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Monotone Positive Radial Solution of Double Index Logarithm Parabolic Equations

Mengru Liu and Lihong Zhang *

School of Mathematics and Computer Science, Shanxi Normal University, Taiyuan 030031, China; liumr_yx@163.com
* Correspondence: zhanglih149@126.com

Abstract: This article mainly studies the double index logarithmic nonlinear fractional g-Laplacian parabolic equations with the Marchaud fractional time derivatives ∂_t^α. Compared with the classical direct moving plane method, in order to overcome the challenges posed by the double non-locality of space-time and the nonlinearity of the fractional g-Laplacian, we establish the unbounded narrow domain principle, which provides a starting point for the moving plane method. Meanwhile, for the purpose of eliminating the assumptions of boundedness on the solutions, the averaging effects of a non-local operator are established; then, these averaging effects are applied twice to ensure that the plane can be continuously moved toward infinity. Based on the above, the monotonicity of a positive solution for the above fractional g-Laplacian parabolic equations is studied.

Keywords: double index logarithm nonlinear parabolic equations; monotonicity; fractional g-Laplacian; averaging effects; Marchaud fractional time derivatives; narrow region principle

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1. Introduction

In this paper, we mainly study the double index logarithmic fractional g-Laplacian parabolic equations with the Marchaud fractional time derivatives

$$\begin{cases}\partial_t^\alpha u(z,t)+(-\Delta_g)^s u(z,t)=\ln|u^q(z,t)+1|+\ln|u^p(z,t)+1|, & \text{in } \mathbb{R}^n_+\times\mathbb{R},\\ u(z,t)=0, & \text{in } \left(\mathbb{R}^n\backslash\mathbb{R}^n_+\right)\times\mathbb{R},\end{cases} \tag{1}$$

where $\mathbb{R}^n_+ := \{z\in\mathbb{R}^n \mid z_1>0\}$ represents the right half space and $p,q\ge 1$.

The Marchaud fractional time derivative ∂_t^α is defined as

$$\partial_t^\alpha u(z,t) := C_\alpha\int_{-\infty}^t \frac{u(z,t)-u(z,\tau)}{(t-\tau)^{1+\alpha}}d\tau,\ \text{for order } \alpha\in(0,1), \tag{2}$$

where $C_\alpha=\frac{\alpha}{\Gamma(1-\alpha)}$ represents the normalized positive constant. To make sense of the integral in Equation (2), let $u(z,t)\in C^1(\mathbb{R})\times L^-_\alpha(\mathbb{R})$, where

$$L^-_\alpha(\mathbb{R}) := \{u\in L^1_{loc}(\mathbb{R})|\int_{-\infty}^t \frac{|u(z,\rho)|}{1+|\rho|^{1+\alpha}}d\rho<+\infty \text{ for any } t\in\mathbb{R}\}.$$

The Marchaud fractional time derivative was introduced by Marchaud in 1927 [1]. The time non-locality explains the historical dependence introduced in dynamics due to abnormally large waiting times. The introduction of the Marchaud fractional time derivative can better describe some complex phenomena in the real world, such as nonlinear effects of media and memory effects. The background and properties of the Marchaud fractional time derivative can be referred to in [2–4]. This fractional time derivative is widely used in various fields. In finance, a fractional time derivative can be used to solve the optimal portfolio problem of investors [5]. In continuum mechanics, a fractional operator has a clear mechanical explanation by the definition of fractional derivatives [6]. In physical phenomena, it is used to describe magneto-thermoelastic heat conduction [7].

Among the non-local nonlinear operators with non-standard growth that occur naturally in fractional Orlicz–Sobolev spaces, the most notable of which is the fractional g-Laplacian. The spatial non-locality of the fractional g-Laplacian explains that the behavior of a point in the system is affected by a distant position in space; that is, there are non-local effects in the system. More background on the fractional g-Laplacian can be found in references [8,9]. The fractional g-Laplacian $(-\Delta_g)^s$ has received increasing attention in recent years because it can simulate the non-power behavior of non-local problems. For some interesting results, we can refer to [10–13] and the references therein.

Define fractional g-Laplacian $(-\Delta_g)^s$ [14],

$$(-\Delta_g)^s u(z,t) := \text{P.V.} \int_{\mathbb{R}^n} g\left(\frac{u(z,t)-u(\vartheta,t)}{|z-\vartheta|^s}\right)\frac{d\vartheta}{|z-\vartheta|^{n+s}}, \tag{3}$$

where P.V. represents the integral principal value and $g = G'$ corresponds to the derivative of a Young function, G. That is,

$$G(t) = \int_0^t g(\tau)d\tau.$$

The properties of g are as follows:

(1) $g(t) > 0$ for $t > 0$;
(2) $g(a) + g(b) \geq cg(a+b)$, where $c > 0$ is a constant;
(3) $g(a) - g(b) \geq dg(a-b)$, where $d > 0$ is a constant;
(4) g is nondecreasing on $(0, \infty)$;
(5) $g(-t) = -g(t)$;
(6) $g' > 0$, since G is convex.

Because of the non-locality of the fractional g-Laplacian $(-\Delta_g)^s$, the behavior of u at infinity needs to be properly controlled when dealing with the operator. We will define

$$L_g(\mathbb{R}^n) := \left\{ u \in L^1_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} g\left(\frac{|u(z,t)|}{1+|z|^s}\right)\frac{dz}{1+|z|^{n+s}} < \infty \right\}.$$

This guarantees that the operator is well defined. In addition, when $G(t) = t^2$, this corresponds to the fractional Laplacian. When $G(t) = t^p$, it is the fractional p-Laplacian.

The moving plane method introduced by Alexandrof is mainly used to study local elliptic and parabolic equations. But due to the non-local property of the fractional Laplace operator, the traditional moving plane method is not suitable for pseudo-differential equations containing a fractional Laplace operator. In order to resolve this dilemma, Caffarelli and Silvestr [15] developed an extended method for converting non-local questions into local questions in high-dimensional spaces, which has been successfully applied to equations with fractional Laplacian (see [16–18] and the references therein). Alternatively, by transforming a given pseudo-differential equation into its equivalent integral equation, the properties of the solutions are studied by applying the moving plane method in the integral form and the regularity lifting. The results of this method can be referred to in [19–22]. However, when employing the extension method or the corresponding integral method, it is necessary to add some additional conditions or restrictions. After that, in [23], Chen, Li, and Li made a further breakthrough in this field by introducing a method of moving the plane directly, thereby eliminating these limitations and simplifying the proof process. Subsequently, this effective direct method has been extensively applied in analyzing the symmetries, monotonicity, and nonexistence of various elliptic equations and systems (see [24–28] and the references therein). But due to the difficulty caused by double non-locality in space-time, the study of the geometric properties of the solutions for space-time fractional nonlinear parabolic equations is very scarce. Until 2023, Chen and Li studied the monotonicity of the solutions for dual fractional nonlinear parabolic equations by using the direct moving plane method in [29]. Inspired by the above ideas, we will investigate a monotone positive radial solution of the double index logarithmic nonlinear fractional g-Laplacian parabolic Equations (1) with the Marchaud fractional time derivatives ∂_t^α. We successfully address the challenges arising from the double non-locality of space-time and the nonlinearity of the fractional g-Laplacian in this equation.

In contrast to the previous approach of taking limits along a subsequence of $\{w_{\Lambda_k}\}$, we utilize the method of average effects to eliminate the assumption of boundedness on the solution. We believe that this method will become a valuable tool in studying unbounded solution sequences. The structure of this article is as follows: In Section 2, we mainly prove the narrow region principle of the antisymmetric function and some maximum principles, which provides a starting point for the moving plane method. In Section 3, for the purpose of eliminating the assumptions of boundedness on the solutions, the averaging effects of the non-local operator are established. In Section 4, the main result of this paper is proved by using the direct moving plane method; that is, the positive solution of Equation (1) is strictly increasing in the z_1-direction for any $t \in \mathbb{R}$.

Notations. The z_1-direction can be any direction.

$$T_\Lambda := \{z = (z_1, z') \in \mathbb{R}^n \mid z_1 = \Lambda \text{ for } \Lambda \in \mathbb{R}\}$$

is the moving planes.

$$\Sigma_\Lambda := \{z \in \mathbb{R}^n \mid z_1 < \Lambda\} \text{ and } \Omega_\Lambda := \{z \in \mathbb{R}^n_+ \mid z_1 < \Lambda\}$$

are the regions to the left of the hyperplane T_Λ in $\mathbb{R}^n$ and in $\mathbb{R}^n_+$, respectively.

$$z^\Lambda := (2\Lambda - z_1, z_2, \cdots, z_n)$$

is the reflection point of z about T_Λ. $u(z,t)$ is a solution of (1) and $u_\Lambda(z,t) := u(z^\Lambda, t)$.

Denote

$$w_\Lambda(z,t) := u_\Lambda(z,t) - u(z,t).$$

2. Maximum Principle

In this section, we mainly prove the following four theorems: the four theorems are the narrow region principle (Theorem 1) and maximum principle (Theorem 2) of an antisymmetric function on an unbounded domain, and the maximum principle (Theorem 3) and maximum principle of an antisymmetric function (Theorem 4) on a bounded domain. From this point on, C represents a constant that may differ between each line, and only the related dependencies are explained later. And C_i is the positive constant throughout the article.

Theorem 1. *Suppose that Ω is an unbounded narrow region contained within $\{z \in \Sigma_\Lambda | \Lambda - 2l < z_1 < \Lambda\}$ for some small l and*

$$w_\Lambda \in (C^{1,1}_{loc}(\Omega) \cap L_g(\mathbb{R}^n)) \times (C^1(\mathbb{R}) \cap L^-_\alpha(\mathbb{R}))$$

is lower semi-continuous with respect to z on $\overline{\Omega}$.

If

$$w_\Lambda(z,t) \geq -C(1 + |z|^\nu) \text{ for some } 0 < \nu < 2s \tag{4}$$

and

$$\begin{cases} \partial_t^\alpha w_\Lambda(z,t) + (-\Delta_g)^s u_\Lambda(z,t) - (-\Delta_g)^s u(z,t) = \left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right) w_\Lambda(z,t), & (z,t) \in \Omega \times \mathbb{R}, \\ w_\Lambda(z,t) \geq 0, & (z,t) \in (\Sigma_\Lambda \backslash \Omega) \times \mathbb{R}, \\ w_\Lambda(z,t) = -w_\Lambda(z^\Lambda, t), & (z,t) \in \Sigma_\Lambda \times \mathbb{R}. \end{cases} \tag{5}$$

where $\xi_1(z,t)$ and $\xi_2(z,t)$ fall in-between $u_\Lambda(z,t)$ and $u(z,t)$, then

$$w_\Lambda(z,t) \geq 0, \ in \ \Sigma_\Lambda \times \mathbb{R}, \tag{6}$$

for l small enough. Moreover, if $w_\Lambda(z^0,t_0) = 0$ for some point $(z^0,t_0) \in \Omega \times \mathbb{R}$, then

$$w_\Lambda(z,t) \equiv 0, \ in \ \mathbb{R}^n \times (-\infty, t_0]. \tag{7}$$

Proof. To obtain Equation (6), we will use proof by contradiction. Since condition (4) may cause $w_\Lambda(z,t)$ to reach negative infinity when $|x| \to \infty$, then $w_\Lambda(z,t)$ may not reach the minimum in z. To solve this difficulty, define

$$h(z) := \left[\left(1 - \frac{(z_1 - (\Lambda - l))^2}{l^2}\right)_+^s + 1\right]\left(1 + |z'|^2\right)^{\frac{\theta}{2}}$$

for some $\nu < \theta < 2s$. Hence, we obtain

$$\lim_{|z|\to\infty} \overline{w}_\Lambda(z,t) := \lim_{|z|\to\infty} \frac{w_\Lambda(z,t)}{h(z)} \geq 0. \tag{8}$$

Assume that there exists a point $z \in \Omega$ such that $\overline{w}_\Lambda(z,t) < 0$ for every fixed $t \in \mathbb{R}$; then, there must be $z(t) \in \Omega$ such that

$$\overline{w}_\Lambda(z(t),t) = \min_{z\in\Omega} \overline{w}_\Lambda(z,t) < 0. \tag{9}$$

By Equation (4), $\nu < \theta$, and the definition of $\overline{w}_\Lambda(z,t)$, it follows that $\overline{w}_\Lambda(z(t),t)$ is bounded.

Therefore, if Equation (6) does not hold, there must exist a constant $m > 0$ such that

$$\inf_{\Omega\times\mathbb{R}} \overline{w}_\Lambda(z,t) = \inf_{\mathbb{R}} \overline{w}_\Lambda(z(t),t) = -m < 0. \tag{10}$$

This means that there is a sequence, $\{(z^k,t_k)\} \subset \Omega \times \mathbb{R}$, and it holds

$$\overline{w}_\Lambda(z^k,t_k) = -m_k \to -m \text{ as } k \to \infty.$$

Let $\varepsilon_k := m - m_k$. It is obvious that $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$.

Because the minimum of $\overline{w}_\Lambda(z(t),t)$ may not be reached when $t \in \mathbb{R}$, to address this challenge, let

$$v_k(z,t) := \overline{w}_\Lambda(z,t) - \varepsilon_k \eta_k(t),$$

where

$$\eta_k(t) = \eta(t - t_k) \in C_0^\infty(-2 + t_k, 2 + t_k),$$

represents a smooth cut-off function, satisfying

$$\eta_k(t)\begin{cases} = 1, & t \in [-1 + t_k, 1 + t_k], \\ \in [0,1], & t \in (-2 + t_k, -1 + t_k) \cup (1 + t_k, 2 + t_k), \\ = 0, & t \notin (-2 + t_k, 2 + t_k). \end{cases}$$

We have

$$v_k(z^k,t_k) = \overline{w}_\Lambda(z^k,t_k) - \varepsilon_k = -m_k - m + m_k = -m.$$

Consequently, there exists $(\bar{z}^k,\bar{t}_k) \in \Omega \times (-2 + t_k, 2 + t_k)$ such that

$$-m - \varepsilon_k \leq v_k(\bar{z}^k,\bar{t}_k) = \inf_{\Omega\times\mathbb{R}} v_k(z,t) \leq -m. \tag{11}$$

It follows that

$$-m \leq \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k) \leq -m + \varepsilon_k = -m_k < 0. \tag{12}$$

By direct calculation, we have

$$\partial_t^\alpha v_k(\overline{z}^k, \overline{t}_k) = C_\alpha \int_{-\infty}^{\overline{t}_k} \frac{v_k(\overline{z}^k, \overline{t}_k) - v_k(\overline{z}^k, \tau)}{(\overline{t}_k - \tau)^{1+\alpha}} d\tau \leq 0.$$

By Lemma 5.1 in [29] and the definition of $v_k(z,t)$, it follows that

$$\partial_t^\alpha \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k) \leq \varepsilon_k \partial_t^\alpha \eta_k(\overline{t}_k) \leq C\varepsilon_k, \tag{13}$$

here, C is a positive constant. In turn, by the properties of g, and Equations (5), (9), and (12), $|\overline{z}^k - \vartheta| < |\overline{z}^k - \vartheta^\Lambda|$ and $h(\vartheta) > h(\vartheta^\Lambda)$ for $\vartheta \in \Sigma_\Lambda$, and we have

$$\begin{aligned}
&(-\Delta_g)^s u_\Lambda(\overline{z}^k, \overline{t}_k) - (-\Delta_g)^s u(\overline{z}^k, \overline{t}_k) \\
=& \text{P.V.} \int_{\mathbb{R}^n} \left[g\left(\frac{u_\Lambda(\overline{z}^k, \overline{t}_k) - u_\Lambda(\vartheta, \overline{t}_k)}{|\overline{z}^k - \vartheta|^s} \right) - g\left(\frac{u(\overline{z}^k, \overline{t}_k) - u(\vartheta, \overline{t}_k)}{|\overline{z}^k - \vartheta|^s} \right) \right] \frac{d\vartheta}{|\overline{z}^k - \vartheta|^{n+s}} \\
=& \text{P.V.} \int_{\mathbb{R}^n} \frac{g'(\xi(\vartheta, \overline{t}_k)) \left(w_\Lambda\left(\overline{z}^k, \overline{t}_k\right) - w_\Lambda(\vartheta, \overline{t}_k) \right)}{\left|\overline{z}^k - \vartheta\right|^{n+2s}} d\vartheta \\
=& \text{P.V.} \int_{\mathbb{R}^n} \frac{g'(\xi(\vartheta, \overline{t}_k)) \left(\overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) h\left(\overline{z}^k\right) - \overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) h(\vartheta) + \overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) h(\vartheta) - \overline{w}_\Lambda(\vartheta, \overline{t}_k) h(\vartheta) \right)}{\left|\overline{z}^k - \vartheta\right|^{n+2s}} d\vartheta \\
=& C_1 \text{P.V.} \int_{\mathbb{R}^n} \frac{h(\vartheta) \left(\overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) - \overline{w}_\Lambda(\vartheta, \overline{t}_k) \right)}{\left|\overline{z}^k - \vartheta\right|^{n+2s}} d\vartheta + C_2 \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k)(-\Delta)^s h(\overline{z}^k) \\
\leq& C_1 \int_{\Sigma_\Lambda} \frac{h(\vartheta) \overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) - w_\Lambda(\vartheta, \overline{t}_k)}{\left|\overline{z}^k - \vartheta^\Lambda\right|^{n+2s}} d\vartheta + C_1 \int_{\Sigma_\Lambda} \frac{h(\vartheta^\Lambda) \overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right) + w_\Lambda(\vartheta, \overline{t}_k)}{\left|\overline{z}^k - \vartheta^\Lambda\right|^{n+2s}} d\vartheta \\
&+ C_2 \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k)(-\Delta)^s h(\overline{z}^k) \\
\leq& C_1 \int_{\Sigma_\Lambda} \frac{2h(\vartheta^\Lambda) \overline{w}_\Lambda\left(\overline{z}^k, \overline{t}_k\right)}{\left|\overline{z}^k - \vartheta^\Lambda\right|^{n+2s}} d\vartheta + C_2 \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k)(-\Delta)^s h(\overline{z}^k) \\
\leq& C_2 \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k)(-\Delta)^s h(\overline{z}^k),
\end{aligned}$$

where

$$\frac{u_\Lambda(\overline{z}^k, \overline{t}_k) - u_\Lambda(\vartheta, \overline{t}_k)}{|\overline{z}^k - \vartheta|^s} < \xi(\vartheta, \overline{t}_k) < \frac{u(\overline{z}^k, \overline{t}_k) - u(\vartheta, \overline{t}_k)}{|\overline{z}^k - \vartheta|^s}$$

and $C_1, C_2 > 0$. Then, by Lemma 2.1 in [30], we can show that there is a positive constant C_3 such that

$$\frac{(-\Delta)^s h(z)}{h(z)} \geq \frac{C_3}{l^{2s}} \text{ for all } \Lambda - 2l < z_1 < \Lambda \text{ with sufficently small } l.$$

We have

$$(-\Delta_g)^s u_\Lambda(\overline{z}^k, \overline{t}_k) - (-\Delta_g)^s u(\overline{z}^k, \overline{t}_k) \leq \frac{C_2 C_3}{l^{2s}} h(\overline{z}^k) \overline{w}_\Lambda(\overline{z}^k, \overline{t}_k). \tag{14}$$

Combining Equations (5), (13), and (14) and the boundedness of $\left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1} \right)$ from above, we obtain

$$
\begin{aligned}
-\frac{C_2C_3}{l^{2s}}m_k h(\bar{z}^k) &\geq \frac{C_2C_3}{l^{2s}}h(\bar{z}^k)\overline{w}_\Lambda(\bar{z}^k,\bar{t}_k) \\
&\geq (-\Delta_g)^s u_\Lambda(\bar{z}^k,\bar{t}_k) - (-\Delta_g)^s u(\bar{z}^k,\bar{t}_k) \\
&= -\partial_t^\alpha w_\Lambda(\bar{z}^k,\bar{t}_k) + \left(\frac{q\xi_1^{q-1}\left(\bar{z}^k,\bar{t}_k\right)}{\xi_1^q\left(\bar{z}^k,\bar{t}_k\right)+1} + \frac{p\xi_2^{p-1}\left(\bar{z}^k,\bar{t}_k\right)}{\xi_2^p\left(\bar{z}^k,\bar{t}_k\right)+1}\right) w_\Lambda(\bar{z}^k,\bar{t}_k) \\
&= -h(\bar{z}^k)\partial_t^\alpha \overline{w}_\Lambda(\bar{z}^k,\bar{t}_k) + \left(\frac{q\xi_1^{q-1}\left(\bar{z}^k,\bar{t}_k\right)}{\xi_1^q\left(\bar{z}^k,\bar{t}_k\right)+1} + \frac{p\xi_2^{p-1}\left(\bar{z}^k,\bar{t}_k\right)}{\xi_2^p\left(\bar{z}^k,\bar{t}_k\right)+1}\right) h(\bar{z}^k)\overline{w}_\Lambda(\bar{z}^k,\bar{t}_k) \\
&\geq -C\varepsilon_k h(\bar{z}^k) - Cmh(\bar{z}^k).
\end{aligned}
$$

We deduce that

$$\frac{C_2C_3}{l^{2s}} \leftarrow \frac{C_2C_3}{l^{2s}}\frac{m_k}{m} \leq \frac{C\varepsilon_k}{m} + C \to C \text{ as } k \to \infty.$$

We obtain the contradiction for l that is small enough. Then, Equation (6) is verified. Next, we prove Equation (7). As a consequence of Equation (6), we can conclude that

$$w_\Lambda(z^0,t_0) = \min_{\Sigma_\Lambda\times\mathbb{R}} w_\Lambda(z,t) = 0.$$

If $w_\Lambda(z,t_0) \not\equiv 0$ in Σ_Λ, by calculation, we obtain

$$\partial_t^\alpha w_\Lambda(z^0,t_0) = -C_\alpha \int_{-\infty}^{t_0} \frac{w_\Lambda(z^0,\tau)}{(t_0-\tau)^{1+\alpha}}d\tau \leq 0$$

and

$$
\begin{aligned}
&(-\Delta_g)^s u_\Lambda(z^0,t_0) - (-\Delta_g)^s u(z^0,t_0) \\
&= \text{P.V.}\int_{\mathbb{R}^n}\left[g\left(\frac{u_\Lambda(z^0,t_0)-u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s}\right) - g\left(\frac{u(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta|^s}\right)\right]\frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
&= \text{P.V.}\int_{\Sigma_\Lambda}\left[g\left(\frac{u_\Lambda(z^0,t_0)-u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s}\right) - g\left(\frac{u(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta|^s}\right)\right]\frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
&\quad + \int_{\Sigma_\Lambda}\left[g\left(\frac{u_\Lambda(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s}\right) - g\left(\frac{u(z^0,t_0)-u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s}\right)\right]\frac{d\vartheta}{|z^0-\vartheta^\Lambda|^{n+s}} \\
&=: I_1 + I_2,
\end{aligned}
$$

here,

$$
\begin{aligned}
I_1 :=& \int_{\Sigma_\Lambda}\left[g\left(\frac{\left(u_\Lambda(z^0,t_0)-u_\Lambda(\vartheta,t_0)\right)}{|z^0-\vartheta|^s}\right) - g\left(\frac{\left(u(z^0,t_0)-u(\vartheta,t_0)\right)}{|z^0-\vartheta|^s}\right)\right] \\
&\left(\frac{1}{|z^0-\vartheta|^{n+s}} - \frac{1}{|z^0-\vartheta^\Lambda|^{n+s}}\right)d\vartheta
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
I_2 :=& \int_{\Sigma_\Lambda}\left[g\left(\frac{u_\Lambda(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s}\right) - g\left(\frac{u(z^0,t_0)-u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s}\right)\right. \\
&\left.+g\left(\frac{u_\Lambda(z^0,t_0)-u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s}\right) - g\left(\frac{u(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta|^s}\right)\right]\frac{d\vartheta}{|z^0-\vartheta^\Lambda|^{n+s}}.
\end{aligned}
\tag{16}
$$

Because of

$$\frac{1}{|z^0-\vartheta|^{n+\alpha}} - \frac{1}{|z^0-\vartheta^\Lambda|^{n+\alpha}} > 0$$

and

$$w_\Lambda(z^0,t_0) - w_\Lambda(\vartheta,t_0) = u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0) - (u(z^0,t_0) - u(\vartheta,t_0)) < 0,$$

we obtain

$$g\left(\frac{(u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0))}{|z^0-\vartheta|^s}\right) - g\left(\frac{(u(z^0,t_0) - u(\vartheta,t_0))}{|z^0-\vartheta|^s}\right) < 0,$$

it follows that $I_1 < 0$. For the other term, we have

$$I_2 \le w_\Lambda(z^0,t_0)\int_{\Sigma_\Lambda} \frac{g'(\zeta_1(\vartheta,t)) + g'(\zeta_2(\vartheta,t))}{|z^0 - \vartheta^\Lambda|^{n+2s}} d\vartheta = 0,$$

with

$$\zeta_1(\vartheta,t) \text{ between } \frac{u_\Lambda(z^0,t_0) - u(\vartheta,t_0)}{|z^0 - \vartheta^\Lambda|^s} \text{ and } \frac{u(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0 - \vartheta^\Lambda|^s}$$

and

$$\zeta_2(\vartheta,t) \text{ between } \frac{u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s} \text{ and } \frac{u(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta|^s}.$$

Combining $I_1 < 0$ and $I_2 = 0$, we obtain

$$(-\Delta_g)^s u_\Lambda(z^0,t_0) - (-\Delta_g)^s u(z^0,t_0) < 0,$$

then

$$\partial_t^\alpha w_\Lambda(z^0,t_0) + (-\Delta_g)^s u_\Lambda(z^0,t_0) - (-\Delta_g)^s u(z^0,t_0) < 0.$$

This contradiction aligns with Equation (5). It holds that $w_\Lambda(z,t_0) \equiv 0$ in Σ_Λ. In addition, by the antisymmetry of $w_\Lambda(z,t)$ in z, we have

$$w_\Lambda(z,t_0) \equiv 0 \text{ in } \mathbb{R}^n.$$

Therefore, for $z \in \Sigma_\Lambda$ such that $w_\Lambda(z,t) \not\equiv 0$ in $(-\infty,t_0)$, by employing similar estimates as above, we obtain

$$\partial_t^\alpha w_\Lambda(z,t_0) = -C_\alpha \int_{-\infty}^{t_0} \frac{w_\Lambda(z,\tau)}{(t_0-\tau)^{1+\alpha}} d\tau < 0$$

and

$$(-\Delta_g)^s u_\Lambda(z,t_0) - (-\Delta_g)^s u(z,t_0) = 0,$$

which also means that

$$\partial_t^\alpha w_\Lambda(z,t_0) + (-\Delta_g)^s u_\Lambda(z,t_0) - (-\Delta_g)^s u(z,t_0) < 0.$$

This contradiction aligns with Equation (5). So, $w_\Lambda(z,t) \equiv 0$ in $\Sigma_\Lambda \times (-\infty,t_0]$. Again, through the antisymmetry of $w_\Lambda(z,t)$ for z, we can conclude that

$$w_\Lambda(z,t) \equiv 0 \text{ in } \mathbb{R}^n \times (-\infty,t_0].$$

This completes the proof. □

Theorem 2. *Assume that $\Omega \subset \Sigma_\Lambda$ is an unbounded domain of finite width in the direction of z_1 and*

$$w_\Lambda \in (C^{1,1}_{loc}(\Omega) \cap L_g(\mathbb{R}^n)) \times (C^1(\mathbb{R}) \cap L^-_\alpha(\mathbb{R}))$$

is lower semi-continuous with respect to z on $\overline{\Omega}$.

If

$$w_\Lambda(z,t) \geq -C(1+|z|^\nu) \; for \; some \; 0<\nu<2s,$$

and

$$\begin{cases} \partial_t^\alpha w_\Lambda(z,t) + (-\Delta_g)^s u_\Lambda(z,t) - (-\Delta_g)^s u(z,t) = \left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right) w_\Lambda(z,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in \; \in \Omega \times \mathbb{R}, \\ w_\Lambda(z,t) \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad in \; (\Sigma_\Lambda \backslash \Omega) \times \mathbb{R}, \\ w_\Lambda(z,t) = -w_\Lambda(z^\Lambda,t), \qquad\qquad\qquad\qquad in \; \Sigma_\Lambda \times \mathbb{R}. \end{cases} \tag{17}$$

where $\xi_1(z,t)$ and $\xi_2(z,t)$ fall in-between $u_\Lambda(z,t)$ and $u(z,t)$,

then

$$w_\Lambda(z,t) \geq 0 \; for \; (z,t) \in \Sigma_\Lambda \times \mathbb{R}. \tag{18}$$

Moreover, if $w_\Lambda(z^0,t_0) = 0$ for a point, $(z^0,t_0) \in \Omega \times \mathbb{R}$, then

$$w_\Lambda(z,t) \equiv 0 \; for \; (z,t) \in \mathbb{R}^n \times (-\infty, t_0].$$

Proof. Because Ω is an unbounded domain of finite width in the direction of z_1, assume that Ω is contained in $\{z \in \Sigma_\Lambda | \Lambda - 2a < z_1 < \Lambda\}$ for some $a > 0$. Here, we choose the auxiliary functions

$$h(z) := \left[\left(1 - \frac{(z_1-(\Lambda-a))^2}{a^2}\right)_+^s + 1\right]\left(1+|bz'|^2\right)^{\frac{\theta}{2}}$$

for some $\nu < \theta < 2s$. Here, b is a small enough positive constant and depends on a. For any $z \in \Omega$ and some constant $C_3 > 0$, we have

$$\frac{(-\Delta)^s h(z)}{h(z)} \geq \frac{C_3}{l^{2s}}.$$

It is clear from the calculations that we can obtain $\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1} \leq p+q$. Similar to the notation and computation in Theorem 1 proofs, if Equation (18) is false, we can finally derive

$$\frac{C_2C_3}{a^{2s}} \leftarrow \frac{C_2C_3}{a^{2s}}\frac{m_k}{m} \leq \frac{C\varepsilon_k}{m} + p + q \to p+q,$$

as $k \to \infty$. By the arbitrariness of a, take $a^{2s} < \frac{C_2C_3}{p+q}$; then, it is a contradiction. Therefore, the validity of Equation (18) is confirmed. The proof is complete. □

Theorem 3. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $[t_1,t_2] \subset \mathbb{R}$ is a finite interval, and*

$$u(z,t) \in (C_{loc}^{1,1}(\Omega) \cap L_g(\mathbb{R}^n)) \times (C^1([t_1,t_2]) \cap L_\alpha^-(\mathbb{R}))$$

is lower semi-continuous with respect to z on $\overline{\Omega}$. If

$$\begin{cases} \partial_t^\alpha u(z,t) + (-\Delta_g)^s u(z,t) \geq 0, & (z,t) \in \Omega \times (t_1,t_2], \\ u(z,t) \geq 0, & (z,t) \in \Omega^c \times (t_1,t_2], \\ u(z,t) \geq 0, & (z,t) \in \Omega \times (-\infty, t_1], \end{cases} \tag{19}$$

then $u(z,t) \geq 0$ in $\Omega \times (t_1,t_2]$.

Proof. If the conclusion is invalid, there exists $(z^0, t_0) \in \Omega \times (t_1, t_2]$ such that

$$u(z^0, t_0) = \min_{\Omega \times (t_1, t_2]} u(z, t) < 0.$$

By Equation (19), we can obtain

$$\begin{aligned}
&\partial_t^\alpha u(z^0, t_0) + (-\Delta_g)^s u(z^0, t_0) \\
=& C_\alpha \int_{-\infty}^{t_0} \frac{u(z^0, t_0) - u(z^0, \tau)}{(t_0 - \tau)^{1+\alpha}} d\tau + \text{P.V.} \int_{\mathbb{R}^n} g\left(\frac{u(z^0, t_0) - u(\vartheta, t_0)}{|z^0 - \vartheta|^s} \right) \frac{d\vartheta}{|z^0 - \vartheta|^{n+s}} \\
=& C_\alpha \int_{-\infty}^{t_1} \frac{u(z^0, t_0) - u(z^0, \tau)}{(t_0 - \tau)^{1+\alpha}} d\tau + C_\alpha \int_{t_1}^{t_0} \frac{u(z^0, t_0) - u(z^0, \tau)}{(t_0 - \tau)^{1+\alpha}} d\tau \\
&+ \text{P.V.} \int_{\Omega} g\left(\frac{u(z^0, t_0) - u(\vartheta, t_0)}{|z^0 - \vartheta|^s} \right) \frac{d\vartheta}{|z^0 - \vartheta|^{n+s}} + \int_{\Omega^c} g\left(\frac{u(z^0, t_0) - u(\vartheta, t_0)}{|z^0 - \vartheta|^s} \right) \frac{d\vartheta}{|z^0 - \vartheta|^{n+s}} \\
<& 0.
\end{aligned}$$

This is inconsistent with Equation (19). So, the proof is completed. □

Theorem 4. *Assume that $\Omega \subset \Sigma_\Lambda$ is a bounded domain, $[t_1, t_2] \subset \mathbb{R}$ is a finite interval, and*

$$w_\Lambda \in (C^{1,1}_{loc}(\Omega) \cap L_g(\mathbb{R}^n)) \times (C^1([t_1, t_2]) \cap L^-_\alpha(\mathbb{R}))$$

is lower semi-continuous for z in $\overline{\Omega}$. If

$$\begin{cases}
\partial_t^\alpha w_\Lambda(z, t) + (-\Delta_g)^s u_\Lambda(z, t) - (-\Delta_g)^s u(z, t) \geq 0, & (z, t) \in \Omega \times (t_1, t_2], \\
w_\Lambda(z, t) \geq 0, & (z, t) \in (\Sigma_\Lambda \backslash \Omega) \times (t_1, t_2], \\
w_\Lambda(z, t) \geq 0, & (z, t) \in \Omega \times (-\infty, t_1], \\
w_\Lambda(z, t) = -w_\Lambda(z^\Lambda, t), & (z, t) \in \Sigma_\Lambda \times \mathbb{R},
\end{cases} \tag{20}$$

then $w_\Lambda(z, t) \geq 0$ in $\Omega \times (t_1, t_2]$.

Proof. If the conclusion is not valid, there exists $(z^0, t_0) \in \Omega \times (t_1, t_2]$ such that

$$w_\Lambda(z^0, t_0) = \min_{\Sigma_\Lambda \times (t_1, t_2]} w_\Lambda(z, t) < 0.$$

By the antisymmetry of $w_\Lambda(z,t)$ in z and (2.17), we have

$$
\begin{aligned}
&\partial_t^\alpha w_\Lambda(z^0,t_0) + (-\Delta_g)^s u_\Lambda(z^0,t_0) - (-\Delta_g)^s u(z^0,t_0) \\
= {}& C_\alpha \int_{-\infty}^{t_0} \frac{w_\Lambda(z^0,t_0) - w_\Lambda(z^0,\tau)}{(t_0-\tau)^{1+\alpha}} d\tau + \mathrm{P.V.} \int_{\mathbb{R}^n} \left[g\left(\frac{u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) \right. \\
&\left. - g\left(\frac{u(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) \right] \frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
= {}& C_\alpha \int_{-\infty}^{t_1} \frac{w_\Lambda(z^0,t_0) - w_\Lambda(z^0,\tau)}{(t_0-\tau)^{1+\alpha}} d\tau + C_\alpha \int_{t_1}^{t_0} \frac{w_\Lambda(z^0,t_0) - w_\Lambda(z^0,\tau)}{(t_0-\tau)^{1+\alpha}} d\tau \\
&+ \mathrm{P.V.} \int_{\Sigma_\Lambda} \left[g\left(\frac{u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) - g\left(\frac{u(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) \right] \frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
&+ \mathrm{P.V.} \int_{\Sigma_\Lambda^c} \left[g\left(\frac{u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) - g\left(\frac{u(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) \right] \frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
< {}& \mathrm{P.V.} \int_{\Sigma_\Lambda} \left[g\left(\frac{u_\Lambda(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) - g\left(\frac{u(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta|^s} \right) \right] \frac{d\vartheta}{|z^0-\vartheta|^{n+s}} \\
&+ \mathrm{P.V.} \int_{\Sigma_\Lambda} \left[g\left(\frac{u_\Lambda(z^0,t_0) - u(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s} \right) - g\left(\frac{u(z^0,t_0) - u_\Lambda(\vartheta,t_0)}{|z^0-\vartheta^\Lambda|^s} \right) \right] \frac{d\vartheta}{|z^0-\vartheta^\Lambda|^{n+s}} \\
= {}& I_1 + I_2 \\
< {}& 0,
\end{aligned}
$$

with I_1 and I_2 being Equations (15) and (16). Since $w_\Lambda(z^0,t_0) < 0$, we obtain $I_1 < 0$ and $I_2 < 0$.

Obviously, the above inequality is contradictory to inequality (20). Then, we have successfully completed the proof. □

3. Averaging Effects

In this section, to prove our main results, we introduce averaging effects (Theorem 5) and averaging effects of antisymmetric functions (Theorem 6) for double non-local operators $\partial_t^\alpha + (-\Delta_g)^s$.

Theorem 5. *Let $D \subset \mathbb{R}^n$. For any $z^0 \in \mathbb{R}^n$ and some $t_0 \in \mathbb{R}$, assume that there exists a radius $r > 0$, satisfying $B_r(z^0) \cap \overline{D} = \varnothing$ as shown in Figure 1, and*

$$u(z,t) \geq C_0 > 0 \; for \; (z,y) \in D \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]. \tag{21}$$

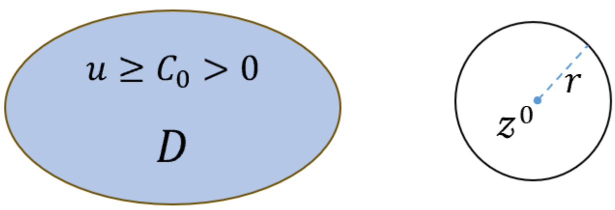

Figure 1. The positional relationship between the region D and the ball $B_r(z^0)$ in $\mathbb{R}^n$.

Suppose that

$$u(z,t) \in (C_{loc}^{1,1}(B_r(z^0)) \cap L_g(\mathbb{R}^n)) \times (C^1([t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]) \cap L_\alpha^-(\mathbb{R}))$$

is lower semi-continuous for z in $\overline{B_r(z^0)}$ *and satisfies*

$$\begin{cases} \partial_t^\alpha u(z,t) + (-\Delta_g)^s u(z,t) \geq -\varepsilon, & in\ B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ u(z,t) \geq 0, & in\ B_r^c(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ u(z,t) \geq 0, & in\ B_r(z^0) \times (-\infty, t_0 - r^{\frac{2s}{\alpha}}], \end{cases} \tag{22}$$

for some small enough positive constant ε. *Consequently, there exists a positive constant* C_1 *such that*

$$u(z^0, t_0) \geq C_1 > 0.$$

Proof. By constructing a sub-solution, we can derive a lower bound estimation. Let

$$\Psi(z,t) := \varphi(z)\eta(t) = C\left(1 - \left|\frac{z - z^0}{r}\right|^2\right)_+^s \eta(t).$$

Here, $\eta(t)$ represents a smooth cut-off function, satisfying

$$\eta(t) \begin{cases} = 1, & t \in [-\frac{r^{\frac{2s}{\alpha}}}{2} + t_0, \frac{r^{\frac{2s}{\alpha}}}{2} + t_0], \\ \in [0,1], & t \in (-r^{\frac{2s}{\alpha}} + t_0, -\frac{r^{\frac{2s}{\alpha}}}{2} + t_0) \cup (\frac{r^{\frac{2s}{\alpha}}}{2} + t_0, r^{\frac{2s}{\alpha}} + t_0), \\ = 0, & t \notin (-r^{\frac{2s}{\alpha}} + t_0, r^{\frac{2s}{\alpha}} + t_0). \end{cases}$$

By choosing a suitable positive constant C, it follows that

$$\begin{cases} (-\Delta)^s \varphi(z) = \frac{1}{r^{2s}}, & \text{in } B_r(z^0), \\ \varphi(z) = 0, & \text{in } B_r^c(z^0). \end{cases} \tag{23}$$

Let

$$\underline{u}(z,t) := u(z,t)\chi_D(z) + \delta\Psi(z,t),$$

where

$$\chi_D(z) = \begin{cases} 1, & z \in D, \\ 0, & z \notin D, \end{cases}$$

and δ is a positive constant that will be determined at a later time.

Next, we will prove that $\underline{u}(z,t)$ is a sub-solution of $u(z,t)$ in $B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]$. Combining Equations (21)–(23), $B_r(z^0) \cap \overline{D} = \varnothing$, the properties of g, and Corollary 5.2 in [29], for $(z,t) \in B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]$, we obtain

$$\begin{aligned}
&\partial_t^\alpha (u(z,t) - \underline{u}(z,t)) + (-\Delta_g)^s u(z,t) - (-\Delta_g)^s \underline{u}(z,t) \\
\geq& -\varepsilon - \delta\varphi(z)\partial_t^\alpha \eta(t) - \text{P.V.} \int_{\mathbb{R}^n} g\left(\frac{\delta\Psi(z,t) - \underline{u}(\vartheta,t)}{|z - \vartheta|^s}\right) \frac{1}{|z - \vartheta|^{n+s}} d\vartheta \\
\geq& -\varepsilon - \delta\varphi(z)\partial_t^\alpha \eta(t) - (-\Delta_g)^s (\delta\Psi(z,t)) \\
&+ \text{P.V.} \int_D \left[g\left(\frac{\delta\Psi(z,t)}{|z - \vartheta|^s}\right) - g\left(\frac{\delta\Psi(z,t) - u(\vartheta,t)}{|z - \vartheta|^s}\right)\right] \frac{1}{|z - \vartheta|^{n+s}} d\vartheta \\
\geq& -\varepsilon - \delta\varphi(z)\partial_t^\alpha \eta(t) - C_2 \text{P.V.} \int_{\mathbb{R}^n} \left[g\left(\frac{\delta\Psi(z,t)}{|z - \vartheta|^s}\right) - g\left(\frac{\delta\Psi(\vartheta,t)}{|z - \vartheta|^s}\right)\right] \frac{1}{|z - \vartheta|^{n+s}} d\vartheta \\
&+ \int_D \frac{g'(\zeta_1(\vartheta,t))u(\vartheta,t)}{|z - \vartheta|^{n+2s}} d\vartheta \\
\geq& -\varepsilon - \delta\varphi(z)\partial_t^\alpha \eta(t) - C_2\delta\eta(t)\text{P.V.} \int_{\mathbb{R}^n} \frac{g'(\zeta_2(\vartheta,t))(\varphi(z) - \varphi(\vartheta))}{|z - \vartheta|^{n+2s}} d\vartheta + C_3 \\
\geq& -\varepsilon - \frac{\delta C_4}{r^{2s}} + C_3,
\end{aligned}$$

where
$$\zeta_1(\vartheta,t) \text{ between } \frac{\delta\Psi(z,t)}{|z-\vartheta|^s} \text{ and } \frac{\delta\Psi(z,t)-u(\vartheta,t)}{|z-\vartheta|^s}$$
and
$$\zeta_2(\vartheta,t) \text{ between } \frac{\delta\Psi(z,t)}{|z-\vartheta|^s} \text{ and } \frac{\delta\Psi(\vartheta,t)}{|z-\vartheta|^s}.$$

By choosing $\varepsilon = \frac{C_3}{2}$ and $\delta = \frac{C_3 r^{2s}}{2C_4}$, we can obtain
$$\partial_t^\alpha(u(z,t)-\underline{u}(z,t)) + (-\Delta_g)^s u(z,t) - (-\Delta_g)^s \underline{u}(z,t) \geq 0,\ (z,t) \in B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}].$$

Due to Equation (22) and the definition of $\Phi(z,t)$, we can deduce that
$$u(z,t) - \underline{u}(z,t) = u(z,t) - u(z,t)\chi_D(z) \geq 0 \text{ in } B_r^c(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}],$$
and
$$u(z,t) - \underline{u}(z,t) = u(z,t) \geq 0 \text{ in } B_r(z^0) \times (-\infty, t_0 - r^{\frac{2s}{\alpha}}].$$

Anyway, we have successfully acquired
$$\begin{cases} \partial_t^\alpha(u(z,t)-\underline{u}(z,t)) + (-\Delta_g)^s u(z,t) - (-\Delta_g)^s \underline{u}(z,t) \geq 0, & \text{in } B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ u(z,t) - \underline{u}(z,t) \geq 0, & \text{in } B_r^c(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ u(z,t) - \underline{u}(z,t) \geq 0, & \text{in } B_r(z^0) \times (-\infty, t_0 - r^{\frac{2s}{\alpha}}]. \end{cases}$$

From Theorem 3, we have
$$u(z,t) \geq \underline{u}(z,t) \text{ in } B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}].$$

Consequently, it can be inferred that
$$u(z^0,t_0) \geq \underline{u}(z^0,t_0) = \delta\varphi(z^0)\eta(t_0) = C\delta =: C_1 > 0.$$

Therefore, we successfully prove Theorem 5. □

Theorem 6. *Let $D \subset \Sigma_\Lambda$. For any $z^0 \in \Sigma_\Lambda$ and some $t_0 \in \mathbb{R}$, assume that there exists a ball, $B_r(z^0) \subset \Sigma_\Lambda$, satisfying $B_r(z^0) \cap \overline{D} = \varnothing$, $r \leq \frac{dist(z^0,T_\Lambda)}{2}$ as shown in Figure 2, and*
$$w_\Lambda(z,t) \geq C_0 > 0 \text{ in } D \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]. \tag{24}$$

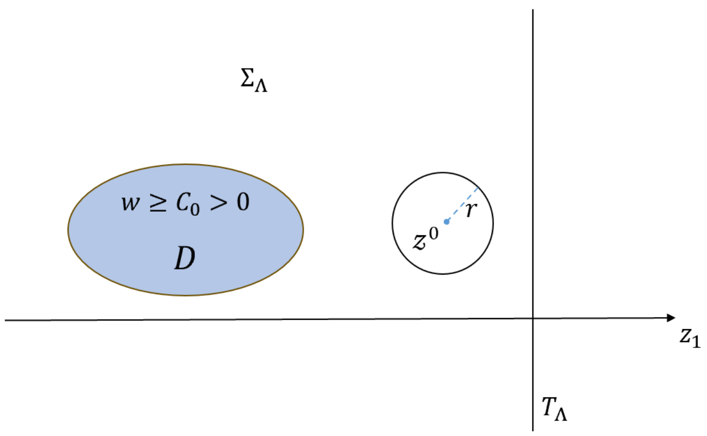

Figure 2. The positional relationship between the region D and the ball $B_r(z^0)$ in Σ_Λ.

Suppose that

$$w_\Lambda(z,t) \in (C^{1,1}_{loc}(B_r(z^0)) \cap L_g(\mathbb{R}^n)) \times (C^1([t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]) \cap L^-_\alpha(\mathbb{R}))$$

is lower semi-continuous for z in $\overline{B_r(z^0)}$ and satisfies

$$\begin{cases} \partial_t^\alpha w_\Lambda(z,t) + (-\Delta_g)^s u_\Lambda(z,t) - (-\Delta_g)^s u(z,t) \geq -\varepsilon, & (z,t) \in B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ w_\Lambda(z,t) \geq 0, & (z,t) \in \left(\Sigma_\Lambda \backslash B_r(z^0)\right) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}], \\ w_\Lambda(z,t) \geq 0, & (z,t) \in B_r(z^0) \times (-\infty, t_0 - r^{\frac{2s}{\alpha}}], \\ w_\Lambda(z,t) = -w_\Lambda(z^\Lambda, t), & (z,t) \in \Sigma_\Lambda \times \mathbb{R}, \end{cases} \tag{25}$$

for some small enough positive constant ε. Consequently, there exists a positive constant C_1 such that

$$w_\Lambda(z^0, t_0) \geq C_1 > 0.$$

Proof. In the process of proving this theorem, the most important step is to construct a sub-solution for $w_\Lambda(z,t)$. Let

$$\varphi(z) = \left(1 - \left|\frac{z - z^0}{r}\right|^2\right)^s_+ \text{ and } \varphi_\Lambda(z) = \left(1 - \left|\frac{z^\Lambda - z^0}{r}\right|^2\right)^s_+.$$

It is easy to obtain an antisymmetric function with respect to the plane T_Λ.

$$\Phi(z) := \varphi(z) - \varphi_\Lambda(z).$$

Denote $\eta(t) \in C_0^\infty\left(t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}\right)$, where $\eta(t)$ represents a smooth cut-off function, satisfying

$$\eta(t) \begin{cases} = 1, & t \in [-\frac{r^{\frac{2s}{\alpha}}}{2} + t_0, \frac{r^{\frac{2s}{\alpha}}}{2} + t_0], \\ \in [0,1], & t \in (-r^{\frac{2s}{\alpha}} + t_0, -\frac{r^{\frac{2s}{\alpha}}}{2} + t_0) \cup (\frac{r^{\frac{2s}{\alpha}}}{2} + t_0, r^{\frac{2s}{\alpha}} + t_0), \\ = 0, & t \notin (-r^{\frac{2s}{\alpha}} + t_0, r^{\frac{2s}{\alpha}} + t_0). \end{cases}$$

Let

$$\underline{w}_\Lambda(z,t) := w_\Lambda(z,t)\chi_{D \cup D^\Lambda}(z) + \delta\Phi(z)\eta(t),$$

where

$$\chi_{D \cup D^\Lambda}(z) = \begin{cases} 1, & z \in D \cup D^\Lambda, \\ 0, & z \notin D \cup D^\Lambda, \end{cases}$$

the domain D^Λ is a reflection of the domain D with respect to the plane T_Λ, and δ is a positive constant that will be determined later.

Next, we will prove that $\underline{w}_\Lambda(z,t)$ is a sub-solution of $w_\Lambda(z,t)$ in $B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]$. This is for $(z,t) \in B_r(z^0) \times (t_0 - r^{\frac{2s}{\alpha}}, t_0 + r^{\frac{2s}{\alpha}}]$. By Equations (23)–(25), Corollary 5.2 in [29], the properties of g, and $r \leq \frac{dist(z^0, T_\Lambda)}{2}$, we obtain

$$
\begin{aligned}
&\partial_t^\alpha(w_\Lambda(z,t)-\underline{w}_\Lambda(z,t))+(-\Delta_g)^s u_\Lambda(z,t)-(-\Delta_g)^s u(z,t)-(-\Delta_g)^s\underline{w}_\Lambda(z,t)\\
\geq&-\varepsilon-\delta\Phi(z)\partial_t^\alpha\eta(t)-\text{P.V.}\int_{\mathbb{R}^n} g\left(\frac{\delta\Phi(z)\eta(t)-\underline{w}_\Lambda(\vartheta,t)}{|z-\vartheta|^s}\right)\frac{1}{|z-\vartheta|^{n+s}}d\vartheta\\
\geq&-\varepsilon-\delta\Phi(z)\partial_t^\alpha\eta(t)-(-\Delta_g)^s(\delta\Phi(z)\eta(t))\\
&+\int_{D\cup D^\Lambda}\left[g\left(\frac{\delta\Phi(z)\eta(t)}{|z-\vartheta|^s}\right)-g\left(\frac{\delta\Phi(z)\eta(t)-w_\Lambda(\vartheta,t)}{|z-\vartheta|^s}\right)\right]\frac{1}{|z-\vartheta|^{n+s}}d\vartheta\\
\geq&-\varepsilon-\delta\Phi(z)\partial_t^\alpha\eta(t)-\text{P.V.}\int_{\mathbb{R}^n} g\left(\frac{\delta\Phi(z)\eta(t)-\delta\Phi(\vartheta)\eta(t)}{|z-\vartheta|^s}\right)\frac{1}{|z-\vartheta|^{n+s}}d\vartheta\\
&+\int_{D\cup D^\Lambda}\frac{g'(\zeta_1(\vartheta,t))w_\Lambda(\vartheta,t)}{|z-\vartheta|^{n+2s}}d\vartheta\\
\geq&-\varepsilon-\delta\Phi(z)\partial_t^\alpha\eta(t)-C_2\delta\eta(t)\text{P.V.}\int_{\mathbb{R}^n}\frac{g'(\zeta_2(\vartheta,t))(\Phi(z)-\Phi(\vartheta))}{|z-\vartheta|^{n+2s}}d\vartheta\\
&+C_3\int_D\left[\frac{1}{|z-\vartheta|^{n+2s}}-\frac{1}{|z-\vartheta^\Lambda|^{n+2s}}\right]d\vartheta\\
\geq&-\varepsilon-\frac{\delta C_4}{r^{2s}}-\delta\eta(t)\int_{B_r((z^0)^\Lambda)}\frac{g'(\zeta_2(\vartheta,t))\varphi_\Lambda(\vartheta)}{|z-\vartheta|^{n+2s}}d\vartheta+C_5\\
\geq&-\varepsilon-\frac{\delta C_6}{r^{2s}}+C_5,
\end{aligned}
$$

where

$$\zeta_1(\vartheta,t)\text{ between }\frac{\delta\Phi(z)\eta(t)}{|z-\vartheta|^s}\text{ and }\frac{\delta\Phi(z)\eta(t)-w_\Lambda(\vartheta,t)}{|z-\vartheta|^s}$$

and

$$\zeta_2(\vartheta,t)\text{ between }\frac{\delta\Phi(z)\eta(t)}{|z-\vartheta|^s}\text{ and }\frac{\delta\Phi(\vartheta)\eta(t)}{|z-\vartheta|^s}.$$

By selecting $\varepsilon=\frac{C_5}{2}$ and $\delta=\frac{C_5 r^{2s}}{2C_6}$, for $(z,t)\in B_r(z^0)\times(t_0-r^{\frac{2s}{\alpha}},t_0+r^{\frac{2s}{\alpha}}]$, we obtain

$$\partial_t^\alpha(w_\Lambda(z,t)-\underline{w}_\Lambda(z,t))+(-\Delta_g)^s u_\Lambda(z,t)-(-\Delta_g)^s u(z,t)-(-\Delta_g)^s\underline{w}_\Lambda(z,t)\geq 0.$$

By Equation (25), we obtain

$$w_\Lambda(z,t)-\underline{w}_\Lambda(z,t)=w_\Lambda(z,t)-w_\Lambda(z,t)\chi_{D\cup D^\Lambda}(z)\geq 0\text{ in }\left(\Sigma_\Lambda\backslash B_r(z^0)\right)\times(t_0-r^{\frac{2s}{\alpha}},t_0+r^{\frac{2s}{\alpha}}].$$

and

$$w_\Lambda(z,t)-\underline{w}_\Lambda(z,t)=w_\Lambda(z,t)\geq 0\text{ in }B_r(z^0)\times(-\infty,t_0-r^{\frac{2s}{\alpha}}].$$

Anyway, we have successfully acquired

$$
\begin{cases}
\partial_t^\alpha(w_\Lambda(z,t)-\underline{w}_\Lambda(z,t))+(-\Delta_g)^s u_\Lambda(z,t)-(-\Delta_g)^s u(z,t)-(-\Delta_g)^s\underline{w}_\Lambda(z,t)\geq 0,\\
\qquad\qquad\qquad\text{in }B_r(z^0)\times(t_0-r^{\frac{2s}{\alpha}},t_0+r^{\frac{2s}{\alpha}}],\\
w_\Lambda(z,t)-\underline{w}_\Lambda(z,t)\geq 0, \quad \text{in }(\Sigma_\Lambda\backslash B_r(z^0))\times(t_0-r^{\frac{2s}{\alpha}},t_0+r^{\frac{2s}{\alpha}}],\\
w_\Lambda(z,t)-\underline{w}_\Lambda(z,t)\geq 0, \quad \text{in }B_r(z^0)\times(-\infty,t_0-r^{\frac{2s}{\alpha}}],\\
w_\Lambda(z,t)-\underline{w}_\Lambda(z,t)=-(w_\Lambda(z^\Lambda,t)-\underline{w}_\Lambda(z^\Lambda,t)), \quad \text{in }\Sigma_\Lambda\times\mathbb{R}.
\end{cases}
$$

From Theorem 4, we obtain

$$w_\Lambda(z,t)\geq\underline{w}_\Lambda(z,t)\text{ in }B_r(z^0)\times(t_0-r^{\frac{2s}{\alpha}},t_0+r^{\frac{2s}{\alpha}}].$$

Consequently, it can be inferred that

$$w_\Lambda(z^0,t_0) \geq \underline{w}_\Lambda(z^0,t_0) = \delta\varphi(z^0)\eta(t_0) = \delta =: C_1 > 0.$$

□

Remark 1. *The average effect shows that the positiveness of the solution of the fractional time-diffusion equation in some region D will spread to any other region B that does not intersect D. The average effect is influenced by the distance between the two regions, and the shorter the distance, the more significant the effect.*

4. Application of Direct Moving Plane Method

In this section, in combination with the maximum principles and the average effects described above, the monotonicity of the positive solution for Equation (26) in half space is established by the direct moving plane method. Consider

$$\begin{cases} \partial_t^\alpha u(z,t) + (-\Delta_g)^s u(z,t) = \ln|u^q(z,t)+1| + \ln|u^p(z,t)+1|, & \text{in } \mathbb{R}^n_+ \times \mathbb{R}, \\ u(z,t) = 0, & \text{in } \left(\mathbb{R}^n \backslash \mathbb{R}^n_+\right) \times \mathbb{R}. \end{cases} \tag{26}$$

where $\mathbb{R}^n_+ := \{z \in \mathbb{R}^n \mid z_1 > 0\}$ represents the right half space and $p, q \geq 1$.

Theorem 7. *Assume that $u(z,t) \in (C^{1,1}_{loc}(\mathbb{R}^n_+) \cap L_g(\mathbb{R}^n)) \times (C^1(\mathbb{R}) \cap L^-_\alpha(\mathbb{R}))$ is a positive solution of Equation (26) and $u(z,t)$ is uniformly continuous in z. If*

$$u(z,t) \leq C(1+|z|^\nu) \text{ for some } 0 < \nu < 2s,$$

then the positive solution $u(z,t)$ of Equation (26) is strictly increasing in half space along the z_1-direction for any $t \in \mathbb{R}$.

Proof. By performing a direct calculation, it follows that

$$\begin{cases} \partial_t^\alpha w_\Lambda(z,t) + (-\Delta_g)^s u_\Lambda(z,t) - (-\Delta_g)^s u(z,t) = \left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right) w_\Lambda(z,t), \\ \qquad (z,t) \in \Omega_\Lambda \times \mathbb{R}, \\ w_\Lambda(z,t) \geq 0, \qquad (z,t) \in (\Sigma_\Lambda \backslash \Omega_\Lambda) \times \mathbb{R}, \\ w_\Lambda(z,t) = -w_\Lambda(z^\Lambda,t), \qquad (z,t) \in \Sigma_\Lambda \times \mathbb{R}. \end{cases} \tag{27}$$

where $\xi_1(z,t)$ and $\xi_2(z,t)$ fall in-between $u_\Lambda(z,t)$ and $u(z,t)$. To obtain our result, we just need to show that

$$w_\Lambda(z,t) > 0 \text{ in } \Omega_\Lambda \times \mathbb{R},$$

for any $\Lambda > 0$. The proof will be divided into three distinct steps.

Step 1: By starting from $z_1 = 0$ and moving the plane T_Λ toward the right along the z_1-axis. With the assumption of Theorem 7, we can apply Theorem 1 to Equation (27) for $\Lambda > 0$ small enough; therefore, we have

$$w_\Lambda(z,t) \geq 0 \text{ for } (z,t) \in \Omega_\Lambda \times \mathbb{R}. \tag{28}$$

Obviously, when $\Lambda > 0$ is small enough, Ω_Λ is a narrow region. The starting point of the moving plane T_Λ is provided by inequality (28).

Step 2: We proceed with the continuous movement of the plane T_Λ along the z_1-axis toward the right, ensuring that inequality (28) remains valid until the plane reaches its limiting position. Let

$$\Lambda_0 = \sup\{\Lambda | w_\mu(z,t) \geq 0, (z,t) \in \Sigma_\mu \times \mathbb{R} \text{ for any } \mu \leq \Lambda\}.$$

Next, our goal is to prove

$$\Lambda_0 = +\infty. \tag{29}$$

Otherwise, if $0 < \Lambda_0 < +\infty$, according to its definition, we can find a sequence, Λ_k, with $\Lambda_k > \Lambda_0$ such that $\Lambda_k \to \Lambda_0$ when $k \to \infty$; then, we have

$$\Sigma^-_{\Lambda_k} \times \mathbb{R} := \{(z,t) \in \Sigma_{\Lambda_k} \times \mathbb{R} | w_{\Lambda_k}(z,t) < 0\}$$

that is nonempty and $\inf_{\Sigma_{\Lambda_k} \times \mathbb{R}} w_{\Lambda_k}(z,t) < 0$. First of all, we need to demonstrate that

$$\inf_{\Sigma_{\Lambda_k} \times \mathbb{R}} w_{\Lambda_k}(z,t) \to 0 \text{ as } k \to \infty. \tag{30}$$

If this assumption does not hold, it follows that there is a positive constant M, satisfying

$$\inf_{\Sigma_{\Lambda_k} \times \mathbb{R}} w_{\Lambda_k}(z,t) < -M < 0.$$

Thus, we can find a sequence, $\{(z^k, t_k)\} \subset \Sigma_{\Lambda_k} \times \mathbb{R}$, such that

$$w_{\Lambda_k}(z^k, t_k) \le -M < 0. \tag{31}$$

It follows that either z^k is between T_{Λ_0} and T_{Λ_k} or $z^k \in \Omega_{\Lambda_0}$. When z^k is between T_{Λ_0} and T_{Λ_k}, then combining $\Lambda_k \to \Lambda_0$ as $k \to \infty$ and the uniform continuity of $u(z,t)$ in z, we can obtain

$$w_{\Lambda_k}(z^k, t_k) = u_{\Lambda_k}(z^k, t_k) - u(z^k, t_k) \to 0 \text{ as } k \to \infty.$$

This contradicts Equation (31).

When $z^k \in \Omega_{\Lambda_0}$, we have

$$w_{\Lambda_k}(z^k, t_k) - w_{\Lambda_0}(z^k, t_k) = u_{\Lambda_k}(z^k, t_k) - u_{\Lambda_0}(z^k, t_k) \to 0 \text{ as } k \to \infty.$$

However, by Equation (31) and $w_{\Lambda_0}(z^k, t_k) \ge 0$, we derive

$$w_{\Lambda_k}(z^k, t_k) - w_{\Lambda_0}(z^k, t_k) \le -M < 0.$$

This is a contradiction. So, Equation (30) is valid, and we infer that

$$\inf_{\Sigma_{\Lambda_k} \times \mathbb{R}} w_{\Lambda_k}(z,t) =: -m_k \to 0 \text{ as } k \to \infty. \tag{32}$$

In the following, denote

$$h_k := \sup_{\Sigma^-_{\Lambda_k} \times \mathbb{R}} \left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1} \right),$$

with $\xi_1(z,t)$ and $\xi_2(z,t)$ falling in-between $u_{\Lambda_k}(z,t)$ and $u(z,t)$. There are two possible cases that arise from the above analysis.

Case 1. If $h_k \le \varepsilon_k \to 0$ as $k \to \infty$, let $\Lambda = \Lambda_k$, and by applying Theorem 2 to problem (27), it follows that

$$w_{\Lambda_k}(z,t) \ge 0 \text{ in } \Sigma_{\Lambda_k} \times \mathbb{R}$$

for a large enough k. This is inconsistent with the definition of Λ_k, hence the contradiction.

Case 2. If $h_k \nrightarrow 0$ when $k \to \infty$, it implies that there exist $\delta_0 > 0$ and a subsequence of $\{h_k\}$ (which we will still denote as $\{h_k\}$), satisfying $h_k \ge \delta_0 > 0$. Since

$(\ln|u^q(z,t)+1|+\ln|u^p(z,t)+1|)' = 0$ when $u(z,t)=0$ and regarding Equation (32), we can infer that there exist $\varepsilon_0>0$ along with a sequence, $\{(z^k,t_k)\}\subset\Sigma^-_{\Lambda_k}\times\mathbb{R}$, such that

$$u(z^k,t_k)\geq\varepsilon_0>0$$

and

$$w_{\Lambda_k}(z^k,t_k)=-m_k+\sigma m_k^2<0,$$

where $\sigma\geq 0$. Combining the fact that $u(z,t)=0$ in $(\mathbb{R}^n\backslash\mathbb{R}^n_+)\times\mathbb{R}$ with the fact that $u(z,t)$ are continuous, there exists $r_0>0$, which is irrespective of k, such that

$$u(z,t)\geq\frac{\varepsilon_0}{2}>0 \text{ for } (z,t)\in B_{r_0}(z^0)\times(t_k-r_0^{\frac{2s}{\alpha}},t_k+r_0^{\frac{2s}{\alpha}}]\subset\mathbb{R}^n_+\times\mathbb{R}. \tag{33}$$

Next, we show that $\delta_k:=dist(z^k,T_{\Lambda_k})=\Lambda_k-z_1^k$ is bounded and $\delta_k\neq 0$ for a large enough k. If not, it follows that $\delta_k\to 0$ as $k\to\infty$. Let

$$v_k(z,t):=w_{\Lambda_k}(z,t)-\sigma m_k^2\eta_k(z,t),$$

where $\eta_k(z,t)$ is a series of smooth cut-off functions,

$$\eta_k(z,t)=\eta(\frac{z-z_k}{\delta_k}-\frac{t-t_k}{\delta_k^{\frac{2s}{\alpha}}})\in C_0^\infty(B_{\delta_k}(z^k)\times(-\delta_k^{\frac{2s}{\alpha}}+t_k,\delta_k^{\frac{2s}{\alpha}}+t_k)),$$

satisfying

$$\eta_k(z,t)\begin{cases}=1, & (z,t)\in B_{\frac{\delta_k}{2}}(z^k)\times(-\frac{\delta_k^{\frac{2s}{\alpha}}}{2}+t_k,\frac{\delta_k^{\frac{2s}{\alpha}}}{2}+t_k),\\ \in[0,1], & (z,t)\in[B_{\delta_k}(z^k)\backslash B_{\frac{\delta_k}{2}}(z^k)]\times[(-\delta_k^{\frac{2s}{\alpha}}+t_k,-\frac{\delta_k^{\frac{2s}{\alpha}}}{2}+t_k)\cup(\frac{\delta_k^{\frac{2s}{\alpha}}}{2}+t_k,r^{\frac{2s}{\alpha}}+t_k)],\\ =0, & (z,t)\notin B_{\delta_k}(z^k)\times(t_k-\delta_k^{\frac{2s}{\alpha}},t_k+\delta_k^{\frac{2s}{\alpha}}).\end{cases}$$

Denote

$$Q_{\delta_k}(z^k,t_k):=B_{\delta_k}(z^k)\times(t_k-\delta_k^{\frac{2s}{\alpha}},t_k+\delta_k^{\frac{2s}{\alpha}}),$$

therefore, direct calculation can be obtained.

$$v_k(z^k,t_k)=w_{\Lambda_k}(z^k,t_k)-m_k^2\eta_k(z^k,t_k)=-m_k+\sigma m_k^2-\sigma m_k^2=-m_k,$$

and

$$v_k(z,t)=w_{\Lambda_k}(z,t)\geq -m_k \text{ in } Q^c_{\delta_k}(z^k,t_k)\cap(\Sigma_{\Lambda_k}\times\mathbb{R}).$$

Then, there exists a point $(\bar{z}^k,\bar{t}^k)$ and $(\bar{z}^k,\bar{t}^k)\in Q_{\delta_k}(z^k,t_k)$ such that

$$-m_k-\sigma m_k^2\leq v_k(\bar{z}^k,\bar{t}^k)=\inf_{\Sigma_{\Lambda_k}\times\mathbb{R}}v_k(z,t)\leq -m_k.$$

By definition of v_k, we have

$$-m_k\leq w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)\leq -m_k+\sigma m_k^2<0.$$

For the minimum point $(\bar{z}^k,\bar{t}^k)$ of $v_k(z,t)$, we have

$$\partial_t^\alpha v_k(\bar{z}^k,\bar{t}^k)=C_\alpha\int_{-\infty}^{\bar{t}^k}\frac{v_k(\bar{z}^k,\bar{t}^k)-v_k(\bar{z}^k,\tau)}{(\bar{t}^k-\tau)^{1+\alpha}}d\tau\leq 0,$$

and

$$
\begin{aligned}
&(-\Delta_g)^s v_k(\bar{z}^k,\bar{t}^k)\\
=&\mathrm{P.V.}\left(\int_{\Sigma_{\Lambda_k}}+\int_{\Sigma^c_{\Lambda_k}}\right) g\left(\frac{v_k(\bar{z}^k,\bar{t}^k)-v_k(\vartheta,\bar{t}^k)}{|\bar{z}^k-\vartheta|^s}\right)\frac{1}{|\bar{z}^k-\vartheta|^{n+s}}d\vartheta\\
\le& C\ \mathrm{P.V.}\int_{\Sigma_{\Lambda_k}} g\left(\frac{2v_k(\bar{z}^k,\bar{t}^k)+\sigma m_k^2\eta_k(\vartheta,\bar{t}^k)}{|\bar{z}^k-\vartheta^{\Lambda_k}|^s}\right)\frac{1}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+s}}d\vartheta\\
\le& C\ \mathrm{P.V.}\int_{\Sigma_{\Lambda_k}} g\left(\frac{2v_k(\bar{z}^k,\bar{t}^k)}{|\bar{z}^k-\vartheta^{\Lambda_k}|^s}\right)\frac{1}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+s}}d\vartheta+C\ \mathrm{P.V.}\int_{\Sigma_{\Lambda_k}} g\left(\frac{\sigma m_k^2\eta_k(\vartheta,\bar{t}^k)}{|\bar{z}^k-\vartheta^{\Lambda_k}|^s}\right)\frac{1}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+s}}d\vartheta\\
\le& -\frac{C}{\delta_k^{2s}}+\frac{Cm_k^2}{\delta_k^{2s}}.
\end{aligned}
$$

By Equation (27), Corollary 5.2 in [29], and properties of g, it follows that

$$
\begin{aligned}
&\partial_t^\alpha v_k(\bar{z}^k,\bar{t}^k)+(-\Delta_g)^s v_k(\bar{z}^k,\bar{t}^k)\\
=&\partial_t^\alpha w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)+(-\Delta_g)^s u_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-(-\Delta_g)^s u(\bar{z}^k,\bar{t}^k)-\sigma m_k^2\partial_t^\alpha\eta_k(\bar{z}^k,\bar{t}^k)\\
&+(-\Delta_g)^s v_k(\bar{z}^k,\bar{t}^k)-[(-\Delta_g)^s u_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-(-\Delta_g)^s u(\bar{z}^k,\bar{t}^k)]\\
\ge&\left(\frac{q\xi_1^{q-1}\left(\bar{z}^k,\bar{t}^k\right)}{\xi_1^q\left(\bar{z}^k,\bar{t}^k\right)+1}+\frac{p\xi_2^{p-1}\left(\bar{z}^k,\bar{t}^k\right)}{\xi_2^p\left(\bar{z}^k,\bar{t}^k\right)+1}\right)w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-\frac{Cm_k^2}{\delta_k^{2s}}\\
&-C\ \mathrm{P.V.}\int_{\mathbb{R}^n}\frac{g'(\zeta(\vartheta,\bar{t}^k))(w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-w_{\Lambda_k}(\vartheta,\bar{t}^k))}{|\bar{z}^k-\vartheta|^{n+2s}}d\vartheta\\
\ge&\left(\frac{q\xi_1^{q-1}\left(\bar{z}^k,\bar{t}^k\right)}{\xi_1^q\left(\bar{z}^k,\bar{t}^k\right)+1}+\frac{p\xi_2^{p-1}\left(\bar{z}^k,\bar{t}^k\right)}{\xi_2^p\left(\bar{z}^k,\bar{t}^k\right)+1}\right)w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-\frac{Cm_k^2}{\delta_k^{2s}}\\
&-C\int_{\Sigma_{\Lambda_k}}\frac{g'(\zeta(\vartheta,\bar{t}^k))(w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-w_{\Lambda_k}(\vartheta,\bar{t}^k))}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+2s}}d\vartheta-C\int_{\Sigma_{\Lambda_k}}\frac{g'(\zeta(\vartheta,\bar{t}^k))(w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)+w_{\Lambda_k}(\vartheta,\bar{t}^k))}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+2s}}d\vartheta\\
\ge&-Cw_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-\frac{Cm_k^2}{\delta_k^{2s}}-C\int_{\Sigma_{\Lambda_k}}\frac{g'(\zeta(\vartheta,\bar{t}^k))2w_{\Lambda_k}(\bar{z}^k,\bar{t}^k)}{|\bar{z}^k-\vartheta^{\Lambda_k}|^{n+2s}}d\vartheta\\
\ge&-Cm_k-\frac{Cm_k^2}{\delta_k^{2s}},
\end{aligned}
$$

where

$$g'(\zeta(\vartheta,\bar{t}^k))\text{ between }\frac{u_{\Lambda_k}(\bar{z}^k,\bar{t}^k)-u_{\Lambda_k}(\vartheta,\bar{t}^k)}{\left|\bar{z}^k-\vartheta\right|^s}\text{ and }\frac{u(\bar{z}^k,\bar{t}^k)-u(\vartheta,\bar{t}^k)}{\left|\bar{z}^k-\vartheta\right|^s},$$

and

$$\xi_1(\bar{z}^k,\bar{t}^k)\text{ and }\xi_2(\bar{z}^k,\bar{t}^k)\text{ fall in between }u_{\Lambda_k}(\bar{z}^k,\bar{t}^k)\text{ and }u(\bar{z}^k,\bar{t}^k).$$

In summary, we obtain

$$-\frac{C}{\delta_k^{2s}}\ge -Cm_k-\frac{Cm_k^2}{\delta_k^{2s}}.$$

The inequality is multiplied by $-\delta_k^{2s}$ on both sides; then, by Equation (32) and the assumption $\lim\limits_{k\to\infty}\delta_k=0$, we have

$$0<C\le Cm_k\delta_k^{2s}+Cm_k^2\to 0\text{ as }k\to\infty.$$

This is a contradiction. So, δ_k is bounded and $\lim\limits_{k\to\infty} \delta_k \neq 0$ for sufficiently large k. In addition, since $\Lambda_k \to \Lambda_0$ when $k \to \infty$, we can infer that there is a subsequence of $\{z^k, t_k\}$ (which we will also denote as $\{z^k, t_k\}$), satisfying $\{z^k, t_k\} \subset \Sigma_{\Lambda_0} \times \mathbb{R}$ and dist $\{z^k, T_{\Lambda_0}\} \geq \delta_0 > 0$. According to Equation (33), we further choose a radius, $r_1 := \min\{r_0, \delta_0\}$, such that

$$u(z,t) \geq \frac{\varepsilon_0}{2} > 0 \text{ in } B_{r_1}(z^k) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}] \subset \Omega_{\Lambda_0} \times \mathbb{R}. \tag{34}$$

Before continuing, let $\bar{z}^k = (2\Lambda_0, (z^k)')$, since $\operatorname{dist}(T_{\Lambda_0}, T_{2\Lambda_0}) = \Lambda_0 > 2r_1$; then, we have $\overline{B_{r_1}(z^k)} \cap B_{2r_1}(\bar{z}^k) = \varnothing$. Next, we demonstrate that there exists a positive constant ε_1 such that

$$u(z,t) \geq \varepsilon_1 > 0 \text{ in } B_{r_1}(\bar{z}^k) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}]. \tag{35}$$

Otherwise, we have

$$u(z,t) < \varepsilon \text{ in } B_{r_1}(\bar{z}^k) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}] \text{ for any } \varepsilon > 0, \tag{36}$$

then, we derive

$$|(\ln|u^q(z,t)+1| + \ln|u^p(z,t)+1|) - (\ln|0^q+1| + \ln|0^p+1|)| \leq C|(u(z,t)) - 0| < C\varepsilon.$$

Furthermore, it follows that

$$\ln|u^q(z,t)+1| + \ln|u^p(z,t)+1| \geq -C\varepsilon \text{ in } B_{r_1}(\bar{z}^k) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}] \text{ for any } \varepsilon > 0.$$

Then, by combining Theorem 5, Equations (26) and (34) and $u(z,t)$ are continuous, and it follows that

$$u(z,t) \geq \varepsilon_1 > 0,\ (z,t) \in B_{\frac{r_1}{2}}(\bar{z}^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right],$$

which contradicts with Equation (36). So, we obtain Equation (35).

Let $\hat{z}^k = (0, (z^k)')$. Due to the fact that $u(z,t) = 0$ in $(\mathbb{R}^n \setminus \mathbb{R}^n_+) \times \mathbb{R}$ and $u(z,t)$ are continuous, there exists $r_2 < \frac{r_1}{2}$, which is irrespective of k, such that

$$u(z,t) \leq \frac{\varepsilon_1}{2},\ (z,t) \in (B_{r_2}(\hat{z}^k) \cap \mathbb{R}^n_+) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}]. \tag{37}$$

For any point $z \in B_{r_2}(\hat{z}^k) \cap \mathbb{R}^n_+$, z^{Λ_0} denotes the reflection point of z about T_{Λ_0}, and $z^{\Lambda_0} \in B_{r_2}(\bar{z}^k) \cap \Sigma_{2\Lambda_0} \subset B_{\frac{r_1}{2}}(\bar{z}^k)$. Combining Equations (35) and (37), we obtain

$$w_{\Lambda_0}(z,t) = u_{\Lambda_0}(z,t) - u(z,t) \geq \varepsilon_1 - \frac{\varepsilon_1}{2} = \frac{\varepsilon_1}{2} > 0,\ (z,t) \in (B_{r_2}(\hat{z}^k) \cap \mathbb{R}^n_+) \times (t_k - r_1^{\frac{2s}{\alpha}}, t_k + r_1^{\frac{2s}{\alpha}}]. \tag{38}$$

Now, we mainly show that there is a positive constant ε_2, such that

$$w_{\Lambda_0}(z,t) \geq \varepsilon_2 > 0 \text{ for } (z,t) \in B_{\frac{r_1}{2}}(z^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right]. \tag{39}$$

If not, we obtain

$$w_{\Lambda_0}(z,t) < \varepsilon \text{ for } (z,t) \in B_{\frac{r_1}{2}}(z^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right] \text{ for any } \varepsilon > 0. \tag{40}$$

Combining Equation (27) and the definition of Λ_0, it follows that

$$\begin{cases} \partial_t^\alpha w_{\Lambda_0}(z,t) + (-\Delta_g)^s u_{\Lambda_0}(z,t) - (-\Delta_g)^s u(z,t) = \left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right) w_{\Lambda_0}(z,t), \\ \qquad\qquad\qquad \text{in } B_{\frac{r_1}{2}}(z^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right], \\ w_{\Lambda_0}(z,t) \geq 0, \qquad \text{in } \left(\Sigma_{\Lambda_0} \backslash B_{\frac{r_1}{2}}(z^k)\right) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right], \\ w_{\Lambda_0}(z,t) \geq 0, \qquad \text{in } B_{\frac{r_1}{2}}(z^k) \times \left(-\infty, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right]. \end{cases}$$

where $\xi_1(z,t)$ and $\xi_2(z,t)$ fall in-between $u_{\Lambda_0}(z,t)$ and $u(z,t)$. Then, by Equation (40) and the boundedness of $\left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right)$, we have

$$\left(\frac{q\xi_1^{q-1}(z,t)}{\xi_1^q(z,t)+1} + \frac{p\xi_2^{p-1}(z,t)}{\xi_2^p(z,t)+1}\right) w_{\Lambda_0}(z,t) > -C\varepsilon,\ (z,t) \in B_{\frac{r_1}{2}}(z^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right]$$

for any $\varepsilon > 0$. Hence, by Equation (38), Theorem 6 and $u(z,t)$ are continuous, and we obtain

$$w_{\Lambda_0}(z,t) \geq \varepsilon_2 > 0,\ (z,t) \in B_{\frac{r_1}{4}}(z^k) \times \left(t_k - (\frac{r_1}{4})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{4})^{\frac{2s}{\alpha}}\right]$$

for some positive constant ε_2, which contradicts with Equation (40). So, we obtain Equation (39). Moreover, by utilizing the continuity of $w_\Lambda(z,t)$, $\Lambda_k \to \Lambda_0$ when $k \to \infty$ and Equation (39), we can ultimately derive

$$w_{\Lambda_k}(z,t) \geq \frac{\varepsilon_2}{2} > 0 \text{ in } B_{\frac{r_1}{2}}(z^k) \times \left(t_k - (\frac{r_1}{2})^{\frac{2s}{\alpha}}, t_k + (\frac{r_1}{2})^{\frac{2s}{\alpha}}\right],$$

which means that $w_{\Lambda_k}(z^k, t_k) \geq \frac{\varepsilon_2}{2} > 0$ for a large enough k. Therefore, this contradicts the assumption that the sequence $\{(z^k, t_k)\} \subset \Sigma_{\Lambda_k}^- \times \mathbb{R}$; then, we must have $\Lambda_0 = +\infty$.

Step 3: Our most critical step is to prove that the positive solution $u(z,t)$ of Equation (26) is strictly increasing in half space along the z_1-direction for any $t \in \mathbb{R}$.

Based on the previous two steps, for any $\Lambda > 0$, we derive

$$w_\Lambda(z,t) \geq 0 \text{ in } \Sigma_\Lambda \times \mathbb{R}.$$

In fact, we just have to prove that

$$w_\Lambda(z,t) > 0 \text{ in } \Omega_\Lambda \times \mathbb{R} \text{ for any } \Lambda > 0. \tag{41}$$

Assume that Equation (41) is invalid; there must be a point, $(z^0, t_0) \in \Omega_{\Lambda_0} \times \mathbb{R}$, and $\Lambda_0 > 0$ such that

$$w_{\Lambda_0}(z^0, t_0) = \min_{\Sigma_{\Lambda_0} \times \mathbb{R}} w_{\Lambda_0}(z,t) = 0.$$

By $w_{\Lambda_0}(z,t_0) \not\equiv 0$ in Σ_{Λ_0}, $u(z,t) = 0$ in $(\mathbb{R}\backslash\mathbb{R}^n_+) \times \mathbb{R}$, $\frac{1}{|z^0 - \vartheta^{\Lambda_0}|^{n+2s}} < \frac{1}{|z^0 - \vartheta|^{n+2s}}$ in Σ_{Λ_0}, and the properties of g, we obtain

$$
\begin{aligned}
&\partial_t^\alpha w_{\Lambda_0}(z^0,t_0)+(-\Delta_g)^s u_{\Lambda_0}(z^0,t_0)-(-\Delta_g)^s u(z^0,t_0)\\
=&C_\alpha\int_{-\infty}^{t_0}\frac{w_{\Lambda_0}(z^0,t_0)-w_{\Lambda_0}(z^0,\tau)}{(t_0-\tau)^{1+\alpha}}d\tau\\
&+\mathrm{P.V.}\int_{\mathbb{R}^n}\left[g\left(\frac{u_{\Lambda_0}(z^0,t_0)-u_{\Lambda_0}(\vartheta,t_0)}{|z^0-\vartheta|^s}\right)-g\left(\frac{u(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta|^s}\right)\right]\frac{1}{|z^0-\vartheta|^{n+s}}d\vartheta\\
\le&\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{g'(\vartheta,t_0)(w_{\Lambda_0}(z^0,t_0)-w_{\Lambda_0}(\vartheta,t_0))}{|z^0-\vartheta|^{n+2s}}d\vartheta\\
\le&\mathrm{P.V.}\int_{\Sigma_{\Lambda_0}}w_{\Lambda_0}(\vartheta,t_0)g'(\vartheta,t_0)\left(\frac{1}{|z^0-\vartheta^{\Lambda_0}|^{n+2s}}-\frac{1}{|z^0-\vartheta|^{n+2s}}\right)d\vartheta\\
<&0,
\end{aligned}
$$

where

$$
g'(\vartheta,t_0)\text{ between }\frac{u_{\Lambda_0}(z^0,t_0)-u_{\Lambda_0}(\vartheta,t_0)}{|z^0-\vartheta|^s}\text{ and }\frac{u(z^0,t_0)-u(\vartheta,t_0)}{|z^0-\vartheta|^s}.
$$

This contradicts

$$
\partial_t^\alpha w_{\Lambda_0}(z^0,t_0)+(-\Delta_g)^s u_{\Lambda_0}(z^0,t_0)-(-\Delta_g)^s u(z^0,t_0)=0.
$$

Then, Equation (41) is valid.

Finally, for every fixed $t\in\mathbb{R}$, utilizing Equation (41), for any $\bar{z}=(\bar{z}_1,z')$ and $\hat{z}=(\hat{z}_1,z')$ in $\mathbb{R}^n_+$, and satisfying $\bar{z}_1<\hat{z}_1$, when we select $\Lambda=\frac{\bar{z}_1+\hat{z}_1}{2}$, we can conclude that

$$
0<w_\Lambda(\bar{z},t)=u_\Lambda(\bar{z},t)-u(\bar{z},t)=u(\hat{z},t)-u(\bar{z},t).
$$

Therefore, we prove that the positive solution $u(z,t)$ of Equation (26) is strictly increasing in half space along the z_1-direction for any $t\in\mathbb{R}$. □

Example 1. *Consider the following equation:*

$$
\begin{cases}
\partial_t^{\frac{1}{4}}u(z,t)+(-\Delta_g)^{\frac{1}{2}}u(z,t)=\ln\left|u^2(z,t)+1\right|+\ln\left|u^3(z,t)+1\right|, & \text{in }\mathbb{R}^2_+\times\mathbb{R},\\
u(z,t)=0, & \text{in }\left(\mathbb{R}^2\backslash\mathbb{R}^2_+\right)\times\mathbb{R}.
\end{cases}
\tag{42}
$$

If $u(z,t)\le C(1+|z|^{\frac{1}{2}})$ is a positive solution to Equation (42), then according to Theorem 7, the positive solution $u(z,t)$ of Equation (42) is strictly increasing in $\mathbb{R}^2_+$ along the z_1-direction for any $t\in\mathbb{R}$.

5. Conclusions

In this paper, we study the double index logarithmic nonlinear fractional g-Laplacian parabolic equations with the Marchaud fractional time derivatives ∂_t^α by using the direct moving plane method. We successfully overcome the difficulties caused by the double non-locality of space-time and the nonlinearity of the fractional g-Laplacian. The results of this paper provide an important tool and method for the study of qualitative properties of solutions, especially for the unbounded solutions of fractional elliptic and parabolic problems. In the future work, we will continue to deeply study other properties and numerical simulations of this class of equations and explore its real-world applications.

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