_\alpha)$ defined as

$$\mu_C(\Theta_1 \times \Theta_2) = \big(\mu_1(\Theta_1), \mu_1(\Theta_2)\big)^T = \begin{pmatrix} w_0^T(\Theta_1) + \sup\limits_{t\in J}\Big\{e^{-\eta\int_0^t \ell_{f_1}(\tau)\,d\tau}\mu(\Theta_1(t))\Big\} \\ w_0^T(\Theta_2) + \sup\limits_{t\in J}\Big\{e^{-\eta\int_0^t \ell_{f_2}(\tau)\,d\tau}\mu(\Theta_2(t))\Big\} \end{pmatrix},$$

for $\Theta = \Theta_1 \times \Theta_2 \subset \mathscr{C}_\alpha$, where μ is the Kuratowski MNC in $\mathcal{X}_\alpha$, $\Theta(t) = \big\{v(t) \in \mathcal{X}_\alpha; v \in \Theta\big\}$, and for a nonempty bounded subset Θ of the space $C(J;\mathcal{X}_\alpha)$, $\mathfrak{u} \in \Theta$, $\varepsilon > 0$, the modulus of continuity, denoted by $w^T(\mathfrak{u},\varepsilon)$, is defined as

$$w^T(\mathfrak{u},\varepsilon) = \sup\Big\{\|\mathfrak{u}(s) - \mathfrak{u}(t)\|;\ t,s \in [0,a] \text{ and } |t-s| \le \varepsilon\Big\}.$$

For any $\Theta \subset \mathcal{X}_\alpha$, we set

$$w^T(\Theta,\varepsilon) = \sup\Big\{w^T(\mathfrak{u},\varepsilon);\ \mathfrak{u} \in \Theta\Big\}, \text{ and } w_0^T(\Theta) = \lim_{t\to\varepsilon} w^T(\Theta,\varepsilon).$$

Now, we introduce the operator $N : \mathscr{C}_\alpha \to \mathscr{C}_\alpha$ in the following manner:

$$(N(\zeta_1,\zeta_2))(t) = \begin{pmatrix} N_1(\zeta_1,\zeta_2)(t) \\ N_2(\zeta_1,\zeta_2)(t) \end{pmatrix}, \quad t \in J,$$

where

$$\begin{cases} N_1(\zeta_1,\zeta_2)(t) = R(t)(\zeta_{0,1} + H_1(\sigma_1(\zeta_1),\zeta_1)) + \int_0^t R(t-s)\,[f_1(s,\zeta_1(s),\zeta_2(s)) + \mathbf{C_1}u_1(s)]\,ds. \\ N_2(\zeta_1,\zeta_2)(t) = R(t)(\zeta_{0,2} + H_2(\sigma_2(\zeta_2),\zeta_2)) + \int_0^t R(t-s)\,[f_2(s,\zeta_1(s),\zeta_2(s)) + \mathbf{C_2}u_2(s)]\,ds. \end{cases}$$

One can see that the fixed points of N correspond to mild solutions of nonlocal system (1). We shall prove that N fulfills all hypotheses stated in Theorem 2.

Let us consider a subset $B_\delta \subseteq \mathscr{C}_\alpha$ such that

$$B_\delta := \{(\zeta_1,\zeta_2) \in \mathscr{C}_\alpha : \|(\zeta_1,\zeta_2)\| \le \delta\},$$

where $\delta = (\delta_1,\delta_2) > 0$. Clearly, B_δ is a closed, bounded, and convex set in $\mathscr{C}_\alpha$.

Initially, we show that $N(B_\delta) \subset B_\delta$. Clearly, it suffices to establish that for any δ, there exists a positive constant $\wp = (\wp_1;\wp_2)$ such that for every $(\zeta_1,\zeta_2) \in B_\delta$, the following holds

$$\|N(\zeta_1,\zeta_2)\|_{\mathscr{C}_\alpha} \quad \le \quad (\wp_1,\wp_2) := \wp.$$

In fact, for each $(\zeta_1, \zeta_2) \in B_\delta$ and $t \in [0, a]$, it yields that

$$\|N_1(\zeta_1,\zeta_2)(t)\|_\alpha \leq \left\|R(t)[\zeta_{0,1} + H_1(\sigma_1(\zeta_1),\zeta_1)]\right\|_\alpha + \int_0^t \|R(t-s)f_1(s,\zeta_1(s),\zeta_2(s))\|_\alpha \, ds + \int_0^t \|R(t-s)\|_\alpha \|C_1\| \|u_1(s)\| \, ds.$$

From the assumptions $(\mathbf{H}_2)(ii)$, $(\mathbf{H}_3)(i)$, $(\mathbf{H}_4)(i)-(ii)$, and Lemma 6, we obtain

$$\begin{aligned}
&\|N_1(\zeta_1,\zeta_2)(t)\|_\alpha \\
&\leq M\Big[\|\zeta_{0,1}\|_\alpha + \big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\Big] + M_\alpha \int_0^t (t-s)^{-\alpha}\Big(\mathcal{P}_1(s)\|\zeta_1(s)\|_\alpha + \mathcal{Q}_1(s)\|\zeta_2(s)\|_\alpha\Big) \, ds \\
&\quad + M_\alpha M_{c_1} \int_0^t (t-s)^{-\alpha}\|u_1(s)\| \, ds \\
&\leq M\Big[\|\zeta_{0,1}\|_\alpha + \big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\Big] \\
&\quad + M_\alpha \Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\Big(\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big) \\
&\quad + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\
&\quad + M\Big(\frac{q-1}{q}\Big)^{q-1} a^{2-q}\Big(\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big] \\
&\leq M\big(\|\zeta_{0,1}\|_\alpha + \aleph_1\big) + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{1,a}\| + M\|\zeta_{0,1}\| + M\|A^{-\alpha}\|\aleph_1\Big] \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}} + M\hbar_1 \\
&\quad + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\big(M\|A^{-\alpha}\|\hbar_1 + M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\big)\Big]\|\zeta_1\|_C \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}} \\
&\quad + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\big(M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\big)\Big]\|\zeta_2\|_C \\
&\leq M\big(\|\zeta_{0,1}\|_\alpha + \aleph_1\big) + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{1,a}\| + M\|\zeta_{0,1}\| + M\|A^{-\alpha}\|\aleph_1\Big] \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}} + M\hbar_1 \\
&\quad + M_\alpha M_{c_1} M_{w_1} M\|A^{-\alpha}\|\hbar_1\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) + M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\big)\Big]\delta_1 \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}} \\
&\quad + M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\big)\Big]\delta_2.
\end{aligned}$$

Thus, we have

$$\|N_1(\zeta_1,\zeta_2)\|_C \quad \leq \quad \wp_1.$$

Similarly, we have

$$
\begin{aligned}
&\|N_2(\zeta_1,\zeta_2)(t)\|_\alpha \\
&\quad\le \Big\|R(t)[\zeta_{0,2}+H_2(\sigma_2(\zeta_2),\zeta_2)]\Big\|_\alpha + \int_0^t \|R(t-s)f_2(s,\zeta_1(s),\zeta_2(s))\|\,ds \\
&\qquad + \int_0^t \|R(t-s)\|_\alpha \|C_2\|\|u_2(s)\|\,ds \\
&\quad\le M\big(\|\zeta_{0,2}\|_\alpha + \aleph_2\big) + M_\alpha M_{c_2} M_{w_2}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{2,a}\| + M\|\zeta_{0,2}\| + M\|A^{-\alpha}\|\aleph_2\Big] \\
&\qquad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}})\Big]\delta_1 \\
&\qquad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M\|A^{-\alpha}\|\hbar_2\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) \\
&\qquad + M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}})\Big]\delta_2.
\end{aligned}
$$

Thus,

$$\|N_2(\zeta_1,\zeta_2)\|_C \quad\le\quad \wp_2.$$

Consequently,

$$\|N(\zeta_1,\zeta_2)\|_{\mathscr{C}_\alpha} \quad\le\quad (\wp_1,\wp_2) := \wp.$$

We further verify that N is continuous. To do so, let $\{\zeta_1^n\}_{n\in\mathbb{N}} \subset B_\delta$ and $\{\zeta_2^n\}_{n\in\mathbb{N}} \subset B_\delta$ be a couple of sequences such that $\zeta_1^n \longrightarrow \zeta_1^*$ and $\zeta_2^n \longrightarrow \zeta_2^*$ in $\mathscr{C}_\alpha$ for some $\zeta_1^*,\zeta_2^* \in B_\delta$, as $n \longrightarrow \infty$. Then, we have

$$\lim_{n\longrightarrow\infty}\zeta_1^n(t) = \zeta_1{}^*(t),\ \lim_{n\longrightarrow\infty}\zeta_2^n(t) = \zeta_2{}^*(t),\ t\in J.$$

Combining this with the conditions $(\mathbf{H}_2)(ii)$ and $(\mathbf{H}_3)(i)$, one obtains that

$$\lim_{n\longrightarrow\infty} H_1(\sigma_1(\zeta_1^n),\zeta_1^n) \longrightarrow H_1(\sigma_1(\zeta_1^*),\zeta_1^*), \tag{9}$$

and

$$\lim_{n\longrightarrow\infty} f_1(s,\zeta_1^n(s),\zeta_2^n(s)) \longrightarrow f_1(s,\zeta_1^*(s),\zeta_2^*(s)),\ \text{ a.e. } s\in[0,a]. \tag{10}$$

By $(\mathbf{H}_2)(ii)$, it yields that for a.e. $s\in[0,t]$, $t\in[0,a]$,

$$(t-s)^{-\alpha}\|f_1(s,\zeta_1^n(s),\zeta_2^n(s)) - f_1(s,\zeta_1^*(s),\zeta_2^*(s))\| \le 2(t-s)^{-\alpha}\Big(\delta_1\mathcal{P}_1(s)+\delta_2\mathcal{Q}_1(s)\Big). \tag{11}$$

Moreover, by virtue of the definition of N_1, it follows that

$$
\begin{aligned}
&\|N_1(\zeta_1^n,\zeta_2^n)(t) - N_1(\zeta_1^*,\zeta_2^*)(t)\|_\alpha \\
&\quad\le M\|H_1(\sigma_1(\zeta_1^n),\zeta_1^n) - H_1(\sigma_1(\zeta_1^*),\zeta_1^*)\|_\alpha \\
&\qquad + \int_0^t \|A^\alpha R(t-s)\|\|f_1(s,\zeta_1^n(s),\zeta_2^n(s)) - f_1(s,\zeta_1^*(s),\zeta_2^*(s))\|\,ds \\
&\qquad + \int_0^t \|C_1\|\|A^\alpha R(t-s)\|\|u_1^n(s) - u_1^*(s)\|\,ds.
\end{aligned}
$$

Therefore, given the Lebesgue integrability of the function $s \longrightarrow 2(t-s)^{-\alpha}\Big(\delta_1\mathcal{P}_1(s)+\delta_2\mathcal{Q}_1(s)\Big)$ for a.e. $s\in[0,t]$, $t\in[0,a]$ and by (9)–(11) and the Lebesgue dominated convergence theorem, we obtain that, for $t\in[0,a]$,

$$\|N_1(\zeta_1^n,\zeta_2^n) - N_1(\zeta_1^*,\zeta_2^*)\|_C \longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Similarly, we obtain

$$\|N_2(\zeta_1^n,\zeta_2^n) - N_2(\zeta_1^*,\zeta_2^*)\|_C \longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

Hence, it follows that the operator N is continuous on $\mathscr{C}_\alpha$.

Now, we need to demonstrate that the operator N maps bounded sets into equicontinuous sets in $\mathscr{C}_\alpha$. To do so, we show that the set $\{N(\zeta_1,\zeta_2),(\zeta_1,\zeta_2)\in B_\delta\}\subset\mathscr{C}_\alpha$ is equicontinuous on $[0,a]$. Let $0<t_1<t_2\le a$ and ε be small enough, such that $0<\varepsilon<t_1<t_2\le a$, then

$$\begin{aligned}
&\|N_1(\zeta_1,\zeta_2)(t_2)-N_1(\zeta_1,\zeta_2)(t_1)\|_\alpha\\
&\quad\le\|R(t_2)-R(t_1)\|\Big(\|\zeta_{0,1}\|_\alpha+\|H_1(\sigma_1(\zeta_1),\zeta_1)\|_\alpha\Big)\\
&\qquad+\int_{t_1}^{t_2}\|A^\alpha R(t_2-s)\|\Big[\|f_1(s,\zeta_1(s),\zeta_2(s))\|+\|C_1\|\|u_1(s)\|\Big]\,ds\\
&\qquad+\int_0^{\varepsilon}\|A^\alpha R(t_2-s)-A^\alpha R(t_1-s)\|\Big[\|f_1(s,\zeta_1(s),\zeta_2(s))\|+\|C_1\|\|u_1(s)\|\Big]\,ds\\
&\qquad+\int_{\varepsilon}^{t_1}\|A^\alpha R(t_2-s)-A^\alpha R(t_1-s)\|\Big[\|f_1(s,\zeta_1(s),\zeta_2(s))\|+\|C_1\|\|u_1(s)\|\Big]\,ds.
\end{aligned}$$

Using assumptions $(\mathbf{H}_2)(ii)$, $(\mathbf{H}_3)(i)$, we obtain

$$\begin{aligned}
&\|N_1(\zeta_1,\zeta_2)(t_2)-N_1(\zeta_1,\zeta_2)(t_1)\|_\alpha\\
&\quad\le\|R(t_2)-R(t_1)\|\Big(\|\zeta_{0,1}\|_\alpha+(\aleph_1+\hbar_1\delta_1)\Big)\\
&\qquad+M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q}(t_2-t_1)^{1-\alpha-q}\Big[\delta_1\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}+\delta_2\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\Big]\\
&\qquad+M_\alpha\|C_1\|\|u_1(s)\|\frac{(t_2-t_1)^{-\alpha+1}}{-\alpha+1}\\
&\qquad+2M\Big[\delta_1\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\Big(\frac{q-1}{q}\Big)^{1-q}\varepsilon^{2-q}+\delta_2\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\Big(\frac{q-1}{q}\Big)^{1-q}\varepsilon^{2-q}+\|C_1\|\|u_1(s)\|\Big]\\
&\qquad+\sup_{s\in[\varepsilon,t_1]}\|A^\alpha R(t_2-s)-A^\alpha R(t_1-s)\|\Big[\delta_1\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\Big(\frac{q-1}{q}\Big)^{1-q}(t_1-\varepsilon)^{2-q}\\
&\qquad+\delta_2\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\Big(\frac{q-1}{q}\Big)^{1-q}(t_1-\varepsilon)^{2-q}\Big];
\end{aligned}\tag{12}$$

by virtue of Lemmas 3 and 4, we readily obtain that the right-hand term of the inequality (12) tends to zero as $t_2\longrightarrow t_1$ and $\varepsilon\longrightarrow0$. As a consequence, we can conclude that N_1 maps B_δ into an equicontinuous family of functions. Similarly, we can establish that N_2 maps B_δ into an equicontinuous family of functions. Consequently, we infer that the set $N(B_\delta)$ is equicontinuous in $\mathscr{C}_\alpha$, which implies that $w_0(N(B_\delta))=0$.

Next, we show that N is generalized μ_C-condensing operator. To prove this, let $\Omega=\Omega_1\times\Omega_2$ be a bounded equicontinuous subset of B_δ. It follows that $N:\Omega\to\Omega$ is a continuous and bounded operator and that $N_i\Omega$, for $i=1,2$, are bounded and equicontinuous (see, Lemma 5). By $(\mathbf{H}_2)(iii)$, $(\mathbf{H}_3)(ii)$, and $(\mathbf{H}_4)(iii)$, for $t\in[0,a]$, we have

$$\begin{aligned}
\mu(N_1(\Omega)(t))&\le\mu\big(\{R(t)[\zeta_{0,1}+H_1(\sigma_1(\zeta_1),\zeta_1)];\ \zeta_1\in\Omega_1\}\big)\\
&\quad+\mu\Big(\Big\{\int_0^t R(t-s)\Big[f_1(s,\zeta_1(s),\zeta_2(s))+C_1u_1(s)\Big]\,ds;\ (\zeta_1,\zeta_2)\in\Omega\Big\}\Big),
\end{aligned}$$

where

$$\begin{aligned}
&\mu(\{u_1(t)\})\\
&=\mu\Big(\mathcal{W}_1^{-1}\Big\{\zeta_{1,a}-R(a)[\zeta_{0,1}+H_1(\sigma_1(\zeta_1),\zeta_1)]-\int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big)\\
&\le k_{w_1}(t)\mu\Big(\Big\{\zeta_{1,a}-R(a)[\zeta_{0,1}+H_1(\sigma_1(\zeta_1),\zeta_1)]-\int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big)\\
&\le k_{w_1}(t)\mu\Big(\Big\{\zeta_{1,a}-R(a)[\zeta_{0,1}+H_1(\sigma_1(\zeta_1),\zeta_1)]\Big\}(t)\Big)\\
&\quad+k_{w_1}(t)\mu\Big(\Big\{\int_0^a R(a-s)f_1(s,\zeta_1(s),\zeta_2(s))\,ds\Big\}(t)\Big)\\
&\le k_{w_1}(t)M\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\}+2k_{w_1}(t)M\int_0^t\ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
\mu((N_1\Omega)(t)) \leq\ & M\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + 2M\int_0^t \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
& +2Mk_{c_1}\int_0^t \Big[k_{w_1}(s)M\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} \\
& +2k_{w_1}(s)M\int_0^a \ell_{f_1}(w)\Big(\mu(\Omega_1(w))+\mu(\Omega_2(w))\Big)\,dw\Big]\,ds \\
\leq\ & M\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + 2M\int_0^t \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
& +2M^2k_{c_1}\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\}\int_0^a k_{w_1}(s)\,ds \\
& +4M^2k_{c_1}\Big(\int_0^a k_{w_1}(s)\,ds\Big)\Big(\int_0^a \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
\leq\ & M\ell_{H_1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + 2M\int_0^t \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
& +2M^2k_{c_1}\ell_{H_1}\|k_{w_1}\|_{L^1}\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} \\
& +4M^2k_{c_1}\|k_{w_1}\|_{L^1}\Big(\int_0^a \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
\leq\ & M\ell_{H_1}\Big(1+2Mk_{c_1}\|k_{w_1}\|_{L^1}\Big)\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} \\
& +2M\Big(1+2Mk_{c_1}\|k_{w_1}\|_{L^1}\Big)\int_0^t \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds
\end{aligned}
$$

Set $\ell_1 = 1+2Mk_{c_1}\|k_{w_1}\|_{L^1}$, we obtain

$$
\begin{aligned}
\mu((N_1\Omega)(t)) \leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + 2M\ell_1\int_0^a \ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
\leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + \int_0^a 2M\ell_1\ell_{f_1}(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds.
\end{aligned}
$$

Set $\mathcal{K}_1(t) = 2M\ell_1\ell_{f_1}(t)$, we have

$$
\begin{aligned}
\mu((N_1\Omega)(t)) \leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + \int_0^a \mathcal{K}_1(s)\Big(\mu(\Omega_1(s))+\mu(\Omega_2(s))\Big)\,ds \\
\leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + \int_0^a \mathcal{K}_1(s)e^{\eta\int_0^t \mathcal{K}_1(\tau)d\tau}e^{-\eta\int_0^t \mathcal{K}_1(\tau)d\tau}\Big(\mu(\Omega_1(s)) \\
& +\mu(\Omega_2(s))\Big)\,ds \\
\leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} \\
& +\Big(\mu_1(\Omega_1)+\mu_1(\Omega_2)\Big)\int_0^t\Big(\frac{e^{\eta\int_0^s \mathcal{K}_1(\tau)\,d\tau}}{\eta}\Big)'\,ds \\
\leq\ & M\ell_{H_1}\ell_1\sup_{t\in J}\Big\{\mu(\Omega_1(t))\Big\} + \Big(\mu_1(\Omega_1)+\mu_1(\Omega_2)\Big)\frac{e^{\eta\int_0^s \mathcal{K}_1(\tau)\,d\tau}}{\eta}.
\end{aligned}
$$

Thus,

$$
\mu_1(N_1(\Omega)) \leq \Big(M\ell_{H_1}\ell_1+\frac{1}{\eta}\Big)\mu_1(\Omega_1)+\frac{1}{\eta}\mu_1(\Omega_2)
$$

Similarly, we obtain

$$
\mu_1(N_2(\Omega)) \leq \frac{1}{\eta}\mu_1(\Omega_1)+\Big(M\ell_{H_2}\ell_2+\frac{1}{\eta}\Big)\mu_1(\Omega_2).
$$

Since $N(\Omega)$ is equicontinuous, we obtain

$$\mu_C(N\Omega) \quad \leq \quad \widetilde{W}\begin{pmatrix} \mu_1(\Omega_1) \\ \mu_1(\Omega_2) \end{pmatrix},$$

such that

$$\widetilde{W} = \begin{pmatrix} M\ell_{H_1}\ell_1 + \dfrac{1}{\eta} & \dfrac{1}{\eta} \\ \dfrac{1}{\eta} & M\ell_{H_2}\ell_2 + \dfrac{1}{\eta} \end{pmatrix}.$$

Now, let us check if $\widetilde{W}$ converges to zero. To do so, let $\ell = \max\Big(\ell_{H_1}\ell_1; \ell_{H_2}\ell_2\Big)$; we calculate

$$\begin{aligned} \det\Big(\widetilde{W} - \lambda I_{2\times 2}\Big) \quad &= \quad \Big(\lambda - \Big(M\ell + \frac{1}{\eta}\Big)\Big)^2 - \frac{1}{\eta^2} \\ &= \quad \Big(\lambda - M\ell - \frac{2}{\eta}\Big)\Big(\lambda - M\ell\Big). \end{aligned}$$

Thus, we obtain $\lambda_1 = M\ell + \dfrac{2}{\eta}$ and $\lambda_2 = M\ell$. Since $M\ell < 1$, there exists $\eta > 0$ such that $\Big(M\ell + \dfrac{2}{\eta}\Big) < 1$. Hence, $\varrho(\widetilde{W}) < 1$, which follows that $\widetilde{W}$ converges to zero. Thus, we conclude that N is a generalized μ_C-condensing operator.

Finally, it remains to show the priori bounds on solutions. In fact, let $(\zeta_1, \zeta_2) \in \mathscr{A}_\lambda$ with $(\zeta_1, \zeta_2) = \lambda N(\zeta_1, \zeta_2)$. Then, $\zeta_1 = \lambda N_1(\zeta_1, \zeta_2)$ and $\zeta_2 = \lambda N_2(\zeta_1, \zeta_2)$. From the assumptions $(\mathbf{H}_2)(ii)$, $(\mathbf{H}_3)(i)$, and $(\mathbf{H}_4)(i) - (ii)$, for $t \in J$ and each $(\zeta_1, \zeta_2) \in \mathscr{A}_\lambda$, we have

$$\begin{aligned} \|\zeta_1(t)\|_\alpha \leq{}& \lambda\|N_1(\zeta_1,\zeta_2)(t)\|_\alpha \\ \leq{}& \big\|R(t)\big[\zeta_{0,1} + H_1(\sigma_1(\zeta_1),\zeta_1)\big]\big\|_\alpha + \int_0^t \big\|R(t-s)f_1(s,\zeta_1(s),\zeta_2(s))\big\|_\alpha\, ds \\ &+ \int_0^t \big\|R(t-s)C_1u_1(s)\big\|_\alpha\, ds \\ \leq{}& M\Big[\|\zeta_{0,1}\|_\alpha + \big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\Big] \\ &+ M_\alpha\int_0^t (t-s)^{-\alpha}\Big(\mathcal{P}_1(s)\|\zeta_1(s)\|_\alpha + \mathcal{Q}_1(s)\|\zeta_2(s)\|_\alpha\Big)\, ds \\ &+ M_\alpha M_{c_1} M_{w_1}\int_0^t (t-s)^{-\alpha}\Big[\|\zeta_{1,a}\| + M\big[\|\zeta_{0,1}\| + \|A^{-\alpha}\|\big(\aleph_1 + \hbar_1\|\zeta_1\|_C\big)\big] \\ &+ M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big]\, ds. \end{aligned}$$

Then,

$$\begin{aligned} \|\zeta_1\|_C \leq{}& M\big(\|\zeta_{0,1}\|_\alpha + \aleph_1\big) + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{1,a}\| + M\|\zeta_{0,1}\| + M\|A^{-\alpha}\|\aleph_1\Big] \\ &+ \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}} + M\hbar_1 \\ &+ M_\alpha M_{c_1} M_{w_1}\hbar_1\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) + M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}}\Big]\|\zeta_1\|_C \\ &+ \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}} \\ &+ M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}\Big]\|\zeta_2\|_C \\ \leq{}& G_0^\Delta + G_1^\Delta\|\zeta_1\|_C + G_2^\Delta\|\zeta_2\|_C, \end{aligned}$$

where

$$G_0^\Delta = M\big(\|\zeta_{0,1}\|_\alpha + \aleph_1\big) + M_\alpha M_{c_1} M_{w_1}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{1,a}\| + M\|\zeta_{0,1}\| + M\|A^{-\alpha}\|\aleph_1\Big],$$

$$G_1^\Delta = M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}} + M\hbar_1$$
$$+ M_\alpha M_{c_1} M_{w_1}\hbar_1\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) + M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_1\|_{L^{\frac{1}{q}}},$$

$$G_2^\Delta = M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_1} M_{w_1} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_1\|_{L^{\frac{1}{q}}}.$$

Similarly, we obtain

$$\begin{aligned}
\|\zeta_2(t)\|_\alpha &\le \lambda\|N_2(\zeta_1,\zeta_2)(t)\|_\alpha \\
&\le \big\|R(t)\big[\zeta_{0,2} + H_2(\sigma_2(\zeta_2),\zeta_2)\big]\big\|_\alpha + \int_0^t \big\|R(t-s)f_2(s,\zeta_1(s),\zeta_2(s))\big\|_\alpha\, ds \\
&\quad + \int_0^t \big\|R(t-s)C_2u_2(s)\big\|_\alpha\, ds \\
&\le M\Big[\|\zeta_{0,2}\|_\alpha + \big(\aleph_2 + \hbar_2\|\zeta_2\|_C\big)\Big] \\
&\quad + M_\alpha\int_0^t (t-s)^{-\alpha}\Big(\mathcal{P}_2(s)\|\zeta_1(s)\|_\alpha + \mathcal{Q}_2(s)\|\zeta_2(s)\|_\alpha\Big)\, ds \\
&\quad + M_\alpha M_{c_2} M_{w_2}\int_0^t (t-s)^{-\alpha}\Big[\|\zeta_{2,a}\| + M\big[\|\zeta_{0,2}\| + \|A^{-\alpha}\|\big(\aleph_2 + \hbar_2\|\zeta_2\|_C\big)\big] \\
&\quad + M\Big(\frac{q-1}{q}\Big)^{1-q} a^{2-q}\Big(\|\mathcal{P}_2\|_{L^{\frac{1}{q}}}\|\zeta_1\|_C + \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}\|\zeta_2\|_C\Big)\Big]\, ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\|\zeta_2\|_C &\le M\big(\|\zeta_{0,2}\|_\alpha + \aleph_2\big) + M_\alpha M_{c_2} M_{w_2}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{2,a}\| + M\|\zeta_{0,2}\| + M\|A^{-\alpha}\|\aleph_2\Big] \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}} \\
&\quad + M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}}\Big]\|\zeta_1\|_C \\
&\quad + \Big[M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M\|A^{-\alpha}\|\hbar_2\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big) \\
&\quad + M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}\Big]\|\zeta_2\|_C \\
&\le \widetilde{G}_0^\Delta + \widetilde{G}_1^\Delta\|\zeta_1\|_C + \widetilde{G}_2^\Delta\|\zeta_2\|_C,
\end{aligned}$$

where

$$\widetilde{G}_0^\Delta = M\big(\|\zeta_{0,2}\|_\alpha + \aleph_2\big) + M_\alpha M_{c_2} M_{w_2}\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)\Big[\|\zeta_{2,a}\| + M\|\zeta_{0,2}\| + M\|A^{-\alpha}\|\aleph_2\Big],$$

$$\widetilde{G}_1^\Delta = M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\frac{a^{3-\alpha-q}}{1-\alpha}\|\mathcal{P}_2\|_{L^{\frac{1}{q}}},$$

$$\widetilde{G}_2^\Delta = M_\alpha\Big(\frac{1-q}{1-\alpha-q}\Big)^{1-q} a^{1-\alpha-q}\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}} + M_\alpha M_{c_2} M_{w_2} M\|A^{-\alpha}\|\hbar_2\Big(\frac{a^{1-\alpha}}{1-\alpha}\Big)$$
$$+ M_\alpha M_{c_2} M_{w_2} M\Big(\frac{q-1}{q}\Big)^{q-1}\Big(\frac{a^{3-\alpha-q}}{1-\alpha}\Big)\|\mathcal{Q}_2\|_{L^{\frac{1}{q}}}.$$

Hence,

$$\begin{bmatrix} \|\zeta_1\|_C \\ \|\zeta_2\|_C \end{bmatrix} \le \widetilde{\mathcal{Z}}\begin{bmatrix} \|\zeta_1\|_C \\ \|\zeta_2\|_C \end{bmatrix} + \begin{bmatrix} G_0^\Delta \\ \widetilde{G}_0^\Delta \end{bmatrix},$$

where

$$\widetilde{\mathcal{Z}} = \begin{bmatrix} G_1^{\Delta} & G_2^{\Delta} \\ \widetilde{G}_1^{\Delta} & \widetilde{G}_2^{\Delta} \end{bmatrix}.$$

It implies that

$$(I - \widetilde{\mathcal{Z}}) \begin{bmatrix} \|\zeta_1\|_C \\ \|\zeta_2\|_C \end{bmatrix} \leq \begin{bmatrix} G_0^{\Delta} \\ \widetilde{G}_0^{\Delta} \end{bmatrix}. \tag{13}$$

Since $(I - \widetilde{\mathcal{Z}})$ satisfies all conditions of Lemma 2, $(I - \widetilde{\mathcal{Z}})^{-1}$ is order-preserving. Applying $(I - \widetilde{\mathcal{Z}})^{-1}$ to both parts of the inequality (13) yields

$$\begin{bmatrix} \|\zeta_1\|_C \\ \|\zeta_2\|_C \end{bmatrix} \leq (I - \widetilde{\mathcal{Z}})^{-1} \begin{bmatrix} G_0^{\Delta} \\ \widetilde{G}_0^{\Delta} \end{bmatrix}.$$

As a result of Theorem 1, we conclude that N has at least one fixed point, thereby implying the controllability of the coupled system (1). □

4. An Example

To illustrate the applicability of the theoretical findings, we consider in this section the following system of PFIDEs with state-dependent nonlocal conditions:

$$\begin{cases} \dfrac{\partial z_1(t,\xi)}{\partial t} = \dfrac{\partial^2 z_1(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \eta(t-s) \dfrac{\partial^2 z_1(s,\xi)}{\partial \xi^2}\, ds + h_1\Big(t, \dfrac{\partial z_1(t,\xi)}{\partial \xi}, \dfrac{\partial z_2(t,\xi)}{\partial \xi}\Big) \\ \qquad + \varkappa_1 u_1(t,\xi), \quad \xi \in [0,\pi], t \in [0,1], \\ \dfrac{\partial z_2(t,\xi)}{\partial t} = \dfrac{\partial^2 z_2(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \eta(t-s) \dfrac{\partial^2 z_2(s,\xi)}{\partial \xi^2}\, ds + h_2\Big(t, \dfrac{\partial z_1(t,\xi)}{\partial \xi}, \dfrac{\partial z_2(t,\xi)}{\partial \xi}\Big) \\ \qquad + \varkappa_2 u_2(t,\xi), \quad \xi \in [0,\pi], t \in [0,1], \\ z_1(t,0) = z_1(t,\pi) = 0, \quad 0 \leq t \leq 1, \\ z_2(t,0) = z_2(t,\pi) = 0, \quad 0 \leq t \leq 1, \\ z_1(0,\xi) = z_{0,1}(\xi) + \displaystyle\int_0^1 \int_0^{\pi} \psi_1(s) \phi_1\Big(\xi, \dfrac{\partial z_1}{\partial x}(s,x)\Big)\, dx\, ds, \quad 0 \leq \xi \leq \pi, \\ z_2(0,\xi) = z_{0,2}(\xi) + \displaystyle\int_0^1 \int_0^{\pi} \psi_2(s) \phi_2\Big(\xi, \dfrac{\partial z_2}{\partial x}(s,x)\Big)\, dx\, ds, \quad 0 \leq \xi \leq \pi, \end{cases} \tag{14}$$

where $z_1(t,\xi)$ and $z_2(t,\xi)$ represent the state variables, $z_{0,1}(\xi), z_{0,2}(\xi) \in \mathcal{X} := L^2([0,\pi])$ represent the initial functions, $u_i \in L^2([0,\pi])$ for $i = 1,2$, $\eta : [0,a] \to \mathbb{R}$, $\varkappa_i > 0$ for $i = 1,2$. The description of the functions ψ_i and ϕ_i is provided below.

Now, we assume the following conditions, for $i = 1,2$:

(a_1) The functions $h_i : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous for $i = 1,2$; there exists $0 \leq q < 1-\alpha$ and functions $a_i(\cdot), b_i(\cdot) \in L^{\frac{1}{q}}$ such that for any $x_1, x_2 \in \mathbb{R}$

$$|h_i(t,x_1,x_2)| \leq a_i(t)|x_1| + b_i(t)|x_2|.$$

(a_2) The functions $\psi_i \in C([0,1],\mathbb{R})$ and satisfies for $i = 1,2$; there exists some positive constant c_i such that $|\psi_i(t)| \leq c_i$.

(a_3) The functions $\phi_i \in C^1([0,\pi] \times \mathbb{R})$ for $i = 1,2$ and satisfy that, there exist $k_i, \bar{k}_i > 0$ such that, for $x \in \mathbb{R}$ and $\xi \in [0,\pi]$,

$$\Big|\frac{\partial}{\partial \xi} \phi_i(\xi,x)\Big| \leq k_i|x| + \bar{k}_i.$$

(a_4) $\eta(\cdot)$ satisfies $g_1(\lambda) := 1 + \eta^*(\lambda) \neq 0$ with $\lambda g_1(\lambda) \in \Lambda$, for $\lambda \in \Lambda$, and moreover, if $\eta^*(\lambda) \to 0$, as $|\lambda| \to +\infty$, $\lambda \in \Lambda$.

It is worth mentioning that the nonlocal system (14) can describe and model systems that have frequently arisen in the context of heat flow in materials [37]. The controllability of such coupled systems is of great practical significance, as external control inputs are applied to steer the system to any desired temperatures. First, we are required to rewrite this system in the form of (1) to apply our controllability results. To do so, let $\mathcal{X} = U := L^2([0,\pi])$. We define the operator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ by

$$Az_i = z_i'', \; i = 1,2,$$

with

$$D(A) = \Big\{ z_i(\cdot) \in \mathcal{X} : z_i', z_i'' \in \mathcal{X}, z_i(0) = z_i(\pi) = 0 \Big\}.$$

It is clear that the operator A generates a strongly continuous C_0-semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$, which is analytic and compact [38]. Moreover, A has a discrete spectrum given by its eigenvalues, which are $\{-n^2, \; n \in \mathbb{N}\}$. The related normalized eigenvectors are $e_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$, for $n \in \mathbb{N}$. Consequently, the following well-known properties hold for $i = 1,2$:

(p_1) If $z_i \in D(A)$, then $(-A)z_i = \sum_{n=1}^{\infty} n^2 < z_i, e_n > e_n$.

(p_2) If $\alpha = \frac{1}{2}$ and $z_i \in \mathcal{X}$, $(-A)^{-\frac{1}{2}} z_i = \sum_{n=1}^{\infty} \frac{1}{n} < z_i, e_n > e_n$. Particularly, $\|A^{-1}\| \leq 1$ and $\|A^{-\frac{1}{2}}\| \leq 1$.

(p_3) The operator $A^{\frac{1}{2}}$ is defined as

$$(-A)^{\frac{1}{2}} z_i = \sum_{n=1}^{\infty} n < z_i, e_n > e_n, \;\; z_i \in \mathcal{X} \text{ on } D(A^{\frac{1}{2}}),$$

with

$$\begin{aligned} D(A^{\frac{1}{2}}) &= \left\{ z_i(\cdot) \in \mathcal{X} : \sum_{n=1}^{\infty} n < z_i, e_n > e_n \in \mathcal{X} \right\} \\ &= \{ z_i(\cdot) \in \mathcal{X}, z_i' \in \mathcal{X}, z_i(0) = z_i(\pi) = 0 \}. \end{aligned}$$

Lemma 7 ([39]). *If $z \in \mathcal{X}_{\frac{1}{2}}$, then z is absolutely continuous, $z' \in \mathcal{X}$, and*

$$\|z\|_{\frac{1}{2}} = \|z'\| = \|A^{\frac{1}{2}} z\|.$$

Next, set

$$\begin{cases} \zeta_1(t)(\xi) = z_1(t,\xi), & 0 \leq t \leq 1, 0 \leq \xi \leq \pi, \\ \zeta_2(t)(\xi) = z_2(t,\xi), & 0 \leq t \leq 1, 0 \leq \xi \leq \pi, \\ \zeta_1(0)(\xi) = z_1(0,\xi), & 0 \leq \xi \leq \pi, \\ \zeta_2(0)(\xi) = z_2(0,\xi), & 0 \leq \xi \leq \pi. \end{cases}$$

We define also the operator

$$\big(\Upsilon(t)w\big)(\xi) = B(t)w''(\xi), \; t \in [0,a] \text{ and } \xi \in [0,\pi].$$

First, we know that for the operator $(A, D(A))$, there exists $b \in (0, \frac{\pi}{2})$ such that

$$\Lambda = \Big\{ \lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + b \Big\} \subset \varrho(A),$$

where $\varrho(A)$ represents the resolvent set of the operator A. Thus, based on (a_1), the assumptions $(V_1') - (V_3')$ stated in [13] are satisfied. Consequently, the linear system associated with (14) possesses an analytic resolvent operator denoted by $(R(t))_{t\geq 0}$, with $R(0) = I$, and

$$R(t)z = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda I - A - \Upsilon^*(\lambda))^{-1} z \; d\lambda, \; \lambda > 0,$$

for $z \in \mathcal{X}$, where Γ is specified in Section 2.

Since the nonlinear function f includes a term involving the partial derivative, we should discuss it in the space $\mathcal{X}_\alpha$. For this purpose, we select $\alpha = \frac{1}{2}$. We define the functions $f_i : [0,1] \times \mathcal{X}_{\frac{1}{2}} \times \mathcal{X}_{\frac{1}{2}} \to \mathcal{X}$, for $i = 1,2$, by

$$f_i(t, \zeta_1, \zeta_2)(\xi) = h_i(t, {\zeta_1}'(\xi), {\zeta_2}'(\xi)),$$

and the functions $H_i : [0,1] \times C([0,1], \mathcal{X}_{\frac{1}{2}}) \to \mathcal{X}_{\frac{1}{2}}$, $i = 1,2$, be defined as

$$H_1(\sigma_1(\zeta_1), \zeta_1)(\xi) = \int_0^1 \int_0^\pi \psi_1(s)\phi_1\Big(\xi, \zeta_1'(x)\Big)\, dx\, ds,$$
$$H_2(\sigma_2(\zeta_2), \zeta_2)(\xi) = \int_0^1 \int_0^\pi \psi_2(s)\phi_2\Big(\xi, \zeta_2'(x)\Big)\, dx\, ds,$$

and also $\sigma_i(\cdot) : C([0,1]; \mathcal{X}_{\frac{1}{2}}) \to [0,1]$, for $i = 1,2$, be defined as

$$\sigma_1(\zeta_1)(t) = \psi_1(t) \text{ and } \sigma_2(\zeta_2)(t) = \psi_2(t).$$

Thus, following these notations and definitions, we may rewrite system (14) in the abstract form (1).

Now, it can readily be checked that the above functions $f_i(\cdot,\cdot,\cdot), H_i(\cdot,\cdot), \sigma_i(\cdot)$, for $i = 1,2$, satisfy the conditions stated in Theorem 2.
First, assumption (a_1) ensures that the functions f_i meet the hypothesis $(\mathbf{H}_2)(i) - (ii)$. In fact, by applying Lemma 7 and the Cauchy-Schwarz inequality, for $\zeta_1, \zeta_2 \in \mathcal{X}_{\frac{1}{2}}$, $i = 1,2$, and $t \in [0,1]$, we have

$$\begin{aligned}
\|f_i(t,\zeta_1,\zeta_2)\|^2 &= \int_0^\pi \Big|h_i(t, \zeta_1'(\xi), \zeta_2'(\xi))\Big|^2 d\xi \\
&\leq \int_0^\pi \Big(a_i(t)|\zeta_1'(\xi)| + b_i(t)|\zeta_2'(\xi)|\Big)^2 d\xi \\
&\leq \Big(|a_i(t)|^2 \int_0^\pi |\zeta_1'(\xi)|^2\, d\xi + 2|a_i(t)||b_i(t)| \int_0^\pi |\zeta_1'(\xi)||\zeta_2'(\xi)|\, d\xi \\
&\quad + |b_i(t)|^2 \int_0^\pi |\zeta_2'(\xi)|^2\, d\xi\Big) \\
&\leq \Big(|a_i(t)|^2 \|\zeta_1\|_{\frac{1}{2}}{}^2 + 2|a_i(t)||b_i(t)|\|\zeta_1\|_{\frac{1}{2}}\|\zeta_2\|_{\frac{1}{2}} + |b_i(t)|^2\|\zeta_2\|_{\frac{1}{2}}{}^2\Big) \\
&\leq \Big(|a_i(t)|\|\zeta_1\|_{\frac{1}{2}} + |b_i(t)|\|\zeta_2\|_{\frac{1}{2}}\Big)^2.
\end{aligned}$$

Thus, we obtain

$$\|f_i(t,\zeta_1,\zeta_2)\| \leq \mathcal{P}_i(t)\|\zeta_1\|_{\frac{1}{2}} + \mathcal{Q}_i(t)\|\zeta_2\|_{\frac{1}{2}},$$

where

$$\mathcal{P}_i(t) = |a_i(t)|, \quad \mathcal{Q}_i(t) = |b_i(t)|.$$

Moreover, the state-dependent functions H_i, $i = 1,2$, clearly satisfy the hypothesis $(\mathbf{H}_3)$, which is guaranteed by the conditions (a_2) and (a_3). Now, it remains to verify that the matrix $(I - \widetilde{\mathcal{Z}})^{-1}$ is order-preserving to draw our conclusions. To do so, let

$$\mathcal{P}_1(t) = \frac{1}{100}e^{-t^2}, \qquad \mathcal{Q}_1(t) = \frac{1}{500(\sqrt{t}+1)}e^{-t^2},$$

$$\mathcal{P}_2(t) = \frac{1}{94}\Big(\frac{\ln(t+1)}{1+t^2}\Big), \qquad \mathcal{Q}_2(t) = \frac{1}{94}\Big(\frac{\sin(t)}{1+\sqrt{t}}\Big).$$

For $q = \frac{2}{5}$, we calculate

$$\|\mathcal{P}_1\|_{L^{\frac{1}{q}}} \simeq 0.00785, \quad \|\mathcal{Q}_1\|_{L^{\frac{1}{q}}} \simeq 0.001062,$$
$$\|\mathcal{P}_2\|_{L^{\frac{1}{q}}} \simeq 0.003165, \quad \|\mathcal{Q}_2\|_{L^{\frac{1}{q}}} \simeq 0.003134.$$

Additionally, by setting $M_\alpha = M = M_{c_i} = M_{w_i} = 1, i = 1,2$, we obtain

$$G_1^\Delta \simeq 0.02592, \quad G_2^\Delta \simeq 0.003505,$$
$$\widetilde{G}_1^\Delta \simeq 0.010451, \quad \widetilde{G}_2^\Delta \simeq 0.010351.$$

Therefore, we have

$$(I - \widetilde{\mathcal{Z}})^{-1} = \begin{pmatrix} 1.026569 & 0.003635 \\ 0.010841 & 1.010272 \end{pmatrix},$$

which implies that $(I - \widetilde{\mathcal{Z}})^{-1}$ is order-preserving. Hence, all conditions $(\mathbf{H}_1) - (\mathbf{H}_3)$ stated in Theorem 2 hold.

Furthermore, we define the operators $\mathbf{C_i} : L^2([0,\pi]) \to L^2([0,\pi])$ for $i = 1, 2$ by $\mathbf{C_i} u_i = \varkappa_i u_i(t,\xi)$, for $\xi \in [0,\pi]$, the operators $\mathcal{W}_i$ are defined by

$$(\mathcal{W}_i u_i)(\xi) = \int_0^1 R(1-s)\varkappa_i u_i(s,\xi)\, ds.$$

We assume that the operators $\mathcal{W}_i$ fulfill the hypothesis $(\mathbf{H}_4)$; thus, all hypotheses of Theorem 2 hold. Hence, it can be inferred that the coupled system (14) is controllable on the interval $[0,1]$.

5. Conclusions

This paper presents novel sufficient conditions for establishing controllability in the α-norm for a system of PFIDEs with nonlocal conditions in GBS. To derive our findings, we employ the resolvent operator as defined by Grimmer, generalized measures of noncompactness, fractional power operators, and Schaefer's fixed-point theorem for condensing operators. We emphasize that the issue addressed in the current setting is novel and contributes additional insight into studying nonlocal and nonlinear coupled problems. In forthcoming research, we aim to extend these findings to systems with discrete nonlocal initial conditions on an infinite interval.

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