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Wavefield Evolution and Arrival Behavior of Elastic Wave Propagation in Two-Dimensional Fractional Brownian Fields

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Abstract: The fractional Brownian field is often used to reproduce the fractal properties of complex heterogeneous media, which closely represent real-world geological materials. Studying elastic wave transport in this type of heterogeneous media is essential for advancing knowledge in geophysics, seismology, and rock mechanics. In this paper, we numerically investigate the wavefield evolution and arrival behavior of elastic wave propagation in a two-dimensional fractional Brownian field characterized by the standard deviation (σ) and the Hurst exponent (H). Using a high-fidelity finite element model, we quantify the influence of these parameters on wavefront morphology, wave arrival synchronization, and energy decay. Our results reveal that increased matrix heterogeneity with higher σ and lower H values leads to pronounced wavefront roughness, asynchronous arrival phenomena, and increscent energy decay, attributed to enhanced scattering and modulus variability. For smaller H values, rougher modulus distributions scatter wave energy more intensely, producing more coda waves and distorted wavefronts, while smoother fields with larger H fields promote smoother wave propagation. Higher σ amplifies these effects by increasing modulus variability, resulting in more attenuated wave energy and substantial wavefield disturbance. This study contributes to a quantitative understanding of how fractal heterogeneity modulates wave transport and energy attenuation in random media. Our findings hold practical significance for geophysical exploration and seismic tomography, as well as aiding in subsurface imaging and structural evaluation within fractured or stratified rock formations.



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Keywords: elastic wave; fractional Brownian field; heterogeneous media; wavefield; arrival behavior

1. Introduction

In geological formations, mechanical properties such as Young's modulus, density, and porosity often display spatial variability [1–3] that follows fractal distributions [4,5], impacting the velocity, attenuation, and scattering of elastic waves [6–8]. The propagation of elastic waves in such heterogeneous media is essential in various geophysical applications, including seismic exploration, subsurface imaging, and hazard assessment [9]. The fractional Brownian field (fBm) is commonly used to model such heterogeneous media due to its ability to represent fractal characteristics observed in natural systems [10,11]. Mandelbrot and Van Ness [10] introduced the concept of fractional Brownian motion, establishing a foundation for modeling self-affine and self-similar structures observed in geological formations. Subsequent studies expanded to identify fractal-based heterogeneity in various geological contexts [4,12]. It has been proven that such fractal characteristics significantly influence wave behavior by inducing scattering, coda waves, and arrival delays [6,13–17]. These findings underscore the importance of fBm-based models for simulating real-world heterogeneity in geological media and understanding how fractal properties affect wave transport.

The investigation on wave propagation through random and fractal media has evolved significantly over the decades. Varadan et al. [6] explored wave scattering and attenuation

in random elastic media, establishing theoretical insights into how fractal heterogeneity influences wave transport. Shapiro and Kneib [13] developed numerical models to investigate seismic wave attenuation due to scattering effects in random media. Frankel and Clayton [18] analyzed wave propagation in random velocity models to characterize scattering and energy attenuation, providing insights into high-frequency seismic wave behavior. Aki and Richards [14] quantified the scattering attenuation of wave transport by introducing the quality factor Q , denoted as the ratio of attenuated energies to incident ones. Sahimi and Tajar [4] demonstrated that the elastic moduli and density in rocks are distributed in a self-affine fractal pattern, identifying these properties' influence on seismic wave velocities. This work underscored the importance of fractional Brownian fields in capturing the spatial variability of natural rock properties and highlighted the need for a detailed analysis of wave transport in fractal environments. Allaei and Sahimi [15] analyzed the shape of wavefronts in heterogeneous media, applying theoretical and numerical approaches to quantify how randomness affects wave coherence and transport patterns. Przybilla et al. [19] developed a Monte Carlo scheme to simulate wave transport in 2D random elastic media with spatially distributed scattering coefficients based on the radiative transfer theory. Sahimi et al. [20] emphasized that the distribution patterns of elastic moduli (correlated versus uncorrelated) play a crucial role in the elastic wave transport in heterogeneous solids, leading to the transition between localization and propagation. Garnier and Sølna [21] underlined the relationship between the damping exponent of a traveling wave and the Hurst parameter of random media, which characterizes the correlation properties of the random field. Hamzehpour et al. [22] simulated wave propagation in 2D heterogeneous fractured media where cracks were randomly distributed and the bulk modulus of the matrix was distributed according to fractional Brownian motion. Zhang et al. [23] employed sensitive kernels to represent the impact of the spatial characteristics of media properties on the wavefield, which is beneficial for fetching new tomographic inversion approaches to map small-scale heterogeneity using scattered waves. Wang et al. [17] numerically investigated the competing roles of matrix modulus heterogeneity and fracture networks in wave propagation and quantified the dependence of wave arrival times on the heterogeneity index and fracture stiffness.

Despite these advancements, several unresolved issues and research gaps remain in the study of wave propagation in fractional Brownian fields. It is essential to build more complex models that can capture fractal heterogeneity and anisotropic behavior in geological materials because such material properties impact wave transport characteristics, such as the wavefront profile, arrival behavior, and energy attenuation [1,4,24,25]. Moreover, appropriate indices representing the heterogeneity of fractional Brownian fields should also be included in the model to quantitatively assess their influence on wave transport. This study aims to address these gaps by exploring elastic wave propagation in a 2D fractional Brownian field, which is characterized by Hurst exponent H and standard deviation σ . We systematically investigate the quantitative dependencies of wave propagation behaviors on these fractal heterogeneity indices.

The organization of the remainder of this paper is as follows. In Section 2, we elaborate on the numerical methodology and model setup, detailing how the fractional Brownian field is constructed and the finite element approach is applied. Section 3 presents the results, highlighting how variations in heterogeneity influence wavefield evolution, wavefront roughness, arrival times, and energy decay. Section 4 discusses the implications of these findings for wave transport in fractal media. Finally, Section 5 concludes with a summary of key findings and suggests directions for future research in wave propagation modeling in fractal-based random media.

2. Numerical Methodology

2.1. Model Setup

A squared study domain of size L is built to simulate a plane incident P wave propagating through the heterogeneous media (Figure 1a). The fractional Brownian field (see

Section 2.3) is placed in this $L \times L$ study domain and the elastic wave (see Section 2.2) is excited at the left boundary (red line in Figure 1a) using a five-cycle Hann windowed tone burst with a force-type signal (Figure 1b) of wavelength $\lambda = L/10$. In the study domain, we record the x -direction displacement, aligned with the propagation direction of the plane incident wave, using an array of 301×301 arranged receivers. The receivers are spaced at a spatial interval of $\delta = L/300$. To mimic infinite wide-plane wave propagation, we use the following two schemes. First, we generate an auxiliary mirrored domain below the study domain with the same modulus distribution symmetric about the mirror boundary. Meanwhile, two periodic boundaries are separately applied at the bottom of the mirrored domain and the top of the study domain. The random fields in these two areas conjointly constitute a unit cell [26] that is capable of accurately reproducing an infinite plane wave transporting in the x direction and eliminating boundary effects at both the top and bottom of the study domain. Second, we place absorbing layers (see Section 2.2) to the left and right boundaries of the whole model to absorb unwanted reflected waves from these two boundaries. In the middle region (study and mirrored domains), we use a spatial mesh of Delaunay triangular elements with an average element size of $l_{\text{ele}} = \lambda/15$ (Figure 1c). For the absorbing layers, mapping elements (Figure 1d) with the same size are generated. To capture the temporal evolution of the wavefield, the time domain is discretized using a timestep $dt = l_{\text{ele}}/V_P \times \text{CFL}$, where V_P is the velocity of the incident P wave and CFL is the Courant–Friedrichs–Lewy (CFL) number [27] (taking 0.1 in our numerical model).

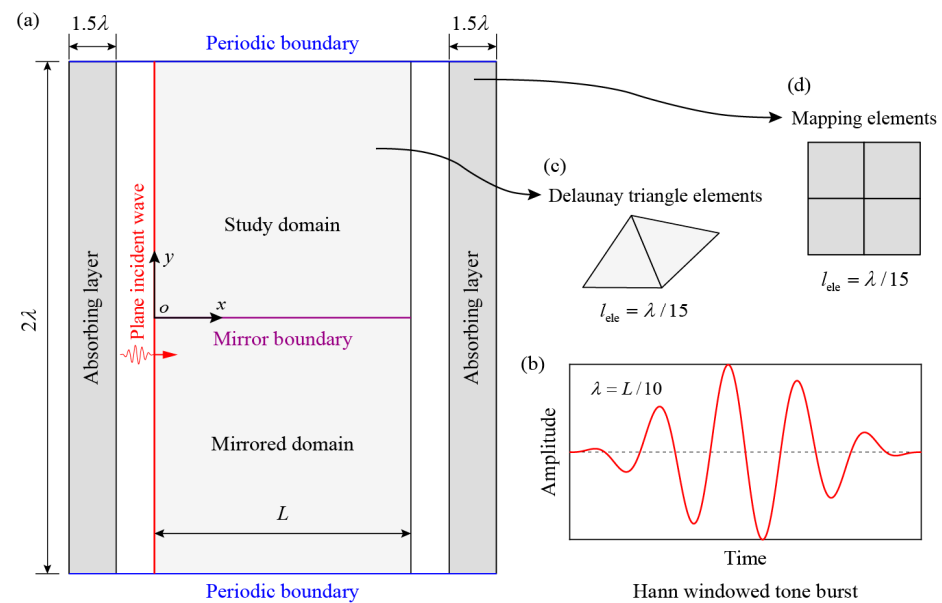


Figure 1. Geometrical model setup for simulating elastic wave transportation in solids: (a) overall configuration of numerical model; (b) Hann windowed tone burst of wavelength $\lambda = L/10$; (c) Delaunay triangular elements in study and mirrored domains; (d) mapping elements in absorbing layers.

2.2. Governing Equations

The elastic wave propagation in two-dimensional media is governed by the time-domain equation of dynamic equilibrium as

$$\mathbf{M} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{C}^* \frac{\partial \mathbf{u}}{\partial t} + \mathbf{K}^* \mathbf{u} = \mathbf{F} \quad (1)$$

where $\mathbf{M}$ is the mass and stiffness matrix, $\mathbf{u}$ stands for the displacement, t is the time, $\mathbf{F}$ represents the external force, and $\mathbf{C}^* = \eta \mathbf{M}$ and $\mathbf{K}^* = \zeta \mathbf{K}$ are the damping and stiffness matrices, respectively. Herein, η and ζ are coefficients representing the mass proportional damping and stiffness reduction, respectively. When waves transport in the study domain,

no damping or stiffness reduction is introduced; thus, the damping is set to zero, and the stiffness matrix remains as $\mathbf{K}^* = \mathbf{K}$. To mimic wave transport in a boundless domain, we employ the stiffness reduction method [27] to mitigate a needless reflected wave at the boundaries of the finite-sized model by implementing absorbing layers at the side boundaries. The absorbing layer has a thickness of 1.5λ . In the absorbing layer, the mass proportional damping coefficient $\eta(\chi)$ is derived as $\omega(\chi/(1.5\lambda))^3$, where χ stands for the x -direction distance of a point away from the boundary of the study or mirrored domain; ω denotes the angular frequency of the incident wave. The stiffness reduction coefficient $\zeta(\chi)$ is derived as $\exp(-\psi(\chi)k\chi)$, where k represents the incident wavenumber; $\psi(\chi) = \psi_{\max}(\chi/(1.5\lambda))^3$ denotes the attenuation factor with $\psi_{\max} = -\ln(\varepsilon/(1.5\lambda k))$ being the maximum attenuation factor. Here, ε is a constant with a small value, usually using 0.01. In this paper, we use the finite element method with the commercial software COMSOL version 5.5 [28] to solve the above elastic wave equations. We conducted a thorough comparison with analytical solutions to assess the validity and precision of our numerical model for simulating wavefield evolution in unbounded heterogeneous media [17,29].

2.3. Generation of Fractional Brownian Fields

Fractional Brownian motion (fBm) is one of the most useful mathematical models for representing the random fractals in nature [10,11,30], which is fundamentally important for understanding wave transport and anomalous diffusion on fractals [31]. Fractional Brownian motion $V_H(t)$ is a single value function of t , of which the increment $V_H(t_{i+1}) - V_H(t_i)$ obeys a Gaussian distribution with the following variance [31]:

$$\langle |V_H(t_{i+1}) - V_H(t_i)|^2 \rangle \propto |t_{i+1} - t_i|^{2H} \quad (2)$$

where the brackets $\langle \rangle$ represent the average over all the samples of $V_H(t)$, and H is the Hurst exponent with a value ranging from 0 to 1. Several methods [31,32] have been developed to generate fractional Brownian motion in different dimensions. In this paper, we use the midpoint displacement method [31] to generate 2D fields of fractional Brownian motion due to it being independent of the grid size. The detailed algorithm of generating 2D fractional Brownian fields is referred to in pages 100–101 by Barnsley et al. [31].

It has been found that material properties, e.g., Young's modulus, Poisson's ratio, and density, may be spatially variable in rock masses [4,33], and the distribution usually follows a fBm [4]. Here, we consider the spatial variation in Young's modulus E in our model (Figure 1a) as a 2D fractional Brownian field, having a mean value $\bar{E}$ and a standard deviation σ . We assume that the heterogeneous media are a typical limestone with a material properties list in Table 1. To explore the influence of field heterogeneity on the propagation of elastic waves, two indices, the standard deviation of Young's modulus σ and the Hurst exponent H , are included, taking the values listed in Table 1. The spatial variability of Young's modulus is presented in each numerical element (Figure 2). The probability density of Young's modulus in the random fields is also given in Figure 3. It can be seen that, when σ is smaller, the modulus is more evenly distributed in the field and the magnitudes are more concentrated at the mean value $\bar{E}$. With larger H values, the distribution of E tends to be smoother while the probability density becomes more dispersive with a more obvious divergence to the corresponding conventional Gaussian distribution.

Table 1. Material properties in the fractional Brownian fields.

Properties	Value	Unit
Mean value of Young's modulus, $\bar{E}$	30	GPa
Standard deviation of Young's modulus, σ	1, 2, 4, 8	GPa
Density, ρ	2700	kg/m ³
Poisson's ratio, ν	0.27	--
Hurst exponent, H	0.1, 0.3, 0.5, 0.7, 0.9	--

"--" indicates no unit for this parameter.

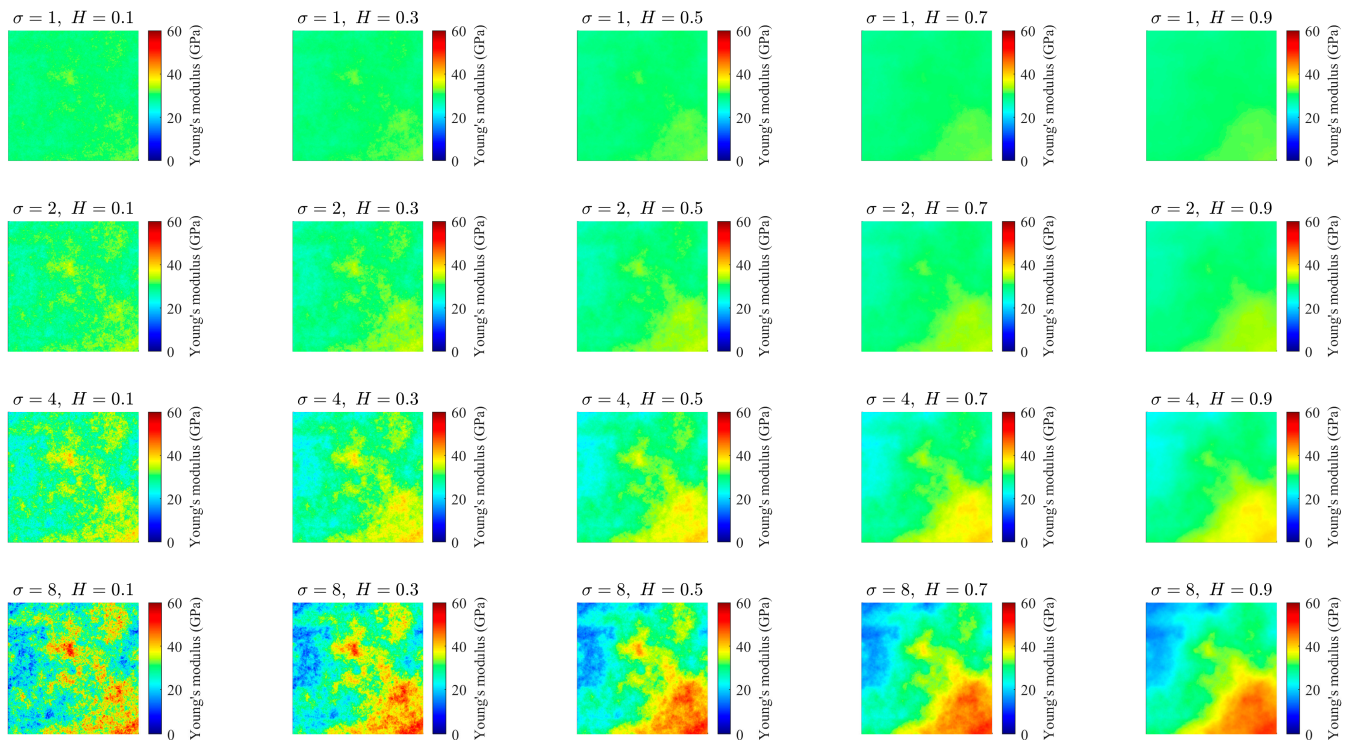


Figure 2. Variability of Young's modulus in the rock matrix characterized by 2D fractional Brownian fields with different values of standard deviation σ and Hurst exponent H .

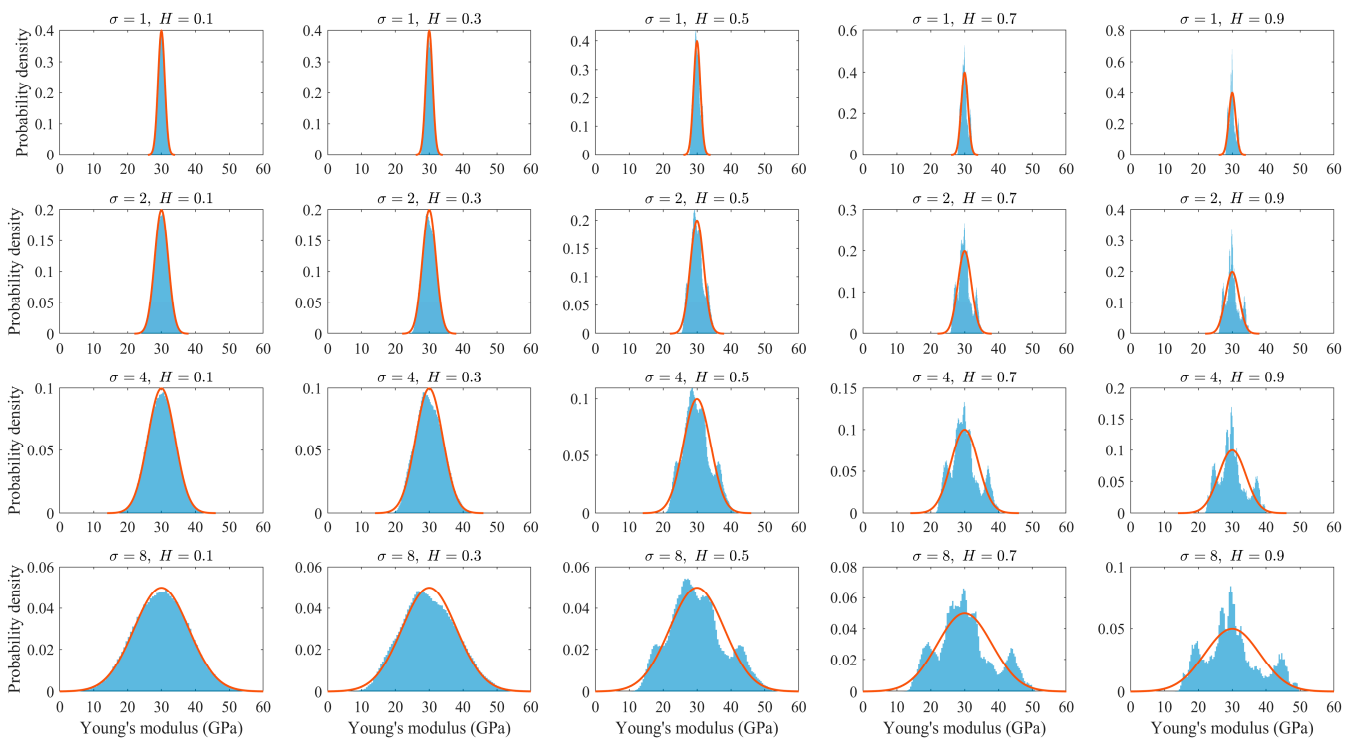


Figure 3. Probability density of Young's modulus in the rock matrix characterized by 2D fractional Brownian fields with different values of standard deviation σ and Hurst exponent H . The red curve in each subplot is the PDF of a Gaussian distribution with the expectation and standard deviation being the same as Young's modulus.

2.4. Characterization of Wave Arrival Behavior

We utilize the wave amplitude to identify the leading front of wave energy to describe the elastic wave arrival behavior, which is called the front of the first arrival wave, abbreviated as FFAW [29]. It is constituted by the points farthest from the source line with the same displacement-to-amplitude ratio, which is denoted as

$$\frac{|U(t_i, x_i, y_i)|}{A(x_i, y_i)} = \frac{|U^0(t^0, x_i, y_i)|}{A^0(x_i, y_i)} \quad (3)$$

where $U(t, x_i, y_i)$ and $U^0(t^0, x_i, y_i)$ represent the displacements of a receiver denoted as (x_i, y_i) at time t_i in the heterogeneous medium and t^0 in the homogeneous medium, respectively; $A^0(x_i, y_i)$ are the wave amplitudes recorded by the receiver (x_i, y_i) in the homogeneous medium while $A(x_i, y_i)$ are the ones in the heterogeneous medium. Specially, t^0 and t_i stand for the times when waves arrive at the receiver (x_i, y_i) in the homogeneous and heterogeneous media, respectively. We can use the same time to derive

$$\frac{|U(t_i, x_j, y_j)|}{A(x_j, y_j)} = \frac{|U(t_i, x_i, y_i)|}{A(x_i, y_i)} \quad (4)$$

and then identify a collection of points to construct the FFAW at a given time t_i in the heterogeneous media. In the same way, we can determine the time at which some point at the right boundary is activated for the first time, i.e., the wave energy reaches the right boundary for the first time, with the corresponding time defined as the breakthrough time $\tilde{t}_b$. When all the receivers at the right boundary have recorded distinguishable wave signals for the first time, the corresponding time is called the final arrival time $\tilde{t}_f$.

At a given dimensionless time $\tilde{t}$, we denote $d(\tilde{t}, y_i)$ as the distance of a point on the FFAW from the source line (Figure 1a), where y_i is the y -direction coordinate. $d(\tilde{t}, y_i)$ can be derived from the seismograms recorded by the receiver array. We calculate the roughness $R(\tilde{t})$ of the FFAW (Barabási and Stanley 1995 [30]) as

$$R(\tilde{t}) = \sqrt{\frac{1}{N_y} \sum_{i=1}^{N_y} [d(\tilde{t}, y_i) - \overline{d(\tilde{t})}]^2} \quad (5)$$

where N_y is the number of sample points on the y axis (i.e., 301 receivers), and $\overline{d(\tilde{t})}$ stands for the average value of point-to-source distances of the FFAW at $\tilde{t}$. We then normalize the roughness by dividing the domain size, namely, $\tilde{R} = R/L$.

2.5. Quantification of Wave Energy

The decay of elastic waves in random fields is identified by the inverse quality factor Q^{-1} [14] derived as

$$Q^{-1} = -\frac{2\bar{V}}{\omega x} \ln\left(\frac{A}{A^0}\right) \quad (6)$$

where $\bar{V}$ is the mean velocity of elastic waves in heterogeneous media, ω is the angular frequency of incident waves, x is the traveling distance of waves equaling the x -direction distance between the receiver and source line, and A^0 and A are wave amplitudes in homogeneous and heterogeneous media, respectively. The wave velocity is derived from the seismograms, i.e., the normalized x -direction displacement recorded by the receivers at the right boundary of the study domain, which is further averaged over the number of these receivers (301 as indicated in Section 2.1) to determine the mean velocity. We use a fast Fourier transform analysis to transform wave signals recorded by receivers to the frequency domain and average the spectral amplitudes of receivers equidistant from the source line to obtain the mean value, which is then used to derive A . Thus, we can fit a straight line to derive the slope of the curve determined by $\ln(A/A^0)$ versus source-to-receiver distance x .

Consequently, the inverse quality factor Q^{-1} can be calculated by the mean velocity and the slope of $\ln(A/A^0)$ versus x .

3. Results

3.1. Spatiotemporal Wavefield Evolution

Herein, the x -direction displacement is normalized by the displacement amplitude of the incident wave, and the traveling time t of elastic waves in heterogeneous media is normalized to be dimensionless as $\tilde{t} = t/t_0$, where t_0 is the traveling time of elastic waves propagating from the source line to the right boundary of a homogeneous study domain, taking the derivation of $t_0 = L/V_p$. Figure 4 demonstrates the spatial distribution of the elastic wavefield represented by the normalized x displacement at dimensionless $\tilde{t} = 1$. The temporal snapshots during wave propagation from $\tilde{t} = 0$ to 3 are presented as animations in. Figure 5 illustrates the seismograms recorded by the receiver column located at the right side of the study area. When σ is small, e.g., 1, the wavefront keeps a planar shape during the early propagation stage ($\tilde{t} < 1$ in S1). In cases of smaller H values, the fractional Brownian field has a rougher surface of modulus distribution (Figure 2), inducing a rougher profile of the wavefront (Figure 4) and a more fluctuant waveform (Figure 5) in the later propagation stage ($\tilde{t} > 1$). Slight coda waves can also be observed behind the frontal wave (Figure 5 and S1). However, due to such a random E distribution, wave energies arrive at the right boundary at almost the same time (Figures 4 and 5). As H increases, larger magnitudes of Young's modulus are more concentrated in the lower right of the study domain (Figure 2), generating larger wave velocity in this region. As a result, waves in the lower region transport faster than those in the upper. Thus, the wavefield becomes inclined in a counterclockwise way at $\tilde{t} = 1$ (Figure 4) and the following timesteps (S1). In these larger- cases, the distribution of E is smoother and the induced waveform is barely perturbed in the later propagation stage (Figure 5); i.e., no obvious coda wave is detected.

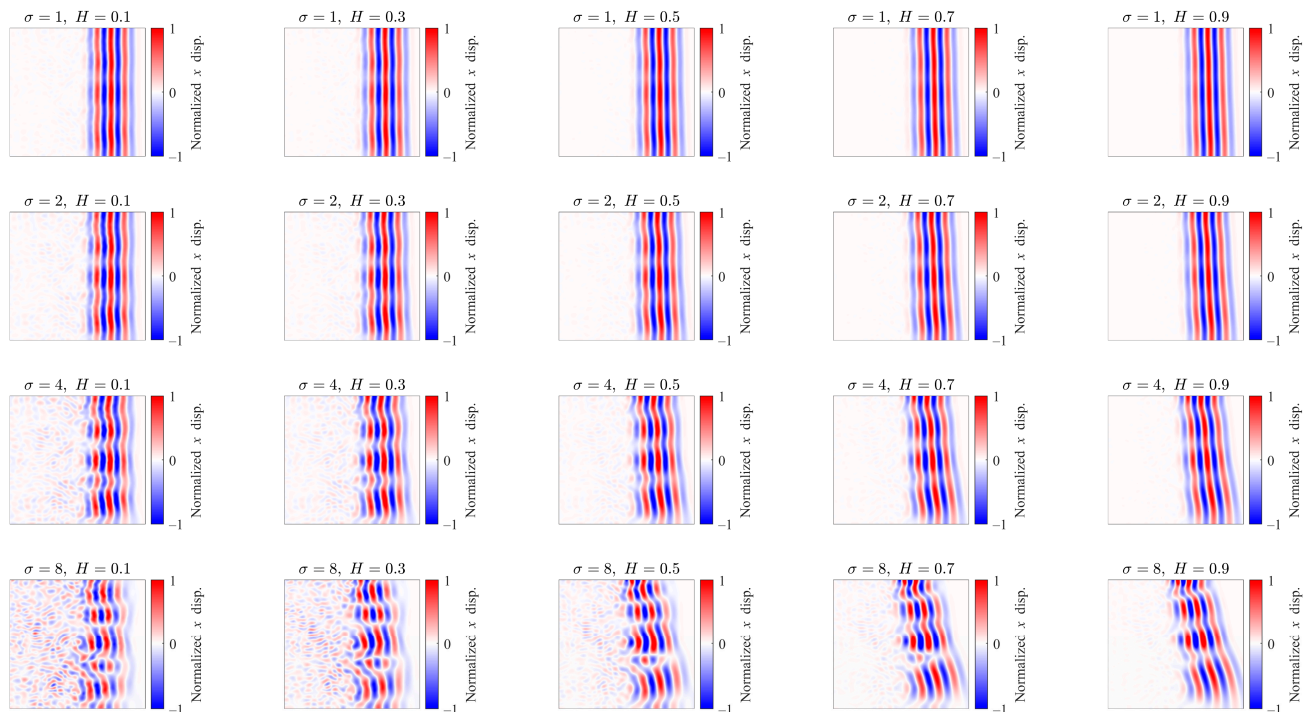


Figure 4. Spatial evolution of elastic wavefield according to normalized x displacement at dimensionless time $\tilde{t} = 1$.

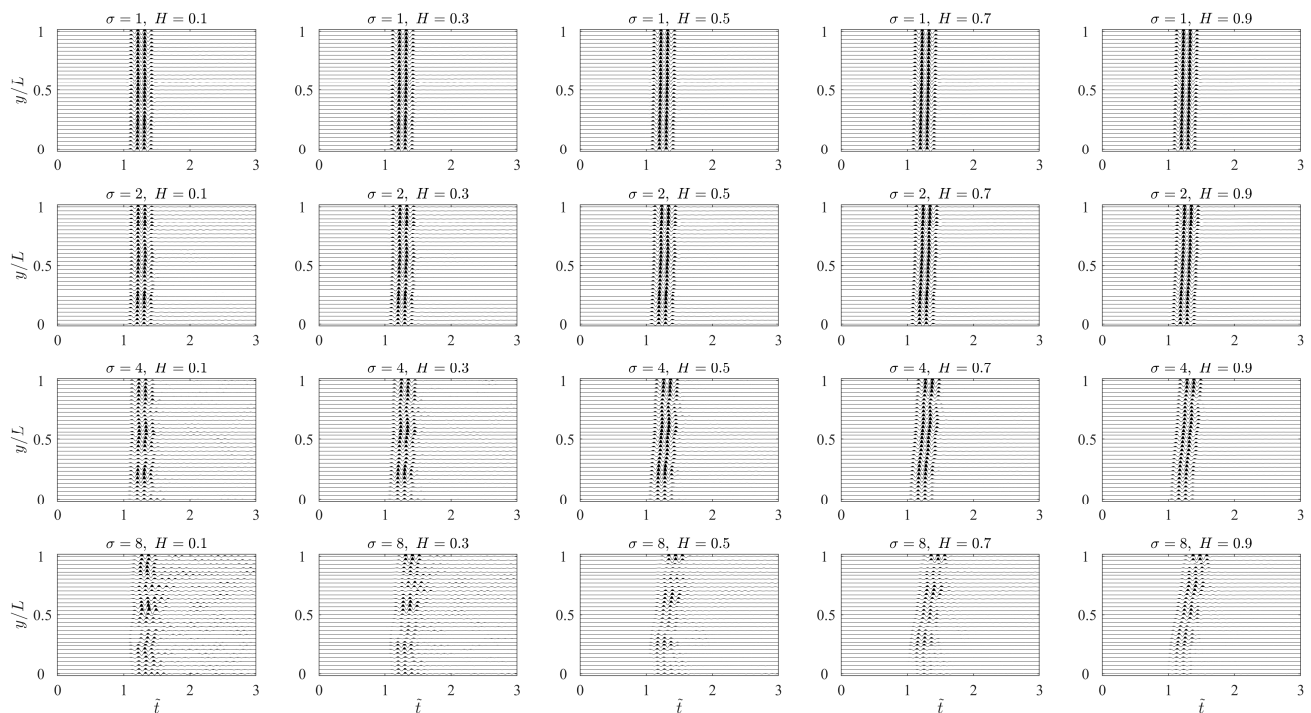


Figure 5. Seismograms according to normalized x -direction displacements recorded by the receiver array at the right boundary of the study domain.

For $\sigma = 2$, the waveform is distorted even in the early propagation stage (Figure 5 and $\tilde{t} < 0.5$ in S1) and more wave energies are scattered by the matrix heterogeneity (Figures 4 and 5). The rough pattern of the wavefront and the asynchronous arrival phenomenon are more recognizable due to the stronger deviation of E . The distortion of the waveform and roughness of the wavefront become more obvious with decreasing H , whereas the skewness of the wavefront (i.e., asynchronous arrival phenomenon) is more apparent with increasing H . As σ becomes larger (≥ 4), the wavefield tends to be rougher and more waves are scattered into coda (Figures 4 and 5). The dependence of the wavefront and arrival behavior on the Hurst exponent remain the same as the cases of smaller σ : small H generates coda waves and distorts the wavefield; large H separates fast and slow waves and thus causes different arrival times.

As Figures 4 and 5 illustrate, the wavefield evolution is strongly related to the statistical parameters of the fractional Brownian field, which dominate the global and local distribution of E in the field. As shown in Figure 2, a smaller H generates a rougher distribution of E , like a rough surface with spins (large E) and slots (small E), which are distributed in a short-range correlation; i.e., only elements with short distance may hold a similar modulus. As a result, the aggregation of elements with a similar modulus is infrequent in the field. The spins and slots serve as barriers and scatterers, occupying the propagation path of elastic waves, producing scattered coda waves and a distorted wavefield, which is highly different from the wavefield in homogeneous media. On the contrary, larger H generates a smooth surface with a concentrated region of a contiguous modulus as the moduli are distributed in a pattern of long-range correlation. The smoothly distributed E provides an unobstructed and gliding path for elastic wave transport, leaving few coda waves behind the anterior wave, and more energy can reach the right boundary. The wave generally keeps its planar shape similar to its incident appearance. However, the conspicuous concentration region causes a significantly asynchronous arrival phenomenon; i.e., the wavefront is tilted. The standard deviation of Young's modulus σ is a key factor influencing the magnitude of the modulus. Larger σ generates a wider range of E values, inducing a greater probability of any two elements in the field having larger differences in the modulus. Consequently, such higher heterogeneity is more visible to the wave, causing

a clearer observation of anomalous wave transport, e.g., the generation of coda waves, rough profile of the wavefront, and asynchronous wave arrival.

3.2. Arrival Behavior of Elastic Wave

To eliminate the randomness caused by different random seeds in generating fractional Brownian fields, we generated 10 realizations for each combination of σ and H ; i.e., in total, 200 fields were created. Figure 6 shows the temporal variation in the dimensionless roughness $\tilde{R}$ of FFAW versus $\tilde{t}$ for each case, where the red line in the subfigure is a result of the corresponding fractional Brownian field in Figure 2 and gray lines are the results of the other nine realizations. It can be clearly seen that the roughness of FFAW generally increases as the wave propagates through the random field across all realizations and parameter combinations, which is a consequence of the incremental effect of matrix heterogeneity on the wave transport due to the increasing propagation distance. When the field variability is small ($\sigma = 1$), the roughness evolves slowly and remains relatively low throughout the wave propagation, suggesting that minor heterogeneity has a limited effect on wavefront distortion. In addition, changes in H have minimal influence on roughness evolution, leading to similar roughness curves, which indicates that the spatial correlation of the modulus (H) plays a minor role in the wave arrival behavior when the field variability is low. As σ increases ($\sigma \geq 2$), the roughness evolution becomes significantly steeper with time, indicating a stronger scattering effect and more prominent wavefront distortion caused by larger modulus variability. In these cases with higher field variability, the modulus distribution (H) tends to amplify scattering effects, leading to a faster growth of roughness. We then plot the profile of FFAW at the breakthrough time $\tilde{t}_b$ for each realization in Figure 7. When σ is small (≤ 1), the width and roughness of the profile barely change with H while FFAW becomes wider and rougher with increasing H when σ exceeds 2. As for FFAWs with the same H , the shape changes more violently with larger σ .

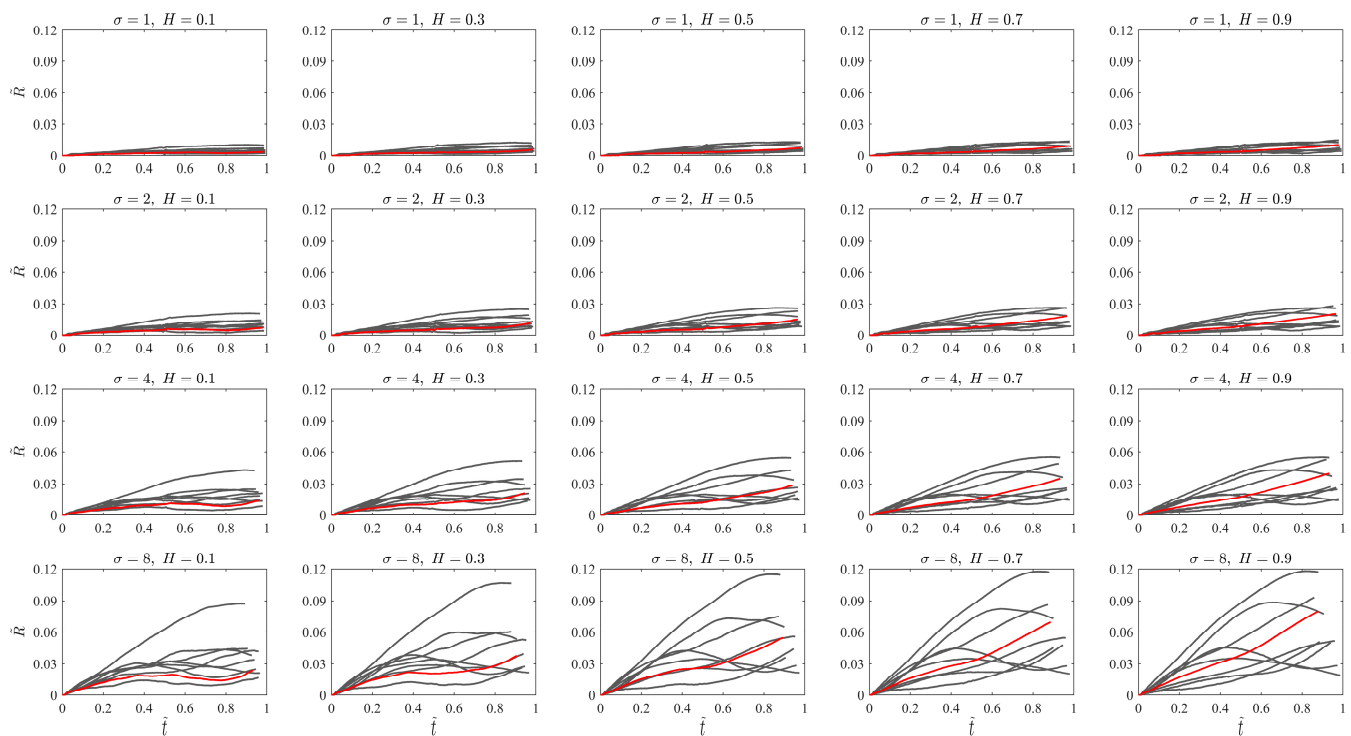


Figure 6. The temporal evolution of the dimensionless roughness $\tilde{R}$ of FFAW versus the dimensionless time $\tilde{t}$. The red line results from the fractional Brownian field in Figure 2, and gray lines result from the other 9 realizations.

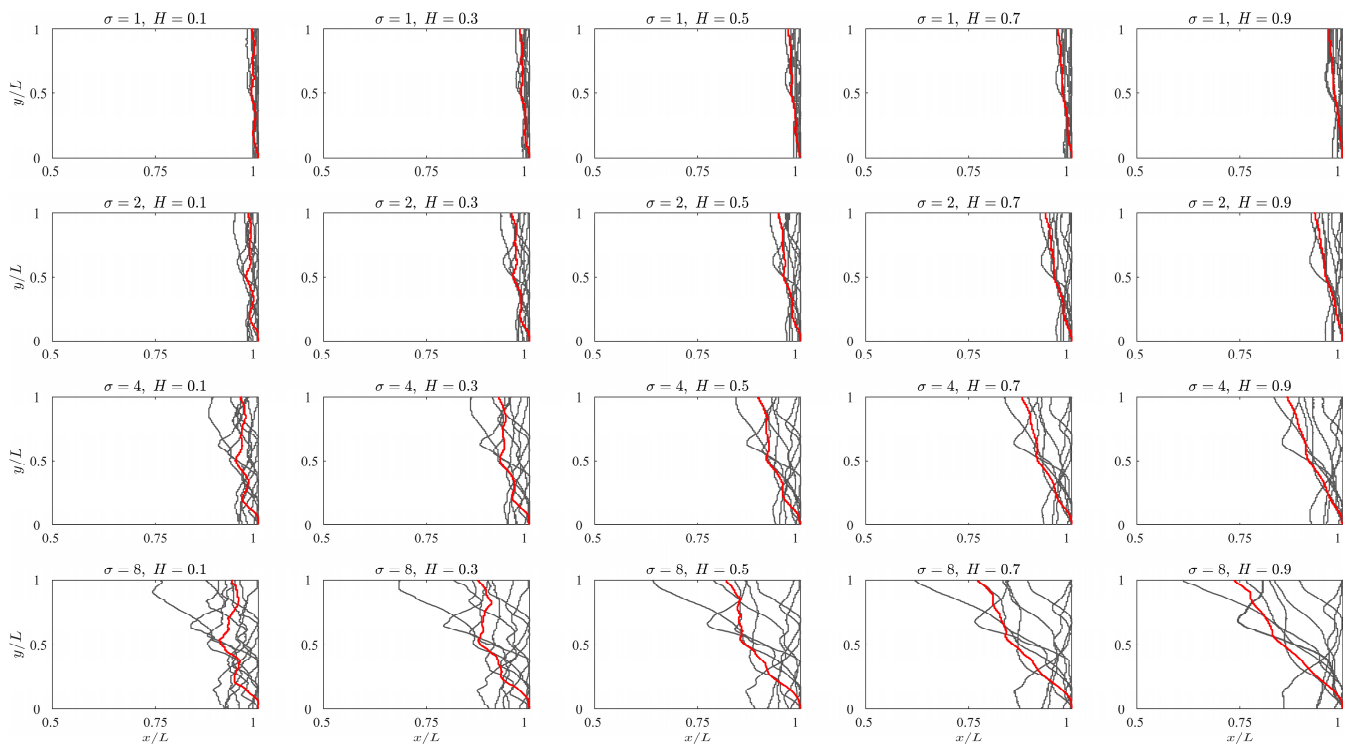


Figure 7. FFAW at the breakthrough time $\tilde{t}_b$. The red line results from the fractional Brownian field in Figure 2, and gray lines result from the other 9 realizations.

We further averaged the data over 10 realizations to show a universal tendency. Figure 8 gives the dimensionless roughness $\tilde{R}$ of FFAW at $\tilde{t}_b$ versus H and σ . The roughness of FFAW shows a generally positive relationship with the Hurst exponent H when $\sigma \geq 2$, whereas the dependence in the cases in which $\sigma = 1$ is not well marked. In Figure 8b, $\tilde{R}$ is linearly dependent on σ and the slope also increases. Such dependent manners are in accordance with the phenomenon demonstrated in Figures 6 and 7. We then determine the dimensionless breakthrough time $\tilde{t}_b$ and final arrival time $\tilde{t}_f$ versus H and σ , as plotted in Figure 9. $\tilde{t}_b < 1$ and $\tilde{t}_f > 1$ for all the cases because the randomly distributed Young's moduli provide interlaced channels that expedite or slow the wave speed during propagation (Figure 2), resulting in the arrival of some pioneering wave energy at the right boundary with some slowed energy still remaining in the study region. The opposite dependencies of $\tilde{t}_b$ and $\tilde{t}_f$ on σ (and H) also date from the transport patterns of wave energy in diverse random fields: higher modulus variability and concentricity produce more of a heterogeneous wavefield (Figure 4), a more fluctuant seismogram (Figure 5), and a rougher wavefront (Figures 4, 7 and 8), making the asynchronous arrival phenomenon more significant (Figures 4, 5 and 7). Thus, the difference derived from $\tilde{t}_f - \tilde{t}_b$ shows a positive relationship with these two heterogeneity indices, as shown in Figure 10. It should be noted that this dependence at $\sigma = 1$ is not evident, similar to Figures 8a and 9a, which is also consistent with the phenomena demonstrated in Figures 4 and 5.

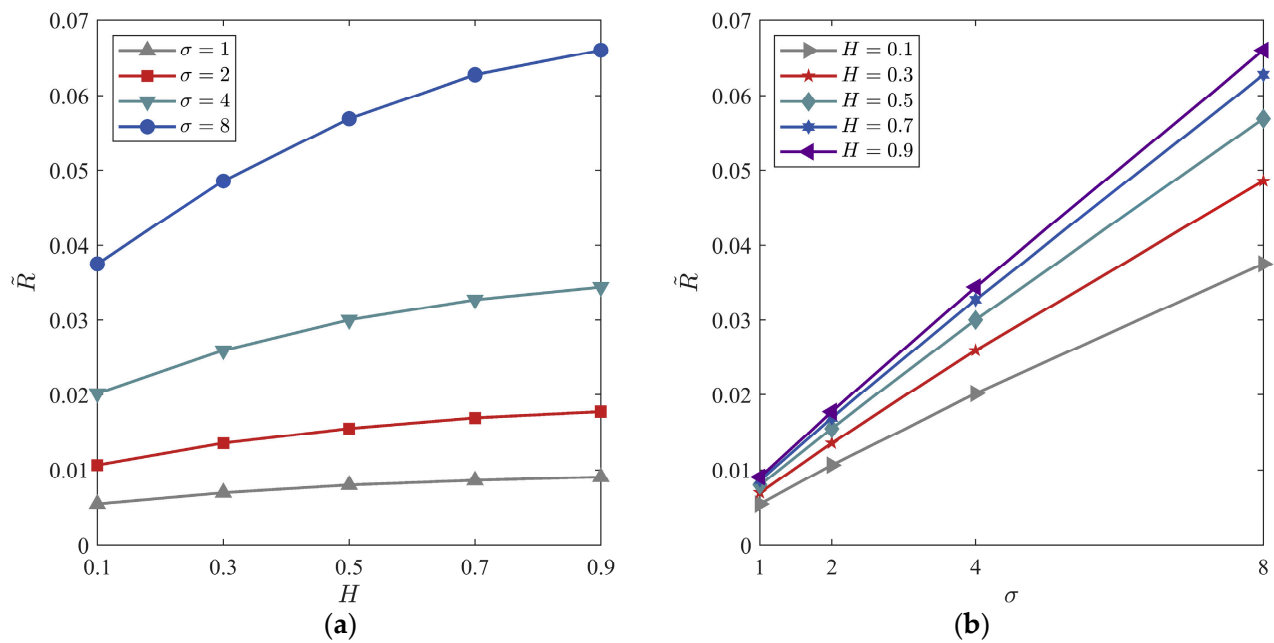


Figure 8. Dimensionless roughness $\tilde{R}$ of FFAW at the breakthrough time $\tilde{t}_b$ versus (a) H and (b) σ . Each marker is the averaged value over 10 realizations.

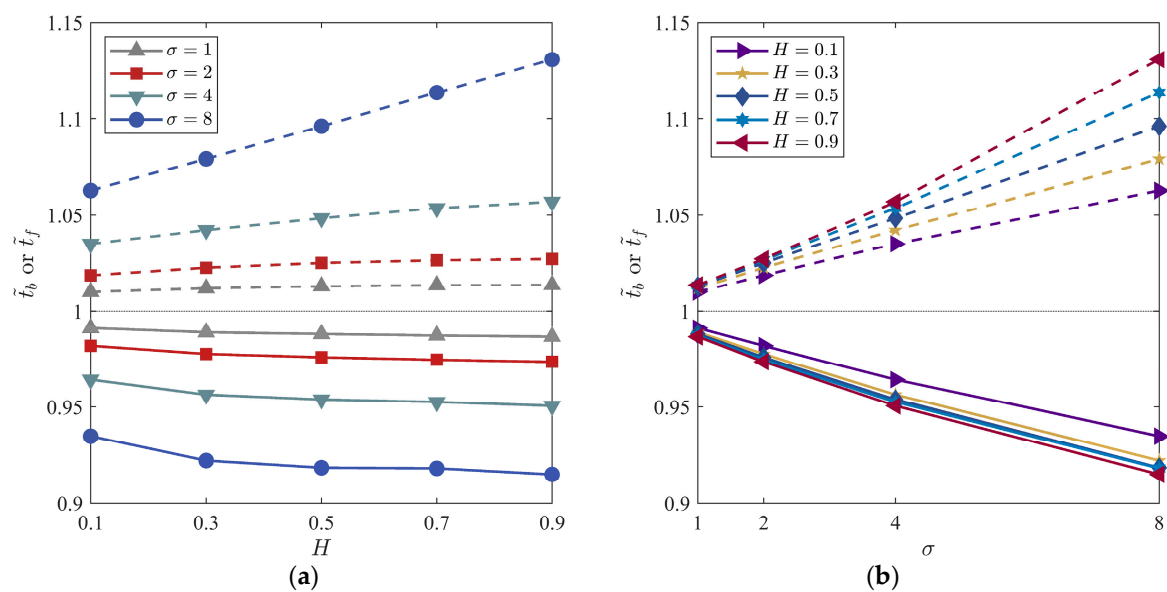


Figure 9. Breakthrough time $\tilde{t}_b$ (solid lines) and final arrival time $\tilde{t}_f$ (dashed lines) as a function of (a) H and (b) σ . Each marker is the averaged value over 10 realizations. The horizontal dotted line represents $\tilde{t}_b = 1$, i.e., the breakthrough time of waves passing through a blank model without Young's modulus being randomly distributed.

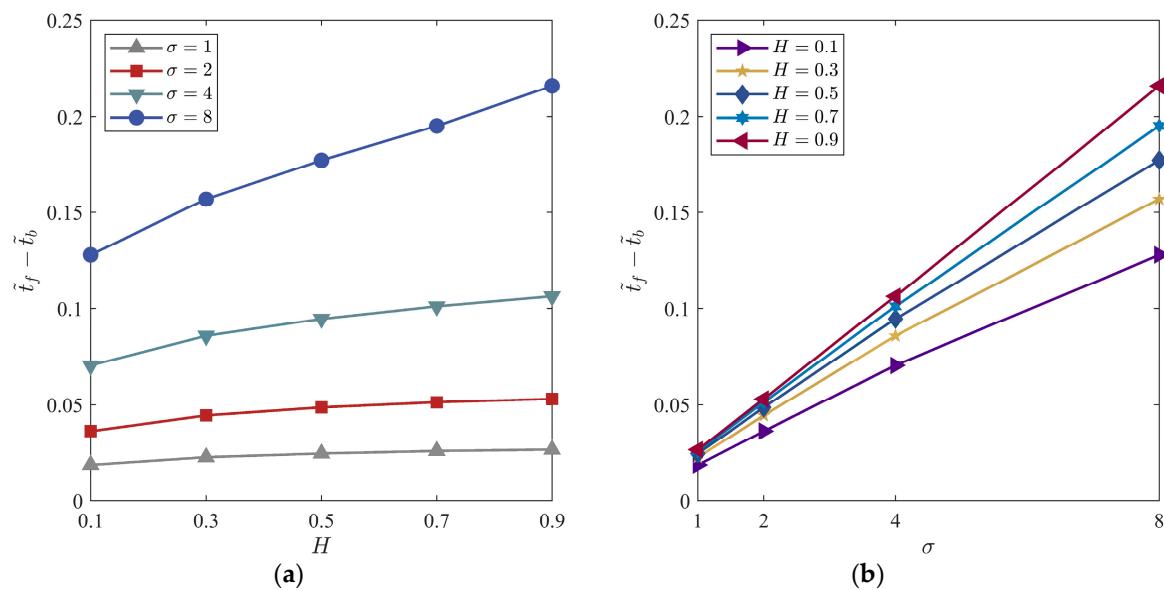


Figure 10. Difference in final arrival time $\tilde{t}_f$ minus breakthrough time $\tilde{t}_b$ as a function of (a) H and (b) σ . Each marker is the averaged value over 10 realizations.

3.3. Decay of Elastic Wave Energy

According to Equation (6), the relative spectral amplitudes as a function of the normalized traveling distance x/L from the source line are plotted in Figure 11. For a small σ , e.g., 1, the decay of amplitude is not significant and the slope of $\ln(A/A^0)$ versus x/L does not change when H changes. For a larger $\sigma \geq 2$, the inverse relationship between the wave amplitude and traveling distance becomes more obvious, showing that the matrix heterogeneity increasingly alters the wavefield as waves across the random field, leaving more energy scattered into the coda. This phenomenon agrees well with the wavefield evolution shown in Figure 4 and S1. We can also see that the decay of wave energy (denoted by the slope of the curve) exhibits an increasing relationship with σ , which is a result of the inducement of the higher variability of random fields with a larger σ . For the cases with the same σ , the decay of wave energy is negatively dependent on the traveling distance, showing that a smoother spatial distribution of Young's modulus is beneficial for energy transport and may produce less coda waves. We then calculate the inverse quality factor Q^{-1} based on the fitting lines of $\ln(A/A^0)$ versus x/L and plot its variation with H and σ in Figure 12. Here, we used the range of normalized distance x/L from 0.25 to 0.75 to derive the slope of $\ln(A/A^0)$ versus x/L because the wave transport observed in Figure 4 and S1 is generally in the propagation regime. It can be seen that Q^{-1} linearly decreases with increasing H showing that wave energies are more attenuated in random fields with a smaller H , and a smoother modulus distribution with larger H facilitates the transport of wave energies across the random media. The variability of Young's modulus σ seems to have no significant effect on the decreasing trend of Q^{-1} versus H . The inverse quality factor Q^{-1} is positively dependent on σ , while H does not change the variation trend. By comparing the variation patterns of Q^{-1} versus H and σ , the influence of matrix variability is more significant than that of the spatial distribution.

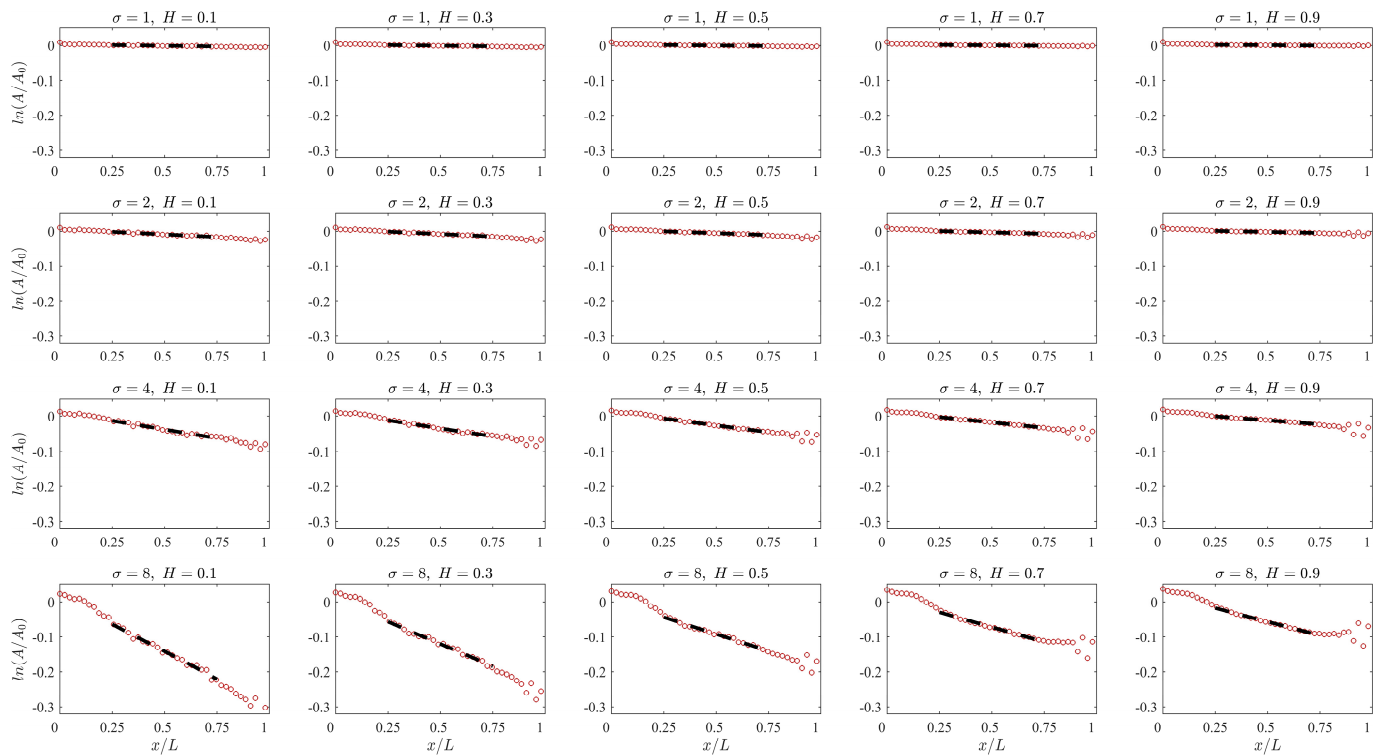


Figure 11. Relative spectral amplitude $\ln(A/A^0)$ varies with the normalized distance x/L from the source for different combinations of σ and H . Each scatter datum is calculated by averaging the mean values of data recorded by each column of 301 receivers equidistant from the source over 10 realizations. The black dashed line in each subfigure is the corresponding fitting line of data with normalized distance x/L from 0.25 to 0.75.

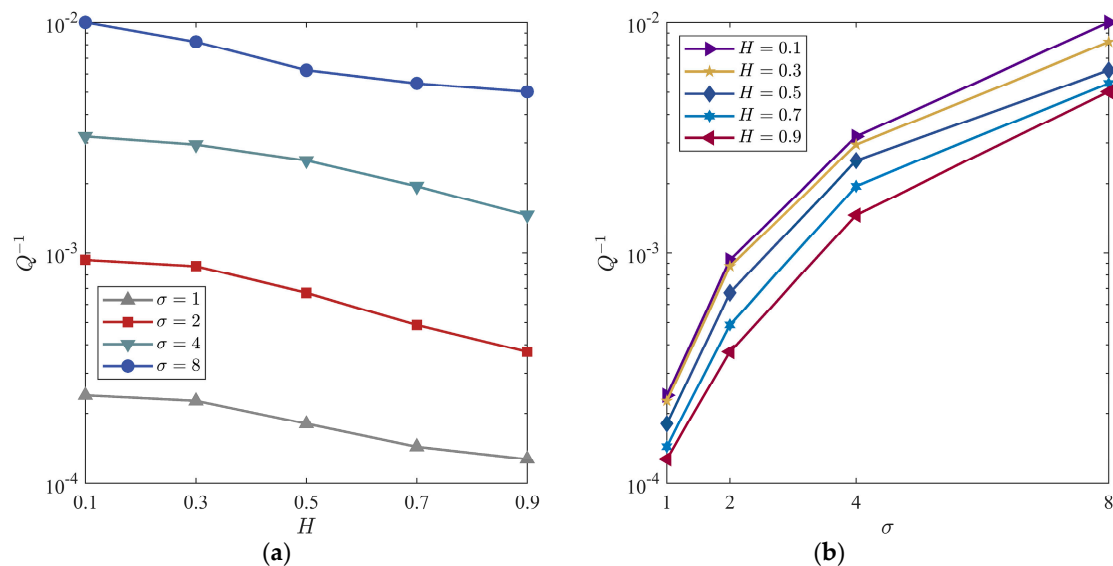


Figure 12. Inverse quality factor Q^{-1} varies with (a) H and (b) σ . Each marker is the averaged value over 10 realizations.

The opposite dependencies of Q^{-1} on H and σ , i.e., negative and positive, respectively, give a straightforward revelation for the wavefield evolution shown in Figure 4 and S1. Higher σ induces stronger differences between element moduli, among which the elements with a smaller modulus slow down and scatter wave energies, intercepting their traveling paths at the right boundary. When H is small, the modulus distribution is rough and thus

exacerbates the scattering effect. A smoother distribution with larger H provides more even channels for wave energies across the field in the pattern of gathering moduli with similar values in concentration clusters. The smoothness of connecting paths between clusters is also positively related to H . Such spatial characteristics of fractional Brownian fields generate fewer codas during waves passing through these smooth paths and clusters. Therefore, Q^{-1} increases and more wave energy decays faster in the fields with increasing σ and decreasing H .

4. Discussion

Our numerical results shed light on the effects of fractal heterogeneity on wavefield evolution, wavefront roughness, arrival behavior, and energy decay, aligning with and extending findings in the existing literature. The observed wavefront distortion and roughness at higher levels of Young's modulus variability (σ) and lower Hurst exponent (H) values underline the previous findings that such heterogeneity amplifies scattering and energy dispersion [4,6]. However, our study further quantifies this relationship by identifying specific parameter regimes in which wavefront irregularities and asynchronous arrivals become most pronounced, suggesting that fields with lower H and higher σ are particularly preferential to complex wave behavior.

The asynchronous arrival behavior observed in this study aligns with the demonstration of wave scattering in heterogeneous media causing delayed and spatially uneven arrivals [13] and acceleration in energy decay [22]. Our results further illustrate that, as H decreases, the heterogeneity becomes increasingly disruptive to wave coherence, producing asynchronous arrivals across the leading wave and emphasizing the significant influence of fractal correlation on wave behavior. This behavior has practical implications in geophysical applications, such as seismic imaging, where accurate arrival time measurements are crucial for subsurface mapping [34]. Specifically, understanding how fractal heterogeneity affects arrival behavior could inform strategies to account for scattering-induced delays, improving the precision of seismic models used in oil exploration [35] and fault detection [36].

In terms of energy decay, our findings indicate that higher σ values lead to increased attenuation due to scattering, consistent with Aki and Richards [14], who noted that random media often result in significant energy dissipation, similar to the effect of media porosity [37]. We expand on this by demonstrating a dual dependency on both σ and H : while high σ amplifies scattering and energy decay, higher H values reduce these effects by producing smoother modulus distributions, allowing more coherent wave transmission across the field. This relationship between heterogeneity parameters and energy decay is particularly relevant for rock mechanics and stability analyses, where accurate predictions of wave energy distribution are essential for assessing material integrity and failure risks in heterogeneous rock formations [38].

Our results also suggest potential applications in geological tomography. The sensitivity of wavefield evolution and energy decay to fractal characteristics could improve modeling of wave interactions in complex geological settings, aiding in the reconstruction of detailed subsurface images. Sahimi and Tadjer [4] emphasized that fractal heterogeneity often dictates transport properties in natural materials; our findings build on this concept by showing how heterogeneity parameters (σ and H) jointly influence transport dynamics, especially wave attenuation patterns. This understanding is vital for accurately imaging subsurface structures in regions with heterogeneous geological compositions, such as fractured reservoirs or fault zones [39–41]. Numerical models that incorporate these insights can improve the accuracy of tomographic reconstructions, which rely on the correct interpretation of wave energy distribution and arrival times.

Furthermore, the dependence of wave behavior on fractal parameters in this study underscores the importance of characterizing geological heterogeneity for seismic tomography. The strong influence of σ and H on wavefield distortion and energy decay implies that highly heterogeneous regions with low H values could pose challenges in predicting

seismic wave behavior, leading to amplified scattering and energy attenuation that could impact the accuracy of seismogram recognition. Future studies may benefit from exploring similar fractal heterogeneity effects in 3D models, as well as investigating additional rock mechanical properties, to develop more comprehensive frameworks for seismic analyses in complex media. As the Hurst exponent H is also an index manifesting the correlation of parameter distribution, another area of focus could be the influence of the correlation length of random fields [42] on wave transport.

5. Conclusions

To conclude, this study enhances the understanding of elastic wave propagation in heterogeneous fractal media by numerically modeling the effects of statistically heterogeneous parameters (Hurst exponent H and standard deviation σ) in a 2D fractional Brownian field. Our findings reveal that higher σ values, generating greater variability in Young's modulus, intensify wave scattering and cause the wavefront to become increasingly distorted. This effect is particularly notable when combined with lower H values, which produce rougher modulus distributions that act as barriers to coherent wave transmission. The asynchronous arrival phenomenon becomes more significant with increasing H and σ , as the time difference in leading waves and delayed ones is positively dependent on these parameters. We also found the strong dependence of energy decay on σ and H : higher σ leads to more scattered energy and faster decay, while higher H moderates this effect by providing smoother paths for wave propagation.

These insights contribute to the broader understanding of wave propagation in random, fractal-based media and offer a solid basis for potential implications for seismic analyses and geophysical exploration in heterogeneous geological media. Experimental verification or comparisons with analytical models are a valuable avenue for future study, e.g., laboratory-scale experiments or field-scale observations of wave transport in heterogeneous rocks or geological media. Another future work may extend this model to three-dimensional configuration and explore additional material parameters to capture a wider array of wave transport phenomena in naturally heterogeneous media in geological applications.

Supplementary Materials: The following supporting information can be downloaded at <https://www.mdpi.com/article/10.3390/fractalfract8120750/s1>, Animation S1: Wavefield evolution from $\tilde{t} = 0$ to 3.

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