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Article

A Unified Approach to Solvability and Stability of Multipoint BVPs for Langevin and Sturm–Liouville Equations with CH–Fractional Derivatives and Impulses via Coincidence Theory

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Abstract: The Langevin equation is a model for describing Brownian motion, while the Sturm–Liouville equation is an important mechanical model. This paper focuses on the solvability and stability of nonlinear impulsive Langevin and Sturm–Liouville equations with Caputo–Hadamard (CH) fractional derivatives and multipoint boundary value conditions. To unify the two types of equations, we investigate a general nonlinear impulsive coupled implicit system. By cleverly constructing relevant operators involving impulsive terms, we establish the coincidence degree theory and obtain the solvability. We explore the stability of solutions using nonlinear analysis and inequality techniques. As the most direct application, we naturally obtained the solvability and stability of the Langevin and Sturm–Liouville equations mentioned above. Finally, an example is provided to demonstrate the validity and availability of our major findings.

Keywords: impulsive coupled implicit system; CH–fractional derivative; multipoint BVP; solvability and stability; coincidence degree theory

MSC: 34B16; 34B37; 34K32; 34K37

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1. Introduction

In this paper, we delve into the following nonlinear impulsive CH–fractional differential Langevin Equation (1) and Sturm–Liouville Equation (2) under the m-point boundary value conditions

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}-\lambda]\mathcal{U}(x)=F(x,\mathcal{U}(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2+\theta_1}\mathcal{U}(x)), x\in[\alpha,\beta]\setminus\{x_l\}_{l=0}^{m}, \\ \Delta\mathcal{U}(x_l)=I_l(\mathcal{U}(x_l)),\ \Delta[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l)=J_l(\mathcal{U}(x_l)),\ 1\le l\le m, \\ \sum_{l=1}^{m+1}a_l\mathcal{U}(\xi_l)=c,\ \sum_{l=1}^{m+1}b_l[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](\xi_l)=d, \end{cases} \tag{1}$$

where $0<\alpha<\beta$, $0<\theta_1,\theta_2\le 1$, $\lambda>0$, $\alpha=x_0=\xi_1<x_1<\xi_2<x_2<\xi_3<x_3<\ldots<x_m<\xi_m=x_{m+1}=\beta$, $a_l,b_l,c,d\in\mathbb{R}$, $\sum_{l=1}^{m}a_l\neq 0$, $\sum_{l=1}^{m}b_l\neq 0$, ${}^{\mathrm{CH}}\mathcal{D}_{x_l}$ is the CH–fractional derivative, $F\in C([\alpha,\beta]\times\mathbb{R}^2,\mathbb{R}^+)$, $I_l,J_l\in C(\mathbb{R},\mathbb{R})$, $\Delta\mathcal{U}(x_l)=\mathcal{U}(x_l^+)-\mathcal{U}(x_l^-)$, $\Delta[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l)=[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l^+)-[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l^-)$, $[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l^-)=[{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l)$, $\mathcal{U}(x_l^-)=\mathcal{U}(x_l)$, $1\le l\le m$, $i=1,2$.

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}[p(x)\,{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}]\mathcal{U}(x)=F(x,\mathcal{U}(x)), x\in[\alpha,\beta]\setminus\{x_l\}_{l=0}^{m}, \\ \Delta\mathcal{U}(x_l)=I_l(\mathcal{U}(x_l)),\ \Delta[p(\cdot)\,{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](x_l)=J_l(\mathcal{U}(x_l)),\ 1\le l\le m, \\ \sum_{l=1}^{m+1}a_l\mathcal{U}(\xi_l)=c,\ \sum_{l=1}^{m+1}b_l[p(\cdot)\,{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(\cdot)](\xi_l)=d, \end{cases} \tag{2}$$

where $p \in C([\alpha,\beta],(0,+\infty))$, $F \in C([\alpha,\beta] \times \mathbb{R}, \mathbb{R}^+)$. The other conditions are the same as (1).

Remark 1. *In (1) and (2), the impulse functions I_l, J_l are only related to $\mathcal{U}(x_l)$ since ${}^{\mathrm{CH}}\mathcal{D}^{*}_{x_l}\mathcal{U}(x_l) \equiv 0$. In addition, (1) is implicit and (2) is explicit.*

As is well known, the Langevin equation is a famous mathematical model that describes the random motion of particles annihilating in a fluid due to collisions between particles and fluid molecules. Compared with the integer-order Langevin equation, the fractional-order Langevin equation is more accurate in describing the random motion of particles in complex viscoelastic fluids. In recent years, many papers dealing with the fractional Langevin equations have been published. For example, Ahmadova and Mahmudov [1] studied the explicit analytical solutions for several families of Langevin differential equations with general fractional orders. Salem et al. [2] applied Darbo's fixed-point theorem to investigate the existence of solutions for the three-point boundary value problem of a fractional Langevin equation in the noncompact Hausdorff space. Zhao, in [3–5], discussed the stability of several types of nonlinear fractional Langevin equations with delays and controls. In [6–8], the authors explored the controllability problem of fractional Langevin equations. Other papers are [9–11] concerned with the dynamics of stochastic Langevin equations.

Furthermore, the Sturm–Liouville equation, which includes the Helmholtz equation, Bessel equation, and Legendre equation, also represents another important class of mathematical and physical equations. Therefore, study of the fractional Sturm–Liouville equation has also become a hot topic in recent years. Afarideh et al. [12] used the pseudospectral method and Chebyshev cardinal functions to solve the Caputo fractional Sturm–Liouville eigenvalue problems. Sadabad and Akbarfam [13] provided an efficient numerical method to estimate the eigenvalues and eigenfunctions of the fractional Sturm–Liouville equation. Allahverdiev et al. [14] obtained a completeness theorem of singular dissipative conformable fractional Sturm–Liouville operators. Goel et al. [15] probed the numerical calculation of mixed boundary value problems for the generalized fractional Sturm–Liouville system. Kumar and Mehra [16] adopted the wavelet method to solve the Sturm–Liouville fractional optimal control problem. In fact, there are many research achievements on fractional Langevin and Sturm–Liouville equations. We will not elaborate further here. However, previous works in the literature have studied the two types of equations separately, and there is rarely a unified approach. Accordingly, it is novel and fascinating to unify Equations (1) and (2) for research purposes.

To address the solvability and stability of Equations (1) and (2) together, we consider a general system including (1) and (2) as follows:

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x) = F_1(x,\mathcal{U}_1(x),\mathcal{U}_2(x),{}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x)), \\ {}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x) = F_2(x,\mathcal{U}_1(x),\mathcal{U}_2(x),{}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x)), \\ \Delta\mathcal{U}_1(x_l) = I_l(\mathcal{U}_1(x_l),\mathcal{U}_2(x_l)),\ \Delta\mathcal{U}_2(x_l) = J_l(\mathcal{U}_1(x_l),\mathcal{U}_2(x_l)), \\ \sum_{l=1}^{m+1} a_l\mathcal{U}_1(\xi_l) = c,\ \sum_{l=1}^{m+1} b_l\mathcal{U}_2(\xi_l) = d, \end{cases} \tag{3}$$

where $F_i \in C([\alpha,\beta] \times \mathbb{R}^4, \mathbb{R}^+)$, $I_l, J_l \in C(\mathbb{R}^2, \mathbb{R})$, $\Delta\mathcal{U}_i(x_l) = \mathcal{U}_i(x_l^+) - \mathcal{U}_i(x_l^-)$ with $\mathcal{U}_i(x_l^-) = \mathcal{U}_i(x_l)$, $1 \le l \le m$, $i = 1,2$. The other conditions are the same as (1).

Remark 2. *When boundary conditions $\sum_{l=1}^{m+1} a_l\mathcal{U}_1(\xi_l) = c$, $\sum_{l=1}^{m+1} b_l\mathcal{U}_2(\xi_l) = d$ degenerate to $\mathcal{U}_1(\alpha) + \mathcal{U}_1(\beta) = 0$, $\mathcal{U}_2(\alpha) + \mathcal{U}_2(\beta) = 0$, system (3) becomes an impulsive implicit antiperiodic boundary value problem.*

The Hadamard fractional calculus proposed by Hadamard in 1892 [17] is a direct and effective extension of Riemann–Liouville (RL) fractional calculus. Its prominent feature is that the logarithmic kernel $H(x,s) = (\log \frac{x}{s})^{\vartheta-1}$ replaces the polynomial kernel

$G(x,s) = (x-s)^{\vartheta-1}$ in the RH–calculus definition. These two kernels maintain certain mathematical commonalities. For example, they have singularity when $0 < \vartheta \leq 1$, that is, $G(x,s) \to \infty$, $H(x,s) \to \infty$, as $s \to x$. Some of their properties are also significantly distinguished. For instance, $\forall\, \zeta > 0$, $H(\zeta x, \zeta s) = H(x,s)$, but $G(\zeta x, \zeta s) = \zeta^{\vartheta-1}G(x,s) \neq G(x,s)$. As an important class of differential equations, the theory and application of Hadamard-type (H-type) fractional differential equations (FDEs) have received extensive and in-depth research, which has achieved fruitful results (see [18–31]). Until now, the exploration of various dynamic properties of H-type fractional differential equations has been a very lively research topic. For example, in [32], the authors discussed the logarithmic decay stability of an H-type fractional equation. Rao et al. [33] considered the problem of multiplicity of solutions for a mixed H–fractional Laplacian system. Zhao [34,35] thought about the approximation and Hyers–Ulam-type stability of two classes of H–fractional boundary value problems. In [36,37], the authors studied the numerical calculation problem of H–fractional equations. Ortigueira et al. [38] explored the unification of H–calculus and RL–calculus. Dhawan et al. [39] applied the upper and lower solution method to analyze a neutral H–fractional equation. Ahmad et al. [40] investigated a coupled system of Hilfer–Hadamard fractional equations. Ben Makhlouf et al. [41] studied the existence, uniqueness, and averaging principle for Hadamard Ito–Doob stochastic delay fractional integral equations. Briefly, the properties, research approaches, and generalization of the concept of H–derivative, as well as the effects of delay, impulse, and random factors on H–fractional differential systems, have always attracted the attention of scholars. We further refer to [42–47].

Generally speaking, the study of implicit forms of differential equations is relatively more difficult than explicit forms. Therefore, the research achievements on implicit differential equations are also rarer than those on explicit differential equations. Only a small number of published papers deal with the solvability and stability of implicit Hadamard fractional differential equations (see [48–56]). Some academic researchers have applied the theory of coincidence degree to study the solvability of integer-order nonlinear functional differential equations and have achieved fruitful results (see [57–68]). In the theory of coincidence degree, the construction of relevant operators is highly skilled, which brings difficulties to the application of this method. Consequently, there are relatively few works [69–73] on the existence of solutions to fractional differential equations via coincidence degree theory.

Owed to the aforementioned, it is fascinating and challenging to investigate the solvability of system (3) by coincidence degree theory. The highlights of this paper mainly comprise the following. (a) Our work enriches and fills the gap in the study of nonlocal boundary value problems for implicit and impulsive fractional coupled systems. (b) In the establishment of coincidence degree theory, we cleverly constructed and proved the complete continuity of the relevant operators for the first time in the study of impulsive fractional differential equations. (c) As an important application of our basic results, we obtained the solvability and stability of the Langevin system and Sturm–Liouville system.

The remaining content of this paper is arranged as follows. Some necessary concepts and lemmas are stated in Section 2. Section 3 studies the existence, uniqueness, and stability of solutions to (3). Section 4 discusses the solvability and stability of the Langevin system (1) and Sturm–Liouville system (2), and gives an example to check the validity and availability of our basic findings. Finally, we provide a simple conclusion of research approaches, results, and significance in Section 5.

2. Preliminaries

This section mainly introduces some basic knowledge required for this article. We first state an important result of the coincidence theory for solving operator equations as follows.

Lemma 1 (Mawhin [74])**.** *Let $\mathbb{E}, \mathbb{F}$ be Banach spaces, $\emptyset \neq \Theta \subset \mathbb{E}$, a bounded open subset. If $\mathcal{L} : \mathbb{E} \to \mathbb{F}$ is a 0-index Fredholm operator, and $\mathcal{N} : \mathbb{E} \times [0,1] \to \mathbb{F}$ is $\mathcal{L}$-compact on $\overline{\Theta} \times [0,1]$,* *then there has to be at least* $\mathcal{X}^* \in \overline{\Theta} \cap \mathrm{Dom}\,\mathcal{L}$ *s.t.* $\mathcal{L}\mathcal{X}^* = \mathcal{N}(\mathcal{X}^*, 1)$ *provided that the following is true:*

(a1) *If* $\mathcal{X}$ *solves* $\mathcal{L}\mathcal{X} = \eta \cdot \mathcal{N}(\mathcal{X}, \eta)$*, then* $\mathcal{X} \notin \partial\Theta \cap \mathrm{Dom}\,\mathcal{L}$*,* $\forall\, \eta \in (0, 1)$*;*
(a2) $\mathcal{Q}\mathcal{N}(\mathcal{X}, 0)\mathcal{X} \neq 0$, $\forall\, \mathcal{X} \in \partial\Theta \cap \mathrm{Ker}\,\mathcal{L}$*;*
(a3) $\deg(\mathcal{J}\mathcal{Q}\mathcal{N}(\mathcal{X}, 0), \Theta \cap \mathrm{Ker}\,\mathcal{L}, 0) \neq 0$*;*

where $\mathcal{Q}, \mathcal{J} : \mathbb{F} \to \mathbb{F}$ *are projected and homotopy, respectively.*

Next, we need to review the basic concepts and results of Caputo–Hadamard fractional calculus.

Definition 1 ([75])**.** *Let* $\theta > 0$, $0 < \alpha < \beta < \infty$*, and* $\mathcal{U} : [\alpha, \beta] \to \mathbb{R}$*; the definition of* θ*-order Hadamard fractional integral of* $\mathcal{U}$ *is*

$$^{\mathrm{H}}\mathcal{J}^{\theta}_{\alpha^+}\mathcal{U}(x) = \frac{1}{\Gamma(\theta)} \int_\alpha^x \left(\log \frac{x}{z}\right)^{\theta-1} \mathcal{U}(z)\frac{dz}{z}.$$

Definition 2 ([75])**.** *Let* $\theta > 0$, $0 < \alpha < \beta < \infty$, $m = [\theta] + 1$*, and* $\left(x\frac{d}{dx}\right)^{m-1}\mathcal{U}(x) \in AC[\alpha, \beta]$*; the definition of* θ*-order Caputo–Hadamard fractional derivative of* $\mathcal{U}$ *is*

$$^{\mathrm{CH}}\mathcal{D}^{\theta}_{\alpha^+}\mathcal{U}(x) = \frac{1}{\Gamma(m-\theta)} \int_\alpha^x \left(\log \frac{x}{z}\right)^{m-\theta-1} \left(z\frac{d}{dz}\right)^m \mathcal{U}(z)\frac{dz}{z}.$$

Lemma 2 ([75])**.** *Let* $\theta > 0$, $0 < \alpha < \beta < \infty$, $m = [\theta] + 1$*, and* $\left(x\frac{d}{dx}\right)^{m-1}\mathcal{U}(x) \in AC[\alpha, \beta]$*, then*

$$^{\mathrm{H}}\mathcal{J}^{\theta}_{\alpha^+}(^{\mathrm{CH}}\mathcal{D}^{\theta}_{\alpha^+}\mathcal{U}(x)) = \mathcal{U}(x) + \sum_{j=0}^{m-1} \frac{\left(z\frac{d}{dz}\right)^j \mathcal{U}(z)\big|_{z=\alpha}}{j!} \left(\log \frac{x}{\alpha}\right)^j.$$

To obtain a prior estimate of the solution to BVP (3), the following lemma is required.

Lemma 3. *Let* $0 < \alpha < \beta$, $0 < \theta_1, \theta_2 \leq 1$, $a_l, b_l, c, d \in \mathbb{R}$, $\sum_{l=1}^m a_l \neq 0$, $\sum_{l=1}^m b_l \neq 0$, $F_i \in C([\alpha, \beta] \times \mathbb{R}^4, \mathbb{R}^+)$, $I_l, J_l \in C(\mathbb{R}^2, \mathbb{R})$, $1 \leq l \leq m$, $i = 1, 2$*. If* $\mathcal{U}(x) = (\mathcal{U}_1(x), \mathcal{U}_2(x))^T$ *solves the BVP (3), then* $\mathcal{U}(x) = (\mathcal{U}_1(x), \mathcal{U}_2(x))^T$ *also solves the following integral equation*

$$\begin{cases} \mathcal{U}_1(x) = c_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_\alpha^x \left(\log \frac{x}{z}\right)^{\theta_1-1} \phi_{\mathcal{U}_1}(z)\frac{dz}{z}, \ x \in [\alpha, x_1], \\ \mathcal{U}_2(x) = d_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_\alpha^x \left(\log \frac{x}{z}\right)^{\theta_2-1} \psi_{\mathcal{U}_2}(z)\frac{dz}{z}, \ x \in [\alpha, x_1], \\ \mathcal{U}_1(x) = c_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_{x_l}^x \left(\log \frac{x}{z}\right)^{\theta_1-1} \phi_{\mathcal{U}_1}(z)\frac{dz}{z}, \ x \in (x_l, x_{l+1}], \\ \mathcal{U}_2(x) = d_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_{x_l}^x \left(\log \frac{x}{z}\right)^{\theta_2-1} \psi_{\mathcal{U}_2}(z)\frac{dz}{z}, \ x \in (x_l, x_{l+1}], \end{cases} \tag{4}$$

where $1 \leq l \leq m$, $a =: \sum_{l=1}^m a_l$, $b =: \sum_{l=1}^m b_l$*,*

$$\begin{cases} \phi_{\mathcal{U}_1}(x) = F_1(x, \mathcal{U}_1(x), \mathcal{U}_2(x), \phi_{\mathcal{U}_1}(x), \psi_{\mathcal{U}_2}(x)), \\ \psi_{\mathcal{U}_2}(x) = F_2(x, \mathcal{U}_1(x), \mathcal{U}_2(x), \phi_{\mathcal{U}_1}(x), \psi_{\mathcal{U}_2}(x)), \end{cases}$$

$$\begin{cases} c_0^{\mathcal{U}} = \frac{1}{a}\Big[c - \sum\limits_{l=1}^{m} a_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_1)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} \\ \quad + I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k))\Big) - \frac{1}{\Gamma(\theta_1)} \sum\limits_{l=1}^{m} a_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}\Big], \\ d_0^{\mathcal{U}} = \frac{1}{b}\Big[d - \sum\limits_{l=1}^{m} b_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_2)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} \\ \quad + J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k))\Big) - \frac{1}{\Gamma(\theta_2)} \sum\limits_{l=1}^{m} b_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}\Big], \\ c_l^{\mathcal{U}} = c_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)), \\ d_l^{\mathcal{U}} = d_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)). \end{cases}$$

Proof. For $x \in [\xi_1, x_1] = [\alpha, x_1]$, by Lemma 2 and (3), we have

$$\begin{cases} \mathcal{U}_1(x) = c_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_\alpha^x (\log \frac{x}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}, \\ \mathcal{U}_2(x) = d_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_\alpha^x (\log \frac{x}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}. \end{cases} \tag{5}$$

For $x \in (x_1, x_2]$, similar to (5), we obtain

$$\begin{cases} \mathcal{U}_1(x) = c_1^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_{x_1}^x (\log \frac{x}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}, \\ \mathcal{U}_2(x) = d_1^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_{x_1}^x (\log \frac{x}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}. \end{cases} \tag{6}$$

From (5) and (6) and the impulsive conditions of (3), we yield that

$$\begin{cases} c_1^{\mathcal{U}} - c_0^{\mathcal{U}} = \frac{1}{\Gamma(\theta_1)} \int_\alpha^{x_1} (\log \frac{x_1}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} + I_1(\mathcal{U}_1(x_1), \mathcal{U}_2(x_1)), \\ d_1^{\mathcal{U}} - d_0^{\mathcal{U}} = \frac{1}{\Gamma(\theta_2)} \int_\alpha^{x_1} (\log \frac{x_1}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} + J_1(\mathcal{U}_1(x_1), \mathcal{U}_2(x_1)). \end{cases} \tag{7}$$

In the same manner, for $x \in (x_l, x_{l+1}], 2 \le l \le m$, we obtain

$$\begin{cases} \mathcal{U}_1(x) = c_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_{x_l}^x (\log \frac{x}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}, \\ \mathcal{U}_2(x) = d_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_{x_l}^x (\log \frac{x}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}, \end{cases} \tag{8}$$

and

$$\begin{cases} c_l^{\mathcal{U}} - c_{l-1}^{\mathcal{U}} = \frac{1}{\Gamma(\theta_1)} \int_{x_{l-1}}^{x_l} (\log \frac{x_l}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} + I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)), \\ d_l^{\mathcal{U}} - d_{l-1}^{\mathcal{U}} = \frac{1}{\Gamma(\theta_2)} \int_{x_{l-1}}^{x_l} (\log \frac{x_l}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} + J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)). \end{cases} \tag{9}$$

In view of (7) and (9), we derive that, for $1 \le l \le m$,

$$\begin{cases} c_l^{\mathcal{U}} = c_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)), \\ d_l^{\mathcal{U}} = d_0^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)). \end{cases} \tag{10}$$

It follows from the boundary value conditions in (3) that

$$\begin{cases} \sum\limits_{l=1}^{m} a_l \Big[c_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_1)} \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}\Big] = c, \\ \sum\limits_{l=1}^{m} b_l \Big[d_l^{\mathcal{U}} + \frac{1}{\Gamma(\theta_2)} \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}\Big] = d. \end{cases} \tag{11}$$

From (10) and (11), we have

$$\begin{cases} c_0^{\mathcal{U}} = \frac{1}{a}\Big[c - \sum\limits_{l=1}^{m} a_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_1)} \int_{x_{k-1}}^{x_k} \big(\log \frac{x_k}{z}\big)^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z} + I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k))\Big) \\ \qquad - \frac{1}{\Gamma(\theta_1)} \sum\limits_{l=1}^{m} a_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_1 - 1} \phi_{\mathcal{U}_1}(z) \frac{dz}{z}\Big], \\ d_0^{\mathcal{U}} = \frac{1}{b}\Big[d - \sum\limits_{l=1}^{m} b_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_2)} \int_{x_{k-1}}^{x_k} \big(\log \frac{x_k}{z}\big)^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} + J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k))\Big) \\ \qquad - \frac{1}{\Gamma(\theta_2)} \sum\limits_{l=1}^{m} b_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z}\Big]. \end{cases} \tag{12}$$

From (5), (8), and (12), we gain the integral Equation (4). The proof is completed. □

3. Solvability and Stability of (3)

In this section, we first establish the theory of coincidence degree for BVP (3), and apply Lemma 1 to explore its solvability. Let

$$PC[\alpha,\beta] = C[\alpha, x_1] \cup \{\mathcal{W} \in C(x_l, x_{l+1}] : \mathcal{W}(x_l^-) = \mathcal{W}(x_l), \text{ and } \mathcal{W}(x_l^+) \text{ exists}, 1 \le l \le m\},$$

$$\mathbb{X} = \left\{\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2)^T : \mathcal{U}_i \in PC[\alpha,\beta], \sum_{l=1}^{m+1} a_l \mathcal{U}_1(\xi_l) = c, \sum_{l=1}^{m+1} b_l \mathcal{U}_2(\xi_l) = d, i = 1,2\right\},$$

$$\mathbb{Y} = \{\mathcal{V} = (\mathcal{U}, h_1, \ldots, h_m) : \mathcal{U} \in \mathbb{X}, h_l = (h_{l1}, h_{l2})^T \in \mathbb{R}^2, 1 \le l \le m\}.$$

Define some norms as follows:

$$\|\mathcal{W}\|_{PC} = \max_{0 \le l \le m} \sup_{x_l \le x \le x_{l+1}} |\mathcal{W}(x)|, \ \forall\, \mathcal{W} \in PC[\alpha,\beta],$$

$$\|\mathcal{U}\|_X = \max\{\|\mathcal{U}_1\|_{PC}, \|\mathcal{U}_2\|_{PC}\}, \ \forall\, \mathcal{U} \in \mathbb{X},$$

$$\|\mathcal{V}\|_Y = \max\{\|\mathcal{U}\|_X, |h_{11}|, |h_{12}|, \ldots, |h_{m1}|, |h_{m2}|\}, \ \forall\, \mathcal{V} \in \mathbb{Y}.$$

Consequently, $(PC[\alpha,\beta], \|\cdot\|_{PC})$, $(\mathbb{X}, \|\cdot\|_X)$, $(\mathbb{Y}, \|\cdot\|_Y)$ are the Banach spaces. Define two operators $\mathcal{L} : \mathrm{Dom}\,\mathcal{L} = \mathbb{X} \to \mathbb{Y}$ and $\mathcal{N} : \mathbb{X} \times [0,1] \to \mathbb{Y}$ as

$$\mathcal{L}\mathcal{U} = \left(\begin{pmatrix} {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1 \\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2 \end{pmatrix}, \begin{pmatrix} \Delta\mathcal{U}_1(x_1) \\ \Delta\mathcal{U}_2(x_1) \end{pmatrix}, \ldots, \begin{pmatrix} \Delta\mathcal{U}_1(x_m) \\ \Delta\mathcal{U}_2(x_m) \end{pmatrix} \right), \tag{13}$$

$$\mathcal{N}(\mathcal{U}, \eta) = \left(\begin{pmatrix} \mathcal{F}_1(x,\eta) \\ \mathcal{F}_2(x,\eta) \end{pmatrix}, \begin{pmatrix} I_1(\mathcal{U}(x_1)) \\ J_1(\mathcal{U}(x_1)) \end{pmatrix}, \ldots, \begin{pmatrix} I_m(\mathcal{U}(x_m)) \\ J_m(\mathcal{U}(x_m)) \end{pmatrix} \right), \tag{14}$$

where $I_l(\mathcal{U}(x_l)) = I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l))$, $J_l(\mathcal{U}(x_l)) = J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l))$, $1 \le l \le m$,

$$\mathcal{F}_j(x,\eta) = F_j(x, \mathcal{U}_1(x), \mathcal{U}_2(x), \eta \cdot {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x), \eta \cdot {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)), j = 1,2.$$

Lemma 4. *$\mathcal{L}$ defined by (13) is a 0-index Fredholm operator.*

Proof. $\mathcal{L}$ is obviously linear. The kernel of $\mathcal{L}$, $\mathrm{Ker}\,\mathcal{L}$ is defined by

$$\mathrm{Ker}\,\mathcal{L} = \{\overline{\mathcal{U}} = (\overline{\mathcal{U}}_1, \overline{\mathcal{U}}_2)^T \in \mathrm{Dom}\,\mathcal{L} : \mathcal{L}\overline{\mathcal{U}} = \mathbf{0}\}. \tag{15}$$

From (13) and (15), it is similar to Lemma 3 that

$$\begin{cases} \overline{\mathcal{U}}_1(x) = \overline{c}_0,\ \overline{\mathcal{U}}_2(x) = \overline{d}_0,\ x \in [\alpha, x_1], \\ \overline{\mathcal{U}}_1(x) = \overline{c}_l,\ \overline{\mathcal{U}}_2(x) = \overline{d}_l,\ x \in (x_l, x_{l+1}],\ 1 \le l \le m, \end{cases} \tag{16}$$

and

$$\begin{cases} \sum\limits_{l=1}^{m} a_l \overline{c}_l = c,\ \sum\limits_{l=1}^{m} b_l \overline{d}_l = d, \\ \overline{c}_l = \overline{c}_0,\ \overline{d}_l = \overline{d}_0,\ 1 \le l \le m. \end{cases} \tag{17}$$

We derive from (16) and (17) that $\operatorname{Ker}\mathcal{L} = \{\overline{\mathcal{U}} = (\frac{c}{a}, \frac{d}{b})^T\}$. Therefore, $\dim(\operatorname{Ker}\mathcal{L}) = 0$. The image set of $\mathcal{L}$, $\operatorname{Im}\mathcal{L}$ is defined by

$$\operatorname{Im}\mathcal{L} = \{\mathcal{V} \in \mathbb{Y} : \exists \mathcal{U} \in \operatorname{Dom}\mathcal{L}\ s.t.\ \mathcal{L}\mathcal{U} = \mathcal{V}\}. \tag{18}$$

Obviously, $\operatorname{Im}\mathcal{L} \subset \mathbb{Y}$. For all $\mathcal{V} \in \mathbb{Y}$,

$$\mathcal{V} = \left(\begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{pmatrix}, \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}, \ \cdots, \begin{pmatrix} h_{m1} \\ h_{m2} \end{pmatrix} \right),$$

it follows from $\mathcal{L}\mathcal{U} = \mathcal{V}$ that

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x) = \mathcal{V}_1(x),\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x) = \mathcal{V}_2(x),\ x \in [\alpha, \beta] \setminus \{x_l\}, \\ \Delta\mathcal{U}_1(x_l) = h_{l1},\ \Delta\mathcal{U}_2(x_l) = h_{l2},\ 1 \le l \le m, \\ \sum_{l=1}^{m+1} a_l \mathcal{U}_1(\xi_l) = c,\ \sum_{l=1}^{m+1} b_l \mathcal{U}_2(\xi_l) = d. \end{cases} \tag{19}$$

Similar to Lemma 3, Equation (19) allows a unique solution $\mathcal{U}^* = (\mathcal{U}_1^*, \mathcal{U}_2^*)^T$ as follows:

$$\begin{cases} \mathcal{U}_1^*(x) = c_0^* + \frac{1}{\Gamma(\theta_1)} \int_\alpha^x (\log\frac{x}{z})^{\theta_1 - 1}\mathcal{V}_1(z)\frac{dz}{z},\ x \in [\alpha, x_1], \\ \mathcal{U}_2^*(x) = d_0^* + \frac{1}{\Gamma(\theta_2)} \int_\alpha^x (\log\frac{x}{z})^{\theta_2 - 1}\mathcal{V}_2(z)\frac{dz}{z},\ x \in [\alpha, x_1], \\ \mathcal{U}_1^*(x) = c_l^* + \frac{1}{\Gamma(\theta_1)} \int_{x_l}^x (\log\frac{x}{z})^{\theta_1 - 1}\mathcal{V}_1(z)\frac{dz}{z},\ x \in (x_l, x_{l+1}], \\ \mathcal{U}_2^*(x) = d_l^* + \frac{1}{\Gamma(\theta_2)} \int_{x_l}^x (\log\frac{x}{z})^{\theta_2 - 1}\mathcal{V}_2(z)\frac{dz}{z},\ x \in (x_l, x_{l+1}], \end{cases} \tag{20}$$

where

$$\begin{cases} c_0^* = \frac{1}{a}\Big[c - \sum\limits_{l=1}^{m} a_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_1)} \int_{x_{k-1}}^{x_k} (\log\frac{x_k}{z})^{\theta_1-1}\mathcal{V}_1(z)\frac{dz}{z} + h_{k1}\Big) \\ \qquad -\frac{1}{\Gamma(\theta_1)} \sum\limits_{l=1}^{m} a_l \int_{x_l}^{\xi_l} \Big(\log\frac{\xi_l}{z}\Big)^{\theta_1-1} \mathcal{V}_1(z)\frac{dz}{z}\Big], \\ d_0^* = \frac{1}{b}\Big[d - \sum\limits_{l=1}^{m} b_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_2)} \int_{x_{k-1}}^{x_k} (\log\frac{x_k}{z})^{\theta_2-1}\mathcal{V}_2(z)\frac{dz}{z} + h_{k2}\Big) \\ \qquad -\frac{1}{\Gamma(\theta_2)} \sum\limits_{l=1}^{m} b_l \int_{x_l}^{\xi_l} \Big(\log\frac{\xi_l}{z}\Big)^{\theta_2-1} \mathcal{V}_2(z)\frac{dz}{z}\Big], \\ c_l^* = c_0^* + \frac{1}{\Gamma(\theta_1)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log\frac{x_k}{z})^{\theta_1-1}\mathcal{V}_1(z)\frac{dz}{z} + \sum\limits_{k=1}^{l} h_{k1}, \\ d_l^* = d_0^* + \frac{1}{\Gamma(\theta_2)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log\frac{x_k}{z})^{\theta_2-1}\mathcal{V}_2(z)\frac{dz}{z} + \sum\limits_{k=1}^{l} h_{k2}. \end{cases} \tag{21}$$

Clearly, $\mathcal{U}^* = (\mathcal{U}_1^*, \mathcal{U}_2^*)^T \in \operatorname{Dom}\mathcal{L}$. Thereby, $\mathbb{Y} \subset \operatorname{Im}\mathcal{L}$. Thus, we claim that $\mathbb{Y} = \operatorname{Im}\mathcal{L}$ and $\operatorname{Im}\mathcal{L}$ is closed, as well as

$$\operatorname{codim}(\operatorname{Im}\mathcal{L}) = \dim(\mathbb{Y}/\operatorname{Im}\mathcal{L}) = \dim(\mathbb{Y}/\mathbb{Y}) = 0 = \dim(\operatorname{Ker}\mathcal{L}).$$

Based on the definition of 0-index Fredholm operator, we know that Lemma 4 is true. The proof is completed. □

$\mathcal{P} : \mathbb{X} \to \mathbb{X}$ is defined by

$$\mathcal{P}\mathcal{U} = \mathcal{P}\begin{pmatrix} \mathcal{U}_1(x) \\ \mathcal{U}_2(x) \end{pmatrix} = \begin{pmatrix} \frac{c}{a} \\ \frac{d}{b} \end{pmatrix}. \tag{22}$$

Obviously, $\mathcal{P}^2 = \mathcal{P}$ and $\operatorname{Ker}\mathcal{P} = \mathbb{X}$. Noticing that $\operatorname{Ker}\mathcal{L}$ is zero space, we yield that $\operatorname{Im}\mathcal{P} = \operatorname{Ker}\mathcal{L}$ and $\mathbb{X} = \operatorname{Ker}\mathcal{L} \oplus \operatorname{Ker}\mathcal{P}$. Therefore, $\mathcal{L}|_{\operatorname{Dom}\mathcal{L}\cap\operatorname{Ker}\mathcal{P}} : \mathbb{X} = \operatorname{Dom}\mathcal{L} \cap \operatorname{Ker}\mathcal{P} \to \operatorname{Im}\mathcal{L} = \mathbb{Y}$; there exists an inverse operator $\mathcal{K}_P$. For each $\mathcal{V} \in \mathbb{Y}$, $\mathcal{K}_P\mathcal{V} = (\mathcal{U}_1^*(x), \mathcal{U}_2^*(x))^T \in \mathbb{X}$ is defined as (19) and (20). Define $\mathcal{Q} : \mathbb{Y} \to \mathbb{Y}$ as

$$\mathcal{Q}\left(\begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{pmatrix}, \left\{ \begin{pmatrix} h_{l1} \\ h_{l2} \end{pmatrix} \right\}_{l=1}^{m} \right) = \left(\begin{pmatrix} \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{V}_1(x)dx \\ \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{V}_2(x)dx \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{l=1}^{m} \right), \tag{23}$$

then $\mathcal{Q}^2 = \mathcal{Q}$, $\operatorname{Ker}\mathcal{Q} = \operatorname{Im}\mathcal{L} = \mathbb{Y}$ and $\mathbb{Y} = \operatorname{Im}\mathcal{L} \oplus \operatorname{Im}\mathcal{Q}$. In addition,

$$\mathcal{Q}\mathcal{N}(\mathcal{U},\eta) = \left(\begin{pmatrix} \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{F}_1(x,\eta)dx \\ \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{F}_2(x,\eta)dx \end{pmatrix}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}_{l=1}^{m} \right).$$

By substituting $\mathcal{N}(\mathcal{U},\eta) - \mathcal{Q}\mathcal{N}(\mathcal{U},\eta)$ into (20) and (21), we can obtain the expression of $\mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\mathcal{U},\eta)$.

Lemma 5. *$\mathcal{N}$ defined by (14) is $\mathcal{L}$-compact.*

Proof. For all bounded subsets $\Theta \subset \mathbb{X}$, it suffices to prove that $\overline{\mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\overline{\Theta})}$ is relatively compact. Indeed, it follows from the continuity of F_i, I_{il} and $J_{il}(i = 1,2; 1 \le l \le m)$ that $\mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\mathcal{U},\eta)$ is uniformly bounded. Set a sequence of functions, $n \in \mathbb{N}$,

$$\mathcal{H}(\mathcal{U}_n,\eta)(x) = \begin{cases} \mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\mathcal{U}_n,\eta)(x), \ x \in [\alpha, x_1] \cup (x_1, x_{l+1}], \ 1 \le l \le m, \\ \mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\mathcal{U}_n,\eta)(x^+), \ x = x_l, \ 1 \le l \le m, \end{cases}$$

then $\{\mathcal{H}_l\mathcal{U}_n =: \mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\mathcal{U}_n,\eta)(x)|_{[x_l,x_{l+1}]}\}_{n\in\mathbb{N}}$ is uniformly bounded on $[x_l, x_{l+1}](0 \le l \le m)$. Therefore, $\{\mathcal{H}_l\mathcal{U}_n\}$ has a uniformly convergent subsequence $\{\mathcal{H}_l\mathcal{U}_{n_1^{(1)}}\}$ on $[x_0, x_1] = [\alpha, x_1]$. Similarly, $\{\mathcal{H}_l\mathcal{U}_{n_1^{(1)}}\}$ has a uniformly convergent subsequence $\{\mathcal{H}_l\mathcal{U}_{n_2^{(2)}}\}$ on $[x_1, x_2]$. By repeating the above, $\{\mathcal{H}_l\mathcal{U}_{n_{m-1}^{(m-1)}}\}$ has a uniformly convergent subsequence $\{\mathcal{H}_l\mathcal{U}_{n_m^{(m)}}\}$ on $[x_m, x_{m+1}] = [x_m, \beta]$. Thus, $\{\mathcal{H}_l\mathcal{U}_{n_m^{(m)}}\}$ is a uniformly convergent subsequence on $[\alpha, \beta]$, which means that $\overline{\mathcal{K}_P(I-\mathcal{Q})\mathcal{N}(\overline{\Theta})}$ is relatively compact. The proof is completed. □

Theorem 1. *BVP (3) permits a unique solution $\widetilde{\mathcal{U}}(x) = (\widetilde{\mathcal{U}}_1(x), \widetilde{\mathcal{U}}_2(x))^T \in \mathbb{X}$, provided that the following conditions* (A1)–(A4) *are true.*

(A1) *Assume that* $0 < \alpha < \beta$, $0 < \theta_1, \theta_2 \le 1$, $\alpha = x_0 = \xi_1 < x_1 < \xi_2 < x_2 < \xi_3 < x_3 < \ldots < x_m < \xi_m = x_{m+1} = \beta$, $a_l, b_l, c, d \in \mathbb{R}$, $a = \sum_{l=1}^m a_l \ne 0$, $b = \sum_{l=1}^m b_l \ne 0$, $F_i \in C([\alpha,\beta] \times \mathbb{R}^4, \mathbb{R}^+)$, $I_l, J_l \in C(\mathbb{R}^2, \mathbb{R})$, $1 \le l \le m$, $i = 1,2$.

(A2) $\forall\, u = (u_1,u_2,u_3,u_4)^T, \overline{u} = (\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)^T \in \mathbb{R}^4$, $\exists\, L_{i1}, L_{i2}, L_{i3}, L_{i4} > 0$ *s.t.*

$$|F_i(x,u) - F_i(x,\overline{u})| \le \sum_{j=1}^4 L_{ij}|u_j - \overline{u}_j|, \ \forall\, x \in [\alpha,\beta], \ i = 1,2.$$

(A3) $\forall\, v = (v_1,v_2)^T, \overline{v} = (\overline{v}_1,\overline{v}_2)^T \in \mathbb{R}^2$, $\exists\, M_{l1}, M_{l2}, N_{l1}, N_{l2} > 0$ *s.t.*

$$|I_l(v) - I_l(\overline{v})| \le \sum_{i=1}^2 M_{li}|v_i - \overline{v}_i|, \ |J_l(v) - J_l(\overline{v})| \le \sum_{i=1}^2 N_{li}|v_i - \overline{v}_i|, \ 1 \le l \le m.$$

(A4) $D > 0, 0 < L_{13}, L_{24}, \rho_1, \rho_2 < 1$, *where* $a^+ = \sum_{l=1}^m |a_l|$, $b^+ = \sum_{l=1}^m |b_l|$,

$$\rho_1 = \frac{1}{|a|}\left[\frac{2D_1 a^+}{D\Gamma(\theta_1+1)}\sum_{k=1}^m\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1+1)}\sum_{k=1}^m |a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1}\right.$$
$$\left. + 2a^+\sum_{k=1}^m (M_{k1}+M_{k2})\right] + \frac{D_1}{D\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1},$$

$$\rho_2 = \frac{1}{|b|}\left[\frac{2D_2 b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^m\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^m |b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2}\right.$$
$$\left. + 2b^+\sum_{k=1}^m (N_{k1}+N_{k2})\right] + \frac{D_2}{D\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2},$$

$$D = \begin{vmatrix} 1-L_{13} & -L_{14} \\ -L_{23} & 1-L_{24} \end{vmatrix}, \; D_1 = \begin{vmatrix} L_{11}+L_{12} & -L_{14} \\ L_{21}+L_{22} & 1-L_{24} \end{vmatrix}, \; D_2 = \begin{vmatrix} 1-L_{13} & L_{11}+L_{12} \\ -L_{23} & L_{21}+L_{22} \end{vmatrix}.$$

Proof. We will apply Lemma 1 to prove Theorem 1. To this end, define $\mathcal{L}$, $\mathcal{N}$, $\mathcal{P}$, and $\mathcal{Q}$ as (13), (14), (22) and (23), respectively. From Lemmas 4 and 5, we claim that $\mathcal{L}$ is a 0-index Fredholm operator, and $\mathcal{N}$ is $\mathcal{L}$-compact. If $\mathcal{U}(x) = (\mathcal{U}_1(x), \mathcal{U}_2(x))^T \in \mathbb{X}$ is a solution of BVP (3), then we know from Lemma 3 that $\mathcal{U}(x) = (\mathcal{U}_1(x), \mathcal{U}_2(x))^T \in \mathbb{X}$ satisfies (4). Noticing that $F_i(x,0,0,0,0) = 0 (i = 1,2)$, we derive from (A2) that

$$\begin{aligned} |\phi_{\mathcal{U}_1}(x)| =& |F_1(x, \mathcal{U}_1(x), \mathcal{U}_2(x), \phi_{\mathcal{U}_1}(x), \psi_{\mathcal{U}_2}(x)) - F_1(x,0,0,0,0)| \\ \le& L_{11}|\mathcal{U}_1(x)| + L_{12}|\mathcal{U}_2(x)| + L_{13}|\phi_{\mathcal{U}_1}(x)| + L_{14}|\psi_{\mathcal{U}_2}(x)|, \\ \le& (L_{11}+L_{12})\|\mathcal{U}\|_X + L_{13}|\phi_{\mathcal{U}_1}(x)| + L_{14}|\psi_{\mathcal{U}_2}(x)|, \end{aligned} \tag{24}$$

and

$$\begin{aligned} |\psi_{\mathcal{U}_2}(x)| =& |F_2(x, \mathcal{U}_1(x), \mathcal{U}_2(x), \phi_{\mathcal{U}_1}(x), \psi_{\mathcal{U}_2}(x)) - F_2(x,0,0,0,0)| \\ \le& L_{21}|\mathcal{U}_1(x)| + L_{22}|\mathcal{U}_2(x)| + L_{23}|\phi_{\mathcal{U}_1}(x)| + L_{24}|\psi_{\mathcal{U}_2}(x)| \\ \le& (L_{21}+L_{22})\|\mathcal{U}\|_X + L_{23}|\phi_{\mathcal{U}_1}(x)| + L_{24}|\psi_{\mathcal{U}_2}(x)|. \end{aligned} \tag{25}$$

Equations (24) and (25) lead to

$$\begin{cases} (1-L_{13})|\phi_{\mathcal{U}_1}(x)| - L_{14}|\psi_{\mathcal{U}_2}(x)| \le (L_{11}+L_{12})\|\mathcal{U}\|_X, \\ -L_{23}|\phi_{\mathcal{U}_1}(x)| + (1-L_{24})|\psi_{\mathcal{U}_2}(x)| \le (L_{21}+L_{22})\|\mathcal{U}\|_X. \end{cases} \tag{26}$$

From (A4) and (26), we have

$$|\phi_{\mathcal{U}_1}(x)| \le \frac{D_1}{D}\|\mathcal{U}\|_X, \;\; |\psi_{\mathcal{U}_2}(x)| \le \frac{D_2}{D}\|\mathcal{U}\|_X, \;\; x \in [\alpha, \beta]. \tag{27}$$

In addition, by (A3), we yield

$$\begin{aligned} &|I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l))| \le |I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)) - I_l(0,0)| + |I_l(0,0)| \\ \le& M_{l1}|\mathcal{U}_1(x_l)| + M_{l2}|\mathcal{U}_2(x_l)| + |I_l(0,0)| \le (M_{l1}+M_{l2})\|\mathcal{U}\|_X + |I_l(0,0)|. \end{aligned} \tag{28}$$

It is similar to obtain

$$|J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l))| \le (N_{l1}+N_{l2})\|\mathcal{U}\|_X + |J_l(0,0)|. \tag{29}$$

In view of (4), (27), and (28), we obtain

$$
\begin{aligned}
|c_0^{\mathcal{U}}| \leq & \frac{1}{|a|}\Bigg[|c| + \sum_{l=1}^{m}|a_l|\sum_{k=1}^{m}\Bigg(\frac{1}{\Gamma(\theta_1)}\int_{x_{k-1}}^{x_k}\left(\log\frac{x_k}{z}\right)^{\theta_1-1}|\phi_{\mathcal{U}_1}(z)|\frac{dz}{z} \\
& + |I_k(\mathcal{U}_1(x_k),\mathcal{U}_2(x_k))|\Bigg) + \frac{1}{\Gamma(\theta_1)}\sum_{l=1}^{m}|a_l|\int_{x_{l-1}}^{\xi_l}\left(\log\frac{\xi_l}{z}\right)^{\theta_1-1}|\phi_{\mathcal{U}_1}(z)|\frac{dz}{z}\Bigg] \\
\leq & \frac{1}{|a|}\Bigg[|c| + \sum_{l=1}^{m}|a_l|\sum_{k=1}^{m}\Bigg(\frac{D_1}{D\Gamma(\theta_1)}\|\mathcal{U}\|_X\int_{x_{k-1}}^{x_k}\left(\log\frac{x_k}{z}\right)^{\theta_1-1}\frac{dz}{z} + |I_k(0,0)| \\
& + (M_{k1}+M_{k2})\|\mathcal{U}\|_X\Bigg) + \frac{D_1}{D\Gamma(\theta_1)}\|\mathcal{U}\|_X\sum_{l=1}^{m}|a_l|\int_{x_{l-1}}^{\xi_l}\left(\log\frac{\xi_l}{z}\right)^{\theta_1-1}\frac{dz}{z}\Bigg] \\
= & \frac{1}{|a|}\Bigg[|c| + \sum_{l=1}^{m}|a_l|\sum_{k=1}^{m}\Bigg(\frac{D_1}{D\Gamma(\theta_1+1)}\|\mathcal{U}\|_X\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + |I_k(0,0)| \\
& + (M_{k1}+M_{k2})\|\mathcal{U}\|_X\Bigg) + \frac{D_1}{D\Gamma(\theta_1+1)}\|\mathcal{U}\|_X\sum_{l=1}^{m}|a_l|\left(\log\frac{\xi_l}{x_{l-1}}\right)^{\theta_1}\Bigg] \\
= & \frac{1}{|a|}\Bigg[\frac{D_1a^+}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1} \\
& + a^+\sum_{k=1}^{m}(M_{k1}+M_{k2})\Bigg]\cdot\|\mathcal{U}\|_X + \frac{1}{|a|}\Bigg[|c| + a^+\sum_{k=1}^{m}|I_k(0,0)|\Bigg],
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
|c_l^{\mathcal{U}}| \leq & |c_0^{\mathcal{U}}| + \frac{1}{\Gamma(\theta_1)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\left(\log\frac{x_k}{z}\right)^{\theta_1-1}|\phi_{\mathcal{U}_1}(z)|\frac{dz}{z} + \sum_{k=1}^{m}|I_k(\mathcal{U}_1(x_k),\mathcal{U}_2(x_k))| \\
\leq & |c_0^{\mathcal{U}}| + \frac{D_1}{D\Gamma(\theta_1+1)}\|\mathcal{U}\|_X\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \sum_{k=1}^{m}[(M_{k1}+M_{k2})\|\mathcal{U}\|_X + I_k(0,0)] \\
\leq & \frac{1}{|a|}\Bigg[\frac{2D_1a^+}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1} \\
& + 2a^+\sum_{k=1}^{m}(M_{k1}+M_{k2})\Bigg]\cdot\|\mathcal{U}\|_X + \frac{1}{|a|}\Bigg[|c| + 2a^+\sum_{k=1}^{m}|I_k(0,0)|\Bigg].
\end{aligned}
\tag{31}
$$

Similar to (30) and (31), we have

$$
\begin{aligned}
|d_0^{\mathcal{U}}| \leq & \frac{1}{|b|}\Bigg[\frac{D_2b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2} \\
& + b^+\sum_{k=1}^{m}(N_{k1}+N_{k2})\Bigg]\cdot\|\mathcal{U}\|_X + \frac{1}{|b|}\Bigg[|d| + b^+\sum_{k=1}^{m}|J_k(0,0)|\Bigg],
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
|d_l^{\mathcal{U}}| \leq & \frac{1}{|b|}\Bigg[\frac{2D_2b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2} \\
& + 2b^+\sum_{k=1}^{m}(N_{k1}+N_{k2})\Bigg]\cdot\|\mathcal{U}\|_X + \frac{1}{|b|}\Bigg[|d| + 2b^+\sum_{k=1}^{m}|J_k(0,0)|\Bigg].
\end{aligned}
\tag{33}
$$

In addition, for $0 \leq l \leq m$, we obtain

$$
g_1(x) = \frac{1}{\Gamma(\theta_1)}\int_{x_l}^{x}\left(\log\frac{x}{z}\right)^{\theta_1-1}|\phi_{\mathcal{U}_1}(z)|\frac{dz}{z} \leq \frac{D_1}{D\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1}\cdot\|\mathcal{U}\|_X,
\tag{34}
$$

and

$$g_2(x)=\frac{1}{\Gamma(\theta_2)}\int_{x_l}^{x}\left(\log\frac{x}{z}\right)^{\theta_2-1}|\psi_{\mathcal{U}_2}(z)|\frac{dz}{z}\leq\frac{D_2}{D\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2}\cdot\|\mathcal{U}\|_X. \tag{35}$$

From (4), (30)–(35) and (A4), we yield

$$|\mathcal{U}_1(x)|\leq|c_l^{\mathcal{U}}|+g_1(x)\leq\rho_1\|\mathcal{U}\|_X+\varpi_1\leq\rho\|\mathcal{U}\|_X+\varpi,\ l=0,1,2,\ldots,m,$$

$$|\mathcal{U}_2(x)|\leq|d_l^{\mathcal{U}}|+g_2(x)\leq\rho_2\|\mathcal{U}\|_X+\varpi_2\leq\rho\|\mathcal{U}\|_X+\varpi,\ l=0,1,2,\ldots,m,$$

which implies that

$$\|\mathcal{U}\|_X\leq\rho\|\mathcal{U}\|_X+\varpi, \tag{36}$$

where $\rho=\max\{\rho_1,\rho_2\}$, $\varpi=\max\{\varpi_1,\varpi_2\}$, and

$$\varpi_1=\frac{1}{|a|}\left[|c|+2a^+\sum_{k=1}^{m}|I_k(0,0)|\right],\ \varpi_2=\frac{1}{|b|}\left[|d|+2b^+\sum_{k=1}^{m}|J_k(0,0)|\right].$$

Equation (36) derives that

$$\|\mathcal{U}\|_X\leq\frac{\varpi}{1-\rho}. \tag{37}$$

Based on (37), we choose $R>\frac{\varpi}{1-\rho}$ and $\Omega=\{\mathcal{U}\in\mathbb{X}:\|\mathcal{U}\|_X<R\}$; then, $\Omega\subset\mathbb{X}$ is a nonempty bounded open subset. Now, we shall verify that (a1) in Lemma 1 is true. In fact, if $\mathcal{U}^{**}\in\mathbb{X}$ is a solution of $\mathcal{L}\mathcal{U}=\eta\cdot\mathcal{N}(\mathcal{U},\eta)$, then, similar to Lemma 3, it can be expressed in the form of an integral equation. A derivation similar to (24)–(37) yields that $\|\mathcal{U}^{**}\|_X\leq\frac{\varpi}{1-\rho}$, which means that $\mathcal{U}^{**}\in\Omega$. Noticing that $\partial\Omega\cap\operatorname{Ker}\mathcal{L}=\{(\frac{c}{a},\frac{d}{b})^T\}$ and $F_i\in C([\alpha,\beta]\times\mathbb{R}^4,\mathbb{R}^+)$, we claim that $\mathcal{QN}((\frac{c}{a},\frac{d}{b})^T,0)(\frac{c}{a},\frac{d}{b})^T>0$, which means that (a2) in Lemma 1 holds. Let $\mathcal{J}$ be an identity; then, we have

$$\deg(\mathcal{J}\mathcal{QN}(\mathcal{U},0),\Omega\cap\operatorname{Ker}\mathcal{L},0)=\deg(\mathcal{J}\mathcal{QN}(\mathcal{U},0),(c/a,d/b)^T,0)=1.$$

Thus, (a3) in Lemma 1 is also true. Therefore, it follows from Lemma 4 that BVP (3) has at least a solution $\widetilde{\mathcal{U}}(x)=(\widetilde{\mathcal{U}}_1(x),\widetilde{\mathcal{U}}_2(x))^T\in\mathbb{X}$.

Next, we will prove the uniqueness of the solution. Let $\widehat{\mathcal{U}}(x)=(\widehat{\mathcal{U}}_1(x),\widehat{\mathcal{U}}_2(x))^T\in\mathbb{X}$ be another solution to (3), and denote $\mathcal{W}(x)=\widetilde{\mathcal{U}}(x)-\widehat{\mathcal{U}}(x)$; then, it follows from Lemma 3 that

$$\begin{cases}\mathcal{W}_1(x)=c_0^{\mathcal{W}}+\frac{1}{\Gamma(\theta_1)}\int_\alpha^x(\log\frac{x}{z})^{\theta_1-1}\Phi(z)\frac{dz}{z},\ x\in[\alpha,x_1],\\ \mathcal{W}_2(x)=d_0^{\mathcal{W}}+\frac{1}{\Gamma(\theta_2)}\int_\alpha^x(\log\frac{x}{z})^{\theta_2-1}\Psi(z)\frac{dz}{z},\ x\in[\alpha,x_1],\\ \mathcal{W}_1(x)=c_l^{\mathcal{W}}+\frac{1}{\Gamma(\theta_1)}\int_{x_l}^x(\log\frac{x}{z})^{\theta_1-1}\Phi(z)\frac{dz}{z},\ x\in(x_l,x_{l+1}],\\ \mathcal{W}_2(x)=d_l^{\mathcal{W}}+\frac{1}{\Gamma(\theta_2)}\int_{x_l}^x(\log\frac{x}{z})^{\theta_2-1}\Psi(z)\frac{dz}{z},\ x\in(x_l,x_{l+1}],\end{cases} \tag{38}$$

$$\begin{cases} c_0^{\mathcal{W}} = \frac{1}{a}\Big[-\sum\limits_{l=1}^{m} a_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_1)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} \Phi(z) \frac{dz}{z} + \mathcal{I}_k(x_k) \Big) \\ \quad - \frac{1}{\Gamma(\theta_1)} \sum\limits_{l=1}^{m} a_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z} \Big)^{\theta_1 - 1} \Phi(z) \frac{dz}{z} \Big], \\ d_0^{\mathcal{W}} = \frac{1}{b}\Big[-\sum\limits_{l=1}^{m} b_l \sum\limits_{k=1}^{l} \Big(\frac{1}{\Gamma(\theta_2)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} \Psi(z) \frac{dz}{z} + \mathcal{J}_k(x_k) \Big) \\ \quad - \frac{1}{\Gamma(\theta_2)} \sum\limits_{l=1}^{m} b_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z} \Big)^{\theta_2 - 1} \Psi(z) \frac{dz}{z} \Big], \\ c_l^{\mathcal{W}} = c_0^{\mathcal{W}} + \frac{1}{\Gamma(\theta_1)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} \Phi(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} \mathcal{I}(x_k), \\ d_l^{\mathcal{W}} = d_0^{\mathcal{W}} + \frac{1}{\Gamma(\theta_2)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} \Psi(z) \frac{dz}{z} + \sum\limits_{k=1}^{l} \mathcal{J}(x_k), \end{cases}$$

where

$$\Phi(z) = \phi_{\widetilde{\mathcal{U}}_1}(z) - \phi_{\widehat{\mathcal{U}}_1}(z), \ \Psi(z) = \phi_{\widetilde{\mathcal{U}}_1}(z) - \phi_{\widehat{\mathcal{U}}_1}(z),$$

$$\mathcal{I}_k(x_k) = I_k(\widetilde{\mathcal{U}}_1(x_k), \widetilde{\mathcal{U}}_2(x_k)) - I_k(\widehat{\mathcal{U}}_1(x_k), \widehat{\mathcal{U}}_2(x_k)),$$

$$\mathcal{J}_k(x_k) = J_k(\widetilde{\mathcal{U}}_1(x_k), \widetilde{\mathcal{U}}_2(x_k)) - J_k(\widehat{\mathcal{U}}_1(x_k), \widehat{\mathcal{U}}_2(x_k)).$$

It is similar to (24)–(27) that

$$|\Phi(x)| = |\phi_{\widetilde{\mathcal{U}}_1}(x) - \phi_{\widehat{\mathcal{U}}_1}(x)| \le \frac{D_1}{D} \|\widetilde{\mathcal{U}} - \widehat{\mathcal{U}}\|_X, \ x \in [\alpha, \beta], \tag{39}$$

$$|\Psi(x)| = |\psi_{\widetilde{\mathcal{U}}_2}(x) - \psi_{\widehat{\mathcal{U}}_2}(x)| \le \frac{D_2}{D} \|\widetilde{\mathcal{U}} - \widehat{\mathcal{U}}\|_X, \ x \in [\alpha, \beta], \tag{40}$$

$$|\mathcal{I}_k(x_l)| = |I_l(\widetilde{\mathcal{U}}_1(x_l), \widetilde{\mathcal{U}}_2(x_l)) - I_l(\widehat{\mathcal{U}}_1(x_l), \widehat{\mathcal{U}}_2(x_l))| \le (M_{l1} + M_{l2}) \|\widetilde{\mathcal{U}} - \widehat{\mathcal{U}}\|_X, \tag{41}$$

$$|\mathcal{J}_k(x_l)| = |I_l(\widetilde{\mathcal{U}}_1(x_l), \widetilde{\mathcal{U}}_2(x_l)) - I_l(\widehat{\mathcal{U}}_1(x_l), \widehat{\mathcal{U}}_2(x_l))| \le (N_{l1} + N_{l2}) \|\widetilde{\mathcal{U}} - \widehat{\mathcal{U}}\|_X. \tag{42}$$

Similar to (30)–(36), we derive from (38)–(42) that $\|\mathcal{W}\|_{\mathbb{X}} \le \rho \|\mathcal{W}\|_{\mathbb{X}}$. Noting that $0 < \rho < 1$, we know that $\|\mathcal{W}\|_{\mathbb{X}} = 0$, which implies that $\widetilde{\mathcal{U}}(x) = \widehat{\mathcal{U}}(x)$. The proof is completed. □

Now, we discuss the Hyers–Ulam (HU) stability of BVP (3). The concept of HU–stability of (3) is given as follows.

Definition 3. *BVP* (3) *is HU–stable if* $\forall\, \zeta > 0$; *there exists a unique solution* $\widetilde{\mathcal{U}}(x) = (\widetilde{\mathcal{U}}_1(x), \widetilde{\mathcal{U}}_2(x)) \in \mathbb{X}$ *solving* (3) *s.t.*

$$\|\mathcal{U}(x) - \widetilde{\mathcal{U}}(x)\|_X \le B \cdot \zeta,$$

where $B > 0$ *is a constant and* $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{X}$ *is any solution of the following inequality*

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x) - F_1(x, \mathcal{U}_1(x), \mathcal{U}_2(x), {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x), {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)) \le \zeta, \\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x) - F_2(x, \mathcal{U}_1(x), \mathcal{U}_2(x), {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x), {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)) \le \zeta, \\ \Delta\mathcal{U}_1(x_l) = I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)), \ \Delta\mathcal{U}_2(x_l) = J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)), \\ \sum_{l=1}^{m+1} a_l \mathcal{U}_1(\xi_l) = c, \ \sum_{l=1}^{m+1} b_l \mathcal{U}_2(\xi_l) = d. \end{cases} \tag{43}$$

Theorem 2. *BVP* (3) *is HU–stable provided that the conditions* (A1)–(A4) *are true.*

Proof. $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2) \in \mathbb{X}$ solves the inequality (43) if it also solves the following system

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x) = F_1(x, \mathcal{U}_1(x), \mathcal{U}_2(x), {}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x), {}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x)) + \omega_1(x), \\ {}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x) = F_2(x, \mathcal{U}_1(x), \mathcal{U}_2(x), {}^{\mathrm{CH}}\mathcal{D}^{\theta_1}_{x_l}\mathcal{U}_1(x), {}^{\mathrm{CH}}\mathcal{D}^{\theta_2}_{x_l}\mathcal{U}_2(x)) + \omega_2(x), \\ \Delta\mathcal{U}_1(x_l) = I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)),\ \Delta\mathcal{U}_2(x_l) = J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)), \\ \sum_{l=1}^{m+1} a_l \mathcal{U}_1(\xi_l) = c,\ \sum_{l=1}^{m+1} b_l \mathcal{U}_2(\xi_l) = d, \end{cases} \tag{44}$$

where $\omega(x) = (\omega_1(x), \omega_2(x)) \in \mathbb{X}$ and $|\omega_i(x)| \leq \zeta\ (i = 1, 2), \forall\, x \in [\alpha, \beta]$.

From Lemma 3 and (44), any solution of inequality (43) is represented by

$$\begin{cases} \mathcal{U}_1(x) = c_0^{\omega} + \frac{1}{\Gamma(\theta_1)} \int_{\alpha}^{x} (\log \frac{x}{z})^{\theta_1 - 1} [\phi_{\mathcal{U}_1}(z) + \omega_1(z)] \frac{dz}{z},\ x \in [\alpha, x_1], \\ \mathcal{U}_2(x) = d_0^{\omega} + \frac{1}{\Gamma(\theta_2)} \int_{\alpha}^{x} (\log \frac{x}{z})^{\theta_2 - 1} [\psi_{\mathcal{U}_2}(z) + \omega_2(z)] \frac{dz}{z},\ x \in [\alpha, x_1], \\ \mathcal{U}_1(x) = c_l^{\omega} + \frac{1}{\Gamma(\theta_1)} \int_{x_l}^{x} (\log \frac{x}{z})^{\theta_1 - 1} [\phi_{\mathcal{U}_1}(z) + \omega_1(z)] \frac{dz}{z},\ x \in (x_l, x_{l+1}], \\ \mathcal{U}_2(x) = d_l^{\omega} + \frac{1}{\Gamma(\theta_2)} \int_{x_l}^{x} (\log \frac{x}{z})^{\theta_2 - 1} [\psi_{\mathcal{U}_2}(z) + \omega_2(z)] \frac{dz}{z},\ x \in (x_l, x_{l+1}], \end{cases} \tag{45}$$

where

$$\begin{cases} c_0^{\omega} = \frac{1}{a}\Bigg[c - \sum\limits_{l=1}^{m} a_l \sum\limits_{k=1}^{l} \Bigg(\frac{1}{\Gamma(\theta_1)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} [\phi_{\mathcal{U}_1}(z) + \omega_1(z)] \frac{dz}{z} \\ \qquad + I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)) \Bigg) - \frac{1}{\Gamma(\theta_1)} \sum\limits_{l=1}^{m} a_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_1 - 1} [\phi_{\mathcal{U}_1}(z) + \omega_1(z)] \frac{dz}{z} \Bigg], \\ d_0^{\omega} = \frac{1}{b}\Bigg[d - \sum\limits_{l=1}^{m} b_l \sum\limits_{k=1}^{l} \Bigg(\frac{1}{\Gamma(\theta_2)} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} [\psi_{\mathcal{U}_2}(z) + \omega_2(z)] \frac{dz}{z} \\ \qquad + J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)) \Bigg) - \frac{1}{\Gamma(\theta_2)} \sum\limits_{l=1}^{m} b_l \int_{x_l}^{\xi_l} \Big(\log \frac{\xi_l}{z}\Big)^{\theta_2 - 1} \psi_{\mathcal{U}_2}(z) \frac{dz}{z} \Bigg], \\ c_l^{\omega} = c_0^{\omega} + \frac{1}{\Gamma(\theta_1)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_1 - 1} [\phi_{\mathcal{U}_1}(z) + \omega_1(z)] \frac{dz}{z} + \sum\limits_{k=1}^{l} I_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)), \\ d_l^{\omega} = d_0^{\omega} + \frac{1}{\Gamma(\theta_2)} \sum\limits_{k=1}^{l} \int_{x_{k-1}}^{x_k} (\log \frac{x_k}{z})^{\theta_2 - 1} [\psi_{\mathcal{U}_2}(z) + \omega_2(z)] \frac{dz}{z} + \sum\limits_{k=1}^{l} J_k(\mathcal{U}_1(x_k), \mathcal{U}_2(x_k)). \end{cases}$$

From Theorem 1, the unique solution $\widetilde{\mathcal{U}} = (\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2) \in \mathbb{X}$ is written as (4). Similar to (39)–(42), we have

$$|\phi_{\mathcal{U}_1}(x) + \omega_1(x) - \phi_{\widetilde{\mathcal{U}}_1}(x)| \leq |\phi_{\mathcal{U}_1}(x) - \phi_{\widetilde{\mathcal{U}}_1}(x)| + |\omega_1(x)| \leq \frac{D_1}{D} \|\mathcal{U} - \widetilde{\mathcal{U}}\|_X + \zeta, \tag{46}$$

$$|\psi_{\mathcal{U}_2}(x) + \omega_2(x) - \phi_{\widetilde{\mathcal{U}}_2}(x)| \leq |\psi_{\mathcal{U}_2}(x) - \psi_{\widetilde{\mathcal{U}}_2}(x)| + |\omega_2(x)| \leq \frac{D_2}{D} \|\mathcal{U} - \widetilde{\mathcal{U}}\|_X + \zeta, \tag{47}$$

$$|I_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)) - I_l(\widetilde{\mathcal{U}}_1(x_l), \widetilde{\mathcal{U}}_2(x_l))| \leq (M_{l1} + M_{l2}) \|\mathcal{U} - \widetilde{\mathcal{U}}\|_X, \tag{48}$$

$$|J_l(\mathcal{U}_1(x_l), \mathcal{U}_2(x_l)) - J_l(\widetilde{\mathcal{U}}_1(x_l), \widetilde{\mathcal{U}}_2(x_l))| \leq (N_{l1} + N_{l2}) \|\mathcal{U} - \widetilde{\mathcal{U}}\|_X. \tag{49}$$

In the same way as (30)–(33), we yield from (45)–(49) that

$$\begin{aligned} |c_0^{\omega} - c_0^{\widetilde{\mathcal{U}}}| \leq & \frac{1}{|a|} \Bigg[\frac{D_1 a^+}{D\Gamma(\theta_1 + 1)} \sum_{k=1}^{m} \Big(\log \frac{x_k}{x_{k-1}} \Big)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1 + 1)} \sum_{k=1}^{m} |a_k| \Big(\log \frac{\xi_k}{x_{k-1}} \Big)^{\theta_1} \\ & + a^+ \sum_{k=1}^{m} (M_{k1} + M_{k2}) \Bigg] (\|\mathcal{U} - \widetilde{\mathcal{U}}\|_X + \zeta), \end{aligned} \tag{50}$$

$$|c_l^{\omega} - c_l^{\widetilde{\mathcal{U}}}| \leq \frac{1}{|a|}\Bigg[\frac{2D_1 a^+}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1} + 2a^+\sum_{k=1}^{m}(M_{k1}+M_{k2})\Bigg](\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta), \tag{51}$$

$$|d_0^{\omega} - d_0^{\widetilde{\mathcal{U}}}| \leq \frac{1}{|b|}\Bigg[\frac{D_2 b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2} + b^+\sum_{k=1}^{m}(N_{k1}+N_{k2})\Bigg](\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta), \tag{52}$$

$$|d_l^{\omega} - d_l^{\widetilde{\mathcal{U}}}| \leq \frac{1}{|b|}\Bigg[\frac{2D_2 b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2} + 2b^+\sum_{k=1}^{m}(N_{k1}+N_{k2})\Bigg](\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta). \tag{53}$$

Additionally, we have

$$\begin{aligned} h_1(x) =& \frac{1}{\Gamma(\theta_1)}\int_{x_l}^{x}\left(\log\frac{x}{z}\right)^{\theta_1-1}|\phi_{\mathcal{U}_1}(x)+\omega_1(x)-\phi_{\widetilde{\mathcal{U}}_1}(x)|\frac{dz}{z} \\ \leq& \frac{D_1}{D\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1}(\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta), \end{aligned} \tag{54}$$

$$\begin{aligned} h_2(x) =& \frac{1}{\Gamma(\theta_2)}\int_{x_l}^{x}\left(\log\frac{x}{z}\right)^{\theta_2-1}|\psi_{\mathcal{U}_2}(x)+\omega_2(x)-\phi_{\widetilde{\mathcal{U}}_2}(x)|\frac{dz}{z} \\ \leq& \frac{D_2}{D\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2}(\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta). \end{aligned} \tag{55}$$

We derive from (4), (45) and (50)–(55) that

$$|\mathcal{U}_1(x)-\widetilde{\mathcal{U}}_1(x)| \leq |c_l^{\omega}-c_l^{\widetilde{\mathcal{U}}}| + h_1(x) \leq \rho_1(\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta),\ l=0,1,2,\ldots,m, \tag{56}$$

$$|\mathcal{U}_2(x)-\widetilde{\mathcal{U}}_2(x)| \leq |d_l^{\omega}-d_l^{\widetilde{\mathcal{U}}}| + h_2(x) \leq \rho_2(\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta),\ l=0,1,2,\ldots,m. \tag{57}$$

Equations (56) and (57) lead to

$$\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X \leq \rho(\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X+\zeta), \tag{58}$$

which implies that

$$\|\mathcal{U}-\widetilde{\mathcal{U}}\|_X \leq \frac{\rho}{1-\rho}\cdot\zeta, \tag{59}$$

where $0<\rho=\max\{\rho_1,\rho_2\}<1$. From Definition 3, we know that BVP (3) is HU–stable. The proof is completed. □

4. Solvability and Stability of (1) and (2)

In this section, we apply our main methods and results to discuss the existence, uniqueness, and HU–stability of solutions for the Langevin system (1) and Sturm–Liouville system (2).

Theorem 3. *The Langevin Equation (1) has a unique solution in $PC[\alpha,\beta]$ which is HU–stable, provided that the following conditions* $(\mathrm{A}'1)$–$(\mathrm{A}'4)$ *are fulfilled.*

$(\mathrm{A}'1)$ *Assume that* $0<\alpha<\beta$, $0<\theta_1,\theta_2\le 1$, $\lambda>0$, $\alpha=x_0=\xi_1<x_1<\xi_2<x_2<\xi_3<x_3<\ldots<x_m<\xi_m=x_{m+1}=\beta$, $a_l,b_l,c,d\in\mathbb{R}$, $a=\sum_{l=1}^{m}a_l\neq 0$, $b=\sum_{l=1}^{m}b_l\neq 0$, $F\in C([\alpha,\beta]\times\mathbb{R}^2,\mathbb{R}^+)$, $I_l,J_l\in C(\mathbb{R},\mathbb{R})$, $1\le l\le m$.

$(\mathrm{A}'2)$ $\forall\, u=(u_1,u_2)^T,\overline{u}=(\overline{u}_1,\overline{u}_2)^T\in\mathbb{R}^2$, $\exists\,\mathfrak{L}_1,\mathfrak{L}_2>0$ *s.t.*

$$|F(x,u)-F(x,\overline{u})|\le\sum_{j=1}^{2}\mathfrak{L}_j|u_j-\overline{u}_j|,\ \forall\ x\in[\alpha,\beta].$$

$(\mathrm{A}'3)$ $\forall\, v,\overline{v}\in\mathbb{R}$, $\exists\,\mathfrak{M}_l,\mathfrak{N}_l>0$ *s.t.*

$$|I_l(v)-I_l(\overline{v})|\le\mathfrak{M}_l|v-\overline{v}|,\ |J_l(v)-J_l(\overline{v})|\le\mathfrak{N}_l|v-\overline{v}|,\ 1\le l\le m.$$

$(\mathrm{A}'4)$ $0<\mathfrak{L}_2,\rho_1',\rho_2'<1$, *where* $a^+=\sum_{l=1}^{m}|a_l|$, $b^+=\sum_{l=1}^{m}|b_l|$,

$$\rho_1'=\frac{1}{|a|}\left[\frac{2a^+}{\Gamma(\theta_1+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1}+\frac{1}{\Gamma(\theta_1+1)}\sum_{k=1}^{m}|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1}+2a^+\sum_{k=1}^{m}\mathfrak{M}_k\right]+\frac{1}{\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1},$$

$$\rho_2'=\frac{1}{|b|}\left[\frac{2(\lambda+\mathfrak{L}_2)b^+}{(1-\mathfrak{L}_2)\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2}+\frac{\lambda+\mathfrak{L}_2}{(1-\mathfrak{L}_2)\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2}+2b^+\sum_{k=1}^{m}\mathfrak{N}_k\right]+\frac{\lambda+\mathfrak{L}_2}{(1-\mathfrak{L}_2)\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2},$$

Proof. Let $\mathcal{U}(x)=\mathcal{U}_1(x)$, ${}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(x)=\mathcal{U}_2(x)$; then, the Langevin system (1) becomes

$$\begin{cases}{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x)=\mathcal{U}_2(x),\ x\in[\alpha,\beta]\setminus\{x_l\},\\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)=\lambda\mathcal{U}_1(x)+F(x,\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)),\ x\in[\alpha,\beta]\setminus\{x_l\},\\ \Delta\mathcal{U}_1(x_l)=I_l(\mathcal{U}_1(x_l)),\ \Delta\mathcal{U}_2(x_l)=J_l(\mathcal{U}_1(x_l)),\ 1\le l\le m,\\ \sum_{l=1}^{m+1}a_l\mathcal{U}_1(\xi_l)=c,\ \sum_{l=1}^{m+1}b_l\mathcal{U}_2(\xi_l)=d.\end{cases}\tag{60}$$

Therefore, the solvability of the Langevin system (1) and BVP (60) is equivalent. It suffices to discuss the existence of solutions for BVP (60). Indeed, let $F_1=\mathcal{U}_2(x)$, $F_2=\lambda\mathcal{U}_1(x)+F(x,\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x))$; then, BVP (60) is transformed into the form of (3). Condition $(\mathrm{A}'1)$ and Condition (A1) correspond exactly. From $(\mathrm{A}'2)$ and $(\mathrm{A}'3)$, a simple calculation provides that $L_{11}=L_{13}=L_{14}=L_{22}=L_{23}=0$, $L_{12}=1$, $L_{21}=\lambda+\mathfrak{L}_1$, $L_{24}=\mathfrak{L}_2$, $M_{l2}=N_{l2}=0$, $M_{l1}=\mathfrak{M}_l$, $N_{l2}=\mathfrak{N}_l$. Substituting these values into Condition (A4) yields Condition $(\mathrm{A}'4)$. From Theorems 1 and 2, we declare that BVP (60) has a unique solution in $\mathbb{X}$ which is HU–stable. The proof is completed. □

Theorem 4. *The Sturm–Liouville system (2) has a unique solution in $PC[\alpha,\beta]$ which is HU–stable, provided that the following conditions* $(\mathrm{A}''1)$–$(\mathrm{A}''4)$ *are fulfilled.*

$(\mathrm{A}''1)$ *Assume that* $0<\alpha<\beta$, $0<\theta_1,\theta_2\le 1$, $\alpha=x_0=\xi_1<x_1<\xi_2<x_2<\xi_3<x_3<\ldots<x_m<\xi_m=x_{m+1}=\beta$, $a_l,b_l,c,d\in\mathbb{R}$, $a=\sum_{l=1}^{m}a_l\neq 0$, $b=\sum_{l=1}^{m}b_l\neq 0$, $p\in C([\alpha,\beta],(0,+\infty))$, $F\in C([\alpha,\beta]\times\mathbb{R},\mathbb{R}^+)$, $I_l,J_l\in C(\mathbb{R},\mathbb{R})$, $1\le l\le m$.

$(\mathrm{A}''2)$ $\forall\, u,\overline{u}\in\mathbb{R}$, $\exists\,\mathfrak{L}>0$ *s.t.*

$$|F(x,u)-F(x,\overline{u})|\le\mathfrak{L}|u-\overline{u}|,\ \forall\ x\in[\alpha,\beta].$$

$(\mathrm{A}''3)$ $\forall\, v,\overline{v}\in\mathbb{R}$, $\exists\,\mathfrak{M}_l,\mathfrak{N}_l>0$ *s.t.*

$$|I_l(v)-I_l(\overline{v})|\le\mathfrak{M}_l|v-\overline{v}|,\ |J_l(v)-J_l(\overline{v})|\le\mathfrak{N}_l|v-\overline{v}|,\ 1\le l\le m.$$

(A$''$4) $0<\rho_1'',\rho_2''<1$, *where* $a^+=\sum_{l=1}^m|a_l|$, $b^+=\sum_{l=1}^m|b_l|$, $p^-=\min\{p(x):\alpha\le x\le\beta\}$,

$$\begin{aligned}\rho_1''=&\frac{1}{|a|}\left[\frac{2a^+}{p^-\Gamma(\theta_1+1)}\sum_{k=1}^m\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1}+\frac{1}{p^-\Gamma(\theta_1+1)}\sum_{k=1}^m|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1}\right.\\&\left.+2a^+\sum_{k=1}^m\mathfrak{M}_k\right]+\frac{1}{p^-\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1},\end{aligned}$$

$$\begin{aligned}\rho_2''=&\frac{1}{|b|}\left[\frac{2\mathfrak{L}b^+}{\Gamma(\theta_2+1)}\sum_{k=1}^m\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2}+\frac{\mathfrak{L}}{\Gamma(\theta_2+1)}\sum_{k=1}^m|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2}\right.\\&\left.+2b^+\sum_{k=1}^m\mathfrak{N}_k\right]+\frac{\mathfrak{L}}{\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2},\end{aligned}$$

Proof. Let $\mathcal{U}(x)=\mathcal{U}_1(x)$, $p(x)\,{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}(x)=\mathcal{U}_2(x)$; then, the Sturm–Liouville system (1) changes into

$$\begin{cases}{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_1}\mathcal{U}_1(x)=\frac{1}{p(x)}\mathcal{U}_2(x),\ x\in[\alpha,\beta]\setminus\{x_l\},\\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{\theta_2}\mathcal{U}_2(x)=F(x,\mathcal{U}_1(x)),\ x\in[\alpha,\beta]\setminus\{x_l\},\\ \Delta\mathcal{U}_1(x_l)=I_l(\mathcal{U}_1(x_l)),\ \Delta\mathcal{U}_2(x_l)=J_l(\mathcal{U}_1(x_l)),\ 1\le l\le m,\\ \sum_{l=1}^{m+1}a_l\mathcal{U}_1(\xi_l)=c,\ \sum_{l=1}^{m+1}b_l\mathcal{U}_2(\xi_l)=d.\end{cases}\tag{61}$$

So it suffices to discuss the existence of solutions for BVP (61). In fact, let $F_1=\frac{1}{p(x)}\mathcal{U}_2(x)$, $F_2=F(x,\mathcal{U}_1(x))$; then, BVP (61) is transformed into the form of (3). Condition (A$''$1) and Condition (A1) correspond exactly. From (A$''$2) and (A$''$3), a simple computation gives that $L_{11}=L_{13}=L_{14}=L_{22}=L_{23}=L_{24}=0$, $L_{12}=\frac{1}{p^-}$, $L_{21}=\mathfrak{L}$, $M_{l2}=N_{l2}=0$, $M_{l1}=\mathfrak{M}_l$, $N_{l2}=\mathfrak{N}_l$. Substituting these values into Condition (A4) yields Condition (A$''$4). From Theorems 1 and 2, we declare that BVP (61) has a unique solution in $\mathbb{X}$ which is HU–stable. The proof is completed. □

To illustrate the availability and correctness of Theorem 1, we provide an example of the three-point boundary value problem with two impulse points as follows.

Example 1. *Consider the following nonlinear impulsive coupled implicit system*

$$\begin{cases}{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.9}\mathcal{U}_1(x)=F_1(x,\mathcal{U}_1(x),\mathcal{U}_2(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.9}\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.7}\mathcal{U}_2(x)),\ x\in[1,e]\setminus\{x_l\},\\ {}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.7}\mathcal{U}_2(x)=F_2(x,\mathcal{U}_1(x),\mathcal{U}_2(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.9}\mathcal{U}_1(x),{}^{\mathrm{CH}}\mathcal{D}_{x_l}^{0.7}\mathcal{U}_2(x)),\ x\in[1,e]\setminus\{x_l\},\\ \Delta\mathcal{U}_1(x_l)=I_l(\mathcal{U}_1(x_l),\mathcal{U}_2(x_l)),\ \Delta\mathcal{U}_2(x_l)=J_l(\mathcal{U}_1(x_l),\mathcal{U}_2(x_l)),\ l=1,2,\\ \frac{1}{2}\mathcal{U}_1(\xi_1)+\frac{1}{3}\mathcal{U}_1(\xi_2)+\frac{1}{6}\mathcal{U}_1(\xi_3)=2,\ \frac{1}{4}\mathcal{U}_2(\xi_1)+\frac{1}{5}\mathcal{U}_2(\xi_2)+\frac{11}{20}\mathcal{U}_2(\xi_3)=-3,\end{cases}\tag{62}$$

where $\alpha=\xi_1=1$, $\beta=\xi_3=e$, $\theta_1=0.9$, $\theta_2=0.7$, $m=2$, $x_1=\frac{e+3}{4}$, $x_2=\frac{3e+1}{4}$, $\xi_2=\frac{e+1}{2}$, $a_1=\frac{1}{2}$, $a_2=\frac{1}{3}$, $a_3=\frac{1}{6}$, $b_1=\frac{1}{4}$, $b_2=\frac{1}{5}$, $b_3=\frac{11}{20}$, $c=2$, $d=-3$, $F_1(x,u_1,u_2,u_3,u_4)=x^2+\frac{2+e^{-u_1^2}+\cos(u_2)+\sin(u_3+u_4)}{100}$, $F_2(x,u_1,u_2,u_3,u_4)=\frac{u_1+u_2}{100}\arctan(u_1+u_2)+\frac{1}{200}\log[1+(u_1+u_2)^2]+\frac{x}{100}[1+\cos(u_3,u_4)]$, $I_1(v_1,v_2)=I_2(v_1,v_2)=\frac{\sin(v_1+v_2)}{100}$. $J_1(v_1,v_2)=J_2(v_1,v_2)=\frac{(v_1+v_2)\arccos(v_1+v_2)+\sqrt{1-(v_1+v_2)^2}}{100}$. *Obviously,* $a=a_1+a_2+a_3=1$, $b=b_1+b_2+b_3=1$, $F_1,F_2\in C([1,e]\times\mathbb{R}^4,\mathbb{R}^+)$, $I_1,I_2,J_1,J_2\in C(\mathbb{R}^2,\mathbb{R})$. *Therefore, the condition* (A1) *holds. We perform a simple calculation to yield that* $\frac{\partial F_1}{\partial u_1}=-\frac{u_1}{50}e^{-u_1^2}$, $\frac{\partial F_1}{\partial u_2}=-\frac{\sin(u_2)}{100}$, $\frac{\partial F_1}{\partial u_3}=\frac{\partial F_1}{\partial u_4}=\frac{\cos(u_3+u_4)}{100}$, $\frac{\partial F_2}{\partial u_1}=\frac{\partial F_2}{\partial u_2}=\frac{\arctan(u_1+u_2)}{100}$, $\frac{\partial F_2}{\partial u_3}=\frac{\partial F_2}{\partial u_4}=-\frac{x\sin(u_3+u_4)}{100}$, $\frac{\partial I_i}{\partial v_1}=\frac{\partial I_i}{\partial v_2}=\frac{\cos(v_1+v_2)}{100}(i=1,2)$, $\frac{\partial J_i}{\partial v_1}=\frac{\partial J_i}{\partial v_2}=\frac{\arccos(v_1+v_2)}{100}(i=1,2)$. *Hence, we derive that*

$$|F_1(x,u_1,u_2,u_3,u_4)-F_1(x,\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)|\le\frac{\sqrt{2}e^{-0.5}}{100}|u_1-\overline{u}_1|+\frac{1}{100}\sum_{j=2}^4|u_j-\overline{u}_j|,$$

$$|F_2(x,u_1,u_2,u_3,u_4) - F_2(x,\overline{u}_1,\overline{u}_2,\overline{u}_3,\overline{u}_4)| \leq \frac{\pi}{200}\sum_{i=1}^{2}|u_i - \overline{u}_i| + \frac{e}{100}\sum_{j=2}^{4}|u_j - \overline{u}_j|,$$

$$|I_i(v_1,v_2)| - I_i(\overline{v}_1,\overline{v}_2) \leq \frac{1}{100}\sum_{j=1}^{2}|v_j - \overline{v}_j|, \ |J_i(v_1,v_2)| - J_i(\overline{v}_1,\overline{v}_2) \leq \frac{\pi}{100}\sum_{j=1}^{2}|v_j - \overline{v}_j|,$$

$$L_{11} = \frac{\sqrt{2}e^{-0.5}}{100}, L_{12} = L_{13} = L_{14} = \frac{1}{100}, L_{21} = L_{22} = \frac{\pi}{200}, L_{23} = L_{24} = \frac{e}{100},$$

$$M_{11} = M_{12} = M_{21} = M_{22} = \frac{1}{100}, N_{11} = N_{12} = N_{21} = N_{22} = \frac{\pi}{100},$$

$$a^+ = \sum_{l=1}^{3}|a_l| = 1, b^+ = \sum_{l=1}^{3}|b_l| = 1, D = \begin{vmatrix} 1 - L_{13} & -L_{14} \\ -L_{23} & 1 - L_{24} \end{vmatrix} \approx 0.9628 < 1,$$

$$D_1 = \begin{vmatrix} L_{11} + L_{12} & -L_{14} \\ L_{21} + L_{22} & 1 - L_{24} \end{vmatrix} \approx 0.0184, \ D_2 = \begin{vmatrix} 1 - L_{13} & L_{11} + L_{12} \\ -L_{23} & L_{21} + L_{22} \end{vmatrix} \approx 0.0316.$$

$$\begin{aligned}\rho_1 =& \frac{1}{|a|}\left[\frac{2D_1a^+}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_1} + \frac{D_1}{D\Gamma(\theta_1+1)}\sum_{k=1}^{m}|a_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_1}\right.\\ &\left.+ 2a^+\sum_{k=1}^{m}(M_{k1}+M_{k2})\right] + \frac{D_1}{D\Gamma(\theta_1+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_1} \approx 0.1466 < 1,\end{aligned}$$

$$\begin{aligned}\rho_2 =& \frac{1}{|b|}\left[\frac{2D_2b^+}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta_2} + \frac{D_2}{D\Gamma(\theta_2+1)}\sum_{k=1}^{m}|b_k|\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta_2}\right.\\ &\left.+ 2b^+\sum_{k=1}^{m}(N_{k1}+N_{k2})\right] + \frac{D_2}{D\Gamma(\theta_2+1)}\left(\log\frac{\beta}{\alpha}\right)^{\theta_2} \approx 0.3950 < 1.\end{aligned}$$

Thus, the conditions (A2)–(A4) *are also satisfied. From Theorems 1 and 2, we conclude that system* (60) *admits a unique solution, which is HU–stable.*

5. Conclusions

This section first provides further analysis and then discussion of our main results. In Theorems 1–4, the most important condition is that $0 < \rho_1, \rho_2, \rho'_1, \rho'_2, \rho''_1, \rho''_2 < 1$. This condition is determined by the response functions F_1, F_2, pulse functions I_l, J_l, impulsive points x_l, boundary value points α, β, ξ_l, and coefficients a_l, b_l together, $l = 1, 2, \ldots, m$. The more complex calculation to verify this condition is that $\sum_{k=1}^{m}\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta}$ and $\sum_{k=1}^{m}\left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta}$, where $\theta \in \{\theta_1, \theta_2\}$. Since $x_{k-1} < \xi_k < x_k (k = 1, 2, \ldots, m)$, the positions of adjacent impulsive points x_{k-1} and x_k will result in two different cases for verifying this condition. Case 1: when $\frac{x_k}{x_{k-1}} < e$, then $0 < \left(\log\frac{x_k}{x_{k-1}}\right)^{\theta} < 1$ and $0 < \left(\log\frac{\xi_k}{x_{k-1}}\right)^{\theta} < 1$. Thus, the condition $0 < \rho_1, \rho_2, \rho'_1, \rho'_2, \rho''_1, \rho''_2 < 1$ is relatively easy to satisfy. Example 1 belongs to this case. Case 2: when $\frac{x_k}{x_{k-1}} > e$, then $\left(\log\frac{x_k}{x_{k-1}}\right)^{\theta} > 1$. So, the condition $0 < \rho_1, \rho_2, \rho'_1, \rho'_2, \rho''_1, \rho''_2 < 1$ is more difficult to satisfy. This requires controlling the values of L_{ij}, M_{ij}, N_{ij}, a_l and b_l, $i, j = 1, 2$; $l = 1, 2, \ldots, m$. In addition, since this paper considers the CH–fractional differential equations with certain singularities, the ODE toolboxes in MATHLAB cannot be applied in numerical simulations. This requires the design of new numerical algorithms, which is also one of our future research directions.

Next, we make a brief summary. Hadamard fractional calculus in the Caputo sense is an important type of fractional calculus, which is a generalization of RL–fractional calculus in the Caputo sense. The CH–fractional differential equation is used to solve many practical problems and has become a popular object of concern for many academic researchers. There have been some good results in the study of CH–fractional differential explicit systems. However, studies on the solvability and stability of CH–fractional coupled implicit systems under impulsive influence are relatively rare because it is difficult to estimate the existence region of the solution. Additionally, the theory of coincidence degree is an important route to solve the existence of solutions to nonlinear differential equations. In this paper, we creatively establish a framework for the coincidence degree theory for system (3) with impulsive effects and prove the existence of a solution. Simultaneously, our main results are applied to solve the solvability of the Langevin system (1) and Sturm–Liouville system (2). Our research objects and findings enrich the theory of CH–fractional differential equations. Our approach also provides a paradigm for uniformly solving such problems.

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