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New Study on the Controllability of Non-Instantaneous Impulsive Hilfer Fractional Neutral Stochastic Evolution Equations with Non-Dense Domain

Gunasekaran Gokul [1], Barakah Almarri [2,*], Sivajiganesan Sivasankar [1], Subramanian Velmurugan [3] and Ramalingam Udhayakumar [1,*]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India; gokul.g2022a@vitstudent.ac.in (G.G.); sivasankar.s2020@vitstudent.ac.in (S.S.)
[2] General Studies Department, Jubail Industrial College, 8244 Rd Number 6, Al Huwaylat, Al Jubail 35718, Saudi Arabia
[3] Department of Mathematics, School of Arts, Sciences, Humanities and Education, SASTRA Deemed to be University, Thanjavur 613 401, Tamil Nadu, India; velmurugan@maths.sastra.edu
* Correspondence: marribj@rcjy.edu.sa (B.A.); udhayakumar.r@vit.ac.in (R.U.)

Abstract: The purpose of this work is to investigate the controllability of non-instantaneous impulsive (NII) Hilfer fractional (HF) neutral stochastic evolution equations with a non-dense domain. We construct a new set of adequate assumptions for the existence of mild solutions using fractional calculus, semigroup theory, stochastic analysis, and the fixed point theorem. Then, the discussion is driven by some suitable assumptions, including the Hille–Yosida condition without the compactness of the semigroup of the linear part. Finally, we provide examples to illustrate our main result.

Keywords: Hilfer fractional derivative; stochastic evolution equations; controllability; non-dense domain

MSC: 26A33; 60H10; 93B05

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1. Introduction

Due to their widespread applications in numerous significant applied fields, including diffusion theory, electromagnetism, population dynamics, fluid dynamics, seepage flow in porous media, heat conduction in materials with memory, autonomous mobile robots, and traffic models, fractional differential equations (FDEs) have drawn a lot of attention. Classical theory and applications of FDEs are discussed in the novels [1–6] and papers [7–10]. Hilfer [11] introduced a fractional derivative, which is a generalization of both R–L and the Caputo fractional derivatives, known as the Hilfer fractional derivative (HFD). Some authors [12–16] examined the existence of mild solution results of FDEs by utilizing HFD. Recently, the researcher in [17] investigated the HF neutral stochastic differential equations (SDEs) with NII by employing the Mönch fixed-point method.

The mathematical control theory includes the notion of controllability. A dynamical system is said to be controllable if it can be guided, by the set of admissible inputs, from an arbitrary initial state to an arbitrary final state. Several writers have explored controllability difficulties for various types of dynamical systems (see [18–23]) and the references therein. The researcher Wang et al. [24] established the controllability of Hilfer fractional NII semilinear differential inclusions with nonlocal conditions. Recently, the researchers [25] investigated the controllability of nonlocal HF delay dynamic inclusions with NII and a non-dense domain.

Since noise and fluctuating systems are frequent and inherent in both artificial and natural systems, stochastic models ought to be investigated rather than deterministic ones.

SDEs capture some occurrences in a way that makes them mathematically unpredictable. For an extensive overview of SDEs and their uses, one can refer to [26–31]. All physical systems evolving with respect to time experience abrupt changes called impulses. These impulses can be split into two distinct types: (i) instantaneous impulses and (ii) non-instantaneous impulses (NII). In a system, impulse occurs for a short time period, which is negligible when comparing the overall time period with an instantaneous impulse. Impulsive disturbance, which starts at any time and remains active over a finite time period is a non-instantaneous impulse. These NII are observed in lasers and in the intravenous introduction of drugs into the bloodstream. In 2016, Gautam and Dabas [32] established mild solutions for a class of neutral fractional functional differential equations with NII. Nowadays, most researchers [18,21,24,30,33–36] study non-instantaneous impulses with the HFD. Researchers delve into the study of non-densely defined operators to tackle the complexities of control and ensure the efficient operation of a wide range of systems, from robotics and autonomous vehicles to power grids and biological networks [9,10,14,19,22,25]. As far as we are aware, no research has been published on the subject of controllability in NII HF neutral stochastic evolution equations with a non-dense domain.

Consider the controllability of NII HF neutral stochastic evolution equations with a non-dense domain:

$$
{}^{H}D_{0^+}^{\mathfrak{l},\mathfrak{m}}[\mathrm{y}(\varpi)-\hbar(\varpi,\mathrm{y}(\varpi))] = \mathscr{A}[\mathrm{y}(\varpi)-\hbar(\varpi,\mathrm{y}(\varpi))] + \mathscr{B}\mathrm{u}(\varpi) + \mathcal{F}(\varpi,\mathrm{y}(\varpi))dW(\varpi),
$$

$$
\varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}+1}] \subset \mathscr{V}' = (0,c],\ \mathtt{k} = 0,1,2,\cdots,\mathbb{N}, \tag{1}
$$

$$
\mathrm{y}(\varpi) = \mathcal{G}_{\mathtt{k}}(\varpi,\mathrm{y}(\varpi)),\ \varpi \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}],\ \mathtt{k} = 1,2,\cdots,\mathbb{N}, \tag{2}
$$

$$
I_{0^+}^{(1-\gamma)}[\mathrm{y}(0)-\hbar(0,\mathrm{y}(0))] = \mathrm{y}_0, \quad \gamma = \mathfrak{l}+\mathfrak{m}-\mathfrak{l}\mathfrak{m}, \tag{3}
$$

where ${}^{H}D_{0^+}^{\mathfrak{l},\mathfrak{m}}$ stands for the HFD of order $\mathfrak{m} \in (0,1)$ and type $\mathfrak{l} \in [0,1]$. Here $\mathscr{V} = [0,c]$, and $\mathscr{V}' = (0,c]$ represent the time intervals. The fixed points $\varpi_{\mathtt{k}}$ and $\varepsilon_{\mathtt{k}}$ satisfy $\varepsilon_{\mathtt{k}} < \varpi_{\mathtt{k}+1} < \varepsilon_{i+1}$, $i = 0,1,\cdots,\mathbb{N}$. The operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{Y} \to \mathscr{Y}$ is a non-densely closed linear operator and generates an integrated semigroup $\{\mathbf{T}(\varpi)\}_{\varpi\geq 0}$ in Hilbert space (HS) $\mathscr{Y}$ with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. The control function $\mathrm{u}(\cdot)$ is provided in $L^2(\mathscr{V},\mathfrak{U})$, an HS of admissible control function with $\mathfrak{U}$ an HS, $\mathcal{F} : \mathscr{V} \times \mathscr{Y} \to \mathscr{Y}$ is the appropriate function. Let $\mathscr{K}$ be another distinct HS with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$.

The primary contributions of this article are as follows:

1. This manuscript focuses on the controllability of NII HF neutral stochastic evolution equations with a non-dense domain.
2. To show the relatively compact requirements, the Hausdorff measure of noncompactness (MNC) is used.
3. The main result is motivated in abstract space by applying the theory of fractional calculus, semigroup operators, and methods based on the fixed-point theorem.
4. The discussion is driven by some suitable assumptions, including the Hille–Yosida condition without the compactness of the semigroup of the linear part.
5. An illustration has been provided to demonstrate the efficiency of the obtained findings.

The structure of our article is as follows: A few key conclusions and terminology related to the fixed-point theorem, stochastic analysis, semigroup theory, and fractional calculus are found in Section 2. We develop the controllability results in Section 3. Lastly, an example demonstrating the established results is provided in Section 4.

2. Preliminaries

In this section, we introduce some fundamental terminology, definitions, and some earlier results that are used in this manuscript.

The symbols $(\mathscr{Y},\|\cdot\|)$ and $(\mathscr{K},\|\cdot\|)$ represent the two real HS. Consider the complete probability space $(\Sigma,\mathscr{E},\mathscr{P})$ connected to an entire set of right continuous increasing sub σ-algebra $\{\mathscr{E}_\varpi : \varpi \in \mathscr{V}\}$ such that $\mathscr{E}_\varpi \subset \mathscr{E}$. Consider a Q-Wiener process $W = (W_\varpi)_{\varpi\geq 0}$, identified on $(\Sigma, \mathscr{E}, \mathscr{P})$, with the covariance operator Q such that $Tr(Q) < \infty$. $L_2(\Sigma, \mathscr{Y})$ is the set of all square-integrable, strongly measurable $\mathscr{Y}$-valued arbitrary components with a Banach space connected with $(E\|\cdot, W\|^2_{\mathscr{Y}})^{\frac{1}{2}}$, which is equal to $\|\mathrm{y}(\cdot)\|_{L_2(\Sigma,\mathscr{Y})}$.

The space of all bounded linear operators from $\mathscr{K} \to \mathscr{Y}$, whenever $\mathscr{Y} = \mathscr{K}$, is defined by $L(\mathscr{K}, \mathscr{Y})$, which is represented by $L(\mathscr{K})$. One may express a non-negative self-adjoint operator as $Q \in L(\mathscr{K})$. Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathscr{K}, \mathscr{Y})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathscr{K} \to \mathscr{Y}$, $\phi \in L_2^0$, which is said to be a Q-Hilbert–Schmidt operator. For $a \in [0, c)$ and $\gamma \in [0, 1]$, consider the weighted spaces of continuous functions

$$C_\gamma([a,c], L_2(\Sigma, \mathscr{Y})) = \{\mathrm{y} \in C([a,c], L_2(\Sigma, \mathscr{Y})) : (\varpi - a)^\gamma \mathrm{y}(\varpi) \in C([a,c], L_2(\Sigma, \mathscr{Y}))\}.$$

Now, we specify $C([a,c], L_2(\Sigma, \mathscr{Y}))$ is a Banach space

$$E\|\mathrm{y}\|_{C([a,c],L_2(\Sigma,\mathscr{Y}))} = (\sup_{\varpi\in[a,c]} (\varpi - a)^\gamma \|\mathrm{y}(\varpi)\|^2).$$

Let $\mathscr{V}_m = (\varepsilon_m, \varpi_{m+1}]$, $\bar{\mathscr{V}}_m = [\varepsilon_m, \varpi_{m+1}](m = 0, 1, 2, \cdots, \mathbb{N})$, $\mathcal{G}_{\mathtt{k}} = (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]$, $\bar{\mathcal{G}}_{\mathtt{k}} = [\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}](\mathtt{k} = 1, 2, \cdots, \mathbb{N})$. Let $H = PC_{1-\gamma}(\mathscr{V}, L_2(\Sigma, \mathscr{Y})) = \{\mathrm{y} : (\varpi - \varepsilon_m)^{1-\gamma}\mathrm{y} \in \mathscr{V}_m$, $\lim_{\varpi\to\varepsilon_m^+}(\varpi - \varepsilon_m)^{1-\gamma}\mathrm{y}(\varpi)$, $\mathrm{y} \in C(\mathcal{G}_{\mathtt{k}}, L_2(\Sigma, \mathscr{Y}))$ and $\lim_{\varpi\to\varpi_{\mathtt{k}}^+} \mathrm{y}(\varpi)$ exists, $m = 0, 1, 2, \cdots, \mathbb{N}$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$, with

$$\begin{aligned}\|\cdot\|_H &= \{E\|\mathrm{y}(\varpi)\|^2_{PC_{1-\gamma}(\mathscr{V},L_2(\Sigma,\mathscr{Y}))}\}^{\frac{1}{2}} \\ &= \max\left\{(\max_{m=0,1,2,\cdots,\mathbb{N}} \sup_{\varpi\in\mathscr{V}_m} E\|(\varpi - \varepsilon_m)^{1-\gamma}\mathrm{y}(\varpi)\|^2)^{\frac{1}{2}}, (\max_{\mathtt{k}=1,2,\cdots,\mathbb{N}} \sup_{\varpi\in\mathcal{G}_{\mathtt{k}}} E\|\mathrm{y}(\varpi)\|^2)^{\frac{1}{2}}\right\}.\end{aligned}$$

Now, we introduce some assumptions for further analysis:

(A_1) $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{Y} \to \mathscr{Y}$ fulfils the Hille–Yosida presumption, i.e., there exists $M_0 > 0$ and $v \in \mathbb{R}$ such that $v \in (0, +\infty) \subset \rho(\mathscr{A})$ and

$$\|(\alpha I - \mathscr{A})^{-n}\| \le \frac{M_0}{(\alpha - v)^n}, \quad n \ge 1.$$

Set $D(\mathscr{A}) = \mathscr{Y}_0$. Assume $\mathscr{A}_0$ to be a part of $\mathscr{A}$ in $D(\mathscr{A})$ classified as $\mathscr{A}_0\mathrm{y} = \mathscr{A}\mathrm{y}$, $\{\mathrm{y} \in D(\mathscr{A}) : \mathscr{A}\mathrm{y} \in D(\mathscr{A})\}$ as the domain of $\mathscr{A}_0$. Subsequently, by referring to [4], the component $\mathscr{A}_0$ of $\mathscr{A}$ represents a strongly continuous semigroup $\{\mathbf{T}(\varpi)\}_{\varpi\ge0}$ on $\mathscr{Y}_0$ with $\|\mathbf{T}(\varpi)\| \le Me^{v\varpi}$, where M and v are constants. Describe $\sup_{\varpi\in[0,c]} \mathbf{T}(\varpi) \le M$.

Assume $\mathscr{B}_\alpha = \alpha R(\alpha, \mathscr{A}) := \alpha(\alpha I - \mathscr{A})^{-1}$ with I, the identity operator on $\mathscr{Y}$, then for any $\mathrm{y} \in \mathscr{Y}_0$, we obtain $\mathscr{B}_\alpha\mathrm{y} = \mathrm{y}$ as $\alpha \to \infty$ and $\lim_{\alpha\to\infty} \|\mathscr{B}_\alpha\mathrm{y}\| \le M_0\|\mathrm{y}\|$. Assume $\gamma = \mathfrak{l} + \mathfrak{m} - \mathfrak{l}\mathfrak{m}$, then $(1 - \gamma) = (1 - \mathfrak{l})(1 - \mathfrak{m})$. Describe $C_\gamma(\mathscr{V}, \mathscr{Y}_0) : \{\mathrm{y} \in C_\gamma(\mathscr{V}, \mathscr{Y}_0) : \lim_{\varpi\to0} \varpi^{(1-\gamma)}\mathrm{y}(\varpi)$ exist and finite$\}$ equipped with $\{\|\mathrm{y}\|_\gamma = \sup_{\varpi\in(0,c]} \|\varpi^{(1-\gamma)}\mathrm{y}(\varpi)\| : \gamma = \mathfrak{l} + \mathfrak{m} - \mathfrak{l}\mathfrak{m}\}$. Undoubtedly, $C_\gamma(\mathscr{V}, \mathscr{Y}_0)$ is a HS. Note $\mathrm{y}(\varpi) = \varpi^{(1-\gamma)}\mathrm{y}(\varpi)$ for $\varpi \in (0, c]$ and $\mathrm{y} \in C_\gamma(\mathscr{V}, \mathscr{Y}_0)$ iff $\mathrm{y} \in C_\gamma(\mathscr{V}, \mathscr{Y}_0)$.

The Wright function is explained as follows

$$W_{\mathfrak{m}}(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta^{n-1})}{(n-1)!\Gamma(1 - \mathfrak{m}n)},\ 0 < \mathfrak{m} < 1,\ \theta \in \mathbb{C},$$

which fulfils

$$\int_0^\infty \theta^\tau W_{\mathfrak{m}}(\theta)d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\mathfrak{m}\tau)},\ \theta \ge 0.$$

Definition 1. *[17,24] An $\mathscr{E}_\varpi$-adapted stochastic process* $\mathsf{y}(\varpi)$ *is called a mild solution of the system (1)–(3) if the succeeding integral equation is fulfilled:*

$$\mathsf{y}(\varpi) = \begin{cases} \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathsf{y}_0 + \hbar(\varpi,\mathsf{y}(\varpi)) + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in [0,\varpi_1], \\ \mathcal{G}_{\mathrm{k}}(\varpi,\mathsf{y}(\varpi)), \quad \text{for } \varpi \in (\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}], \\ \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})[\mathcal{G}_{\mathrm{k}}(\varpi,\mathsf{y}(\varepsilon_{\mathrm{k}}))] + \hbar(\varpi,\mathsf{y}(\varepsilon_{\mathrm{k}})) + \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathrm{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathrm{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in (\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}], \end{cases}$$

where $\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi) = I_{0^+}^{\mathfrak{l}(1-\mathfrak{m})}\mathbf{Q}_{\mathfrak{m}}(\varpi)$, $\mathbf{Q}_{\mathfrak{m}}(\varpi) = \varpi^{\mathfrak{m}-1}\mathbf{K}_{\mathfrak{m}}(\varpi)$, $\mathbf{K}_{\mathfrak{m}}(\varpi) = \int_0^{\infty} \mathfrak{m}\theta W_{\mathfrak{m}}(\theta)\mathbf{T}(\varpi^{\mathfrak{m}}\theta)d\theta$.

Lemma 1. *[13]*

(*i*) $\mathbf{T}(\varpi)$ *is continuous in the uniform operator topology for* $\varpi > 0$.
(*ii*) $\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)$ *and* $\mathbf{Q}_{\mathfrak{m}}(\varpi)$*are strongly continuous for* $\varpi > 0$.
(*iii*) *For the linear operators* $\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)$ *and* $\mathbf{Q}_{\mathfrak{m}}(\varpi)$, $\varpi > 0$ *and for every* $\mathsf{y} \in \mathscr{Y}_0$, *we obtain*

$$\|\mathbf{Q}_{\mathfrak{m}}(\varpi)\mathsf{y}\| \le \frac{M\varpi^{\mathfrak{m}-1}}{\Gamma(\mathfrak{m})}\|\mathsf{y}\|, \ \|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathsf{y}\| \le \frac{M\varpi^{\gamma-1}}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\|\mathsf{y}\|.$$

Now, we introduce the definition and some basic characteristics of Hausdorff MNC [37,38].

Definition 2. *[39] The Hausdorff MNC* μ *of the set* $\mathscr{D}$ *in the HS* $\mathscr{Y}$ *is specified as*

$$\mu(\mathscr{D}) = \inf\{\epsilon > 0 : \mathscr{D} \text{ has a finite } \epsilon - \text{net in } \mathscr{Y}\},$$

for each bounded subset $\mathscr{D}$ *in the HS* $\mathscr{Y}$.

Definition 3. *[37] A continuous and bounded map* $\Psi : D \subseteq \mathbb{X} \to \mathbb{X}$ *is said to be* μ*-contraction if there exists a constant* $0 < \kappa < 1$ *such that*

$$\mu(\Psi(\mathscr{D})) \le \kappa\mu(\mathscr{D}),$$

for every noncompact bounded subset $\mathscr{D} \subset D$, *where* $\mathbb{X}$ *is a Banach space.*

Lemma 2. *[37] If* $\{\mathsf{y}_n\}_{n=1}^{\infty} : \mathscr{V} \to \mathscr{Y}$ *is a series of Bochner integrable functions with the measurement* $\|\mathsf{y}_n\| \le \beta(\varpi)$, *for all* $\varpi \in \mathscr{V}$ *and for* $n \ge 1$, *where* $\beta \in L^1(\mathscr{V},\mathbb{R})$, *then the function* $\psi(\varpi) = \mu(\{\mathsf{y}_n\}_{n=1}^{\infty})$ *in* $L^1(\mathscr{V},\mathbb{R})$ *and fulfils*

$$\mu\left(\left\{\int_0^{\varpi} \mathsf{y}_n(\varepsilon)d\varepsilon : n \ge 1\right\}\right) \le 2\int_0^{\varpi} \psi(\varepsilon)d\varepsilon.$$

Lemma 3. *[37] Let* $\mathscr{D} \to \mathbb{X}$ *be a bounded set; then, a countable set* $\mathscr{D}_0 \to \mathscr{D}$ *exists such that* $\mu(\mathscr{D}) \le 2\mu(\mathscr{D}_0)$.

Definition 4. *[24] System (1)–(3) is said to be controllable on the interval* $\mathscr{V} = [0,c]$ *if for each* $\mathsf{y}_0,\mathsf{y}_1 \in \mathscr{Y}$, *there exists a control function* $\mathsf{u} \in L^2(\mathscr{V},\mathfrak{U})$ *such that any corresponding mild solution of* $\mathsf{y}(\varpi)$ *for the system (1)–(3) must satisfy the condition* $I_{0^+}^{(1-\gamma)}[\mathsf{y}(0) - \hbar(0,\mathsf{y}(0))] = \mathsf{y}_0$ *and* $\mathsf{y}(c) = \mathsf{y}_1$.

Theorem 1. *(Darbo–Sadovskii) [37] If* $\mathscr{D} \subseteq \mathbb{X}$ *be closed, bounded and convex. If the continuous map* $\Psi : \mathbb{X} \to \mathbb{X}$ *is a* μ*-contraction, then* Ψ *has a fixed-point in* $\mathscr{D}$.

3. Controllability

In this section, we will demonstrate the existence result, which is based on the Darbo–Sadovskii fixed-point method; for this, we have the succeeding presumptions:

(H_1) The operator $\mathbf{T}(\varpi)$, $\varpi > 0$ in $\mathscr{Y}$ such that $\|\mathbf{T}(\varpi)\| \leq M$ where, $M \geq 0$ is a constant.

(H_2) (a) The function $\hbar : \mathscr{V} \times \mathscr{Y} \to \mathscr{Y}$ is continuous and there exists constants $M_\hbar > 0$ for all $\varpi \in \mathscr{V}$, $\mathsf{y}, \mathsf{z} \in \mathscr{Y}$

$$E\|\hbar(\varpi, \mathsf{y}(\varpi)) - \hbar(\varpi, \mathsf{z}(\varpi))\|^2 \leq M_\hbar(\|\mathsf{y} - \mathsf{z}\|^2),$$
$$E\|\hbar(\varpi, \mathsf{y}(\varpi))\|^2 \leq M_\hbar(1 + \|\mathsf{y}\|^2).$$

(b) There exists a function $\Phi_1 \in L(\mathscr{V}, \mathbb{R}^+)$ and $\hbar^* > 0$ with $\sup_{\varpi \in \mathscr{V}} \Phi_1(\varpi) = \hbar^*$ such that for each bounded subset $\mathbb{E} \subset \mathscr{Y}$,

$$\mu(\hbar(\varpi, \mathsf{y})) \leq \Phi_1(\varpi)[\sup_{\varpi \in \mathscr{V}} \mu(\mathbb{E}(\varpi))].$$

(H_3) The function $\mathcal{F} : \mathscr{V} \times \mathscr{Y} \to \mathscr{Y}$ satisfies

(a) $\mathsf{y} \to \mathcal{F}(\varpi, \mathsf{y})$ is continuous for a.e $\varpi \in \mathscr{V}$, and $\varpi \to \mathcal{F}(\varpi, \mathsf{y})$ is strongly measurable for $\mathsf{y} \in \mathscr{Y}$.

(b) There exists a function $M_{\mathcal{F}}(\varpi) \in L(\mathscr{V}, \mathbb{R}^+)$ and a continuous increasing function $\Phi_2 : [0, \infty) \to (0, \infty)$ such that for every $\mathsf{y} \in \mathscr{Y}$ and $\varpi \in \mathscr{V}$,

$$E\|\mathcal{F}(\varpi, \mathsf{y}(\varpi))\|^2 \leq M_{\mathcal{F}}(\varpi)\Phi_2(\|\mathsf{y}(\varpi)\|^2).$$

(c) There exists a function $\Phi_3 \in L(\mathscr{V}, \mathbb{R}^+)$ and $\mathcal{F}^* > 0$ with $\sup_{\varpi \in \mathscr{V}} \Phi_3(\varpi) = \mathcal{F}^*$ such that for every bounded subset $\mathbb{E} \subset \mathscr{Y}$,

$$\mu(\mathcal{F}(\varpi, \mathsf{y})) \leq \Phi_3(\varpi)[\sup_{\varpi \in \mathscr{V}} \mu(\mathbb{E}(\varpi))].$$

(H_4) The functions $\mathcal{G}_{\mathtt{k}} : (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}] \times \mathscr{Y} \to \mathscr{Y}$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$ are continuous and fulfil the preceding requirements:

(a) For $\mathfrak{r} > 0$, there exists positive functions $\varrho_{\mathtt{k}}(\mathfrak{r})$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$ dependent on $\mathfrak{r}$ such that

$$E\|\mathcal{G}_{\mathtt{k}}(\varpi, \mathsf{y}(\varpi))\|^2 \leq \varrho_{\mathtt{k}}(\mathfrak{r}).$$

(b) There exists constants $\bar{\varrho_{\mathtt{k}}} > 0$ such that for any bounded subset $\mathbb{E} \subset \mathscr{Y}$,

$$\mu(\mathcal{G}_{\mathtt{k}}(\varpi, \mathsf{y})) \leq \bar{\varrho_{\mathtt{k}}} \sup_{\varpi \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]} \mu(\mathbb{E}(\varpi)),\ \mathtt{k} = 1, 2, \cdots, \mathbb{N}.$$

(H_5) (a) The function $\mathscr{B} : L^2(\mathscr{V}, \mathfrak{U}) \to L(\mathscr{V}, \mathscr{Y})$ is bounded, $\mathscr{W} : L^2(\mathscr{V}, \mathfrak{U}) \to \mathscr{Y}$ represented by $\mathscr{W}\mathsf{u} = \lim_{\lambda \to \infty} \int_0^c \mathbf{Q}_{\mathtt{m}}(c - \varepsilon)\mathscr{B}_\lambda \mathsf{u}(\varepsilon)d\varepsilon$, and it has an inverse operator $\mathscr{W}^{-1} : \mathscr{Y} \to L^2(\mathscr{V}, \mathfrak{U})/ker\mathscr{W}$, and there exist two positive constants $M_{\mathscr{B}}$ and $M_{\mathscr{W}}$ such that $\|\mathscr{B}\|_{L(\mathfrak{U}, \mathbb{E})} \leq M_{\mathscr{B}}$, $\|\mathscr{W}^{-1}\|_{L(\mathbb{E}, \mathfrak{U}/ker\mathscr{W})} \leq M_{\mathscr{W}}$.

(b) There exists a function $\Phi_4 \in L(\mathscr{V}, \mathbb{R}^+)$ and $\mathscr{W}^* > 0$ with $\sup_{\varpi \in \mathscr{V}} \Phi_4(\varpi) = \mathscr{W}^*$ such that for each bounded subset $\mathbb{E} \subset \mathscr{Y}$,

$$\mu((\mathscr{W}^{-1}Q)(\varpi)) \leq \Phi_4(\varpi)[\sup_{\varpi \in \mathscr{V}} \mu(Q(\varpi))].$$

Theorem 2. *If* (H_0)-(H_5) *holds, then the noninstantaneous impulsive HF neutral stochastic evolution of Equations* (1)–(3) *has a mild solution on* $\mathscr{V}$.

Proof. Depending on hypothesis $H_5(a)$, we can define the control function $\mathsf{u}(\varpi)$, as follows:

$$\begin{aligned}\mathsf{u}(\varpi) = \mathscr{W}^{-1}\Big\{ & \mathsf{y}_1 - \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c-\varepsilon_{\mathbb{N}})[\mathcal{G}_{\mathbb{N}}(c,\mathsf{y}(\varepsilon_{\mathbb{N}}))] - \hbar(c,\mathsf{y}(\varepsilon_{\mathbb{N}})) \\ & - \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & - \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon)\Big\} \end{aligned} \tag{4}$$

Using this control, we will show that the operator $\Psi : \mathbb{X} \to \mathbb{X}$ is defined by:

$$(\Psi\mathsf{y})(\varpi) = \begin{cases} \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathsf{y}_0 + \hbar(\varpi,\mathsf{y}(\varpi)) + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in [0,\varpi_1], \\ \mathcal{G}_{\mathtt{k}}(\varpi,\mathsf{y}(\varpi)), \quad \text{for } \varpi \in (\varepsilon_{\mathtt{k}},\varpi_{\mathtt{k}}],\ \mathtt{k} = 1,2,\ldots\mathbb{N}, \\ \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})[\mathcal{G}_{\mathtt{k}}(\varpi,\mathsf{y}(\varepsilon_{\mathtt{k}}))] + \hbar(\varpi,\mathsf{y}(\varepsilon_{\mathtt{k}})) + \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in (\varepsilon_{\mathtt{k}},\varpi_{\mathtt{k}+1}],\ \mathtt{k} = 1,2,\ldots\mathbb{N}. \end{cases}$$

Let us show that using the control function defined by (4), any fixed point for Ψ is a mild solution for (1)–(3) and satisfies $\mathsf{y}(0) = \mathsf{y}_0$ and $\mathsf{y}(c) = \mathsf{y}_1$. Infact, if y is a fixed point for Ψ, then from (4), we have

$$\begin{aligned}\mathsf{y}(c) =& \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c-\varepsilon_{\mathbb{N}})[\mathcal{G}_{\mathbb{N}}(c,\mathsf{y}(\varepsilon_{\mathbb{N}}))] + \hbar(c,\mathsf{y}(\varepsilon_{\mathbb{N}})) \\ & + \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & + \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & + \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ & + \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\ =& \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c-\varepsilon_{\mathbb{N}})[\mathcal{G}_{\mathbb{N}}(c,\mathsf{y}(\varepsilon_{\mathbb{N}}))] + \hbar(c,\mathsf{y}(\varepsilon_{\mathbb{N}})) \\ & + \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & + \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) + \mathscr{W}\mathsf{u}(\varpi) \\ =& \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c-\varepsilon_{\mathbb{N}})[\mathcal{G}_{\mathbb{N}}(c,\mathsf{y}(\varepsilon_{\mathbb{N}}))] + \hbar(c,\mathsf{y}(\varepsilon_{\mathbb{N}})) \\ & + \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & + \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) + \mathsf{y}_1 \\ & - \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c-\varepsilon_{\mathbb{N}})[\mathcal{G}_{\mathbb{N}}(c,\mathsf{y}(\varepsilon_{\mathbb{N}}))] - \hbar(c,\mathsf{y}(\varepsilon_{\mathbb{N}})) \\ & - \int_0^{\varepsilon_{\mathbb{N}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathbb{N}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ & - \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathsf{y}(\varepsilon))dW(\varepsilon) \\ =& \mathsf{y}_1. \end{aligned}$$

We now prove, using Theorem 1, that Ψ has a fixed point.

Step 1: $\Psi : B_{\mathfrak{r}} \to B_{\mathfrak{r}}$ in $\mathbb{X}$.

Indeed, it is enough to demonstrate for every $\mathfrak{r} > 0$, there exists $P > 0$ such that for $\mathrm{y} \in B_{\mathfrak{r}} = \{\mathrm{y} \in \mathbb{X},\ \|\mathrm{y}\|^2_{\mathbb{X}} < \mathfrak{r}\}$, we have $\|\Psi \mathrm{y}\|^2_{\mathbb{X}} < L$.

For $\varpi \in [0, \varpi_1]$,

$$\begin{aligned}\sup_{\varpi\in[0,\varpi_1]} \varpi_1^{2(1-\gamma)} E\|(\Psi \mathrm{y})(\varpi)\|^2_{\mathbb{X}} \leq 4 \sup_{\varpi\in[0,\varpi_1]} \varpi_1^{2(1-\gamma)} \Big\{ & E\|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathrm{y}_0\|^2 + E\|\hbar(\varpi, \mathrm{y}(\varpi))\|^2 \\ &+ E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \right\|^2 \\ &+ E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \right\|^2 \Big\} \\ \leq 4 \sum_{m=1}^{4} I_m. \end{aligned} \tag{5}$$

By Lemma 1, we have

$$\begin{aligned} I_1 &= E\|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathrm{y}_0\|^2 \\ &\leq \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 \varpi^{2(\gamma-1)} E\|\mathrm{y}_0\|^2. \end{aligned}$$

According to Lemma 1 and (H_2), we obtain

$$\begin{aligned} I_2 &= E\|\hbar(\varpi, \mathrm{y}(\varpi))\|^2 \\ &\leq M_\hbar(1 + \|\mathrm{y}\|^2) \\ &\leq M_\hbar(1 + \mathfrak{r}). \end{aligned}$$

According to Lemma 1 and $(H_5)(a)$, we have

$$\begin{aligned} I_3 &= E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \right\|^2 \\ &\leq E\Bigg\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathscr{W}^{-1}\big(\mathrm{y}_1 - \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c)\mathrm{y}_0 - \hbar(c, \mathrm{y}(c)) \\ &\quad - \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\omega)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\omega, \mathrm{y}(\omega))dW(\omega)\big)d\varepsilon \Bigg\|^2 \\ &\leq 4\|\alpha(\alpha I - \mathscr{A})^{-1}\|^2\|\mathscr{B}\|^2\|\mathscr{W}^{-1}\|^2\|\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\|^2\big(E\|\mathrm{y}_1\|^2 + E\|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(c)\mathrm{y}_0\|^2 + E\|\hbar(c, \mathrm{y}(c))\|^2 \\ &\quad + E\| \int_0^{c} \mathbf{Q}_{\mathfrak{m}}(c-\omega)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\omega, \mathrm{y}(\omega))dW(\omega)\|^2\big)d\varepsilon \end{aligned}$$

$$\begin{aligned} I_3 \leq 4M_0^2 M_{\mathscr{B}}^2 M_{\mathscr{W}}^2 \left(\frac{M}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \Bigg[& \|\mathrm{y}_1\|^2 + \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 \varpi^{2(\gamma-1)} E\|\mathrm{y}_0\|^2 \\ &+ M_\hbar(1+\mathfrak{r}) + Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \left(\int_0^{c} M_{\mathcal{F}}(\omega)d\omega\right)\Phi_2(\mathfrak{r})\Bigg]. \end{aligned}$$

By using Lemma 1 and $(H_3)(b)$, we obtain

$$\begin{aligned} I_4 &= E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \right\|^2 \\ &\leq Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \left(\int_0^{\varpi} M_{\mathcal{F}}(\varepsilon)d\varepsilon\right)\Phi_2(\mathfrak{r}). \end{aligned}$$

From the above, (5) becomes,

$$
\begin{aligned}
&\sup_{\varpi\in[0,\varpi_1]} \varpi_1^{2(1-\gamma)} E\|(\Psi \mathrm{y})(\varpi)\|_{\mathbb{X}}^2 \\
&\leq 4 \sup_{\varpi\in[0,\varpi_1]} \varpi_1^{2(1-\gamma)} \Bigg\{ \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 \varpi^{2(\gamma-1)} E\|\mathrm{y}_0\|^2 + M_{\hbar}(1+\mathfrak{r}) \\
&\quad + 4M_0^2 M_{\mathscr{B}}^2 M_{\mathscr{W}}^2 \left(\frac{M}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \Bigg[\|\mathrm{y}_1\|^2 + \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 \varpi^{2(\gamma-1)} E\|\mathrm{y}_0\|^2 \\
&\quad + M_{\hbar}(1+\mathfrak{r}) + Tr(Q) \left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \left(\int_0^c M_{\mathcal{F}}(\omega)d\omega\right) \Phi_2(\mathfrak{r}) \Bigg] \\
&\quad + Tr(Q) \left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)^2 \left(\int_0^{\varpi} M_{\mathcal{F}}(\varepsilon)d\varepsilon\right) \Phi_2(\mathfrak{r}) \Bigg\} \\
&= L_1.
\end{aligned}
$$

Next, for $\varpi \in (\varepsilon_{\mathrm{k}}, \varpi_{\mathrm{k}}]$, $\mathrm{k} = 1, 2, \cdots, \mathbb{N}$,

$$
\begin{aligned}
\sup_{\varpi\in[\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}]} E\|(\Psi \mathrm{y})(\varpi)\|_{\mathbb{X}}^2 &\leq \sup_{\varpi\in[\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}]} \left\{ E\|\mathcal{G}_{\mathrm{k}}(\varpi, \mathrm{y}(\varpi))\|^2 \right\} \\
&\leq \varrho_{\mathrm{k}}(\mathfrak{r}) \\
&= L_2.
\end{aligned}
$$

Similarly, for every $\varpi \in (\varepsilon_{\mathrm{k}}, \varpi_{\mathrm{k}+1}]$ $\mathrm{k} = 1, 2, \cdots, \mathbb{N}$, one can estimate,

$$
\begin{aligned}
&\sup_{\varpi\in[\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}]} (\varpi-\varepsilon_{\mathrm{k}})^{2(1-\gamma)} E\|(\Psi \mathrm{y})(\varpi)\|_{\mathbb{X}}^2 \\
&\leq 6 \sup_{\varpi\in[\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}]} (\varpi-\varepsilon_{\mathrm{k}})^{2(1-\gamma)} \Bigg\{ E\|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})[\mathcal{G}_{\mathrm{k}}(\varepsilon_{\mathrm{k}}, \mathrm{y}(\varepsilon_{\mathrm{k}}))]\|^2 + E\|\hbar(\varpi, \mathrm{y}(\varpi))\|^2 \\
&\quad + E\left\| \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathrm{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathrm{k}}-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \right\|^2 \\
&\quad + E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \right\|^2 \Bigg\} \\
&\leq 6 \sup_{\varpi\in[\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}}]} (\varpi-\varepsilon_{\mathrm{k}})^{2(1-\gamma)} \Bigg\{ \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 (\varpi-\varepsilon_{\mathrm{k}})^{2(\gamma-1)} \varrho_{\mathrm{k}}(\mathfrak{r}) + M_{\hbar}(1+\mathfrak{r}) \\
&\quad + 4M_0^2 M_{\mathscr{B}}^2 M_{\mathscr{W}}^2 \left(\frac{M}{\Gamma(\mathfrak{m})}\right)^2 \varepsilon_{\mathrm{k}}^{2(\mathfrak{m}-1)} \Bigg[\|\mathrm{y}_1\|^2 + \left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2 (\varpi-\varepsilon_{\mathrm{k}})^{2(\gamma-1)} E\|\mathrm{y}_0\|^2 \\
&\quad + M_{\hbar}(1+\mathfrak{r}) + Tr(Q) \left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \varepsilon_{\mathrm{k}}^{2(\mathfrak{m}-1)} \left(\int_0^c M_{\mathcal{F}}(\omega)d\omega\right) \Phi_2(\mathfrak{r}) \Bigg] \\
&\quad + Tr(Q) \left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2 \varepsilon_{\mathrm{k}}^{2(\mathfrak{m}-1)} \left(\int_0^{\varpi_{\mathrm{k}}} M_{\mathcal{F}}(\varepsilon)d\varepsilon\right) \Phi_2(\mathfrak{r})
\end{aligned}
$$

$$+4M_0^2M_{\mathscr{B}}^2M_{\mathscr{W}}^2\left(\frac{M}{\Gamma(\mathfrak{m})}\right)^2\varpi^{2(\mathfrak{m}-1)}\Bigg[\|\mathrm{y}_1\|^2+\left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2(\varpi-\varepsilon_{\mathrm{k}})^{2(\gamma-1)}E\|\mathrm{y}_0\|^2$$
$$+M_{\hbar}(1+\mathfrak{r})+Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2\varpi^{2(\mathfrak{m}-1)}\left(\int_0^c M_{\mathcal{F}}(\omega)d\omega\right)\Phi_2(\mathfrak{r})\Bigg]$$
$$+Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2\varpi^{2(\mathfrak{m}-1)}\left(\int_0^{\varpi} M_{\mathcal{F}}(\varepsilon)d\varepsilon\right)\Phi_2(\mathfrak{r})\Bigg\}$$
$$\leq 6c^{2(1-\gamma)}\Bigg\{\left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2c^{2(\gamma-1)}\varrho_{\mathrm{k}}(\mathfrak{r})+M_{\hbar}(1+\mathfrak{r})$$
$$+2\Bigg[4M_0^2M_{\mathscr{B}}^2M_{\mathscr{W}}^2\left(\frac{M}{\Gamma(\mathfrak{m})}\right)^2c^{2(\mathfrak{m}-1)}\Bigg[\|\mathrm{y}_1\|^2+\left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2c^{2(\gamma-1)}E\|\mathrm{y}_0\|^2$$
$$+M_{\hbar}(1+\mathfrak{r})+Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2c^{2(\mathfrak{m}-1)}\left(\int_0^c M_{\mathcal{F}}(\omega)d\omega\right)\Phi_2(\mathfrak{r})\Bigg]$$
$$+Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2c^{2(\mathfrak{m}-1)}\left(\int_0^c M_{\mathcal{F}}(\varepsilon)d\varepsilon\right)\Phi_2(\mathfrak{r})\Bigg]\Bigg\}$$
$$=L_3.$$

Let $L=\max\{L_1,L_2,L_3\}$ then for any $\mathrm{y}\in B_{\mathfrak{r}}$, we obtain $\|(\Psi\mathrm{y})(\varpi)\|^2_{\mathbb{X}}\leq L$.

Step 2: Ψ is continuous on $B_{\mathfrak{r}}$.

Let $\{\mathrm{y}^n(\varpi)\}_{n=1}^{\infty}\subset B_{\mathfrak{r}}$ with $\varpi^n\to\mathrm{y}$, $(n\to\infty)$ in $B_{\mathfrak{r}}$. Therefore, the continuous functions are $\hbar,\mathscr{B}$ and $\mathcal{F}$ for every $\epsilon>0$, and there exists $\mathbb{N}$ such that for any $n\in\mathbb{N}$,

$$E\|\hbar(\varepsilon,\mathrm{y}^n(\varepsilon))-\hbar(\varepsilon,\mathrm{y}(\varepsilon))\|^2<\epsilon,$$
$$E\|\mathscr{B}\mathrm{u}^n(\varepsilon)-\mathscr{B}\mathrm{u}(\varepsilon)\|^2<\epsilon$$
$$E\|\mathcal{F}(\varepsilon,\mathrm{y}^n(\varepsilon))-\mathcal{F}(\varepsilon,\mathrm{y}(\varepsilon))\|^2<\epsilon,$$

For each $\varpi\in\mathscr{V}$, we obtain

$$E\|\mathscr{B}\mathrm{u}^n(\varepsilon)-\mathscr{B}\mathrm{u}(\varepsilon)\|^2\leq 12M_0^2M_{\mathscr{B}}^2M_{\mathscr{W}}^2\Bigg[\|\mathrm{y}_1\|^2+\left(\frac{M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}\right)^2c^{2(\gamma-1)}E\|\mathrm{y}_0\|^2$$
$$+M_{\hbar}(1+\mathfrak{r})+Tr(Q)\left(\frac{MM_0}{\Gamma(\mathfrak{m})}\right)^2\left(\frac{c^{\mathfrak{m}}}{\mathfrak{m}}\right)^2\left(\int_0^c M_{\mathcal{F}}(\omega)d\omega\right)\Phi_2(\mathfrak{r})\Bigg]$$
$$E\|\mathcal{F}(\varepsilon,\mathrm{y}^n(\varepsilon))-\mathcal{F}(\varepsilon,\mathrm{y}(\varepsilon))\|^2\leq 3Tr(Q)M_{\mathcal{F}}(\varepsilon)\Phi_2(\mathfrak{r})d\varepsilon.$$

By $(H_1)-(H_5)$ Lebesgue Dominated Convergence Theorem, for $\varpi\in[0,\varpi_1]$,

$$\sup_{\varpi\in\mathscr{V}}\varpi^{2(1-\gamma)}E\|(\Psi\mathrm{y}^n)(\varpi)-(\Psi\mathrm{y})(\varpi)\|^2$$
$$\leq 3\sup_{\varpi\in[0,\varpi_1]}\varpi^{2(1-\gamma)}\Bigg\{E\|\hbar(\varepsilon,\mathrm{y}^n(\varepsilon))-\hbar(\varepsilon,\mathrm{y}(\varepsilon))\|^2$$
$$+E\left\|\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}[\mathscr{B}\mathrm{u}^n(\varepsilon)-\mathscr{B}\mathrm{u}(\varepsilon)]d\varepsilon\right\|^2$$
$$+E\left\|\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}[\mathcal{F}(\varepsilon,\mathrm{y}^n(\varepsilon))-\mathcal{F}(\varepsilon,\mathrm{y}(\varepsilon))]dW(\varepsilon)\right\|^2\Bigg\}$$
$$\to 0 \text{ as } n\to\infty.$$

Next, for every $\varpi \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$,

$$\sup_{\varpi \in \mathscr{V}} E\|(\Psi \mathrm{y}^n)(\varpi) - (\Psi \mathrm{y})(\varpi)\|^2 \leq \sup_{\varpi \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]} E\|\mathcal{G}_{\mathtt{k}}(\varpi, \mathrm{y}^n(\varpi)) - \mathcal{G}_{\mathtt{k}}(\varpi, \mathrm{y}(\varpi))\|^2 \to 0 \text{ as } n \to \infty.$$

For any $\varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}+1}]$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$,

$$\begin{aligned}
&\sup_{\varpi \in \mathscr{V}} \varpi^{2(1-\gamma)} E\|(\Psi \mathrm{y}^n)(\varpi) - (\Psi \mathrm{y})(\varpi)\|^2 \\
&\leq 6 \sup_{\varpi \in [\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}}]} (\varpi - \varepsilon_{\mathtt{k}})^{2(1-\gamma)} \Big\{ E\|\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi - \varepsilon_{\mathtt{k}})[\mathcal{G}_{\mathtt{k}}(\varepsilon_{\mathtt{k}}, \mathrm{y}^n(\varepsilon_{\mathtt{k}})) - \mathcal{G}_{\mathtt{k}}(\varepsilon_{\mathtt{k}}, \mathrm{y}(\varepsilon_{\mathtt{k}}))]\|^2 \\
&\quad + E\|\hbar(\varpi, \mathrm{y}^n(\varpi)) - \hbar(\varpi, \mathrm{y}(\varpi))\|^2 \\
&\quad + E\left\| \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}} - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}[\mathscr{B}\mathrm{u}^n(\varepsilon) - \mathscr{B}\mathrm{u}(\varepsilon)]d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}} - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}[\mathcal{F}(\varepsilon, \mathrm{y}^n(\varepsilon)) - \mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))]dW(\varepsilon) \right\|^2 \\
&\quad + E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}[\mathscr{B}\mathrm{u}^n(\varepsilon) - \mathscr{B}\mathrm{u}(\varepsilon)]d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}[\mathcal{F}(\varepsilon, \mathrm{y}^n(\varepsilon)) - \mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))]dW(\varepsilon) \right\|^2 \Big\} \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Then,

$$\sup_{\varpi \in \mathscr{V}} \varpi^{2(1-\gamma)} E\|(\Psi \mathrm{y}^n)(\varpi) - (\Psi \mathrm{y})(\varpi)\|^2 \to 0 \text{ as } n \to \infty.$$

Therefore, Ψ is continuous.

Step 3: Ψ maps bounded sets into equicontinuous sets of $B_{\mathfrak{r}}$.

Let $0 < \iota_1 < \iota_2 < \varpi_1$. For every $\mathrm{y} \in B_{\mathfrak{r}}$,

$$\begin{aligned}
&\sup_{\varpi \in [0, \varpi_1]} \varpi_1^{2(1-\gamma)} E\|(\Psi \mathrm{y})(\iota_2) - (\Psi \mathrm{y})(\iota_1)\|^2 \\
&\leq 4 \sup_{\varpi \in [0, \varpi_1]} \varpi_1^{2(1-\gamma)} \Big\{ E\|[\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\iota_2) - \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\iota_1)]\mathrm{y}_0\|^2 + E\|[\hbar(\iota_2, \mathrm{y}(\iota_2)) - \hbar(\iota_1, \mathrm{y}(\iota_1))]\|^2 \\
&\quad + E\Big\| \int_0^{\iota_1} [\mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon) - \mathbf{Q}_{\mathfrak{m}}(\iota_1 - \varepsilon)]\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \\
&\qquad + \int_{\iota_1}^{\iota_2} \mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathrm{u}(\varepsilon)d\varepsilon \Big\|^2 \\
&\quad + E\Big\| \int_0^{\iota_1} [\mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon) - \mathbf{Q}_{\mathfrak{m}}(\iota_1 - \varepsilon)]\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \\
&\qquad + \int_{\iota_1}^{\iota_2} \mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathrm{y}(\varepsilon))dW(\varepsilon) \Big\|^2 \Big\}.
\end{aligned}$$

For $\iota_1, \iota_2 \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]$, $\iota_1 < \iota_2$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$,

$$E\|(\Psi \mathrm{y})(\iota_2) - (\Psi \mathrm{y})(\iota_1)\|^2 = \sup_{\varpi \in (\varpi_{\mathtt{k}}, \varepsilon_{\mathtt{k}}]} E\|\mathcal{G}_{\mathtt{k}}(\iota_2, \mathrm{y}(\iota_2)) - \mathcal{G}_{\mathtt{k}}(\iota_1, \mathrm{y}(\iota_1))\|^2.$$

Similarly, for $\iota_1, \iota_2 \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}+1}],\ \iota_1 < \iota_2,\ \mathtt{k} = 1, 2, \cdots, \mathbb{N}$,

$$
\begin{aligned}
&\sup_{\varpi\in[0,\varpi_1]} \varpi_1^{2(1-\gamma)} E\|(\Psi\mathsf{y})(\iota_2) - (\Psi\mathsf{y})(\iota_1)\|^2 \\
&\leq 6 \sup_{\varpi\in[\varepsilon_{\mathtt{k}},\varpi_{\mathtt{k}}]} (\varpi - \varepsilon_{\mathtt{k}})^{2(1-\gamma)} \Big\{ E\|[\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\iota_2 - \varepsilon_{\mathtt{k}}) - \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\iota_1 - \varepsilon_{\mathtt{k}})]\mathcal{G}_{\mathtt{k}}(\varepsilon_{\mathtt{k}}, \mathsf{y}(\varepsilon_{\mathtt{k}}))\|^2 \\
&\quad + E\|[\hbar(\iota_2, \mathsf{y}(\iota_2)) - \hbar(\iota_1, \mathsf{y}(\iota_1))]\|^2 \\
&\quad + E\left\| \int_0^{\iota_1} [\mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon) - \mathbf{Q}_{\mathfrak{m}}(\iota_1 - \varepsilon)]\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_{\iota_1}^{\iota_2} \mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \right\|^2 \\
&\quad + E\left\| \int_0^{\iota_1} [\mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon) - \mathbf{Q}_{\mathfrak{m}}(\iota_1 - \varepsilon)]\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathsf{y}(\varepsilon))dW(\varepsilon) \right\|^2 \\
&\quad + E\left\| \int_{\iota_1}^{\iota_2} \mathbf{Q}_{\mathfrak{m}}(\iota_2 - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathsf{y}(\varepsilon))dW(\varepsilon) \right\|^2 \Big\}.
\end{aligned}
$$

The RHS of the aforementioned inequalities $\to 0$ as $\iota_2 \to \iota_1$, and since the operators $\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\cdot)$, $\mathbf{Q}_{\mathfrak{m}}(\cdot)$ are continuous, we obtain that $\|(\Psi\mathsf{y})(\iota_2) - (\Psi\mathsf{y})(\iota_1)\|^2_{\mathbb{X}} \to 0$ independently of $\mathsf{y} \in B_{\mathfrak{r}}$ as $\iota_2 \to \iota_1$, for ϵ sufficiently small. Moreover, $\Psi\mathsf{y},\ \mathsf{y} \in B_{\mathfrak{r}}$ is equicontinuous. Thus, Ψ maps $B_{\mathfrak{r}}$ into a set of equicontinuous.

Step 4: Prove that $\Psi : B_{\mathfrak{r}} \to B_{\mathfrak{r}}$ is a μ-contraction operator.

Let $\mathbb{E} \subseteq B_{\mathfrak{r}}$, then by Lemma 3, there exists a countable set $\mathbb{E}_0 = \{\mathsf{y}_n\}_{n=1}^{\infty} \subset \mathbb{E}$ such that $\mu(\Psi(\mathbb{E})(\varpi)) \leq 2\mu(\{\mathsf{y}_n\}_{n=1}^{\infty})$. By the equicontinuousness of $B_{\mathfrak{r}}$, we know that $\mathbb{E}$ is also equicontinuous. Then, by Lemma 3, we have

$$
\mu_c(\Psi(\mathbb{E}_0)) = \max_{\varpi\in[0,c]} \mu(\Psi(\mathbb{E}_0)).
$$

Now, define

$$
(\Psi\mathbb{E})(\varpi) = \begin{cases}
\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathsf{y}_0 + \hbar(\varpi, \mathsf{y}(\varpi)) + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\
+ \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in [0, \varpi_1],\ i = 0, \\
\mathcal{G}_{\mathtt{k}}(\varpi, \mathsf{y}(\varpi)), \qquad \text{for } \varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}}],\ i \geq 1, \\
\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi - \varepsilon_{\mathtt{k}})[\mathcal{G}_{\mathtt{k}}(\varpi, \mathsf{y}(\varepsilon_{\mathtt{k}}))] + \hbar(\varpi, \mathsf{y}(\varepsilon_{\mathtt{k}})) + \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}} - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\
+ \int_0^{\varepsilon_{\mathtt{k}}} \mathbf{Q}_{\mathfrak{m}}(\varepsilon_{\mathtt{k}} - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathsf{y}(\varepsilon))dW(\varepsilon) \\
+ \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}(\varepsilon)d\varepsilon \\
+ \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathsf{y}(\varepsilon))dW(\varepsilon), \quad \text{for } \varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}+1}],\ i \geq 1.
\end{cases}
$$

Let $\Psi(\mathbb{E}) = \Psi_1(\mathbb{E}) + \Psi_2(\mathbb{E}) + \Psi_3(\mathbb{E})$.

First, we estimate $\Psi_1(\mathbb{E})$, for $\varpi \in [0, \varpi_1]$, and we obtain

$$
\begin{aligned}
\mu_c(\Psi_1(\mathbb{E})) &\leq 2\mu_c(\Psi_1(\mathbb{E}_0)) \\
&= 2 \max_{\varpi\in[0,c]} \mu(\Psi_1(\mathbb{E}_0)(\varpi)) \\
&\leq 2 \max_{\varpi\in[0,c]} \Big[\mu\Big(\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathsf{y}_0 + \hbar(\varpi, \mathbb{E}_0(\varpi)) + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathscr{B}\mathsf{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon \\
&\quad + \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi - \varepsilon)\alpha(\alpha I - \mathscr{A})^{-1}\mathcal{F}(\varepsilon, \mathbb{E}_0(\varepsilon))dW(\varepsilon) \Big) \Big]
\end{aligned}
$$

$$\leq 2 \max_{\varpi\in[0,c]} \Big[\mu\Big(\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi)\mathrm{y}_0\Big) + \mu\Big(\hbar(\varpi,\mathbb{E}_0(\varpi))\Big) + \mu\Big(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon\Big) + \mu\Big(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))dW(\varepsilon)\Big)\Big].$$

Since, $\mathbf{S}_{\mathfrak{l},\mathfrak{m}}$ is relatively compact, we obtain

$$\begin{aligned}
\|\mu_c(\Psi_1(\mathbb{E}))\| &\leq 2 \max_{\varpi\in[0,c]} \Big\|\mu\Big(\hbar(\varpi,\mathbb{E}_0(\varpi))\Big) \\
&\quad + \mu\Big(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon\Big) \\
&\quad + \mu\Big(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))dW(\varepsilon)\Big)\Big\| \\
&\leq 4 \max_{\varpi\in[0,c]} \Big\|\mu\Big(\hbar(\varpi,\mathbb{E}_0(\varpi))\Big) \\
&\quad + \int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Big(\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)\Big)d\varepsilon \\
&\quad + \int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Big(\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))\Big)dW(\varepsilon)\Big\| \\
&\leq 4 \max_{\varpi\in[0,c]} \Big\{\Big\|\mu\Big(\hbar(\varpi,\mathbb{E}_0(\varpi))\Big)\Big\| \\
&\quad + \Big\|\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Big(\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)\Big)d\varepsilon\Big\| \\
&\quad + \Big\|\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Big(\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))\Big)dW(\varepsilon)\Big\|\Big\} \\
&\leq 4\Phi_1(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\quad + 4\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_4(\varpi)\Big[\Phi_1(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\quad + Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_3(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))]\Big] \\
&\quad + 4Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_3(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\leq 4\Big[1+\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_4(\varpi)\Big]\Phi_1(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\quad + 4\Big[1+\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_4(\varpi)\Big] \\
&\qquad \times Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_3(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\leq 4\Big[1+\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_4(\varpi)\Big]\Big[\Phi_1(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))] \\
&\quad + Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\Big(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\Big)\Phi_3(\varpi)[\sup_{\varpi\in[0,\varpi_1]}\mu(\mathbb{E}_0(\varpi))]\Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\left[1+\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]\left[\hbar^*+Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathcal{F}^*\right]\mu(\mathbb{E}_0(\varpi))\\
&\leq \Xi_1^*\mu(\mathbb{E}_0(\varpi)),
\end{aligned}
$$

where

$$
\Xi_1^* = 4\left[1+\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]\left[\hbar^*+Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi_1^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathcal{F}^*\right].
$$

For $\varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}}]$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$, we obtain

$$
\begin{aligned}
\mu_c(\Psi_2(\mathbb{E})) &\leq 2\mu_c(\Psi_2(\mathbb{E}_0))\\
&= 2\max_{\varpi\in[0,c]}\mu(\Psi_2(\mathbb{E}_0)(\varpi))\\
&\leq 2\max_{\varpi\in[0,c]}\mu[\mathcal{G}_{\mathtt{k}}(\varpi,(\mathbb{E}_0)(\varpi))]\\
&\leq 2\varrho_{\mathtt{k}}\mu((\mathbb{E}_0)(\varpi))\\
&\leq 2\Xi_2^*\mu((\mathbb{E}_0)(\varpi)),
\end{aligned}
$$

where $\varrho_{\mathtt{k}} = 2\Xi_2^*$.

For $\varpi \in (\varepsilon_{\mathtt{k}}, \varpi_{\mathtt{k}+1}]$, $\mathtt{k} = 1, 2, \cdots, \mathbb{N}$, we obtain

$$
\begin{aligned}
\mu_c(\Psi_3(\mathbb{E})) &\leq 2\mu_c(\Psi_3(\mathbb{E}_0))\\
&= 2\max_{\varpi\in[0,c]}\mu(\Psi_3(\mathbb{E}_0)(\varpi))\\
&\leq 2\max_{\varpi\in[0,c]}\Bigg[\mu\Bigg(\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})[\mathcal{G}_{\mathtt{k}}(\varpi,\mathbb{E}_0(\varepsilon_{\mathtt{k}}))]+\hbar(\varpi,\mathbb{E}_0(\varpi))\\
&\quad+\int_0^{\varepsilon_{\mathtt{k}}}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathsf{u}_{\mathbb{E}_0}(\varepsilon_{\mathtt{k}})d\varepsilon_{\mathtt{k}}\\
&\quad+\int_0^{\varepsilon_{\mathtt{k}}}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon_{\mathtt{k}},\mathbb{E}_0(\varepsilon_{\mathtt{k}}))dW(\varepsilon_{\mathtt{k}})\\
&\quad+\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathsf{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon\\
&\quad+\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))dW(\varepsilon)\Bigg)\Bigg]\\
&\leq 2\max_{\varpi\in[0,c]}\Bigg[\mu\Bigg(\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})[\mathcal{G}_{\mathtt{k}}(\varpi,\mathsf{y}(\varepsilon_{\mathtt{k}}))]\Bigg)+\mu\Bigg(\hbar(\varpi,\mathbb{E}_0(\varpi))\Bigg)\\
&\quad+\mu\Bigg(\int_0^{\varepsilon_{\mathtt{k}}}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathsf{u}_{\mathbb{E}_0}(\varepsilon_{\mathtt{k}})d\varepsilon_{\mathtt{k}}\Bigg)\\
&\quad+\mu\Bigg(\int_0^{\varepsilon_{\mathtt{k}}}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathtt{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon_{\mathtt{k}},\mathbb{E}_0(\varepsilon_{\mathtt{k}}))dW(\varepsilon_{\mathtt{k}})\Bigg)\\
&\quad+\mu\Bigg(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathsf{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon\Bigg)\\
&\quad+\mu\Bigg(\int_0^{\varpi}\mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))dW(\varepsilon)\Bigg)\Bigg].
\end{aligned}
$$

$$\begin{aligned}
\|\mu_c(\Psi_3(\mathbb{E}))\| \leq\ & 2 \max_{\varpi\in[0,c]} \Bigg\| \mu\Bigg(\mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})[\mathcal{G}_{\mathrm{k}}(\varpi,\mathrm{y}(\varepsilon_{\mathrm{k}}))]\Bigg) + \mu\Bigg(\hbar(\varpi,\mathbb{E}_0(\varpi))\Bigg) \\
&+ \mu\Bigg(\int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon_{\mathrm{k}})d\varepsilon_{\mathrm{k}}\Bigg) \\
&+ \mu\Bigg(\int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon_{\mathrm{k}},\mathbb{E}_0(\varepsilon_{\mathrm{k}}))dW(\varepsilon_{\mathrm{k}})\Bigg) \\
&+ \mu\Bigg(\int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)d\varepsilon\Bigg) \\
&+ \mu\Bigg(\int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))dW(\varepsilon)\Bigg)\Bigg\| \\
\leq\ & 4 \max_{\varpi\in[0,c]} \Bigg\{ \Bigg\| \mathbf{S}_{\mathfrak{l},\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})\mu\Bigg([\mathcal{G}_{\mathrm{k}}(\varpi,\mathrm{y}(\varepsilon_{\mathrm{k}}))]\Bigg)\Bigg\| + \Bigg\|\mu\Bigg(\hbar(\varpi,\mathbb{E}_0(\varpi))\Bigg)\Bigg\| \\
&+ \Bigg\| \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mu\Bigg(\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon_{\mathrm{k}})\Bigg)d\varepsilon_{\mathrm{k}}\Bigg\| \\
&+ \Bigg\| \int_0^{\varepsilon_{\mathrm{k}}} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon_{\mathrm{k}})\alpha(\alpha I-\mathscr{A})^{-1}\mu\Bigg(\mathcal{F}(\varepsilon_{\mathrm{k}},\mathbb{E}_0(\varepsilon_{\mathrm{k}}))\Bigg)dW(\varepsilon_{\mathrm{k}})\Bigg\| \\
&+ \Bigg\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Bigg(\mathscr{B}\mathrm{u}_{\mathbb{E}_0}(\varepsilon)\Bigg)d\varepsilon\Bigg\| \\
&+ \Bigg\| \int_0^{\varpi} \mathbf{Q}_{\mathfrak{m}}(\varpi-\varepsilon)\alpha(\alpha I-\mathscr{A})^{-1}\mu\Bigg(\mathcal{F}(\varepsilon,\mathbb{E}_0(\varepsilon))\Bigg)dW(\varepsilon)\Bigg\|\Bigg\} \\
\leq\ & \frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}} \sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu((\mathbb{E}_0)(\varpi)) \\
&+ 4\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))] \\
&+ 4\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varepsilon_{\mathrm{k}}^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_4(\varpi)\Bigg[\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))] \\
&+ Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varepsilon_{\mathrm{k}}^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))]\Bigg] \\
&+ 4Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varepsilon_{\mathrm{k}}^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))] \\
&+ 4\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_4(\varpi)\Bigg[\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))] \\
&+ Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))]\Bigg] \\
&+ 4Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]} \mu(\mathbb{E}_0(\varpi))]
\end{aligned}$$

$$\leq \frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}} \sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu((\mathbb{E}_0)(\varpi)) + 4\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]$$
$$+ 8\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_4(\varpi)\left[\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]\right.$$
$$\left. + Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]\right]$$
$$+ 8Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]$$
$$\leq \frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}} \sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu((\mathbb{E}_0)(\varpi))$$
$$+ \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_4(\varpi)\right]4\Phi_1(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]$$
$$+ \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_4(\varpi)\right]Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\Phi_3(\varpi)[\sup_{\varpi\in(\varepsilon_{\mathrm{k}},\varpi_{\mathrm{k}+1}]}\mu(\mathbb{E}_0(\varpi))]$$
$$\leq \left\{\frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}}\right.$$
$$+ \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]4\hbar^*$$
$$\left. + \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathcal{F}^*\right\}\mu((\mathbb{E}_0)(\varpi))$$
$$\leq \left\{\frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}}\right.$$
$$\left. + \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]\left[4\hbar^* + Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathcal{F}^*\right\}\mu((\mathbb{E}_0)(\varpi))$$
$$\leq \Xi_3^*\mu((\mathbb{E}_0)(\varpi)),$$

where

$$\Xi_3^* = \left\{\frac{4M}{\Gamma(\mathfrak{l}(1-\mathfrak{m})+\mathfrak{m})}(\varpi-\varepsilon_{\mathrm{k}})^{\gamma-1}\varrho_{\mathrm{k}}\right.$$
$$\left. + \left[1+2\frac{MM_0M_{\mathscr{B}}M_{\mathscr{W}}}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathscr{W}^*\right]\left[4\hbar^* + Tr(Q)\frac{MM_0}{\Gamma(\mathfrak{m})}\left(\frac{\varpi^{\mathfrak{m}}}{\mathfrak{m}}\right)\mathcal{F}^*\right]\right\}.$$

$$\begin{aligned}\mu(\Psi\mathbb{E})(\varpi) &= \mu(\Psi_1\mathbb{E})(\varpi) + \mu(\Psi_2\mathbb{E})(\varpi) + \mu(\Psi_3\mathbb{E})(\varpi)\\ &\leq \Xi_1^*\mu((\mathbb{E}_0)(\varpi)) + \Xi_2^*\mu((\mathbb{E}_0)(\varpi)) + \Xi_3^*\mu((\mathbb{E}_0)(\varpi))\\ &\leq [\Xi_1^* + \Xi_2^* + \Xi_3^*]\mu((\mathbb{E}_0)(\varpi))\\ &\leq \Xi^*\mu((\mathbb{E}_0)(\varpi)).\end{aligned}$$

Thus, by Definition 1, Ψ is a μ_c-contraction operator. Hence, Ψ has at least one fixed-point from Theorem 1, and the mild solution also exists.
The results are proved. □

4. Examples

4.1. Example I

Consider the partial Hilfer fractional derivative system,

$$
\begin{aligned}
D_{0^+}^{\mathfrak{l},\mathfrak{m}}\left[\mathrm{y}(\varpi,\zeta)-\frac{|\mathrm{y}(\varpi,\zeta)|}{100}\right] &= \frac{\partial^2}{\partial \mathrm{y}^2}\left[\mathrm{y}(\varpi,\zeta)-\frac{|\mathrm{y}(\varpi,\zeta)|}{100}\right]+\mathscr{B}\mathrm{u}(\varpi)\\
&\quad+\frac{e^{-2\varpi}}{10(1+5e^{-\varpi})}\mathrm{y}(\varpi,\zeta)dW(\varpi), \quad \varpi\in(0,1/3]\cup(2/3,1],\\
\mathrm{y}(\varpi,\zeta) &= \frac{|\mathrm{y}(\varpi,\zeta)|}{5e^{2\varpi}}, \quad \varpi\in(1/3,2/3],\\
\mathrm{y}(\varpi,0) &= \mathrm{y}(\varpi,\pi)=0, \quad \varpi\in[0,\pi],\\
I_{0^+}^{1-\gamma}[\mathrm{y}(0)-\hbar(0,\mathrm{y}(0))] &= \mathrm{y}_0, \qquad (6)
\end{aligned}
$$

where $D_{0^+}^{\mathfrak{l},\mathfrak{m}}$ is the HFD of order $\mathfrak{l}=\frac{6}{10},\mathfrak{m}=\frac{1}{8}$. Let $W(\varpi)$ be a one-dimensional standard Brownian motion in $\mathscr{Y}$ represented by $\|\cdot\|_{\mathscr{Y}}$ on the filtered probability space $(\Sigma,\mathscr{E},\mathscr{P})$. Consider $\mathscr{Y}=\mathtt{C}([0,\pi],\mathbb{R})$ equipped with the uniform topology, and let the operator $\mathscr{A}:D(\mathscr{A})\subseteq\mathscr{Y}\to\mathscr{Y}$ be classified by

$$D(\mathscr{A})=\{\mathrm{y}\in\mathtt{C}^2([0,\pi],\mathbb{R}):\mathrm{y}(0)=\mathrm{y}(\pi)=0\},\ \mathscr{A}\mathrm{y}=\mathrm{y}''.$$

Then, we have

$$\overline{D(\mathscr{A})}=\{\mathrm{y}\in\mathtt{C}([0,\pi],\mathbb{R}):\mathrm{y}(0)=\mathrm{y}(\pi)=0\}\neq\mathscr{Y}.$$

As we know from [40], $\mathscr{A}$ fulfils the Hille–Yosida condition with $(0,\infty)\subseteq\rho(\mathscr{A})$ and for $\alpha>0$, $\|R(\alpha,\mathscr{A})\|\le\frac{1}{\alpha}$. Also, if $\mathscr{A}_0\mathrm{y}=\mathscr{A}\mathrm{y}$ is taken for $\mathrm{y}\in D(\mathscr{A})$, by Hille–Yosida condition, $\mathscr{A}_0$ produces a C_0-semigroup $\mathbf{T}(\cdot)$, which is evidenced by

$$\mathbf{T}(\varpi)\mathrm{y}=\sum_{n=1}^{\infty}e^{-n^2\varpi}\langle\mathrm{y},e_n\rangle e_n,$$

where $e_n=\sqrt{\frac{2}{\pi}}\sin(n\mathrm{y}),\ n\in\mathbb{N}$ is a complete orthonormal basis in $\mathscr{Y}$. Clearly, $\|\mathbf{T}(\varpi)\|\le1$.

Now, define an infinite dimensional space $\mathfrak{U}$ by

$$\mathfrak{U}=\left\{\mathrm{u}\mid\mathrm{u}=\sum_{n=2}^{\infty}\mathrm{u}_n e_n\text{ with }\sum_{n=2}^{\infty}\mathrm{u}_n^2<\infty\right\}.$$

We shall define a norm in $\mathfrak{U}$ by $\|\mathrm{u}\|_{\mathfrak{U}}=\left(\sum_{n=2}^{\infty}\mathrm{u}_n^2\right)^{\frac{1}{2}}$.

Define a mapping $\mathscr{B}\in L(\mathfrak{U},\mathscr{Y})$ as follows:

$$\mathscr{B}\mathrm{u}=2\mathrm{u}_2e_1+\sum_{n=2}^{\infty}\mathrm{u}_n e_n,\ \text{for }\mathrm{u}\in\mathfrak{U}.$$

Obviously, $\|\mathscr{B}\|_{L(\mathfrak{U},\mathscr{Y})}\le\sqrt{5}$.

Now, we represent the system (6) in the abstract form (1)–(3) by setting

$$
\begin{aligned}
\hbar(\varpi,\mathrm{y}(\varpi))(\zeta)&=\frac{1}{100}|\mathrm{y}(\varpi,\zeta)|,\\
\mathcal{F}(\varpi,\mathrm{y}(\varpi))(\zeta)&=\frac{e^{-2\varpi}}{10(1+5e^{-\varpi})}|\mathrm{y}(\varpi,\zeta)|,\\
\mathcal{G}_{\mathrm{k}}(\varpi,\mathrm{y}(\varpi))(\zeta)&=\frac{1}{5e^{2\varpi}}|\mathrm{y}(\varpi,\zeta)|.
\end{aligned}
$$

Then, for any bounded set $B_{\mathfrak{r}} \subseteq X$, we estimate

$$\|\hbar(\varpi, \mathrm{y}(\varpi))\| = \frac{1}{100}\|\mathrm{y}(\varpi,\zeta)\|,$$
$$\|\mathcal{F}(\varpi, \mathrm{y}(\varpi))\| = \frac{e^{-2\varpi}}{10(1+5e^{-\varpi})}\|\mathrm{y}(\varpi,\zeta)\|,$$
$$\|\mathcal{G}_{\mathrm{k}}(\varpi, \mathrm{y}(\varpi))\| = \frac{1}{5e^{2\varpi}}\|\mathrm{y}(\varpi,\zeta)\|.$$

Also, it is easy to verify that,

$$\mu(\hbar(\varpi, \mathbb{E}_0(\varpi))) = \frac{1}{100}\mu(\mathbb{E}_0),$$
$$\mu(\mathcal{F}(\varpi, \mathbb{E}_0(\varpi))) = \frac{e^{-2\varpi}}{10(1+5e^{-\varpi})}\mu(\mathbb{E}_0),$$
$$\mu(\mathcal{G}_{\mathrm{k}}(\varpi, \mathbb{E}_0(\varpi))) = \frac{1}{5e^{2\varpi}}\mu(\mathbb{E}_0).$$

Hence, we have that the functions $\hbar, \mathcal{F}, \mathcal{G}_{\mathrm{k}}$ satisfy the hypotheses $(H_2)-(H_4)$. Further, we assume that the linear operator $\mathscr{W} : L^2(\mathscr{V}, \mathfrak{U}) \to \mathscr{Y}$ defined by

$$\mathscr{W}\mathrm{u} = \lim_{\lambda\to\infty}\int_0^c \mathbf{Q}_{\frac{1}{8}}(c-\varepsilon)\mathscr{B}_\lambda \mathrm{u}(\varepsilon)d\varepsilon,$$

admits an invertible operator and satisfies (H_5).

Hence, all the requirements of Theorem 2 are fulfilled. Therefore, system (6) has at least one mild solution. Furthermore, system (6) can be steered from the initial state y_0 to the final state y_1. Thus, system (6) is controllable.

4.2. Example II

Consider the following partial non-instantaneous impulsive Hilfer fractional neutral stochastic evolution system of the form

$$\begin{aligned}
D_{0^+}^{\mathfrak{l},\mathfrak{m}}\left[\mathrm{y}(\varpi,\zeta) - \frac{\sin(\mathrm{y}(\varpi,\zeta))}{40}\right] &= \frac{\partial^2}{\partial \mathrm{y}^2}\left[\mathrm{y}(\varpi,\zeta) - \frac{\sin(\mathrm{y}(\varpi,\zeta))}{40}\right] + \mathscr{B}\mathrm{u}(\varpi) \\
&\quad + e^{-\varpi}\sin\varpi dW(\varpi), \quad \varpi \in (0,1/3] \cup (2/3,1], \\
\mathrm{y}(\varpi,\zeta) &= \frac{\cos\varpi|\mathrm{y}(\varpi,\zeta)|}{25+|\mathrm{y}(\varpi,\zeta)|}, \quad \varpi \in (1/3,2/3], \\
\mathrm{y}(\varpi,0) = \mathrm{y}(\varpi,\pi) &= 0, \quad \varpi \in [0,\pi], \\
I_{0^+}^{1-\gamma}[\mathrm{y}(0) - \hbar(0,\mathrm{y}(0))] &= \mathrm{y}_0, \qquad (7)
\end{aligned}$$

where $D_{0^+}^{\mathfrak{l},\mathfrak{m}}$ is the HFD of order $\mathfrak{l} = \frac{1}{2}, \mathfrak{m} = \frac{1}{8}$. Let $W(\varpi)$ be a one-dimensional standard Brownian motion in $\mathscr{Y}$ represented by $\|\cdot\|_{\mathscr{Y}}$ on the filtered probability space $(\Sigma, \mathscr{E}, \mathscr{P})$. Consider $\mathscr{Y} = \mathtt{C}([0,\pi], \mathbb{R})$ equipped with the uniform topology and the operator $\mathscr{A} : D(\mathscr{A}) \subseteq \mathscr{Y} \to \mathscr{Y}$ to be classified by

$$D(\mathscr{A}) = \{\mathrm{y} \in \mathtt{C}^2([0,\pi], \mathbb{R}) : \mathrm{y}(0) = \mathrm{y}(\pi) = 0\}, \; \mathscr{A}\mathrm{y} = \mathrm{y}''.$$

Then, we have

$$\overline{D(\mathscr{A})} = \{\mathrm{y} \in \mathtt{C}([0,\pi], \mathbb{R}) : \mathrm{y}(0) = \mathrm{y}(\pi) = 0\} \neq \mathscr{Y}.$$

We know from [40] that $\mathscr{A}$ fulfils the Hille–Yosida condition with $(0,\infty) \subseteq \rho(\mathscr{A})$ and for $\alpha > 0$, $\|R(\alpha,\mathscr{A})\| \leq \frac{1}{\alpha}$. Also, if $\mathscr{A}_0\mathrm{y} = \mathscr{A}\mathrm{y}$ is taken for $\mathrm{y} \in D(\mathscr{A})$, by Hille–Yosida condition, $\mathscr{A}_0$ produces a C_0-semigroup $\mathbf{T}(\cdot)$, which is evidenced by

$$\mathbf{T}(\varpi)\mathrm{y} = \sum_{n=1}^{\infty} e^{-n^2\varpi}\langle \mathrm{y}, e_n\rangle e_n,$$

where $e_n = \sqrt{\frac{2}{\pi}}\sin(n\mathrm{y}),\ n \in \mathbb{N}$ is a complete orthonormal basis in $\mathscr{Y}$.

Clearly, $\|\mathbf{T}(\varpi)\| \leq e^{-1} < 1 = M,\ \varpi \geq 0$.

Now, define an infinite dimensional space $\mathfrak{U}$ by

$$\mathfrak{U} = \left\{ \mathrm{u} \mid \mathrm{u} = \sum_{n=2}^{\infty} \mathrm{u}_n e_n \text{ with } \sum_{n=2}^{\infty} \mathrm{u}_n^2 < \infty \right\}.$$

We shall define a norm in $\mathfrak{U}$ by $\|\mathrm{u}\|_{\mathfrak{U}} = \left(\sum_{n=2}^{\infty} \mathrm{u}_n^2\right)^{\frac{1}{2}}$.

Define a mapping $\mathscr{B} \in L(\mathfrak{U},\mathscr{Y})$ as follows:

$$\mathscr{B}\mathrm{u} = 2\mathrm{u}_2 e_1 + \sum_{n=2}^{\infty} \mathrm{u}_n e_n, \text{ for } \mathrm{u} \in \mathfrak{U}.$$

Now, $\mathbb{E}$ is any bounded subset $B_{\mathfrak{r}} \subseteq X$. Define

$$\begin{aligned} \hbar(\varpi,\mathrm{y}(\varpi))(\zeta) = \hbar(\varpi,\mathrm{y}(\varpi,\zeta)) &= \frac{\sin(\mathrm{y}(\varpi,\zeta))}{40}, \\ \mathcal{F}(\varpi,\mathrm{y}(\varpi))(\zeta) = \mathcal{F}(\varpi,\mathrm{y}(\varpi,\zeta)) &= e^{-\varpi}\sin\varpi, \\ \mathcal{G}_{\mathrm{k}}(\varpi,\mathrm{y}(\varpi))(\zeta) = \mathcal{G}_{\mathrm{k}}(\varpi,\mathrm{y}(\varpi,\zeta)) &= \frac{\cos\varpi|\mathrm{y}(\varpi,\zeta)|}{25 + |\mathrm{y}(\varpi,\zeta)|}. \end{aligned}$$

Hence, we have that the functions $\hbar, \mathcal{F}, \mathcal{G}_{\mathrm{k}}$ satisfy the hypothesis $(H_2) - (H_4)$. Further, we assume that the linear operator $\mathscr{W} : L^2(\mathscr{V},\mathfrak{U}) \to \mathscr{Y}$ defined by

$$\mathscr{W}\mathrm{u} = \lim_{\lambda\to\infty} \int_0^c \mathbf{Q}_{\frac{1}{8}}(c-\varepsilon)\mathscr{B}_\lambda \mathrm{u}(\varepsilon)d\varepsilon,$$

admits an invertible operator and satisfies (H_5).

Hence, all the requirements of Theorem 2 are fulfilled. Therefore, system (7) has at least one mild solution. Furthermore, system (7) can be steered from the initial state y_0 to the final state y_1. Thus, system (7) is controllable.

5. Conclusions

This manuscript deals with the controllability results for NII HF neutral stochastic evolution equations, which are defined in the non-dense domain. The primary outcomes are obtained by employing semigroup theory, fractional calculus, stochastic analysis, and the fixed-point theorem. At the end, we provided an illustration to explain our results. In the future, we will investigate the optimal control of the Sobolev-type hemivariational stochastic HF NII differential system with Poisson jumps and a non-dense domain.

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Abbreviations

The following abbreviations are used in this manuscript:

FDEs	Fractional differential equations
R-L	Riemann–Liouville
HF	Hilfer fractional
HFD	Hilfer fractional derivative
SDEs	stochastic differential equations
NII	non-instantaneous impulse

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