fractal and fractional

Article

Existence of Solutions for the Initial Value Problem with Hadamard Fractional Derivatives in Locally Convex Spaces

Weiwei Liu † and **Lishan Liu** *,†

School of Mathematics Sciences, Qufu Normal University, Qufu 273165, China; liuww1019@qfnu.edu.cn
* Correspondence: lls@qfnu.edu.cn
† These authors contributed equally to this work.

Abstract: In this paper, we investigate an initial value problem for a nonlinear fractional differential equation on an infinite interval. The differential operator is taken in the Hadamard sense and the nonlinear term involves two lower-order fractional derivatives of the unknown function. In order to establish the global existence criteria, we first verify that there exists a unique positive solution to an integral equation based on a class of new integral inequality. Next, we construct a locally convex space, which is metrizable and complete. On this space, applying Schäuder's fixed point theorem, we obtain the existence of at least one solution to the initial value problem.

Keywords: Hadamard fractional differential equation; initial value problem; locally convex space; integral inequality

Citation: Liu, W.; Liu, L. Existence of Solutions for the Initial Value Problem with Hadamard Fractional Derivatives in Locally Convex Spaces. *Fractal Fract.* **2024**, *8*, 191. https://doi.org/10.3390/fractalfract8040191

Academic Editors: Ricardo Almeida, Lihong Zhang, Ravi P. Agarwal, Bashir Ahmad and Guotao Wang

Received: 25 February 2024
Revised: 18 March 2024
Accepted: 22 March 2024
Published: 27 March 2024

1. Introduction

For the purpose of problem-solving in many different domains, such as engineering and control, fractional differential equations, or FDEs, are indispensable tools. Fractional derivatives of the Hadamard, Caputo, Riemann–Liouville, and other varieties are the subject of numerous studies on fractional differential equations. The boundary value and initial value problems for nonlinear fractional equations have been extensively researched. Most of the findings in this field relate to establishing the uniqueness and existence of positive solutions on finite intervals.

The study of initial value problems is a direction that cannot be overlooked in articles dealing with the existence of solutions to fractional differential equations on infinite intervals (see [1–12]).

Take the nonlinear fractional differential equation below as an example

$$D_{0+}^{\alpha}x(t) = f(t, x(t)),\ \ \alpha \in (0,1),\ t \in (0,\infty), \tag{1}$$

where D_{0+}^{α} denotes the Riemann–Liouville fractional derivative. By constructing a special Banach space

$$E = \Big\{x(t)\ \big|\ x(t) \in C_{1-\alpha}(\mathbb{R}^+), \lim_{t\to\infty}\frac{t^{1-\alpha}x(t)}{1+t^2} = 0\Big\}, \tag{2}$$

Kou et al. [1] employed fixed point theorems to obtain the global existence of solutions for Equation (1) supplemented with the initial condition of the form

$$\lim_{t\to 0+} t^{1-\alpha}x(t) = x_0 \tag{3}$$

on $[0,\infty)$. In [11,12], Zhu applied some new fractional integral inequalities to study global existence results for the fractional differential Equation (1) with the initial condition (3). In [3], for the initial value problem (1), (3) was proved to have solutions in $C_{1-\alpha}(\mathbb{R}^+)$ by constructing a special locally convex space and utilizing Schauder's fixed point theorem. Also, for this initial value problem, by using a Bielecki type norm and the Banach fixed point theorem, Tuan et al. [7] proved a Picard–Lindelöf-type theorem on the existence and uniqueness of global solutions.

Zhang and Hu [9] considered the unique existence of an approximate solution to the following initial value problem

$$\begin{cases} D_{0+}^{p(t)}x(t) = f(t, x(t), D_{0+}^{q(t)}),\ 0 < t < \infty, \\ x(0) = 0, \end{cases}$$

where $0 < q(t) < p(t) < 1, D_{0+}^{p(t)}, D_{0+}^{p(t)}$ denote derivatives of variable order $p(t)$ and $q(t)$.

Zhu et al. [6] investigated the existence results for fractional differential equations of the form

$$\begin{cases} D_c^q x(t) = f(t, x(t)) \quad t \in [0, T)(0 < T \leq \infty),\ q \in (1, 2), \\ x(0) = a_0,\ x'(0) = a_1, \end{cases}$$

where D_c^q is a Caputo derivative and f satisfies the Caratheodory condition. Using a fixed point theorem introduced by O'Regan in [13], the authors proved the existence of solutions for the above initial value problem in $C[0, T)$.

In [8], Boucenna et al. studied the following inital value problem of a nonlinear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)), t \in J = (0, \infty), \\ D_{0+}^{\alpha-1}x(0) = x_0, I_{0+}^{2-\alpha}x(0) = x_1, \end{cases}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative of order α, $x_0, x_1 \in \mathbb{R}, 1 < \alpha \leq 2$, $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$. The existence and uniqueness of the solutions were obtained through some fixed point theorems in Sobolev space.

By taking the existing ideas of some of the above articles, we now discuss the existence of solutions for the following initial value problem for the Hadamard fractional differential equation:

$$\begin{cases} {}^H\mathcal{D}_{1+}^{\alpha}x(t) = f(t, x(t), {}^H\mathcal{D}_{1+}^{\beta}x(t), {}^H\mathcal{D}_{1+}^{\nu}x(t)), \quad 1 < t < +\infty, \\ {}^H\mathcal{D}_{1+}^{\alpha-1}x(1) = x_0, \\ {}^H\mathcal{J}_{1+}^{2-\alpha}x(1) = x_1, \end{cases} \tag{4}$$

where $1 < \alpha < 2, 0 < \beta \leq \alpha - 1 < \nu < \alpha$, $f : J \times \mathbb{R}^2 \to \mathbb{R}, J = (1, +\infty)$ and f may be singular at $t = 1$. ${}^H\mathcal{D}_{1+}^{\alpha}$ denotes the Hadamard fractional derivative of order α and is defined by

$$\begin{aligned} {}^H\mathcal{D}_{1+}^{\alpha}g(t) =& \delta^n({}^H\mathcal{J}_{1+}^{n-\alpha}g)(t) \\ =& \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} g(s)\frac{ds}{s},\ \ n-1 < \alpha < n, n \in \mathbb{N}, \\ {}^H\mathcal{D}_{1+}^{\alpha}g(t) =& (\delta^n g)(t),\ \ \alpha = n \in \mathbb{N}, \end{aligned}$$

and ${}^H\mathcal{J}_{1+}^{n-\alpha}$ is the Hadamard fractional integral of order $n - \alpha$, where $\delta = t\frac{d}{dt}$, $\ln(\cdot) = \log_e(\cdot)$, $\mathbb{N}$ denotes the set of positive integers. We study the existence of one solution to the initial value problem (4) in a weighted function space defined as $C_{\lambda,\ln}(J) = \{x \in C(J) | \lim_{t\to 1+}(\ln t)^{\lambda}x(t) \text{ exists}\}$.

According to previous studies, the existence of global solutions of differential equations on infinite intervals is based on two ideas: one is to construct a new function space to obtain the boundedness on infinite intervals, and then to use fixed point theory to obtain the existence of solutions of differential equations (see [1,4,5,8].) The other is to first study the existence of solutions of the differential equation on a finite interval, that is, the existence of local solutions, and then to expand the solutions to infinite intervals in combination with the continuous theorem (see [2,6,7,11,12]).

As stated in [2], the continuation theorems for nonlinear FDEs have not been derived yet. Thus, it is inconvenient, even impossible, to obtain the global existence of solutions by directly using the results on the local existence. In order to prove the existence of global solutions, continuation theorems for the nonlinear fractional initial value problems must be proved.

Integral inequalities play a significant role in discussions of the quantitative and qualitative behavior (such as boundedness, uniqueness, stability, and continuous dependence on the initial or boundary value and parameters of solutions) of solutions to differential equations, integral equations, and difference equations. These inequalities are being studied by an increasing number of scholars due to their richness, and they have been generalized, altered, and expanded in a wide range of ways, as can be seen in [11,12,14–16]. To the best of our knowledge, inequalities with the Hadamard fractional integral have been studied less frequently in the past.

The initial value problem (4) differs from the initial value problems in the references [6,8]. The nonlinear term of the differential equation studied in [6] does not contain any derivatives of lower order, while the nonlinear term of the equation in [8] has only one special $\alpha-1$ order derivative $D^{\alpha-1}_{1+}$. There is also the fact that the conditions in this paper are weaker relative to the literature [6,8].

In Section 2, we prove a weakly singular inequality of the Hadamard fractional integral type with a doubly singular kernel. Avoiding utilizing function spaces like (2) in Section 3, we build a locally convex space which endows the whole space $C_{2-\alpha,\ln}(J)$ with the topology induced by a sufficient family of semi-norms, and introduce some properties in this space, in accordance with the idea of [3]. The inequality in Section 2 allows us to prove the existence and uniqueness of the positive solution to the linear integral equation in Section 4, after which we identify the existence of one solution to the initial value problem in the space generated in Section 3.

2. Some Preliminaries and Lemmas

In this section, we present the preliminary results needed in our proofs later. From here on, for a non-negative real number β, we use ϑ_β to denote the function defined on J or $J_1 = [1, \infty)$ by $\vartheta_\beta(t) = (\ln t)^{\beta-1}/\Gamma(\beta)$. First of all, we list some basic lemmas about Hadamard fractional derivatives and integrals.

Lemma 1 ([17,18])**.** *For $\alpha > 0, n = [\alpha] + 1$ and $x \in C(J) \cap L^1(J)$, the solution of the Hadamard fractional differential equation ${}^H\mathcal{D}^\alpha_{1+}x(t) = 0$ is*

$$x(t) = \sum_{i=1}^{n} c_i (\ln t)^{\alpha-i},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$.

Lemma 2 ([17,18])**.** *If $u \in C(J)$ and ${}^H\mathcal{D}^\alpha_{1+}u \in L^1(J)$, then*

$${}^H\mathcal{J}^\alpha_{1+}({}^H\mathcal{D}^\alpha_{1+}u)(t) = u(t) + c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \cdots + c_n(\ln t)^{\alpha-n},$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n), n = [\alpha] + 1$.

Lemma 3 ([17,18])**.** *Let $\alpha > 0$. If $u \in L^1(J)$, then the equality ${}^H\mathcal{D}^\alpha_{1+}({}^H\mathcal{J}^\alpha_{1+}u)(t) = u(t)$ holds a.e. on J.*

Lemma 4 ([17,18])**.** *If $\beta, \gamma > 0$, then*

(1) ${}^H\mathcal{J}^\gamma_{1+}\vartheta_\beta(t) = \vartheta_{\beta+\gamma}(t)$.

(2) ${}^{H}\mathcal{D}^{\gamma}_{1+}\vartheta_{\beta}(t) = \vartheta_{\beta-\gamma}(t)$, *provided that* $\beta - \gamma > 0$.
(3) ${}^{H}\mathcal{D}^{\gamma}_{1+}\vartheta_{\gamma-j+1}(t) = 0, n-1 < \gamma < n, j = 1, 2, \dots, n$.
(4) ${}^{H}\mathcal{D}^{\beta}_{1+}\,{}^{H}\mathcal{J}^{\gamma}_{1+}u(t) = {}^{H}\mathcal{J}^{\gamma-\beta}_{1+}u(t)$, *provided that* $\gamma - \beta > 0$.

Lemma 5 ([19]). *Let $n-1 < \alpha < n, n \in \mathbb{N}$ and $I = [a, b]$ be a finite or infinite interval. Assume that $\{f_k\}_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a, b]$ and ${}^{H}\mathcal{D}^{\alpha}_{1+}f_k$ exist for every k. Moreover assume that $\{{}^{H}\mathcal{D}^{\alpha}_{1+}f_k\}_{k=1}^{\infty}$ converge uniformly on $[a+\epsilon, b]$ for every $\epsilon > 0$. Then, for every $x \in [a, b]$, we have $\lim\limits_{k\to\infty} {}^{H}\mathcal{D}^{\alpha}_{1+}f_k(x) = {}^{H}\mathcal{D}^{\alpha}_{1+} \lim\limits_{k\to\infty} f_k(x)$.*

The following inequality in Lemma 6 plays an important role in proving the uniqueness of the solution to the integral Equation (20) corresponding to the initial value problem (Theorem 7 in Section 4). The method of proof we employ is similar in concept to that of [15] to obtain the following inequality involving an integral with a doubly singular kernel.

Lemma 6. *Suppose that $a, b < 1$ and $a + b > 1$, $q(t), g(t) \in L^{\infty}[1, T]$ are non-negative. If $u(t) \in L^{\infty}[1, T](T > 1)$ is non-negative and satisfies*

$$u(t) \leq q(t) + g(t)\int_1^t \left(\ln\frac{t}{s}\right)^{a-1}(\ln s)^{b-1}u(s)\frac{ds}{s},\ \textit{for a.e. } t \in [1, T]. \tag{5}$$

Then,

$$u(t) \leq \frac{q^*(t)b}{a+b-1}\exp\left(\frac{(g^*(t))^{\frac{b}{a+b-1}}}{a+b-1}\left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}}(\ln t)^b\right),\ \textit{for a.e. } t \in [1, T], \tag{6}$$

where $q^*(t) = \sup\limits_{s\in[1,t]} q(s), g^*(t) = \sup\limits_{s\in[1,t]} g(s)$.

Proof of Lemma 6. First, suppose that $q(t) \equiv q, g(t) \equiv g$ are constants; if the non-negative function $u \in L^{\infty}[1, T]$ and satisfies $u(t) \leq q + g\int_1^t (\ln\frac{t}{s})^{a-1}(\ln s)^{b-1}u(s)\frac{ds}{s}$, we claim

$$u(t) \leq \frac{q}{1-r_1}\exp\left(\frac{g(\ln(t_g(r_1)))^{a-1}}{b(1-r_1)}(\ln t)^b\right), \tag{7}$$

where $t_g(r) = \exp\left(((gB(a,b))^{-1}r)^{\frac{1}{a+b-1}}\right)$, $r \geq 0$, $r_1 \leq r_0$, $r_1 < 1$ and $r_0 = gB(a,b)(\ln T)^{a+b-1}$ is fixed. Since $t_g(r)$ is increasing and $t_g(r_0) = T$, then $1 < t_g(r_1) \leq T$.

Let $v(t) = q + g\int_1^t (\ln\frac{t}{s})^{a-1}(\ln s)^{b-1}u(s)\frac{ds}{s}$, then $v \in C[1, T]$, $u(t) \leq v(t)$, and

$$v(t) \leq q + g\int_1^t \left(\ln\frac{t}{s}\right)^{a-1}(\ln s)^{b-1}v(s)\frac{ds}{s},\ t \in [1, T].$$

We need to prove the conclusion (7) holds with v replacing u; it implies that it suffices to suppose that $u \in C[1, T]$. Denote $u^*(t) = \sup\limits_{1\leq s\leq t} u(s), t \in [1, T]$. Let $t \in (1, T]$, $\varsigma \in (1, t]$ is chosen arbitrarily. If $\varsigma \leq t_g(r_1)$, then,

$$\begin{aligned} u(\varsigma) \leq & q + g\int_1^{\varsigma}\left(\ln\frac{\varsigma}{s}\right)^{a-1}(\ln s)^{b-1}u(s)\frac{ds}{s} \\ \leq & q + gB(a,b)(\ln(t_g(r_1)))^{a+b-1}u^*(t) = q + r_1 u^*(t). \end{aligned}$$

If $t_g(r_1) \leq \varsigma \leq t$, we have

$$\begin{aligned} u(\varsigma) \leq & q + g\int_1^{\frac{\varsigma}{t_g(r_1)}} \left(\ln\frac{\varsigma}{s}\right)^{a-1} (\ln s)^{b-1} u(s)\frac{ds}{s} + g\int_{\frac{\varsigma}{t_g(r_1)}}^{\varsigma} \left(\ln\frac{\varsigma}{s}\right)^{a-1} (\ln s)^{b-1} u(s)\frac{ds}{s} \\ \leq & q + g(\ln(t_g(r_1)))^{a-1} \int_1^{\frac{\varsigma}{t_g(r_1)}} (\ln s)^{b-1} u^*(s)\frac{ds}{s} \\ & + g\int_{\frac{\varsigma}{t_g(r_1)}}^{\varsigma} \left(\ln\frac{\varsigma}{s}\right)^{a-1} \left(\ln s - \ln\frac{\varsigma}{t_g(r_1)}\right)^{b-1} u^*(t)\frac{ds}{s} \\ \leq & q + g(\ln(t_g(r_1)))^{a-1} \int_1^{\varsigma} (\ln s)^{b-1} u^*(s)\frac{ds}{s} + gB(a,b)(\ln(t_g(r_1)))^{a+b-1} u^*(t) \\ \leq & q + g(\ln(t_g(r_1)))^{a-1} \int_1^{t} (\ln s)^{b-1} u^*(s)\frac{ds}{s} + r_1 u^*(t). \end{aligned}$$

We have arrived at the following conclusion after synthesizing the above findings,

$$u(\varsigma) \leq q + g(\ln(t_g(r_1)))^{a-1} \int_1^{t} (\ln s)^{b-1} u^*(s)\frac{ds}{s} + r_1 u^*(t), \varsigma \in (1,t].$$

Taking the supremum for $\varsigma \in [1,t]$, we obtain

$$u^*(t) \leq \frac{q}{1-r_1} + \frac{g(\ln(t_g(r_1)))^{a-1}}{1-r_1} \int_1^{t} (\ln s)^{b-1} u^*(s)\frac{ds}{s}.$$

By using the classical Gronwall's inequality, we have

$$u(t) \leq u^*(t) \leq \frac{q}{1-r_1}\exp\left(\frac{g(\ln(t_g(r_1)))^{a-1}}{b(1-r_1)}(\ln t)^b\right), \text{ for a.e. } t \in [1,T].$$

In (7), the parameter r_1 is indefinite; we then attempt to choose an "optimal" parameter to guarantee that the inequality holds and that the term $\exp\left(\frac{g(\ln(t_g(r_1)))^{a-1}}{b(1-r_1)}(\ln t)^b\right)$ is as small as possible. Let

$$\kappa(r) = (1-r)(\ln(t_g(r)))^{1-a} = (1-r)((gB(a,b))^{-1}r)^{\frac{1-a}{a+b-1}},\ r_* = \frac{1-a}{b}.$$

By calculation, we have $\kappa'(r_*) = 0$. r_* is a maximum of $\kappa(r)$ and $r_* < 1$. If $r_* \leq r_0$, that is $gB(a,b)(\ln T)^{a+b-1} \geq \frac{1-a}{b}$, then r_* is the 'optimal' parameter and we obtain

$$\begin{aligned} u(t) \leq & \frac{qb}{a+b-1}\exp\left(\frac{g(\ln(t_g(r_*)))^{a-1}}{a+b-1}(\ln t)^b\right) \\ = & \frac{qb}{a+b-1}\exp\left(\frac{g^{\frac{b}{a+b-1}}}{a+b-1}\left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}}(\ln t)^b\right), \text{ for a.e. } t \in [1,T]. \end{aligned} \quad (8)$$

If $r_* > r_0$, or $gB(a,b)(\ln T)^{a+b-1} < \frac{1-a}{b}$, let $t \in [1,T]$ and choose an arbitrary $\varsigma \in [1,t]$,

$$\begin{aligned} u(\varsigma) \leq & q + g\int_1^{\varsigma} \left(\ln\frac{\varsigma}{s}\right)^{a-1} (\ln s)^{b-1} u(s)\frac{ds}{s} \\ \leq & q + gB(a,b)(\ln T)^{a+b-1} u^*(t) \\ < & q + \frac{1-a}{b} u^*(t). \end{aligned}$$

Taking the supremum for $\varsigma \in [1,t]$ gives $u(t) \leq u^*(t) \leq \frac{qb}{a+b-1},\ t \in [1,T]$. Actually, this is a stronger conclusion than (8).

Lastly, we will show that the conclusion also holds for the general case. For any $\tau \in (1, T]$,

$$u(t) \le q^*(\tau) + g^*(\tau) \int_1^t \left(\ln \frac{t}{s}\right)^{a-1} (\ln s)^{b-1} u(s) \frac{ds}{s}, \text{ for a.e. } t \in [1, \tau].$$

According to the above conclusion, (8) is satisfied in $[1, \tau]$, that is

$$u(t) \le \frac{q^*(\tau) b}{a+b-1} \exp\left(\frac{(g^*(\tau))^{\frac{b}{a+b-1}}}{a+b-1} \left(\frac{1-a}{bB(a,b)} \right)^{\frac{a-1}{a+b-1}} (\ln t)^b \right), \text{ for a.e. } t \in [1, \tau],$$

therefore,

$$u^*(\tau) \le \frac{q^*(\tau) b}{a+b-1} \exp\left(\frac{(g^*(\tau))^{\frac{b}{a+b-1}}}{a+b-1} \left(\frac{1-a}{bB(a,b)} \right)^{\frac{a-1}{a+b-1}} (\ln \tau)^b \right).$$

Since τ is arbitrary in $(1, T]$, then we obtain the conclusion. □

3. The Locally Convex Space

For readers' convenience, first, some basic concepts and properties of locally convex and topological spaces are briefly reviewed (see the monographs [20–23] for further details).

Definition 1 ([20,21]). *If $(X, \mathcal{T})$ is a topology space, a base for $\mathcal{T}$ is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{T} = \{ \bigcup_{B \in \mathcal{G}} B \mid \mathcal{G} \subseteq \mathcal{B} \}$.*

Lemma 7 ([20,21]). *Suppose that X is a non-empty set, $\mathcal{B} \subseteq 2^X$, if $\mathcal{B}$ satisfies*

(1) $X = \bigcup_{B \in \mathcal{B}} B$.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \bigcap B_2$.

Then, there is a unique topology with $\mathcal{B}$ as the topological base.

Definition 2 ([20,21]). *A topological space X is said to be Hausdorff if whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.*

Definition 3 ([20,21]). *A topological space (X, τ) is metrizable if the topology τ is the metric topology τ_ρ for some metric ρ on X.*

Definition 4 ([22,23]). *A real linear topological space (LTS) is a real linear space (vector space) X together with a topology such that, with respect to this topology,*

(1) the map of $X \times X \to X$ defined by $(x, y) \mapsto x + y$ is continuous;

(2) the map of $\mathbb{R} \times X \to X$ defined by $(\alpha, x) \mapsto \alpha x$ is continuous.

Definition 5 ([23]). *A locally convex space (LCS) is an LTS, whose topology is defined by a family of semi-norms $\mathcal{P}$ such that $\bigcap_{p \in \mathcal{P}} \{x \mid p(x) = 0\} = \{0\}$.*

Lemma 8 ([22]). *X is an LTS, $\{p_1, p_2, \ldots\}$ be a sequence of semi-norms on X, such that $\bigcap_{n=1}^{\infty} \{x \mid p_n(x) = 0\} = \{0\}$. For x and y in X, define*

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)}.$$

Then, d is metric on X and the topology on X defined by d is the topology on X defined by the semi-norms $\{p_1, p_2, \ldots\}$. Thus, X is metrizable if, and only if, its topology is determined by a countable family of semi-norms.

From here up to Theorem 5, we always suppose that $0 < \beta < \alpha - 1$. Define a function space as follows:

$$X = \left\{ x \;\middle|\; \begin{array}{r} x, {}^H\mathcal{D}^{\beta}_{1+}x, {}^H\mathcal{D}^{\nu}_{1+}x \in C(J), \lim\limits_{t\to 1+} (\ln t)^{2-\alpha} x(t), \lim\limits_{t\to 1+} (\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}x(t) \\ \text{and } \lim\limits_{t\to 1+} (\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}x(t) \text{ exist} \end{array} \right\}.$$

Consider the family $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ of semi-norms on X, where $p_n : X \to [0, \infty)$ is defined by

$$\begin{aligned} p_n(x) = &\sup_{t\in(1,1+n]} (\ln t)^{2-\alpha}|x(t)| + \sup_{t\in(1,1+n]} (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}x(t)| \\ &+ \sup_{t\in(1,1+n]} (\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}x(t)|. \end{aligned}$$

For every $x \in X$, set $U_{x,n,\varepsilon} := \{y \in X \mid p_n(y - x) < \varepsilon\}$, where $\varepsilon > 0, n \in \mathbb{N}$. All of the finite intersection of elements of $\{U_{x,n,\varepsilon}\}$ form a collection $\mathcal{B}$, that is

$$\mathcal{B} = \{ \bigcap_{\substack{n\in N_x \\ \varepsilon\in S_x}} U_{x,n,\varepsilon} \mid x \in X, N_x \subseteq \mathbb{N}, S_x \subseteq (0, \infty) \text{ have the same finite cardinality}\}.$$

The function space denoted as X will be referenced in Section 4, where we will establish the existence of solutions to the initial value problem (4) on a specific subset of this space. As a result, the subsequent analysis will concentrate on investigating pertinent characteristics of this particular function space.

Theorem 1. *There exists a unique topology $\mathcal{T}$ such that $\mathcal{B}$ is a base for the topology.*

Proof of Theorem 1. The condition (1) in Lemma 7 is clearly satisfied. Next, we will show the condition (2) also holds. For any $B_1, B_2 \in \mathcal{B}$, where

$$B_i = \bigcap_{j=1}^{m_i} (U_{x_i,n_{ij},\varepsilon_{ij}}),\ n_{ij} \in N_i \subseteq \mathbb{N}, \varepsilon_{ij} \in S_i \subseteq \mathbb{R}^+ \text{ have the same finite cardinality } m_i (i = 1, 2).$$

Suppose $x \in B_1 \bigcap B_2$, then $p_{n_{ij}}(x - x_i) < \varepsilon_{ij}, i = 1, 2, j = 1, 2, \ldots, m_1/m_2$. Choose a number ε satisfying

$$0 < \varepsilon \le \min\{\varepsilon_{ij} - p_{n_{ij}}(x - x_i), i = 1, 2, j = 1, 2, \ldots, m_1/m_2\}.$$

Define $B_3 = \bigcap\limits_{n_\mu\in(N_1 \bigcup N_2)} (U_{x,n_\mu,\varepsilon})$, then $x \in B_3 \in \mathcal{B}$. This implies that $B_3 \subseteq B_1 \bigcap B_2$. For each $y \in B_3$, then $p_{n_\mu}(y - x) < \varepsilon, n_\mu \in N_1 \bigcup N_2$. Therefore, for $i = 1, 2, j = 1, 2, \ldots, m_1/m_2$,

$$p_{n_{ij}}(y - x_i) \le p_{n_{ij}}(y - x) + p_{n_{ij}}(x - x_i) < \varepsilon + p_{n_{ij}}(x - x_i) < \varepsilon_{ij}.$$

Hence, $y \in B_1 \bigcap B_2$.

Define

$$\mathcal{T} = \{U \subset X \mid \text{there is a subset } \mathcal{B}_U \subseteq \mathcal{B} \text{ such that } U = \bigcup_{B\in\mathcal{B}_U} B\}.$$

It is clear that $\mathcal{T}$ is a topology with $\mathcal{B}$ as the topological base. The proof of uniqueness can be obtained directly by the definition. □

Remark 1. *In accordance with the theorem, a subset U of X is an open set if, and only if, for every point x_0 in U, there exist $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $\bigcap_{j=1}^{n} \{x \in X \mid p_n(x - x_0) < \varepsilon_j\} \subseteq U$. Moreover, for any $x \in X$, the family $\mathcal{B}_x = \{B_{x,n,\varepsilon} \mid n \in \mathbb{N}, \varepsilon > 0\}$ consisting of ball $B_{x,n,\varepsilon} = \{y \in X \mid p_n(y - x) \leq \varepsilon\}$ is a neighborhood-base at x with respect to $\mathcal{T}$. Therefore, a set V is a neighborhood of x with respect to $\mathcal{T}$ if and only if there exist $n \in \mathbb{N}, \varepsilon > 0$ such that $B_{x,n,\varepsilon} \subseteq V$.*

Theorem 2. *$(X, \mathcal{T})$ is Hausdorff, LCS, and metrizable.*

Proof of Theorem 2. We claim that the conclusion $\bigcap_{n=1}^{\infty} \{x \in X \mid p_n(x) = 0\} = \{0\}$ is sustained. If not, there exists $x_0 \in X, x_0 \neq 0$ such that $p_n(x_0) = 0, \forall n \in \mathbb{N}$, i.e., $\sup_{t \in (1,1+n]} (\ln t)^{2-\alpha} |x_0(t)| = 0$. Then, $x_0(t) = 0, t \in (1, n+1], \forall n \in \mathbb{N}$. Hence, $x_0(t) = 0, t \in (1, \infty)$, which is a contradiction. In light of this, for any $x, y \in X$ and $x \neq y$, there exists $p_n \in \mathcal{P}$ such that $p_n(x - y) \neq 0$. Choose a positive number ϵ satisfying $p_n(x - y) > \epsilon > 0$. Set $U = \{z \mid p_n(x - z) < \frac{\epsilon}{2}\}, V = \{z \mid p_n(y - z) < \frac{\epsilon}{2}\}$; it is apparent that U, V are open sets containing x and y, respectively, and $U \cap V = \varnothing$. Therefore, we know the topology must be Hausdorff.

To prove that $(X, \mathcal{T})$ is an LCS, by Definitions 4 and 5, it suffices to prove $(X, \mathcal{T})$ is an LTS. From the properties of the semi-norm, we can reach the conclusion that the vector space operations (addition and scalar multiplication) are continuous with respect to the topology $\mathcal{T}$.

Define a metric ρ on X by

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)},$$

let $\mathcal{T}_\rho$ be the topology induced by ρ. According to Lemma 8, we know $\mathcal{T}_\rho$ coincides with $\mathcal{T}$. By Theorem 1, the topology is generated by a countable family of semi-norms; it follows that topology $\mathcal{T}$ is metrizable. □

Theorem 3. *A sequence $\{u_k\} \subseteq X$ converges to 0 with respect to $\mathcal{T}$ if, and only if, it satisfies the following conditions:*

(i) $\{u_k\}, \{{}^H\mathcal{D}_{1+}^{\beta} u_k\}, \{{}^H\mathcal{D}_{1+}^{\nu} u_k\}$ *converge uniformly to 0 on any compact set $I \subseteq J$.*

(ii) $\{(\ln t)^{2-\alpha} u_k(t)\}, \{(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}_{1+}^{\beta} u_k(t)\}, \{(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}_{1+}^{\nu} u_k(t)\}$ *converge to 0 when $t \to 1+, k \to \infty$ uniformly with respect to $\mathcal{T}$, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0, t \in (1, 1+\delta)$,*

$$|(\ln t)^{2-\alpha} u_k(t)| \leq \frac{\varepsilon}{3}, \ |(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}_{1+}^{\beta} u_k(t)| \leq \frac{\varepsilon}{3}, \ |(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}_{1+}^{\nu} u_k(t)| \leq \frac{\varepsilon}{3}. \quad (9)$$

Proof of Theorem 3. We first show that the sufficiency holds. Suppose that $\{u_k\} \subseteq X$ satisfies $(i), (ii)$. Let V be an arbitrary neighbourhood of 0 with respect to $\mathcal{T}$. From Remark 1, we have $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $B_{0,n,\varepsilon} \subseteq V$. Due to the condition (ii), we choose $k_0 \in \mathbb{N}, \delta > 0$ such that (9) holds.

For $\delta > n$, then for all $k \geq k_0$, we have

$$\begin{aligned} p_n(u_k) = & \sup_{t \in (1,1+n]} (\ln t)^{2-\alpha} |u_k(t)| + \sup_{t \in (1,1+n]} (\ln t)^{2+\beta-\alpha} |{}^H\mathcal{D}_{1+}^{\beta} u_k(t)| \\ & + \sup_{t \in (1,1+n]} (\ln t)^{2+\nu-\alpha} |{}^H\mathcal{D}_{1+}^{\nu} u_k(t)| \leq \varepsilon, \end{aligned}$$

then, we obtain $u_k \in B_{0,n,\varepsilon} \subseteq V$ for all $k \geq k_0$.

For $\delta \leq n$, let $I = [\delta, n], \varepsilon' = \min\{\frac{\varepsilon}{3}(\ln n)^{\alpha-2}, \frac{\varepsilon}{3}(\ln n)^{\alpha-\beta-2}, \frac{\varepsilon}{3}(\ln n)^{\alpha-\nu-2}\}$. According to the condition (i), there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1, t \in I$, we obtain

$$|u_k(t)| \leq \varepsilon', \ |{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq \varepsilon', \ |{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq \varepsilon'.$$

Consequently, for all $k \geq k_1$ and $t \in I$, we infer

$$(\ln t)^{2-\alpha}|u_k(t)| \leq (\ln t)^{2-\alpha}\varepsilon' \leq (\ln n)^{2-\alpha}\varepsilon' \leq \frac{\varepsilon}{3},$$

$$(\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq (\ln t)^{2+\beta-\alpha}\varepsilon' \leq (\ln n)^{2+\beta-\alpha}\varepsilon' \leq \frac{\varepsilon}{3},$$
$$(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq (\ln t)^{2+\nu-\alpha}\varepsilon' \leq (\ln n)^{2+\nu-\alpha}\varepsilon' \leq \frac{\varepsilon}{3}.$$

These three inequalities imply that $p_n(u_k) \leq \varepsilon$; then, for all $k \geq \max\{k_0, k_1\}$, we conclude that $u_k \in B_{0,n,\varepsilon} \subseteq V$. Hence, $\{u_k\} \subseteq X$ converges to 0 with respect to $\mathcal{T}$.

Next, we will prove the necessity holds. Suppose $\{u_k\} \subseteq X$ converges to 0 with respect to $\mathcal{T}$. In order to show (i) holds, choose an arbitrary compact set $I \subseteq J$, let $l = \min\{t \mid t \in I\}, r = \max\{t \mid t \in I\}$. Then, $1 < l \leq r < \infty$ and $I \subseteq [l, r]$. For $\forall \varepsilon > 0$, let $\varepsilon' = \min\{(\ln l)^{2-\alpha}\varepsilon, (\ln l)^{2+\beta-\alpha}\varepsilon, (\ln l)^{2+\nu-\alpha}\varepsilon\}$, let n be a natural number with $n \geq r$. $B_{0,n,\varepsilon'}$ is a neighborhood of 0 with respect to $\mathcal{T}$, based on the convergence of $\{u_k\}$, there exists $k_0 \in \mathbb{N}$ such that $u_k \in B_{0,n,\varepsilon'}$ for all $k \geq k_0$. Then, for all $k \geq k_0, t \in I$, we have

$$(\ln l)^{2-\alpha}|u_k(t)| \leq (\ln t)^{2-\alpha}|u_k(t)| \leq p_n(u_k) \leq \varepsilon',$$

$$(\ln l)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq p_n(u_k) \leq \varepsilon',$$
$$(\ln l)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq (\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq p_n(u_k) \leq \varepsilon',$$

therefore,

$$|u_k(t)| \leq \varepsilon'(\ln l)^{\alpha-2} \leq \varepsilon, \ |{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq \varepsilon'(\ln l)^{\alpha-\beta-2} \leq \varepsilon,$$
$$|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq \varepsilon'(\ln l)^{\alpha-\nu-2} \leq \varepsilon,$$

these mean that $\{u_k(t)\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k(t)\}, \{{}^H\mathcal{D}^{\nu}_{1+}u_k(t)\}$ converge to 0 on I.

Now, let us prove that (ii) holds. For any $\varepsilon > 0$, since $B_{0,1,\frac{\varepsilon}{3}}$ is a neighborhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $p_1(u_k) \leq \frac{\varepsilon}{3}$ for all $k \geq k_0$. Then, for all $k \geq k_0, t \in (1, 2)$, we have

$$(\ln t)^{2-\alpha}|u_k(t)| \leq p_1(u_k) \leq \frac{\varepsilon}{3}, (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u_k(t)| \leq p_1(u_k) \leq \frac{\varepsilon}{3},$$
$$(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)| \leq p_1(u_k) \leq \frac{\varepsilon}{3}.$$

Thus, the proof is completed. □

Theorem 4. *The metrizable locally convex space $(X, \mathcal{T})$ is complete.*

Proof of Theorem 4. Choose an arbitrary sequence $\{u_k\}$ in $(X, \mathcal{T})$. We will prove that the sequence $\{u_k\}$ is convergent by the following five steps.

Step 1 For any $t \in J, \{u_k(t)\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k(t)\}$ and $\{{}^H\mathcal{D}^{\nu}_{1+}u_k(t)\}$ are Cauchy sequences in $\mathbb{R}$.

Let $t \in J$ be arbitrarily fixed, $\varepsilon > 0, n \in \mathbb{N}$ and $n \geq t$. Set $\varepsilon' = \min\{(\ln t)^{2-\alpha}\varepsilon, (\ln t)^{2+\beta-\alpha}\varepsilon, (\ln t)^{2+\nu-\alpha}\varepsilon\}$. Since $\{u_k\}$ is a Cauchy sequence and $B_{0,n,\varepsilon'}$ is a neighborhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$, for any $k, m \geq k_0$, we have $u_k - u_m \in B_{0,n,\varepsilon'}$. Then,

$$(\ln t)^{2-\alpha}|u_k(t) - u_m(t)| \leq p_n(u_k - u_m) \leq \varepsilon',$$

$$(\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}(u_k(t)-u_m(t))| \le p_n(u_k-u_m) \le \varepsilon',$$

and

$$(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}(u_k(t)-u_m(t))| \le p_n(u_k-u_m) \le \varepsilon'.$$

Whence,

$$\begin{aligned}&|u_k(t)-u_m(t)| \le (\ln t)^{\alpha-2}\varepsilon' \le \varepsilon,\ |{}^H\mathcal{D}^{\beta}_{1+}(u_k(t)-u_m(t))| \le \varepsilon,\\ &|{}^H\mathcal{D}^{\nu}_{1+}(u_k(t)-u_m(t))| \le \varepsilon.\end{aligned}$$

It follows that $\{u_k(t)\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k(t)\}$ and $\{{}^H\mathcal{D}^{\nu}_{1+}u_k(t)\}$ are Cauchy sequences in $\mathbb{R}$. Let $u(t), v(t), w(t)$ be the limits of $\{u_k(t)\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k(t)\}$ and $\{{}^H\mathcal{D}^{\nu}_{1+}u_k(t)\}$, respectively, i.e.,

$$\lim_{k\to\infty} u_k(t) = u(t),\ \lim_{k\to\infty} {}^H\mathcal{D}^{\beta}_{1+}u_k(t) = v(t),\ \lim_{k\to\infty} {}^H\mathcal{D}^{\nu}_{1+}u_k(t) = w(t),\ t \in J. \tag{10}$$

Step 2 $\{u_k-u\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k-v\}$ and $\{{}^H\mathcal{D}^{\nu}_{1+}u_k-w\}$ satisfy the condition (i) in Theorem 3, i.e.,

$$\begin{aligned}&\{u_k-u\}, \{{}^H\mathcal{D}^{\beta}_{1+}u_k-v\} \text{ and } \{{}^H\mathcal{D}^{\nu}_{1+}u_k-w\} \text{ converge uniformly to } 0\\ &\text{on every compact set } I \subseteq J.\end{aligned} \tag{11}$$

Since $\{u_k\}$ is a Cauchy sequence in X, just as the proof of necessity of Theorem 3, the conclusion (11) is satisfied. Meanwhile, by (11), we have $u, v, w \in C(J)$.

Step 3 For any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for $\forall k \ge k_0, t \in (1,2)$, then,

$$\begin{aligned}&(\ln t)^{2-\alpha}|u_k(t)-u(t)| \le \frac{\varepsilon}{3},\ (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u_k(t)-v(t)| \le \frac{\varepsilon}{3},\\ &(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u_k(t)-w(t)| \le \frac{\varepsilon}{3}.\end{aligned} \tag{12}$$

In fact, since $B_{0,1,\frac{\varepsilon}{3}}$ is a neighborhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$, for all $k, m \ge k_0$, $u_k - u_m \in B_{0,1,\frac{\varepsilon}{3}}$; as a result, for all $k \ge k_0, t \in (1,2)$, we have

$$\begin{aligned}&(\ln t)^{2-\alpha}|u_k(t)-u_m(t)| \le \frac{\varepsilon}{3},\ (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}(u_k(t)-u_m(t))| \le \frac{\varepsilon}{3},\\ &(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}(u_k(t)-u_m(t))| \le \frac{\varepsilon}{3}.\end{aligned}$$

The conclusion (12) is obtained let $m \to \infty$.

Step 4 The limits $\lim\limits_{t\to1+}(\ln t)^{2-\alpha}u(t)$, $\lim\limits_{t\to1+}(\ln t)^{2+\beta-\alpha}v(t)$ and $\lim\limits_{t\to1+}(\ln t)^{2+\nu-\alpha}w(t)$ exist.

We first prove the limit $\lim\limits_{t\to1+}(\ln t)^{2-\alpha}u(t)$ exists. In fact, for any $\varepsilon > 0$, it follows from (12) that there exists $k_0 \in \mathbb{N}$ such that

$$(\ln t)^{2-\alpha}|u_k(t)-u(t)| \le \frac{\varepsilon}{3},\ \text{for all } k \ge k_0, t \in (1,2). \tag{13}$$

Choose $k_1 \in \mathbb{N}$ with $k_1 \ge k_0$, since the limit $\lim\limits_{t\to1+}(\ln t)^{2-\alpha}u_{k_1}(t)$ exists, it results that there exist $0 < \epsilon < 1$ and $\delta > 0$, such that for all $t, s \in (1, 1+\epsilon)$ with $|t-s| < \delta$, we have

$$|(\ln t)^{2-\alpha}u_{k_1}(t) - (\ln s)^{2-\alpha}u_{k_1}(s)| \le \frac{\varepsilon}{3}. \tag{14}$$

Then, for $t, s \in (1,2)$ with $|t-s| < \delta$, combining (13) and (14), we have

$$|(\ln t)^{2-\alpha}u(t) - (\ln s)^{2-\alpha}u(s)| \le \varepsilon.$$

Similarly, we infer that $\lim_{t\to 1+}(\ln t)^{2+\beta-\alpha}v(t)$ and $\lim_{t\to 1+}(\ln t)^{2+\nu-\alpha}w(t)$ exist.

Step 5 ${}^{H}\mathcal{D}^{\beta}_{1+}u(t) = v(t)$, ${}^{H}\mathcal{D}^{\nu}_{1+}u(t) = w(t), t \in J$.

For any $k \in \mathbb{N}$ and compact set $I \subseteq J$, we know ${}^{H}\mathcal{D}^{\beta}_{1+}u_k, {}^{H}\mathcal{D}^{\nu}_{1+}u_k \in C(I)$. From (11), we have $\{{}^{H}\mathcal{D}^{\beta}_{1+}u_k - v\}$ and $\{{}^{H}\mathcal{D}^{\nu}_{1+}u_k - w\}$ converge uniformly to 0 on I, according to Lemma 5, then

$$ {}^{H}\mathcal{D}^{\beta}_{1+}u(t) = {}^{H}\mathcal{D}^{\beta}_{1+}(\lim_{k\to\infty}u_k(t)) = \lim_{k\to\infty}{}^{H}\mathcal{D}^{\beta}_{1+}u_k(t) = v(t), t \in I, $$

$$ {}^{H}\mathcal{D}^{\nu}_{1+}u(t) = {}^{H}\mathcal{D}^{\nu}_{1+}(\lim_{k\to\infty}u_k(t)) = \lim_{k\to\infty}{}^{H}\mathcal{D}^{\nu}_{1+}u_k(t) = w(t), t \in I, $$

Since I is arbitrary, we know ${}^{H}\mathcal{D}^{\beta}_{1+}u(t) = v(t)$, ${}^{H}\mathcal{D}^{\nu}_{1+}u(t) = w(t), t \in J$.

Summarizing the above steps, it follows from Theorem 3 that $\{u_k - u\}$ converges to 0 with respect to $\mathcal{T}$. $\square$

Theorem 5. *Let $(X, \mathcal{T})$ be a metrizable locally convex space, then $Y \subseteq X$ is relatively compact in $(X, \mathcal{T})$ provided that it satisfies the following conditions:*

(i) *Y is pointwise bounded on J;*
(ii) *Y is equicontinuous on J;*
(iii) *Y is equiconvergent at $1+$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in Y, t \in (1, 1+\delta)$ one has*

$$ \left|(\ln t)^{2-\alpha}y(t) - \lim_{t\to 1+}(\ln t)^{2-\alpha}y(t)\right| \le \varepsilon, $$

$$ \left|(\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}y(t) - \lim_{t\to 1+}(\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}y(t)\right| \le \varepsilon, $$

and

$$ \left|(\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}y(t) - \lim_{t\to 1+}(\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}y(t)\right| \le \varepsilon. $$

Proof of Theorem 5. Define another linear space

$$ \widetilde{X} = \Big\{\widetilde{u} \in C(J_1) \mid (\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)) \in C(J_1), \\ (\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)) \in C(J_1)\Big\}. $$

Consider a family $\widetilde{\mathcal{P}} = \{\widetilde{p_n}\}_{n\in\mathbb{N}}$ of semi-norms on $\widetilde{X}$, where $\widetilde{p_n} : \widetilde{X} \to [0, \infty)$ is defined by

$$ \widetilde{p_n}(\widetilde{u}) = \max_{[1,1+n]}|\widetilde{u}(t)| + \max_{[1,1+n]}|(\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))| \\ + \max_{[1,1+n]}|(\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))|, \ \forall\widetilde{u} \in \widetilde{X}. $$

Let $\widetilde{\mathcal{T}}$ be the topology induced by the family $\widetilde{\mathcal{P}}$, with the proof of Theorems 1, 2, and 4, we know $(\widetilde{X}, \widetilde{\mathcal{T}})$ is also a locally convex space which is metrizable and complete. Obviously, the convergence $\lim_{k\to\infty}\widetilde{u_k} = \widetilde{u}$ in $(\widetilde{X}, \widetilde{\mathcal{T}})$ is exactly the uniform convergence, then $\lim_{k\to\infty}\widetilde{u_k}(t) = \widetilde{u}(t)$, $\lim_{k\to\infty}(\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u_k}(t)) = (\ln t)^{2+\beta-\alpha}\,{}^{H}\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))$ and $\lim_{k\to\infty}(\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u_k}(t)) = (\ln t)^{2+\nu-\alpha}\,{}^{H}\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))$ on every compact subset $I \subseteq J_1$. Then, $\widetilde{\mathcal{T}}$ is the topology of compact convergence.

For any $u \in X$, associated with this function, we define a new function $\widetilde{u} : J_1 \to \mathbb{R}$ as follows:

$$\widetilde{u}(t) = \begin{cases} (\ln t)^{2-\alpha} u(t), & t \in J, \\ \lim\limits_{t\to 1+} (\ln t)^{2-\alpha} u(t), & t = 1. \end{cases} \tag{15}$$

It is clear that $\widetilde{u} \in C(J_1)$. For any $t \in J$, we have

$$(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)) = (\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+} u(t) \in C(J)$$

and $(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)) = (\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+} u(t) \in C(J)$. If we define $\lim\limits_{t\to 1+}$ $(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+} u(t)$ as the value of $(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))$ at $t = 1$; likewise, we take $\lim\limits_{t\to 1+} (\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+} u(t)$ as the value of $(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t))$ at $t = 1$. Then, we deduce that

$$(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)), (\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{u}(t)) \in C(J_1).$$

Hence, $\widetilde{u} \in \widetilde{X}$.

Choose an arbitrary subset $Y \subseteq X$ satisfying $(i) - (iii)$, let $\widetilde{Y} = \{\widetilde{y} \in \widetilde{X} \mid y \in Y\}$, where $\widetilde{y}$ is defined as the formula (15), then $\widetilde{Y}$ is pointwise bounded and equicontinuous on J_1. By the Arzela–Ascoli theorem, it follows that $\widetilde{Y}$ is relatively compact in $(\widetilde{X}, \widetilde{\mathcal{T}})$. Let $\{y_k\}$ be any sequence in Y, for every $k \in \mathbb{N}$, imitating the formula (15), we rewrite y_k as $\widetilde{y_k}$, then $\{\widetilde{y_k}\} \subseteq \widetilde{Y}$. So, there must be a subsequence $\{\widetilde{y_{k_j}}\}$ and $\widetilde{y} \in \widetilde{X}$ such that $\widetilde{y_{k_j}} \to \widetilde{y}$ with respect to $\widetilde{\mathcal{T}}$. Then,

$$\begin{aligned} &\left\{\widetilde{y_{k_j}} - \widetilde{y}\right\}, \left\{(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{y_{k_j}}(t)) - (\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}\widetilde{y}(t))\right\}, \\ &\text{and } \left\{(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{y_{k_j}}(t)) - (\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}\widetilde{y}(t))\right\} \\ &\text{converge uniformly to 0 on every compact set } I_1 \subseteq J_1. \end{aligned} \tag{16}$$

Let $y(t) = (\ln t)^{\alpha-2}\widetilde{y}(t)$, then $y \in X$. In order to show that $\{y_{k_j} - y\}$ converges to 0 with respect to $\mathcal{T}$, we only need to verify that the sequence satisfies all the conditions of Theorem 3.

For any compact set $I \subseteq J$, there exists $M_I > 0$ such that $\max\limits_{t\in I}\{(\ln t)^{\alpha-2}, (\ln t)^{\alpha-\beta-2},$ $(\ln t)^{\alpha-\nu-2}\} \le M_I$. From (16), we know for any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} &|\widetilde{y_{k_j}}(t) - \widetilde{y}(t)| < \frac{\varepsilon}{M_I}, \ |(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t) - \widetilde{y}(t)))| < \frac{\varepsilon}{M_I}, \\ &|(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t) - \widetilde{y}(t)))| < \frac{\varepsilon}{M_I}, \ \ \forall t \in I, j \ge j_0. \end{aligned}$$

Then, for $\forall t \in I, j \ge j_0$, we have

$$|y_{k_j}(t) - y(t)| = (\ln t)^{\alpha-2}|\widetilde{y_{k_j}}(t) - \widetilde{y}(t)| < \varepsilon,$$

$$|\, {}^H\mathcal{D}^{\beta}_{1+}(y_{k_j}(t) - y(t))| = (\ln t)^{\alpha-\beta-2}|(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t) - \widetilde{y}(t)))| < \varepsilon,$$

$$|\, {}^H\mathcal{D}^{\nu}_{1+}(y_{k_j}(t) - y(t))| = (\ln t)^{\alpha-\nu-2}|(\ln t)^{2+\beta-\nu}\, {}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t) - \widetilde{y}(t)))| < \varepsilon.$$

These imply that

$$\begin{aligned} &\{y_{k_j} - y\}, \{{}^H\mathcal{D}^{\beta}_{1+} y_{k_j} - {}^H\mathcal{D}^{\beta}_{1+} y\}, \{{}^H\mathcal{D}^{\nu}_{1+} y_{k_j} - {}^H\mathcal{D}^{\nu}_{1+} y\} \text{ converge uniformly} \\ &\text{to 0 on compact set } I. \end{aligned} \tag{17}$$

From (16), we know $\{\widetilde{y_{k_j}}(t)-\widetilde{y}(t)\},\{(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t)-\widetilde{y}(t)))\}$ and $\{(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t)-\widetilde{y}(t)))\}$ converge uniformly to 0 on [1,2]. For any $\varepsilon>0$, there exists $j_1\in\mathbb{N}$ such that for any $j\geq j_1, t\in[1,2]$, we have

$$\begin{aligned}&|\widetilde{y_{k_j}}(t)-\widetilde{y}(t)|\leq\frac{\varepsilon}{3},\ |(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t)-\widetilde{y}(t))|\leq\frac{\varepsilon}{3},\\&|(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}((\ln t)^{\alpha-2}(\widetilde{y_{k_j}}(t)-\widetilde{y}(t))|\leq\frac{\varepsilon}{3},\end{aligned}$$

i.e.,

$$\begin{aligned}&(\ln t)^{2-\alpha}|y_{k_j}(t)-y(t)|\leq\frac{\varepsilon}{3},\ (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}(y_{k_j}(t)-y(t))|\leq\frac{\varepsilon}{3},\\&(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}(y_{k_j}(t)-y(t))|\leq\frac{\varepsilon}{3}.\end{aligned}\tag{18}$$

From (17) and (18), all the conditions in Theorem 3 hold; it follows that $\{y_{k_j}-y\}$ converges to 0 with respect to the topology $\mathcal{T}$. Consequently, Y is relatively compact in $(X,\mathcal{T})$. □

Now, we assume that $\beta=\alpha-1$. Define another function space as follows:

$$\begin{aligned}\widehat{X}=\Big\{&x\mid x,\ {}^H\mathcal{D}^{\alpha-1}_{1+}x,\ {}^H\mathcal{D}^{\nu}_{1+}x\in C(J),\ \lim_{t\to1+}(\ln t)^{2-\alpha}x(t),\ \lim_{t\to1+}{}^H\mathcal{D}^{\alpha-1}_{1+}x(t)\text{ and}\\&\lim_{t\to1+}(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}x(t)\text{ exist}\Big\}.\end{aligned}$$

Consider the family $\widehat{\mathcal{P}}=\{\widehat{p_n}\}$ of semi-norms on X, where $\widehat{p_n}:X\to[0,\infty)$ is defined by

$$\begin{aligned}\widehat{p_n}(x)=&\sup_{t\in(1,1+n]}(\ln t)^{2-\alpha}|x(t)|+\sup_{t\in(1,1+n]}|{}^H\mathcal{D}^{\alpha-1}_{1+}x(t)|\\&+\sup_{t\in(1,1+n]}(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}x(t)|,x\in X,n\in\mathbb{N}.\end{aligned}$$

Let $\widehat{U}_{x,n,\varepsilon}=\{y\in\widehat{X}\mid\widehat{p_n}(y-x)<\varepsilon\}$. Just like the set $\mathcal{B}$ and $\mathcal{T}$, applying these sets $\widehat{U}_{x,n,\varepsilon}$, we construct some new sets $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{T}}$. Then, we arrive at the same conclusions as Theorems 2–5, which are fully summarized in the following Theorem 6.

Theorem 6.

(1) *$(\widehat{X},\widehat{\mathcal{T}})$ is Hausdorff, LCS, and metrizable.*

(2) *A sequence $\{u_k\}\subseteq\widehat{X}$ converges to 0 with respect to $\widehat{\mathcal{T}}$ if, and only if, it satisfies the following conditions:*

  (i) *$\{u_k\},\{{}^H\mathcal{D}^{\alpha-1}_{1+}u_k\},\{{}^H\mathcal{D}^{\nu}_{1+}u_k\}$ converge uniformly to 0 on any compact set $I\subseteq J$;*

  (ii) *$\{(\ln t)^{2-\alpha}u_k(t)\},\{{}^H\mathcal{D}^{\alpha-1}_{1+}u_k(t)\},\{(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}u_k(t)\}$ converge to 0 when $t\to1+,k\to\infty$ uniformly with respect to $\mathcal{T}$, i.e., for $\forall\varepsilon>0$, there exists $\delta>0$ and $k_0\in\mathbb{N}$ such that*

$$\begin{aligned}&|(\ln t)^{2-\alpha}u_k(t)|\leq\frac{\varepsilon}{3},\ |{}^H\mathcal{D}^{\alpha-1}_{1+}u_k(t)|\leq\frac{\varepsilon}{3},\\&|(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}u_k(t)|\leq\frac{\varepsilon}{3},\ \text{for all } k\geq k_0,t\in(1,1+\delta).\end{aligned}\tag{19}$$

(3) *The metrizable locally convex space $(\widehat{X},\widehat{\mathcal{T}})$ is complete;*

(4) *Let $(\widehat{X},\widehat{\mathcal{T}})$ be a metrizable locally convex space, then $Y\subseteq\widehat{X}$ is relatively compact in $(\widehat{X},\widehat{\mathcal{T}})$ and satisfies the following conditions:*

  (i) *Y is pointwise bounded on J_1;*

  (ii) *Y is equicontinuous on J_1;*

(iii) *Y is equiconvergent at $1+$, i.e., for every $\varepsilon >$ there exists $\delta > 0$ such that for all $y \in Y, t \in (1, 1+\delta)$ one has*

$$\left|(\ln t)^{2-\alpha}y(t) - \lim_{t\to 1+}(\ln t)^{2-\alpha}y(t)\right| \le \varepsilon, \ \left|{}^H\mathcal{D}^{\alpha-1}_{1+}y(t) - \lim_{t\to 1+}{}^H\mathcal{D}^{\alpha-1}_{1+}y(t)\right| \le \varepsilon,$$

and

$$\left|(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}y(t) - \lim_{t\to 1+}(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}y(t)\right| \le \varepsilon.$$

For convenience, we will uniformly denote the function spaces X and $\widehat{X}$ as X. Depending on the range of β, we will use the space X or $\widehat{X}$ accordingly.

4. Main Results

First, we list the following conditions which will be used in the the subsequent theorems.

Hypothesis 1. *There exist three constants δ, γ, ϱ and non-negative functions $\varphi, \psi, \eta, \omega$, such that $\varphi(t) \in C_{2-\alpha,\ln}(J), \psi(t) \in C_{\delta,\ln}(J), \eta(t) \in C_{\gamma,\ln}(J), \omega(t) \in C_{-\varrho,\ln}(J)$ and $\varphi(t) + \mu_1\vartheta_{\alpha-\nu-1}(t)\,\omega(t) \ge 0, t \in J$, where $0 < \delta < \alpha - 1, 0 < \gamma < \alpha - \beta - 1($ if $0 < \beta < \alpha - 1)$ or $0 < \gamma < \alpha - \nu($ if $\beta = \alpha - 1), \varrho \in (1+\nu-\alpha, 1)$ and $\min\{2\alpha - \nu - \delta, 2\alpha - \nu - \beta - \gamma(0 < \beta < \alpha - 1), 2\alpha - 2\nu + \varrho\} > 2$. In addition, μ_0, μ_1 are two arbitrary positive constants satisfying $\mu_0 \ge |x_0|, \mu_1 \ge |x_1|$.*

Hypothesis 2. *$f(t,x,y,z) : J \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and*

$$\begin{aligned}|f(t,x,y,z)| \le& \varphi(t) + \psi(t)|x| + \eta(t)|y| + \omega(t)(|z - x_1\vartheta_{\alpha-\nu-1}(s)| \\ &+ \mu_1\vartheta_{\alpha-\nu-1}(s)), \forall (t,x,y,z) \in J \times \mathbb{R}^3.\end{aligned}$$

Remark 2. *Since $\alpha - 1 < \nu < \alpha$, $\vartheta_{\alpha-\nu-1}(t)$ just adopts the definition form as above, and according to the properties of the gamma function, its true expression is $\vartheta_{\alpha-\nu-1}(t) = -\frac{1+\nu-\alpha}{\Gamma(\alpha-\nu)}(\ln t)^{\alpha-\nu-2}$, it is obvious that $\vartheta_{\alpha-\nu-1}(t) < 0$, $\forall t \in J$.*

For convenience, we introduce some notations:

$$\begin{aligned}
&M_{\varphi,\iota} = \sup_{t\in(1,j]}(\ln t)^{2-\alpha}\varphi(t), \ \ M_{\psi,\iota} = \sup_{t\in(1,j]}(\ln t)^{\delta}\psi(t), \iota = 2, T,\\
&M_{\eta,\iota} = \sup_{t\in(1,j]}(\ln t)^{\gamma}\eta(t), \ \ M_{\omega,\iota} = \sup_{t\in(1,j]}(\ln t)^{-\varrho}\omega(t), \iota = 2, T,\\
&M_2 = \max\{M_{\varphi,2},\ M_{\psi,2},\ M_{\eta,2},\ M_{\omega,2}\},\ M_T = \max\{M_{\varphi,T},\ M_{\psi,T},\ M_{\eta,T},\ M_{\omega,T}\},\\
&H(t) = \max\{(\ln t)^{\alpha-\delta-2}, (\ln t)^{\alpha-\beta-\gamma-2}, (\ln t)^{\alpha-\nu+\varrho-2}\}(0 < \beta < \alpha - 1),\\
&H_1(t) = \max\{(\ln t)^{\alpha-\delta-2}, (\ln t)^{-\gamma}, (\ln t)^{\alpha-\nu+\varrho-2}\}.
\end{aligned}$$

Remark 3. *By direct calculation, we know if $t \in (1, e]$, then*

$$H(t) = (\ln t)^{\max\{\alpha-\delta-1, \alpha-\beta-\gamma-1, \alpha-\nu+\varrho-1\}-1}.$$

If $t \in [e, \infty)$, $H(t) = (\ln t)^{\min\{\alpha-\delta-1, \alpha-\beta-\gamma-1, \alpha-\nu+\varrho-1\}-1}$. Let

$$\theta = \begin{cases}\max\{\alpha-\delta-1, \alpha-\beta-\gamma-1, \alpha-\nu+\varrho-1\}, & t \in (1, e];\\ \min\{\alpha-\delta-1, \alpha-\beta-\gamma-1, \alpha-\nu+\varrho-1\}, & t \in [e, \infty).\end{cases}$$

Then, $\theta \in (0,1)$ *and we can rewrite the function* $H(t)$ *as* $H(t) = (\ln t)^{\theta-1}$*. Similarly, let*

$$\theta_1 = \begin{cases} \max\{\alpha-\delta-1, 1-\gamma, \alpha-\nu+\varrho-1\}, & t \in (1,e]; \\ \min\{\alpha-\delta-1, 1-\gamma, \alpha-\nu+\varrho-1\}, & t \in [e,\infty). \end{cases}$$

Then, we have $\theta_1 \in (0,1)$ *and* $H_1(t) = (\ln t)^{\theta_1-1}$*.*

Theorem 7. *Assume that Hypothesis 1 holds. Then, the integral equation*

$$\begin{aligned} u(t) =& \mu_0\vartheta_\alpha(t) + \mu_1\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}(\varphi(s)+\psi(s)u(s) \\ &+ \eta(s)^H\mathcal{D}^\beta_{1+}u(s) + \omega(s)|^H\mathcal{D}^\nu_{1+}u(s)|)\frac{ds}{s} \end{aligned} \tag{20}$$

has a unique positive solution $u \in X$*.*

Proof of Theorem 7. For each $T > 1$, when $0 < \beta < \alpha - 1$, we define a Banach space $X[1,T]$ as follows:

$$X[1,T] = \left\{ \begin{array}{l} x \;\Big|\; x,\ ^H\mathcal{D}^\beta_{1+}x,\ ^H\mathcal{D}^\nu_{1+}x \in C(1,T], \lim\limits_{t\to1+}(\ln t)^{2-\alpha}x(t), \\ \quad \lim\limits_{t\to1+}(\ln t)^{2+\beta-\alpha}\ ^H\mathcal{D}^\beta_{1+}x(t) \text{ and } \lim\limits_{t\to1+}(\ln t)^{2+\nu-\alpha}\ ^H\mathcal{D}^\nu_{1+}x(t) \text{ exist} \end{array} \right\},$$

which is equipped with a norm

$$\begin{aligned} \|x\|_{X_{[1,T]}} =& \sup_{t\in(1,T]}(\ln t)^{2-\alpha}|x(t)| + \sup_{t\in(1,T]}(\ln t)^{2+\beta-\alpha}|^H\mathcal{D}^\beta_{1+}x(t)| \\ &+ \sup_{t\in(1,T]}(\ln t)^{2+\nu-\alpha}|^H\mathcal{D}^\nu_{1+}x(t)|. \end{aligned}$$

Similarly, when $\beta = \alpha - 1$, we also denote

$$X[1,T] = \left\{ \begin{array}{l} x \;\Big|\; x,^H\mathcal{D}^{\alpha-1}_{1+}x,^H\mathcal{D}^\nu_{1+}x \in C(1,T], \lim\limits_{t\to1+}(\ln t)^{2-\alpha}x(t), \\ \quad \lim\limits_{t\to1+}\ ^H\mathcal{D}^{\alpha-1}_{1+}x(t) \text{ and } \lim\limits_{t\to1+}(\ln t)^{2+\nu-\alpha}\ ^H\mathcal{D}^\nu_{1+}x(t) \text{ exist} \end{array} \right\}$$

a Banach space by defining a norm

$$\|x\|_{X_{[1,T]}} = \sup_{t\in(1,T]}(\ln t)^{2-\alpha}|x(t)| + \sup_{t\in(1,T]}|^H\mathcal{D}^{\alpha-1}_{1+}x(t)| + \sup_{t\in(1,T]}(\ln t)^{2+\nu-\alpha}|^H\mathcal{D}^\nu_{1+}x(t)|.$$

Denote a set $P \subseteq X[1,T]$ with the form

$$P = \left\{x \in X[1,T] \;\middle|\; x(t) \geq 0,^H\mathcal{D}^\beta_{1+}u(t) \geq 0,^H\mathcal{D}^\nu_{1+}u(t) - \mu_1\vartheta_{\alpha-\nu-1}(t) \geq 0, t \in (1,T]\right\}.$$

Clearly, P is a cone of $X[1,T]$.

Next, we define an operator $A_T : P \to P$ by

$$\begin{aligned} (A_Tu)(t) =& \mu_0\vartheta_\alpha(t) + \mu_1\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\Big(\varphi(s)+\psi(s)u(s) \\ &+ \eta(s)^H\mathcal{D}^\beta_{1+}u(s) + \omega(s)^H\mathcal{D}^\nu_{1+}u(s)\Big)\frac{ds}{s}. \end{aligned} \tag{21}$$

From Lemma 4, applying the operator ${}^H\mathcal{D}^{\beta}_{1+}$ to A_T, we have

$$\begin{aligned}
{}^H\mathcal{D}^{\beta}_{1+}(A_Tu)(t) =&\mu_0\vartheta_{\alpha-\beta}(t)+\mu_1\vartheta_{\alpha-\beta-1}(t)+\frac{1}{\Gamma(\alpha-\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\beta-1}\\
&\Big(\varphi(s)+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s},\\
&(0<\beta<\alpha-1)\\
{}^H\mathcal{D}^{\alpha-1}_{1+}(A_Tu)(t) =&\mu_0+\int_1^t\Big(\varphi(s)+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)\\
&+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s},(\beta=\alpha-1)
\end{aligned}\tag{22}$$

$$\begin{aligned}
{}^H\mathcal{D}^{\nu}_{1+}(A_Tu)(t) =&\mu_0\vartheta_{\alpha-\nu}(t)+\mu_1\vartheta_{\alpha-\nu-1}(t)+\frac{1}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big(\varphi(s)\\
&+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s}.
\end{aligned}\tag{23}$$

When $0<\beta<\alpha-1$, for any $u\in P, t\in(1,2]$, using Hypothesis 1, we have

$$\begin{aligned}
&\Big|\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}(\varphi(s)+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)\\
&\quad+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big|\\
\le&\frac{M_2(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\Big[B(\alpha,\alpha-1)(\ln t)^{2\alpha-2}+\Big(B(\alpha,\alpha-\delta-1)(\ln t)^{2\alpha-2-\delta}\\
&+B(\alpha,\alpha-\beta-\gamma-1)(\ln t)^{2\alpha-2-\beta-\gamma}\\
&+B(\alpha,\alpha-\nu+\varrho-1)(\ln t)^{2\alpha-2-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big]\\
=&\frac{M_2}{\Gamma(\alpha)}\Big[B(\alpha,\alpha-1)(\ln t)^{\alpha}+\Big(B(\alpha,\alpha-\delta-1)(\ln t)^{\alpha-\delta}+B(\alpha,\alpha-\beta-\gamma-1)\\
&(\ln t)^{\alpha-\beta-\gamma}+B(\alpha,\alpha-\nu+\varrho-1)(\ln t)^{\alpha-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big],
\end{aligned}\tag{24}$$

$$\begin{aligned}
&\Big|\frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\beta-1}(\varphi(s)+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)\\
&\quad+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big|\\
\le&\frac{M_2}{\Gamma(\alpha-\beta)}\Big[B(\alpha-\beta,\alpha-1)(\ln t)^{\alpha}+\Big(B(\alpha-\beta,\alpha-\delta-1)(\ln t)^{\alpha-\delta}+B(\alpha-\beta,\\
&\alpha-\beta-\gamma-1)(\ln t)^{\alpha-\beta-\gamma}+B(\alpha-\beta,\alpha-\nu+\varrho-1)(\ln t)^{\alpha-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big],
\end{aligned}\tag{25}$$

$$\begin{aligned}
&\Big|\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}(\varphi(s)+\psi(s)u(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}u(s)\\
&\quad+\omega(s){}^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big|\\
\le&\frac{M_2}{\Gamma(\alpha-\nu)}\Big[B(\alpha-\nu,\alpha-1)(\ln t)^{\alpha}+\Big(B(\alpha-\nu,\alpha-\delta-1)(\ln t)^{\alpha-\delta}+B(\alpha-\nu,\\
&\alpha-\beta-\gamma-1)(\ln t)^{\alpha-\beta-\gamma}+B(\alpha-\nu,\alpha-\nu+\varrho-1)(\ln t)^{\alpha-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big].
\end{aligned}\tag{26}$$

If $1 < T < 2$, then we choose $t \in (1, T]$ and change M_2 to M_T in the above formula. By virtue of (24)–(26), let $t \to 1+$, then,

$$\begin{aligned}
&\lim_{t\to 1+} \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\beta}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s} = 0, \\
&\lim_{t\to 1+} \frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\beta-1} (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\beta}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s} = 0, \\
&\lim_{t\to 1+} \frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\beta}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s} = 0.
\end{aligned} \tag{27}$$

When $\beta = \alpha - 1$, by the same deduction method, we have

$$\begin{aligned}
&\Big|\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big| \\
\leq&\frac{M_2}{\Gamma(\alpha)}\Big[B(\alpha,\alpha-1)(\ln t)^{\alpha} + \Big(B(\alpha,\alpha-\delta-1)(\ln t)^{\alpha-\delta} + B(\alpha,1-\gamma)(\ln t)^{\alpha-\gamma} \\
&+B(\alpha,\alpha-\nu+\varrho-1)(\ln t)^{\alpha-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big], \\
&\Big|\int_1^t (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big| \\
\leq&M_2\Big[\frac{(\ln t)^{\alpha-1}}{\alpha-1} + \Big(\frac{(\ln t)^{\alpha-\delta-1}}{\alpha-\delta-1} + \frac{(\ln t)^{1-\gamma}}{1-\gamma} + \frac{(\ln t)^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1}\Big)\|u\|_{X_{[1,T]}}\Big], \\
&\Big|\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} (\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s))\frac{ds}{s}\Big| \\
\leq&\frac{M_2}{\Gamma(\alpha-\nu)}\Big[B(\alpha-\nu,\alpha-1)(\ln t)^{\alpha} + \Big(B(\alpha-\nu,\alpha-\delta-1)(\ln t)^{\alpha-\delta} \\
&+B(\alpha-\nu,1-\gamma)(\ln t)^{2-\gamma} + B(\alpha-\nu,\alpha-\nu+\varrho-1)(\ln t)^{\alpha-\nu+\varrho}\Big)\|u\|_{X_{[1,T]}}\Big],
\end{aligned}$$

then,

$$\begin{aligned}
&\lim_{t\to 1+} \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big(\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s} = 0, \\
&\lim_{t\to 1+} \int_1^t \Big(\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s} = 0, \\
&\lim_{t\to 1+} \frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} \Big(\varphi(s) + \psi(s)u(s) + \eta(s)^H\mathcal{D}^{\alpha-1}_{1+}u(s) \\
&\qquad + \omega(s)^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s} = 0.
\end{aligned} \tag{28}$$

By (27) and (28), we deduce $\lim_{t\to1+}(\ln t)^{2-\alpha}A_Tu(t)$, $\lim_{t\to1+}(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}A_Tu(t)$ $(0<\beta<\alpha-1)$, $\lim_{t\to1+}{}^H\mathcal{D}^{\alpha-1}_{1+}A_Tu(t)$, and $\lim_{t\to1+}(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}A_Tu(t)$ exist. From (21), for any $u\in P$, we know

$$\begin{aligned} A_Tu(t) =&\mu_0\vartheta_\alpha(t)+\mu_1\vartheta_{\alpha-1}(t)+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big[\varphi(s)+\mu_1\vartheta_{\alpha-\nu-1}(s)\omega(s)\Big]\frac{ds}{s}\\ &+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big[\psi(s)u(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}u(s)\\ &+\omega(s)\Big({}^H\mathcal{D}^{\nu}_{1+}u(s)-\mu_1\vartheta_{\alpha-\nu-1}(s)\Big)\Big]\frac{ds}{s}. \end{aligned}$$

Hypothesis 1 naturally gives the conclusion $A_Tu(t)\geq0,\ t\in J$. Likewise, ${}^H\mathcal{D}^{\beta}_{1+}(A_Tu)(t)\geq 0$ and ${}^H\mathcal{D}^{\nu}_{1+}(A_Tu)(t)-\mu_1\vartheta_{\alpha-\nu-1}(t)\geq0, t\in J$. According to the above assertions, it follows that $A_T:P\to P$ is well-defined. Due to Hypothesis 1, we can easily show that A_T is completely continuous.

Obviously, in order to show that the integral Equation (20) has a unique positive solution, it suffices to show that the equation has a unique positive solution in $X[1,T]$ for each $T>1$, that is, to prove that the operator A_T has a unique fixed point in P. According to the Leray–Schauder alternative theorem, to prove that A_T has a unique fixed point, we need to show that $\mathcal{E}(A_T)=\{x\in P: x=\lambda A_Tx \text{ for some } 0<\lambda<1\}$ is bounded. Suppose that there exists $\lambda\in(0,1)$ such that $u(t)=\lambda A_Tu(t), t\in(1,T], u\in P$. We discuss it separately in the two cases:

(1) $0<\beta<\alpha-1$

In view of (21) and Hypothesis 1, one has

$$\begin{aligned} &(\ln t)^{2-\alpha}|u(t)|=\lambda(\ln t)^{2-\alpha}|A_Tu(t)|\\ \leq&\frac{\mu_0}{\Gamma(\alpha)}\ln t+\frac{\mu_1}{\Gamma(\alpha-1)}+\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}M_T\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big((\ln s)^{\alpha-2}+(\ln s)^{-\delta}u(s)\\ &+(\ln s)^{-\gamma}\,{}^H\mathcal{D}^{\beta}_{1+}u(s)+(\ln s)^{\varrho}\,{}^H\mathcal{D}^{\nu}_{1+}u(s)\Big)\frac{ds}{s}\\ \leq&\frac{\mu_0}{\Gamma(\alpha)}\ln t+\frac{\mu_1}{\Gamma(\alpha-1)}+\frac{B(\alpha,\alpha-1)}{\Gamma(\alpha)}M_T(\ln t)^{\alpha}\\ &+\frac{M_T}{\Gamma(\alpha)}(\ln t)^{2-\alpha}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}H(s)U(s)\frac{ds}{s}\\ \leq&\phi_1(t)+\frac{M_T}{\Gamma(\alpha)}(\ln t)^{2+\nu-\alpha}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}H(s)U(s)\frac{ds}{s}, \end{aligned}\tag{29}$$

where $\phi_1(t)=\frac{\mu_0}{\Gamma(\alpha)}\ln t+\frac{\mu_1}{\Gamma(\alpha-1)}+\frac{B(\alpha,\alpha-1)}{\Gamma(\alpha)}M_T(\ln t)^{\alpha}, U(s)=(\ln s)^{2-\alpha}|u(s)|+(\ln s)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u(s)|+(\ln s)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u(s)|$. Likewise, by (22) and (23), we have

$$\begin{aligned} &(\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}u(t)|\\ \leq&\phi_2(t)+\frac{M_T}{\Gamma(\alpha-\beta)}(\ln t)^{2+\nu-\alpha}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}H(s)U(s)\frac{ds}{s}, \end{aligned}\tag{30}$$

and

$$\begin{aligned} &(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}u(t)|\\ \leq&\phi_3(t)+\frac{M_T}{\Gamma(\alpha-\nu)}(\ln t)^{2+\nu-\alpha}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}H(s)U(s)\frac{ds}{s}, \end{aligned}\tag{31}$$

where $\phi_2(t) = \frac{\mu_0}{\Gamma(\alpha-\beta)} \ln t + \frac{\mu_1}{\Gamma(\alpha-\beta-1)} + \frac{B(\alpha-\beta,\alpha-1)}{\Gamma(\alpha-\beta)} M_T (\ln t)^\alpha$, $\phi_3(t) = \frac{\mu_0}{\Gamma(\alpha-\nu)} \ln t + \frac{\mu_1(1+\nu-\alpha)}{\Gamma(\alpha-\nu)} + \frac{B(\alpha-\nu,\alpha-1)}{\Gamma(\alpha-\nu)} M_T (\ln t)^\alpha$. Adding the left and right sides of (29)–(31), respectively, we obtain an inequality about the function $U(t)$

$$\begin{aligned} U(t) \leq& \phi_1(t) + \phi_2(t) + \phi_3(t) + \left[\frac{M_T}{\Gamma(\alpha)} + \frac{M_T}{\Gamma(\alpha-\beta)} + \frac{M_T}{\Gamma(\alpha-\nu)}\right] \\ & (\ln t)^{2+\nu-\alpha} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} H(s) U(s) \frac{ds}{s} \\ =& q(t) + g(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} (\ln s)^{\theta-1} U(s) \frac{ds}{s}, \end{aligned} \tag{32}$$

where $q(t) = \phi_1(t) + \phi_2(t) + \phi_3(t), g(t) = \left[\frac{M_T}{\Gamma(\alpha)} + \frac{M_T}{\Gamma(\alpha-\beta)} + \frac{M_T}{\Gamma(\alpha-\nu)}\right] (\ln t)^{2+\nu-\alpha}, \theta$ is defined in Remark 3. An application of Lemma 6 yields

$$U(t) \leq \frac{q^*(t) b}{a+b-1} \exp\left(\frac{(g^*(t))^{\frac{b}{a+b-1}}}{a+b-1} \left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}} (\ln t)^b \right), \text{for a.e. } t \in [1, T].$$

Since the functions q and g are increasing, it is immediately seen that

$$\sup_{t\in(1,T]} U(t) \leq \frac{q(T) b}{a+b-1} \exp\left(\frac{(g(T))^{\frac{b}{a+b-1}}}{a+b-1} \left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}} (\ln T)^b \right) \triangleq \mathcal{M}_T.$$

Consequently, we deduce that

$$\begin{aligned} \|u\|_{X_{[1,T]}} =& \sup_{t\in(1,T]} (\ln t)^{2-\alpha} |u(t)| + \sup_{t\in(1,T]} (\ln t)^{2+\beta-\alpha} |{}^H\mathcal{D}_{1+}^{\beta} u(t)| \\ &+ \sup_{t\in(1,T]} (\ln t)^{2+\nu-\alpha} |{}^H\mathcal{D}_{1+}^{\nu} u(t)| \\ \leq& 3 \sup_{t\in(1,T]} U(t) < 3\mathcal{M}_T + 1. \end{aligned}$$

When $\beta = \alpha - 1$, let $U_1(t) = (\ln t)^{2-\alpha} |u(t)| + |{}^H\mathcal{D}_{1+}^{\alpha-1} u(t)| + (\ln t)^{2+\nu-\alpha} |{}^H\mathcal{D}_{1+}^{\nu} u(t)|$, we estimate each of the three terms of the function U_1 separately.

$$\begin{aligned} (\ln t)^{2-\alpha} |u(t)| \leq& \phi_1(t) + \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} M_T \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big((\ln s)^{-\delta} u(s) \\ &+ (\ln s)^{-\gamma}\, {}^H\mathcal{D}_{1+}^{\alpha-1} u(s) + (\ln s)^{\varrho}\, {}^H\mathcal{D}_{1+}^{\nu} u(s) \Big) \frac{ds}{s} \\ \leq& \phi_1(t) + \frac{M_T}{\Gamma(\alpha)} (\ln t)^{2+\nu-\alpha} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} H_1(s) U_1(s) \frac{ds}{s}, \\ |{}^H\mathcal{D}_{1+}^{\alpha-1} u(t)| \leq& \mu_0 + \frac{M_T}{\alpha-1} (\ln t)^{\alpha-1} + M_T \int_1^t H_1(s) U_1(s) \frac{ds}{s} \\ \leq& \mu_0 + \frac{M_T}{\alpha-1} (\ln t)^{\alpha-1} + M_T (\ln t)^{1+\nu-\alpha} \\ &\cdot \int_1^t (\ln s)^{\alpha-\nu-1} H_1(s) U_1(s) \frac{ds}{s}, \\ (\ln t)^{2+\nu-\alpha} |{}^H\mathcal{D}_{1+}^{\nu} u(t)| \leq& \phi_3(t) + \frac{M_T}{\Gamma(\alpha-\nu)} (\ln t)^{2+\nu-\alpha} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} H_1(s) U_1(s) \frac{ds}{s}. \end{aligned}$$

Adding the estimation results of the above three inequalities, we have

$$\begin{aligned}U_1(t) \leq &\phi_1(t) + \mu_0 + \frac{M_T}{\alpha-1}(\ln t)^{\alpha-1} + \phi_3(t) + M_T\Big(\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha)} + (\ln t)^{1+\nu-\alpha} \\ &+ \frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\Big)\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-\nu-1} H_1(s)U_1(s)\frac{ds}{s} \\ =&q_1(t) + g_1(t)\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}(\ln s)^{\theta_1-1}U_1(s)\frac{ds}{s},\end{aligned} \tag{33}$$

where $q_1(t) = \phi_1(t) + \mu_0 + \frac{M_T}{\alpha-1}(\ln t)^{\alpha-1} + \phi_3(t), g_1(t) = M_T\Big(\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha)} + (\ln t)^{1+\nu-\alpha}$ $+\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\Big)$. Applying Lemma 6 again, we have

$$U_1(t) \leq \frac{q_1^*(t)b}{a+b-1}\exp\left(\frac{(g_1^*(t))^{\frac{b}{a+b-1}}}{a+b-1}\left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}}(\ln t)^b\right).$$

Hence, according to the fact that the functions q_1 and g_1 are increasing, we deduce that

$$\begin{aligned}\|u\|_{X_{[1,T]}} = &\sup_{t\in(1,T]}(\ln t)^{2-\alpha}|u(t)| + |{}^H\mathcal{D}_{1+}^{\alpha-1}u(t)| + (\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}_{1+}^{\nu}u(t)| \\ \leq&3\sup_{t\in(1,T]}U_1(t) \\ <&3\frac{q_1(T)b}{a+b-1}\exp\left(\frac{(g_1(T))^{\frac{b}{a+b-1}}}{a+b-1}\left(\frac{1-a}{bB(a,b)}\right)^{\frac{a-1}{a+b-1}}(\ln T)^b\right) + 1.\end{aligned}$$

By virtue of the Leray–Schauder alternative theorem, A_T has a fixed point $u \in P$. Based on the operator's expression and conditions, the fixed point of the operator A_T is confirmed to be positive.

Finally, we will show that the fixed point is unique. Suppose there is another fixed point $v \in X[1,T]$. Let $Z(t) = (\ln t)^{2-\alpha}|u(t) - v(t)| + (\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}_{1+}^{\beta}u(t) - {}^H\mathcal{D}_{1+}^{\beta}v(t)| + (\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}_{1+}^{\nu}u(t) - {}^H\mathcal{D}_{1+}^{\nu}v(t)|, t \in (1,T], (0 < \beta < \alpha-1)$. Similar to the derivation shown above, we obtain

$$\begin{aligned}Z(t) \leq &\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big[\psi(s)|u(s)-v(s)| + \eta(s)|{}^H\mathcal{D}_{1+}^{\beta}u(s) - {}^H\mathcal{D}_{1+}^{\beta}v(s)| \\ &+\omega(s)|{}^H\mathcal{D}_{1+}^{\nu}u(s) - {}^H\mathcal{D}_{1+}^{\nu}v(s)|\Big]\frac{ds}{s} \\ &+ \frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\beta-1}\Big[\psi(s)|u(s)-v(s)| + \eta(s)|{}^H\mathcal{D}_{1+}^{\beta}u(s) - {}^H\mathcal{D}_{1+}^{\beta}v(s)| \\ &+\omega(s)|{}^H\mathcal{D}_{1+}^{\nu}u(s) - {}^H\mathcal{D}_{1+}^{\nu}v(s)|\Big]\frac{ds}{s} \\ &+ \frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big[\psi(s)|u(s)-v(s)| + \eta(s)|{}^H\mathcal{D}_{1+}^{\beta}u(s) - {}^H\mathcal{D}_{1+}^{\beta}v(s)| \\ &+\omega(s)|{}^H\mathcal{D}_{1+}^{\nu}u(s) - {}^H\mathcal{D}_{1+}^{\nu}v(s)|\Big]\frac{ds}{s} \\ \leq&\left(\frac{M_T}{\Gamma(\alpha)} + \frac{M_T}{\Gamma(\alpha-\beta)} + \frac{M_T}{\Gamma(\alpha-\nu)}\right)(\ln t)^{2+\nu-\alpha}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}H(s)Z(s)\frac{ds}{s} \\ =&g(t)\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}(\ln s)^{\theta-1}Z(s)\frac{ds}{s},\end{aligned}$$

where g is defined in (32). For $\beta = \alpha - 1$, let

$$\begin{aligned} Z_1(t) =& (\ln t)^{2-\alpha}|u(t) - v(t)| + |^H\mathcal{D}_{1+}^{\alpha-1}u(t) - {}^H\mathcal{D}_{1+}^{\alpha-1}v(t)| \\ &+ (\ln t)^{2+\nu-\alpha}|^H\mathcal{D}_{1+}^{\nu}u(t) - {}^H\mathcal{D}_{1+}^{\nu}v(t)|, t \in (1, T], \end{aligned}$$

then $Z_1(t) \leq g_1(t) \int_1^t (\ln \frac{t}{s})^{\alpha-\nu-1}(\ln s)^{\theta_1 - 1} Z_1(s) \frac{ds}{s}$, where g_1 is defined in (33). Applying Lemma 6, we derive $Z(t) \leq 0, Z_1(t) \leq 0, \forall t \in (1, T]$, which means that $u = v$. □

Next, we will establish a subset of the function space X by utilizing the unique positive solution of the integral equation as defined in Theorem 7. Subsequently, we will demonstrate the existence of the fixed point of nonlinear integral operators to establish the existence of a solution for the initial value problem.

Theorem 8. *Suppose that Hypotheses 1 and 2 hold. Then, for any $x_0, x_1 \in \mathbb{R}$, the initial value problem (4) has at least one solution $x \in X$.*

Proof of Theorem 8. Obviously, a function $x \in X$ is a solution of the initial value problem (4) if, and only if, it is a solution of the integral equation

$$x(t) = x_0\vartheta_\alpha(t) + x_1\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(s), {}^H\mathcal{D}_{1+}^{\beta}x(s), {}^H\mathcal{D}_{1+}^{\nu}x(s))\frac{ds}{s}. \quad (34)$$

Consider an operator $A : X \to X$ defined by

$$Ax(t) = x_0\vartheta_\alpha(t) + x_1\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(s), {}^H\mathcal{D}_{1+}^{\beta}x(s), {}^H\mathcal{D}_{1+}^{\nu}x(s))\frac{ds}{s}.$$

In order to show that the integral Equation (34) has a solution, it suffices to prove that the operator A has a fixed point. Based on the condition Hypothesis 1 and Theorem 7, we know A is well-defined. Next, we will prove that A has at least one fixed point in X.

The following discussion will proceed under this assumption $0 < \beta < \alpha - 1$. Let $\xi(t)$ be the unique solution to the integral Equation (20). It is easily seen that $\xi(t) > 0, {}^H\mathcal{D}_{1+}^{\beta}\xi(t) > 0, {}^H\mathcal{D}_{1+}^{\nu}\xi(t) - \mu_1\vartheta_{\alpha-\nu-1}(t) > 0, \forall t \in J$. Based on this function, we construct a subset $\Omega \subset X$ as below

$$\Omega = \left\{ x \in X \;\middle|\; \begin{array}{l} |x(t)| \leq \xi(t),\ |^H\mathcal{D}_{1+}^{\beta}x(t)| \leq {}^H\mathcal{D}_{1+}^{\beta}\xi(t), \\ |^H\mathcal{D}_{1+}^{\nu}x(t) - x_1\vartheta_{\alpha-\nu-1}(t)| \leq {}^H\mathcal{D}_{1+}^{\nu}\xi(t) - \mu_1\vartheta_{\alpha-\nu-1}(t),\ \forall t \in J \end{array} \right\}.$$

Then, Ω is a nonempty closed convex subset of X. For any $x \in \Omega, t \in J$, by Hypothesis 2, we have

$$\begin{aligned} |Ax(t)| \leq& |x_0|\vartheta_\alpha(t) + |x_1|\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big[\varphi(s) + \psi(s)|x(s)| \\ &+ \eta(s)|^H\mathcal{D}_{1+}^{\beta}x(s)| + \omega(s)\Big(|^H\mathcal{D}_{1+}^{\nu}x(s) - x_1\vartheta_{\alpha-\nu-1}(s)| + \mu_1\vartheta_{\alpha-\nu-1}(s)\Big)\Big]\frac{ds}{s} \\ \leq& \mu_0\vartheta_\alpha(t) + \mu_1\vartheta_{\alpha-1}(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big(\varphi(s) + \psi(s)\xi(s) \\ &+ \eta(s){}^H\mathcal{D}_{1+}^{\beta}\xi(s) + \omega(s){}^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s} = \xi(t), \end{aligned}$$

$$\begin{aligned} |^H\mathcal{D}_{1+}^{\beta}Ax(t)| =& \Bigg| x_0\vartheta_{\alpha-\beta}(t) + x_1\vartheta_{\alpha-\beta-1}(t) + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\beta-1} f\Big(s, x(s), \\ &{}^H\mathcal{D}_{1+}^{\beta}x(s), {}^H\mathcal{D}_{1+}^{\nu}x(s)\Big)\frac{ds}{s}\Bigg| \end{aligned}$$

$$\leq \mu_0 \vartheta_{\alpha-\beta}(t) + \mu_1 \vartheta_{\alpha-\beta-1}(t) + \frac{1}{\Gamma(\alpha-\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\beta-1} \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s} = {}^H\mathcal{D}_{1+}^{\beta}\xi(t),$$

and

$$\begin{aligned}
&|{}^H\mathcal{D}_{1+}^{\nu}Ax(t) - x_1\vartheta_{\alpha-\nu-1}(t)| \\
=&\left|x_0\vartheta_{\alpha-\nu}(t) + \frac{1}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} f(s, x(s), {}^H\mathcal{D}_{1+}^{\beta}x(s), {}^H\mathcal{D}_{1+}^{\nu}x(s))\frac{ds}{s}\right| \\
\leq& \mu_0\vartheta_{\alpha-\nu}(t) + \frac{1}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} \Big(\varphi(s) + \psi(s)\xi(s) \\
&+ \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s} = {}^H\mathcal{D}_{1+}^{\nu}\xi(t) - \mu_1\vartheta_{\alpha-\nu-1}(t).
\end{aligned}$$

Therefore, $Ax \in \Omega$.

Next, we will show that $A : \Omega \to \Omega$ is continuous. For any $x \in \Omega$, let $\mathbf{V}$ be a neighborhood of Ax with respect to the topology $\mathcal{T}$, by Remark 1, there exist $n \in \mathbb{N}, r > 0$ such that $B_{Ax,n,r} \subseteq \mathbf{V}$. In order to show A is continuous at x, it suffices to find a neighborhood $\mathbf{U}$ of x and prove that $A(\mathbf{U}) \subseteq B_{Ax,n,r}$.

In (24)–(26), replacing the function $u(t)$ with the function $\xi(t)$, these estimates still hold. Let $t \to 1+$, the right-side functions of these estimates give the limit 0. For $r > 0$, there exists $\delta_0 \in (0,1)$ such that

$$\begin{aligned}
&\sup_{t\in(1,1+\delta_0]}\Big|\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) \\
&\qquad + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s}\Big| \leq \frac{r}{12}, \\
&\sup_{t\in(1,1+\delta_0]}\Big|\frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\beta-1} \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) \\
&\qquad + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s}\Big| \leq \frac{r}{12}, \\
&\sup_{t\in(1,1+\delta_0]}\Big|\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) \\
&\qquad + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s}\Big| \leq \frac{r}{12}.
\end{aligned} \tag{35}$$

Let

$$\overline{M}_2 = \max\left\{\begin{array}{l} M_{\varphi,2}, \sup\limits_{t\in(1,2]} (\ln t)^{\delta}\psi(t)(\ln t)^{2-\alpha}\xi(t), \sup\limits_{t\in(1,2]} (\ln t)^{\gamma}\eta(t)(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}_{1+}^{\beta}\xi(t), \\ \sup\limits_{t\in(1,2]} (\ln t)^{-\varrho}\omega(t)(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}_{1+}^{\nu}\xi(t) \end{array}\right\},$$

for $t \in (1,2]$, the following conclusions are satisfied:

$$\begin{aligned}
&\int_1^t \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s} \\
\leq& \overline{M}_2\left[\frac{(\ln t)^{\alpha-1}}{\alpha-1} + \frac{(\ln t)^{\alpha-\delta-1}}{\alpha-\delta-1} + \frac{(\ln t)^{\alpha-\beta-\gamma-1}}{\alpha-\beta-\gamma-1} + \frac{(\ln t)^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1}\right] \to 0,\ t \to 1+. \\
&\frac{1}{\Gamma(\alpha-\nu)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\nu-1} \Big(\varphi(s) + \psi(s)\xi(s) + \eta(s)^H\mathcal{D}_{1+}^{\beta}\xi(s) + \omega(s)^H\mathcal{D}_{1+}^{\nu}\xi(s)\Big)\frac{ds}{s}
\end{aligned}$$

$$\leq \frac{\overline{M}_2}{\Gamma(\alpha-\nu)}\Big[B(\alpha-\nu,\alpha-1)(\ln t)^{2\alpha-\nu-2}+B(\alpha-\nu,\alpha-\delta-1)(\ln t)^{2\alpha-\nu-\delta-2} \\ +B(\alpha-\nu,\alpha-\beta-\gamma-1)(\ln t)^{2\alpha-\nu-\beta-\gamma-2} \\ +B(\alpha-\nu,\alpha-\nu+\varrho-1)(\ln t)^{2\alpha-2\nu+\varrho-2}\Big]\to 0,\ t\to 1+.$$

For the foregoing δ_0, we have

$$\begin{aligned}&\frac{\overline{M}_2\ln(1+n)}{\min\{\Gamma(\alpha),\Gamma(\alpha-\beta)\}}\int_1^{1+\delta_0}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s) \\ &\quad+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\leq\frac{r}{24}, \\ &\frac{(\ln(1+n))^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^{1+\delta_0}\left(\ln\frac{1+\delta_0}{s}\right)^{\alpha-\nu-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s) \\ &\quad+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\leq\frac{r}{24}.\end{aligned}\tag{36}$$

On the other hand, set

$$M_\xi=\max\left\{\begin{aligned}&\max_{t\in[1+\delta_0,1+n]}\xi(t),\max_{t\in[1+\delta_0,1+n]}{}^H\mathcal{D}^{\beta}_{1+}\xi(t), \\ &\max_{t\in[1+\delta_0,1+n]}{}^H\mathcal{D}^{\nu}_{1+}\xi(t)+2\mu_1(\ln t)^{\alpha-\nu-2}\end{aligned}\right\},$$

then f is uniformly continuous on $[1+\delta_0,1+n]\times[-M_\xi,M_\xi]^3$. Hence, choose a positive number ϵ, for any $t\in[1+\delta_0,1+n],u_i,v_i,w_i\in[-M_\xi,M_\xi](i=1,2)$ with $|u_1-u_2|\leq\epsilon,|v_1-v_2|\leq\epsilon,|w_1-w_2|\leq\epsilon$, the following inequality is satisfied

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)|\leq\frac{\Gamma(\alpha+1)r}{12\ln^2(1+n)}.\tag{37}$$

Let $\mathbf{U}=B_{x,n,r_x}\cap\Omega$, where $r_x=\min\left\{[\ln(1+\delta_0)]^{2-\alpha}\epsilon,\ [\ln(1+\delta_0)]^{2+\beta-\alpha}\epsilon,\ [\ln(1+\delta_0)]^{2+\nu-\alpha}\epsilon\right\}$, then $\mathbf{U}$ is a neighborhood of x. It remains to prove that $Ay\in B_{Ax,n,r}$ for any $y\in\mathbf{U}$, i.e., $p_n(Ay-Ax)\leq r$. For any $y\in\Omega,t\in[1+\delta_0,1+n]$, then $|y(t)|\leq M_\xi,|{}^H\mathcal{D}^{\beta}_{1+}y(t)|\leq M_\xi$ and

$$\begin{aligned}|{}^H\mathcal{D}^{\nu}_{1+}y(t)|\leq&{}^H\mathcal{D}^{\nu}_{1+}\xi(t)-\mu_1\vartheta_{\alpha-\nu-1}(t)+|x_1\vartheta_{\alpha-\nu-1}(t)| \\ =&{}^H\mathcal{D}^{\nu}_{1+}\xi(t)+\frac{(\mu_1+|x_1|)(1+\nu-\alpha)}{\Gamma(\alpha-\nu)}(\ln t)^{\alpha-\nu-2}\leq M_\xi.\end{aligned}$$

For any $y\in\mathbf{U},t\in[1+\delta_0,1+n]$, we have $|y(t)-x(t)|\leq r_x(\ln t)^{\alpha-2}\leq\epsilon$, $|{}^H\mathcal{D}^{\beta}_{1+}y(t)-{}^H\mathcal{D}^{\beta}_{1+}x(t)|\leq r_x(\ln t)^{\alpha-\beta-2}\leq\epsilon$ and $|{}^H\mathcal{D}^{\nu}_{1+}y(t)-{}^H\mathcal{D}^{\nu}_{1+}x(t)|\leq r_x(\ln t)^{\alpha-\nu-2}\leq\epsilon$. According to (35)–(37), we deduce that

$$\begin{aligned}&\sup_{t\in(1,1+n]}(\ln t)^{2-\alpha}|Ay(t)-Ax(t)| \\ \leq&\sup_{t\in(1,1+\delta_0]}(\ln t)^{2-\alpha}|Ay(t)-Ax(t)|+\sup_{t\in[1+\delta_0,1+n]}(\ln t)^{2-\alpha}|Ay(t)-Ax(t)| \\ \leq&\sup_{t\in(1,1+\delta_0]}\left|\frac{2(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)\right. \\ &\left.\quad+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\right| \\ &+\sup_{t\in[1+\delta_0,1+n]}\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^{1+\delta_0}\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)\end{aligned}\tag{38}$$

$$
\begin{aligned}
&\quad + \omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\\
&+ \sup_{t\in[1+\delta_0,1+n]} \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_{1+\delta_0}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)\\
&\quad + \omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\\
\le& 2\frac{r}{12}+2\frac{r}{24}+\frac{\ln^2(1+n)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)r}{12\ln^2(1+n)}=\frac{r}{3}.
\end{aligned}
$$

Similarly, we have

$$
\sup_{t\in(1,1+n]}(\ln t)^{2+\beta-\alpha}|{}^H\mathcal{D}^{\beta}_{1+}Ay(t)-{}^H\mathcal{D}^{\beta}_{1+}Ax(t)|\le\frac{r}{3}. \tag{39}
$$

Again, according to (35) and (37), we have

$$
\begin{aligned}
&\sup_{t\in(1,1+\delta_0]}\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big|f\Big(s,y(s),{}^H\mathcal{D}^{\beta}_{1+}y(s),{}^H\mathcal{D}^{\nu}_{1+}y(s)\Big)\\
&\quad - f\Big(s,x(s),{}^H\mathcal{D}^{\beta}_{1+}x(s),{}^H\mathcal{D}^{\nu}_{1+}x(s)\Big)\Big|\frac{ds}{s}\\
\le& \sup_{t\in(1,1+\delta_0]}\frac{2(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)\\
&\quad + \omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\\
\le& 2\frac{r}{12}=\frac{r}{6},\\
&\sup_{t\in[1+\delta_0,1+n]}\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_{1+\delta_0}^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big|f\Big(s,y(s),{}^H\mathcal{D}^{\beta}_{1+}y(s),{}^H\mathcal{D}^{\nu}_{1+}y(s)\Big)\\
&\quad - f\Big(s,x(s),{}^H\mathcal{D}^{\beta}_{1+}x(s),{}^H\mathcal{D}^{\nu}_{1+}x(s)\Big)\Big|\frac{ds}{s}\\
\le& \frac{\ln^2(1+n)}{\Gamma(\alpha-\nu+1)}\frac{\Gamma(\alpha+1)r}{12\ln^2(1+n)}<\frac{r}{12}.
\end{aligned}
$$

Meanwhile, by (36),

$$
\begin{aligned}
&\sup_{t\in[1+\delta_0,1+n]}\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^{1+\delta_0}\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big|f\Big(s,y(s),{}^H\mathcal{D}^{\beta}_{1+}y(s),{}^H\mathcal{D}^{\nu}_{1+}y(s)\Big)\\
&\qquad - f\Big(s,x(s),{}^H\mathcal{D}^{\beta}_{1+}x(s),{}^H\mathcal{D}^{\nu}_{1+}x(s)\Big)\Big|\frac{ds}{s}\\
\le& \frac{2(\ln(1+n))^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^{1+\delta_0}\left(\ln\frac{1+\delta_0}{s}\right)^{\alpha-\nu-1}\Big(\varphi(s)+\psi(s)\xi(s)\\
&\quad + \eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}<\frac{r}{12}.
\end{aligned}
$$

Therefore, synthesizing the above conclusions, one has

$$
\sup_{t\in(1,1+n]}(\ln t)^{2+\nu-\alpha}|{}^H\mathcal{D}^{\nu}_{1+}Ay(t)-{}^H\mathcal{D}^{\nu}_{1+}Ax(t)|\le\frac{r}{3}. \tag{40}
$$

From (38)–(40), we obtain

$$
\begin{aligned}
p_n(Ay - Ax) = & \sup_{t\in(1,1+n]} (\ln t)^{2-\alpha} |Ay(t) - Ax(t)| \\
& + \sup_{t\in(1,1+n]} (\ln t)^{2+\beta-\alpha} |{}^H\mathcal{D}^{\beta}_{1+} Ay(t) - {}^H\mathcal{D}^{\beta}_{1+} Ax(t)| \\
& + \sup_{t\in(1,1+n]} (\ln t)^{2+\nu-\alpha} |{}^H\mathcal{D}^{\nu}_{1+} Ay(t) - {}^H\mathcal{D}^{\nu}_{1+} Ax(t)| \le r,
\end{aligned}
$$

which implies that $Ay \in B_{Ax,n,r} \subseteq \mathbf{V}$, on account of the arbitrariness of y, we have $A(\mathbf{U}) \subseteq \mathbf{V}$ and A is continuous at x.

Finally, we will show that $A(\Omega)$ is relatively compact in X. Due to Theorem 5, what we need to do is to prove that $A(\Omega)$ satisfies those three conditions (i)–(iii) of the theorem. Since $A(\Omega) \subseteq \Omega \subseteq X$, all the functions in $A(\Omega)$ are controlled by $\xi(t)$, then $A(\Omega)$ is pointwise bounded on J. The next task is to show that $A(\Omega)$ satisfies the condition (ii). Let $t_1 \in J$ be arbitrarily chosen, then choose $n_1 \in \mathbb{N}$ and a real number a_1 such that $1 < a_1 < t_1 < 1 + n_1$. Set

$$
\begin{aligned}
M_{n_1} = \max\Bigg\{ & \sup_{t\in(1,1+n_1]} (\ln t)^{2-\alpha}\varphi(t), \sup_{t\in(1,1+n_1]} (\ln t)^{\delta}\psi(t)(\ln t)^{2-\alpha}\xi(t), \\
& \sup_{t\in(1,1+n_1]} (\ln t)^{\gamma}\eta(t)(\ln t)^{2+\beta-\alpha}\, {}^H\mathcal{D}^{\beta}_{1+}\xi(t), \sup_{t\in(1,1+n_1]} (\ln t)^{-\varrho}\omega(t)(\ln t)^{2+\nu-\alpha}\, {}^H\mathcal{D}^{\nu}_{1+}\xi(t) \Bigg\}.
\end{aligned}
$$

For any $z \in A(\Omega)$, there exists $y \in \Omega$ such that $z(t) = Ay(t)$. For any $t_1, t_2 \in [a_1, 1 + n_1]$ with $t_1 < t_2$, one has

$$
\begin{aligned}
& |z(t_2) - z(t_1)| = |Ay(t_2) - Ay(t_1)| \\
\le & \frac{|x_0|}{\Gamma(\alpha)} \Big| (\ln t_2)^{\alpha-1} - (\ln t_1)^{\alpha-1} \Big| + \frac{|x_1|}{\Gamma(\alpha-1)} \Big| (\ln t_2)^{\alpha-2} - (\ln t_1)^{\alpha-2} \Big| \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_1^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} f(s, y(s), {}^H\mathcal{D}^{\beta}_{1+}y(s), {}^H\mathcal{D}^{\nu}_{1+}y(s)) \frac{ds}{s} \\
& \qquad - \int_1^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s, y(s), {}^H\mathcal{D}^{\beta}_{1+}y(s), {}^H\mathcal{D}^{\nu}_{1+}y(s)) \frac{ds}{s} \Big| \\
\le & \frac{|x_0|}{\Gamma(\alpha)} |\ln t_2 - \ln t_1|^{\alpha-1} + \frac{|x_1|}{\Gamma(\alpha-1)} \Big| (\ln t_2)^{\alpha-2} - (\ln t_1)^{\alpha-2} \Big| \\
& + \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right| \Big| f(s, y(s), {}^H\mathcal{D}^{\beta}_{1+}y(s), {}^H\mathcal{D}^{\nu}_{1+}y(s)) \Big| \frac{ds}{s} \qquad (41) \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \Big| f(s, y(s), {}^H\mathcal{D}^{\beta}_{1+}y(s), {}^H\mathcal{D}^{\nu}_{1+}y(s)) \Big| \frac{ds}{s}. \qquad (42)
\end{aligned}
$$

For convenience, the latter two integrals (41) and (42) are estimated separately.

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right| \Big| f(s, y(s), {}^H\mathcal{D}^{\beta}_{1+}y(s), {}^H\mathcal{D}^{\nu}_{1+}y(s)) \Big| \frac{ds}{s} \\
\le & \frac{1}{\Gamma(\alpha)} |\ln t_2 - \ln t_1|^{\alpha-1} \int_1^{t_1} \left(\varphi(s) + \psi(s)\xi(s) + \eta(s)\,{}^H\mathcal{D}^{\beta}_{1+}\xi(s) + \omega(s)\,{}^H\mathcal{D}^{\nu}_{1+}\xi(s) \right) \frac{ds}{s} \\
\le & \frac{M_{n_1}}{\Gamma(\alpha)} |\ln t_2 - \ln t_1|^{\alpha-1} \Bigg[\frac{(\ln(1+n_1))^{\alpha-1}}{\alpha-1} + \frac{(\ln(1+n_1))^{\alpha-\delta-1}}{\alpha-\delta-1} \\
& \qquad + \frac{(\ln(1+n_1))^{\alpha-\beta-\gamma-1}}{\alpha-\beta-\gamma-1} + \frac{(\ln(1+n_1))^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1} \Bigg],
\end{aligned}
$$

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\left|f(s,y(s),^H\mathcal{D}^{\beta}_{1+}y(s),^H\mathcal{D}^{\nu}_{1+}y(s))\right|\frac{ds}{s}\\ \le&\frac{(\ln(1+n_1))^{\alpha-1}}{\Gamma(\alpha)}M_{n_1}\int_{t_1}^{t_2}\Big[(\ln s)^{\alpha-2}+(\ln s)^{\alpha-2-\delta}+(\ln s)^{\alpha-\beta-2-\gamma}\\ &+(\ln s)^{\alpha-\nu-2+\varrho}\Big]\frac{ds}{s}\\ \le&\frac{(\ln(1+n_1))^{\alpha-1}}{\Gamma(\alpha)}M_{n_1}\Big[\frac{|\ln t_2-\ln t_1|^{\alpha-1}}{\alpha-1}+\frac{|\ln t_2-\ln t_1|^{\alpha-\delta-1}}{\alpha-\delta-1}\\ &+\frac{|\ln t_2-\ln t_1|^{\alpha-\beta-\gamma-1}}{\alpha-\beta-\gamma-1}+\frac{|\ln t_2-\ln t_1|^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1}\Big].\end{aligned}$$

Synthesizing the above inequalities and substituting them into (41) and (42), we further obtain

$$\begin{aligned}&|z(t_2)-z(t_1)|=|Ay(t_2)-Ay(t_1)|\\ \le&\frac{|x_0|}{\Gamma(\alpha)}|\ln t_2-\ln t_1|^{\alpha-1}+\frac{|x_1|}{\Gamma(\alpha-1)}\left|(\ln t_2)^{\alpha-2}-(\ln t_1)^{\alpha-2}\right|\\ &+\frac{M_{n_1}}{\Gamma(\alpha)}|\ln t_2-\ln t_1|^{\alpha-1}\Big[\frac{(\ln(1+n_1))^{\alpha-1}}{\alpha-1}+\frac{(\ln(1+n_1))^{\alpha-\delta-1}}{\alpha-\delta-1}\\ &\quad+\frac{(\ln(1+n_1))^{\alpha-\beta-\gamma-1}}{\alpha-\beta-\gamma-1}+\frac{(\ln(1+n_1))^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1}\Big]\\ &+\frac{(\ln(1+n_1))^{\alpha-1}}{\Gamma(\alpha)}M_{n_1}\Big[\frac{|\ln t_2-\ln t_1|^{\alpha-1}}{\alpha-1}+\frac{|\ln t_2-\ln t_1|^{\alpha-\delta-1}}{\alpha-\delta-1}\\ &\quad+\frac{|\ln t_2-\ln t_1|^{\alpha-\beta-\gamma-1}}{\alpha-\beta-\gamma-1}+\frac{|\ln t_2-\ln t_1|^{\alpha-\nu+\varrho-1}}{\alpha-\nu+\varrho-1}\Big].\end{aligned}\tag{43}$$

Let $t_2\to t_1$, by (43), it follows that $|z(t_2)-z(t_1)|\to 0$. In the same way, we have

$$|^H\mathcal{D}^{\beta}_{1+}z(t_2)-{}^H\mathcal{D}^{\beta}_{1+}z(t_1)|\to 0,\ \ t_2\to t_1.\tag{44}$$

As regards $|^H\mathcal{D}^{\nu}_{1+}z(t_2)-{}^H\mathcal{D}^{\nu}_{1+}z(t_1)|$, we first make the following estimates.

$$\begin{aligned}&\frac{1}{\Gamma(\alpha-\nu)}\int_1^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-\nu-1}-\left(\ln\frac{t_1}{s}\right)^{\alpha-\nu-1}\right|\left|f(s,y(s),^H\mathcal{D}^{\beta}_{1+}y(s),^H\mathcal{D}^{\nu}_{1+}y(s))\right|\frac{ds}{s}\\ \le&\frac{M_{n_1}}{\Gamma(\alpha-\nu)}\int_1^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-\nu-1}-\left(\ln\frac{t_1}{s}\right)^{\alpha-\nu-1}\right|\Big[(\ln s)^{\alpha-2}+(\ln s)^{\alpha-2-\delta}\\ &\quad+(\ln s)^{\alpha-\beta-2-\gamma}+(\ln s)^{\alpha-\nu-2+\varrho}\Big]\frac{ds}{s}\\ \le&\frac{M_{n_1}}{\Gamma(\alpha-\nu+1)}\Big[(\ln a_1)^{\alpha-2}+(\ln a_1)^{\alpha-2-\delta}+(\ln a_1)^{\alpha-\beta-2-\gamma}+(\ln a_1)^{\alpha-\nu-2+\varrho}\Big]\\ &\quad\Big[\left(\ln\frac{t_2}{t_1}\right)^{\alpha-\nu}-((\ln t_2)^{\alpha-\nu}-(\ln t_1)^{\alpha-\nu})\Big],\\ &\frac{1}{\Gamma(\alpha-\nu)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-\nu-1}\left|f(s,y(s),^H\mathcal{D}^{\beta}_{1+}y(s),^H\mathcal{D}^{\nu}_{1+}y(s))\right|\frac{ds}{s}\\ \le&\frac{M_{n_1}}{\Gamma(\alpha-\nu+1)}\Big[(\ln a_1)^{\alpha-2}+(\ln a_1)^{\alpha-2-\delta}+(\ln a_1)^{\alpha-\beta-2-\gamma}\\ &+(\ln a_1)^{\alpha-\nu-2+\varrho}\Big]\left(\ln\frac{t_2}{t_1}\right)^{\alpha-\nu}.\end{aligned}$$

Combining the above conclusions, we infer

$$
\begin{aligned}
&|{}^H\mathcal{D}^{\nu}_{1+}z(t_2) - {}^H\mathcal{D}^{\nu}_{1+}z(t_1)| \\
\leq &\frac{|x_0|}{\Gamma(\alpha-\nu)}\Big|(\ln t_2)^{\alpha-\nu-1} - (\ln t_1)^{\alpha-\nu-1}\Big| + \frac{|x_1|(1+\nu-\alpha)}{\Gamma(\alpha-\nu)}\Big|(\ln t_2)^{\alpha-\nu-2} - (\ln t_1)^{\alpha-\nu-2}\Big| \\
&+ \frac{1}{\Gamma(\alpha-\nu)}\int_1^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-\nu-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-\nu-1}\right|\Big|f(s,y(s),{}^H\mathcal{D}^{\beta}_{1+}y(s),{}^H\mathcal{D}^{\nu}_{1+}y(s))\Big|\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha-\nu)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-\nu-1}\Big|f(s,y(s),{}^H\mathcal{D}^{\beta}_{1+}y(s),{}^H\mathcal{D}^{\nu}_{1+}y(s))\Big|\frac{ds}{s} \\
\leq &\frac{|x_0|}{\Gamma(\alpha-\nu)}\Big|(\ln t_2)^{\alpha-\nu-1} - (\ln t_1)^{\alpha-\nu-1}\Big| + \frac{|x_1|(1+\nu-\alpha)}{\Gamma(\alpha-\nu)}\Big|(\ln t_2)^{\alpha-\nu-2} - (\ln t_1)^{\alpha-\nu-2}\Big| \\
&+ \frac{M_{n_1}}{\Gamma(\alpha-\nu+1)}\Big[(\ln a_1)^{\alpha-2} + (\ln a_1)^{\alpha-2-\delta} + (\ln a_1)^{\alpha-\beta-2-\gamma} \\
&\quad + (\ln a_1)^{\alpha-\nu-2+\varrho}\Big]\left[\left(\ln\frac{t_2}{t_1}\right)^{\alpha-\nu} - ((\ln t_2)^{\alpha-\nu} - (\ln t_1)^{\alpha-\nu})\right] \\
&+ \frac{M_{n_1}}{\Gamma(\alpha-\nu+1)}[(\ln a_1)^{\alpha-2} + (\ln a_1)^{\alpha-2-\delta} \\
&+ (\ln a_1)^{\alpha-\beta-2-\gamma} + (\ln a_1)^{\alpha-\nu-2+\varrho}]\left(\ln\frac{t_2}{t_1}\right)^{\alpha-\nu} \to 0,\ \ t_2 \to t_1.
\end{aligned}
\tag{45}
$$

Whence, by virtue of (43)–(45), the condition (ii) is satisfied.

Now let us verify that condition (iii) is also true. Note the fact $\lim_{t\to 1+} \ln t = 0$, combining this limit with (27), we conclude that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for all $t \in (1, 1+\delta_1)$ one has

$$
\frac{|x_0|}{\min\{\Gamma(\alpha),\Gamma(\alpha-\beta)\}}\ln t < \frac{\varepsilon}{2},
$$

and

$$
\begin{aligned}
&\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}(\varphi(s)+\psi(s)\xi(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s){}^H\mathcal{D}^{\nu}_{1+}\xi(s))\frac{ds}{s} < \frac{\varepsilon}{2}, \\
&\frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\beta-1}(\varphi(s)+\psi(s)\xi(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s){}^H\mathcal{D}^{\nu}_{1+}\xi(s))\frac{ds}{s} < \frac{\varepsilon}{2}, \\
&\frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}(\varphi(s)+\psi(s)\xi(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s){}^H\mathcal{D}^{\nu}_{1+}\xi(s))\frac{ds}{s} < \frac{\varepsilon}{2}.
\end{aligned}
$$

For any $y \in \Omega$, by (27) and Hypothesis 2, we infer that

$$
\begin{aligned}
&\lim_{t\to 1+}(\ln t)^{2-\alpha}Ay(t) = \frac{x_1}{\Gamma(\alpha-1)},\ \ \lim_{t\to 1+}(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}Ay(t) = \frac{x_1}{\Gamma(\alpha-\beta-1)}, \\
&\lim_{t\to 1+}(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}Ay(t) = \frac{x_1}{\Gamma(\alpha-\nu-1)}.
\end{aligned}
$$

Consequently, for any $y \in \Omega, t \in (1, 1+\delta_1)$, we have

$$
\begin{aligned}
&|(\ln t)^{2-\alpha}Ay(t) - \lim_{t\to 1+}(\ln t)^{2-\alpha}Ay(t)| \\
\leq &\frac{|x_0|}{\Gamma(\alpha)}\ln t + \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\Big(\varphi(s)+\psi(s)\xi(s)+\eta(s){}^H\mathcal{D}^{\beta}_{1+}\xi(s) \\
&+\omega(s){}^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon, \\
&\Big|(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}Ay(t) - \lim_{t\to 1+}(\ln t)^{2+\beta-\alpha}\,{}^H\mathcal{D}^{\beta}_{1+}Ay(t)\Big|
\end{aligned}
$$

$$\leq \frac{|x_0|}{\Gamma(\alpha-\beta)}\ln t + \frac{(\ln t)^{2+\beta-\alpha}}{\Gamma(\alpha-\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-\beta-1}\Big(\varphi(s)+\psi(s)\xi(s)$$
$$+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\leq\varepsilon,$$

and

$$\left|(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}Ay(t)-\lim_{t\to1+}(\ln t)^{2+\nu-\alpha}\,{}^H\mathcal{D}^{\nu}_{1+}Ay(t)\right|$$
$$\leq \frac{|x_0|}{\Gamma(\alpha-\nu)}\ln t + \frac{(\ln t)^{2+\nu-\alpha}}{\Gamma(\alpha-\nu)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-\nu-1}\Big(\varphi(s)+\psi(s)\xi(s)$$
$$+\eta(s)^H\mathcal{D}^{\beta}_{1+}\xi(s)+\omega(s)^H\mathcal{D}^{\nu}_{1+}\xi(s)\Big)\frac{ds}{s}\leq\varepsilon.$$

When $\beta=\alpha-1$, the continuity and compactness of the operator A are obtained in the same way, omitted here.

To sum up, $A:\Omega\to\Omega$ is continuous and $A(\Omega)$ is relatively compact in X. According to Schäuder's fixed point theorem, A has at least one fixed point in Ω, then the initial value problem (4) has at least one solution. □

The subsequent example will serve to further substantiate the practicality and validity of Theorem 8.

Example 1. *Consider the following initial value problem of the fractional differential equation:*

$$\begin{cases} {}^H\mathcal{D}^{\frac{9}{5}}_{1+}u(t)=f(t,u(t),{}^H\mathcal{D}^{\frac{1}{5}}_{1+}u(t),{}^H\mathcal{D}^{\frac{6}{5}}_{1+}u(t)),\\ {}^H\mathcal{D}^{\frac{4}{5}}_{1+}u(1)=1,\\ {}^H\mathcal{J}^{\frac{1}{5}}_{1+}u(1)=-1, \end{cases}\tag{46}$$

where $f(t,x,y,z)=\frac{\frac{4}{5}}{\Gamma(\frac{3}{5})(\ln t)^{\frac{1}{5}}}(1+\frac{2}{t})-\frac{\frac{4}{5}}{\Gamma(\frac{3}{5})}(\ln t)^{-\frac{1}{5}}\left(1+\frac{1}{t}\right)+\left(\frac{1}{(\ln t)^{\frac{3}{10}}}+t^2\right)|x|^{\frac{1}{2}}\ln(1+|x|^{\frac{1}{2}})+\frac{1+\sqrt{t}}{(\ln t)^{\frac{1}{10}}}|y|^{\frac{2}{3}}\ln(1+|y|^{\frac{1}{3}})+(\ln t)^{\frac{6}{5}}\left(1+\frac{1}{t}\right)|z-\frac{\frac{2}{5}}{\Gamma(\frac{3}{5})}(\ln t)^{-\frac{7}{5}}|$. *Let* $\varphi(t)=\frac{\frac{4}{5}}{\Gamma(\frac{3}{5})(\ln t)^{\frac{1}{5}}}(1+\frac{2}{t})$, $\psi(t)=\frac{1}{(\ln t)^{\frac{3}{10}}}+t^2$, $\eta(t)=\frac{1+\sqrt{t}}{(\ln t)^{\frac{1}{10}}}$, $\omega(t)=(\ln t)^{\frac{6}{5}}\left(1+\frac{1}{t}\right)$, *and* $\alpha=\frac{9}{5},\beta=\frac{1}{5},\nu=\frac{6}{5},\delta=\frac{3}{10},\gamma=\frac{1}{10},\varrho=\frac{9}{10},x_0=1,x_1=-1,\mu_0=1,\mu_1=2$, *then all the parameters* α,β,ν *and the functions* φ,ψ,η,ω *satisfy Hypothesis 1 and* $\min\{2\alpha-\nu-\delta,2\alpha-\nu-\beta-\gamma(0<\beta<\alpha-1),2\alpha-2\nu+\varrho\}>2$. *Moreover,* $|f(t,x,y,z)|\leq\varphi(t)+\psi(t)|x|+\eta(t)|y|+\omega(t)(|z-x_1\vartheta_{\alpha-\nu-1}(t)|+\mu_1\vartheta_{\alpha-\nu-1}(t)),(t,x,y,z)\in J\times\mathbb{R}^3$, *then the hypothesis Hypothesis 2 holds. Theorem 8 guarantees that the initial value problem (46) has at least one solution* $u\in X$.

5. Conclusions

This study investigates the existence of solutions to the initial value problem associated with a Hadamard-type fractional order differential equation on an infinite interval. The equation's nonlinear term incorporates lower-order derivatives of the unknown functions. Initially, a weak singular inequality for Hadamard fractional integrals with a doubly singular kernel is derived, and subsequently applied to demonstrate the existence of a unique solution to the integral equation corresponding to the original initial value problem. Rather than employing the traditional approach of establishing global solutions for differential equations on infinite intervals, a fixed point theorem on a metrizable complete locally convex space is utilized to establish the existence of at least one solution to the initial value problem.

Author Contributions: Conceptualization, W.L.; methodology, W.L. and L.L.; resources, W.L. and L.L.; funding acquisition, L.L.; writing—original draft preparation, W.L.; writing—review and editing, L.L.; supervision, L.L.; project administration, W.L. and L.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Natural Science Foundation of China (11871302) and the ARC Discovery Project Grant (DP230102079).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors thank the reviewers for their useful comments, which led to improvement of the content of the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Kou, C.; Zhou, H.; Yan, Y. Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 5975–5986. [CrossRef]
2. Kou, C.; Zhou, H.; Li, C. Existence and continuation theorems of Riemann–Liouville type fractional differential equations. *Int. J. Bifurcat. Chaos* **2012**, *22*, 1250077. [CrossRef]
3. Trif, T. Existence of solutions to initial value problems for nonlinear fractional differential equations on the semi-axis. *Fract. Calc. Appl. Anal.* **2013**, *16*, 595–612. [CrossRef]
4. Liu, Y. Existence and uniqueness of solutions for a class of initial value problems of fractional differential systems on half lines. *Bull. Sci. Math.* **2013**, *137*, 1048–1071 [CrossRef]
5. Toumi, F.; Zine El Abidine, Z. Existence of multiple positive solutions for nonlinear fractional boundary value problems on the half-line. *Mediterr. J. Math.* **2016**, *13*, 2353–2364. [CrossRef]
6. Zhu, T.; Zhong, C.; Song, C. Existence results for nonlinear fractional differential equations in C [0, T). *J. Appl. Math. Comput.* **2018**, *57*, 57–68. [CrossRef]
7. Tuan, H.T.; Czornik, A.; Nieto, J.J.; Niezabitowski, M. Global attractivity for some classes of Riemann-Liouville fractional differential systems. *J. Integral. Equ. Appl.* **2019**, *31*, 265–282. [CrossRef]
8. Boucenna, D.; Boulfoul, A.; Chidouh, A.; Ben, Makhlouf, A.; Tellab, B. Some results for initial value problem of nonlinear fractional equation in Sobolev space. *J. Appl. Math. Comput.* **2021**, *67*, 605–621. [CrossRef]
9. Zhang, S.Q.; Hu, L. Unique Existence Result of Approximate Solution to Initial Value Problem for Fractional Differential Equation of Variable Order Involving the Derivative Arguments on the Half-Axis. *Mathematics* **2019**, *7*, 286. [CrossRef]
10. Chen, P.; Li, Y.; Chen, Q.; Feng, B. On the initial value problem of fractional evolution equations with noncompact semigroup. *Comput. Math. Appl.* **2014**, *67*, 1108–1115. [CrossRef]
11. Zhu, T. Fractional integral inequalities and global solutions of fractional differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2020**, *5*, 1–16. [CrossRef]
12. Zhu, T. Weakly Singular Integral Inequalities and Global Solutions for Fractional Differential Equations of Riemann–Liouville Type. *Mediterr. J. Math.* **2021**, *18*, 184. [CrossRef]
13. Zhao, X.; Ge, W. Unbounded solutions for a fractional boundary value problems on the infinite interval. *Acta Appl. Math.* **2010**, *109*, 495–505. [CrossRef]
14. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
15. Webb, J.R.L. Weakly singular Gronwall inequalities and applications to fractional differential equations. *J. Math. Anal. Appl.* **2019**, *471*, 692–711. [CrossRef]
16. Mitrinovic, D.S.; Pecaric, J.; Fink, A.M. *Inequalities Involving Functions and Their Integrals and Derivatives*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 1991.
17. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; pp. 110–120.
18. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993; pp. 329–333.
19. Sousa, J.V.; Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
20. Munkres, J.R. *Topology*, 2nd ed; Pearson Education, Inc.: London, UK, 2014; pp. 75–83.
21. Willard, S. *General Topology*; Addison-Wesley Publishing Company: Boston, MA, USA, 1970; pp. 23–39.

22. Conway, J.B. *A course in Functional Analysis*, 2nd ed; Springer: New York, NY, USA, 2007; pp. 99–106.
23. Yosida, B.K. *Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 2012; pp. 23–28.