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Existence and Stability of Neutral Stochastic Impulsive and Delayed Integro-Differential System via Resolvent Operator

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Abstract: In this paper, we present the existence of a mild solution for a class of a neutral stochastic integro-differential system over a Hilbert space. Such systems are influenced by both multiplicative and fractional noise, alongside non-instantaneous impulses, with a Hurst index H in the interval $(\frac{1}{2}, 1)$. Additionally, the systems under consideration feature state-dependent delays (SDDs). To address this, we develop an approach to reformulate the neutral stochastic integro-differential system, incorporating SDDs and non-instantaneous impulses, into an equivalent fixed-point (FP) problem via an appropriate integral operator. By integrating stochastic analysis with the theory of resolvent operators, we employ Banach's FP theorem to establish both the existence and uniqueness of the solution. Furthermore, we explore the Ulam–Hyers–Rassias stability of the system. Lastly, we provide illustrative examples to demonstrate the practical applicability of our results.

Keywords: neutral; existence; uniqueness; stochastic equation; resolvent operator

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1. Introduction

The exploration of SDD differential equations has attracted considerable research attention owing to its practical relevance and the inherent complexity of its qualitative theory, which differs from that of conventional delay differential equations. Consequently, there has been significant interest in investigating various properties associated with this class of equations, as evidenced by the literature [1–5]. Conversely, the utilization of mathematics to model real-world phenomena or human-induced events often results in dynamical systems with inherent randomness. Stochastic differential equations $(\mathcal{SDE}s)$ provide a mathematical framework for describing the time evolution of systems or phenomena that are subject to random influences. In recent years, both finite and infinite-dimensional $\mathcal{SDE}s$ have emerged as focal points of research across diverse fields. These neutral stochastic integro-differential systems have gained significant attention due to their ability to effectively represent diverse phenomena across various scientific and engineering domains. They serve as valuable models for understanding and analyzing population dynamics, physics, electrical engineering, ecology, medicine, biology, and numerous other fields. The versatility of these systems makes them instrumental in studying complex real-world phenomena and addressing practical problems in different disciplines. For a comprehensive overview of $\mathcal{SDE}s$ and their applications, readers are directed to works such as [6–13].

The theory of impulsive differential equations $(\mathcal{IDE}s)$ provides an effective tool for accurately simulating a wide range of real-world scenarios, such as frequency-modulated systems, biological phenomena involving thresholds, and rhythmic bursting models in biology and medicine. In $\mathcal{IDE}s$, impulses are distinguished from continuous evolution processes by abrupt and instantaneous perturbations, which begin abruptly and last for a negligible amount of time relative to the process as a whole. However, instantaneous impulses ($\mathcal{II}$) might not adequately capture the dynamics of evolution in many real-world scenarios. For example, when a patient receives medication that is customized to meet their specific hemodynamic balance, drug absorption happens gradually and continuously. Consequently, such gradual processes are not well represented by $\mathcal{II}$ [14–16].

The influence of drug consumption on memory represents a phenomenon that remains incompletely understood within the framework of the recently proposed impulsive conditions discussed in reference [17]. In these particular cases, the field of fractional calculus ($\mathcal{FC}$) offers a promising approach for gaining deeper insights into the underlying mechanisms [18]. Consequently, there is currently a growing interest among the scientific community in developing robust mathematical models that incorporate historical information inputs. These models aim to provide a more comprehensive understanding of the impact of drug intake on memory function, thereby contributing to advancements in this field of study. In this context, the concept of fractional derivatives ($\mathcal{FD}$) stands out as particularly effective, especially in complex models with hereditary properties. This paper aims to explore the existence outcomes concerning a specific set of impulsive neutral stochastic integro-differential equations with state dependence.

Additionally, stability analysis plays a key role in areas such as $\mathcal{FD}$, numerical analysis, economics, and optimization theory. Various forms of stability are discussed in the literature, including Mittag-Leffler stability, Lyapunov stability, exponential stability, and Hyers–Ulam ($\mathcal{HU}$) stability [19]. Recently, $\mathcal{HU}$ stability has become a topic of growing interest in the analysis of $\mathcal{FDE}s$, with several papers being dedicated to this type of stability [20,21]. We draw inspiration from the work presented in [22] and thoroughly examine the considered model.

$$\begin{cases} d[\Psi(v) - \sum_{i=1}^{n} \Gamma_i(v, \Psi_{\varrho(v,\Psi_v)})] = \mathfrak{A}\Psi(v) + \int_0^v \mathcal{G}(v-u)\Psi(u)du + \mathbf{\Delta}(v, \Psi_{\varrho(v,\Psi_v)}) \\ +\Upsilon(v, \Psi_{\varrho(v,\Psi)}, \int_0^v h(v,u,\Psi_{\varrho(v,\Psi)})du) + \int_0^v \lambda(v,s,\Psi_{\varrho(u,\Psi_)})dB^H(u) \\ + \int_0^v g(v,s,\Psi_{\varrho(u,\Psi)})dB(u), \;\; v \in \bigcup_{i=0}^{n}(u_i, v_{i+1}], \;\; n \in \mathbb{N}, \\ \Psi(v) = \mathfrak{S}_i(v, \Psi_{\varrho(v,\Psi_v)}), \;\; v \in \bigcup_{i=1}^{n}(v_i, u_i], \\ \Psi(v) = \eta(v) \in \mathfrak{B}, \;\; v \in (-\infty, 0], \end{cases} \tag{1}$$

Based on the value of H, the fractional Brownian motion (fBm) process exhibits the following characteristics:

If $H = \frac{1}{2}$, then the process corresponds to a Brownian motion or a Wiener process, which is a special case of fBm. In this case, the increments of the process are uncorrelated and exhibit no long-term dependence.

If $H > \frac{1}{2}$, then the increments of the process are positively correlated. This means that if the process increases (or decreases) over a certain period, it is more likely to continue in the same direction over subsequent periods. This positive correlation indicates a persistent behavior in the process.

If $H < \frac{1}{2}$, then the increments of the process are negatively correlated. This implies that if the process increases (or decreases) over a certain period, it is more likely to reverse its direction and move in the opposite direction over subsequent periods. This negative correlation suggests a mean-reverting behavior in the process.

In summary, depending on the value of H, the fBm process can exhibit characteristics such as uncorrelated increments (Brownian motion), positive correlation (persistent behavior), or negative correlation (mean-reverting behavior).

$\mathcal{SDE}s$ featuring SDD have garnered significant attention in recent years. Numerous researchers have delved into both the qualitative and quantitative aspects of various stochastic systems exhibiting such delays. In the considered framework, the state variable $\Psi(\cdot)$ is defined to be an element of a separable real Hilbert space $(\mathcal{N}, (\cdot,\cdot), |\cdot|)$.

Let us consider the operator $\mathfrak{A} : \mathbf{D}(\mathfrak{A}) \subset \mathcal{N} \to \mathcal{N}$, which is a closed and linear genus, a semigroup denoted as $(T(v))_{t>0}$. Additionally, we have $\mathcal{G}(v)$ as another operator which is closed and linear with its domain $\mathbf{D}(g(v)) \subset \mathbf{D}(\mathfrak{A})$. The history function $\Psi_t : (-\infty, 0] \to \mathcal{N}$ describes the temporal evolution of Ψ from $-\infty$ to the current time t. It is defined as $\Psi_t(\vartheta) = \Psi(v + \vartheta)$ for $\vartheta \in (-\infty, 0]$, and it belongs to the abstract phase space denoted as $\mathcal{B}$. Furthermore, $w(v)$ represents a Brownian motion taking place in Hilbert space (a real separable) $(\mathfrak{K}, (\cdot,\cdot)\mathfrak{K}, |\cdot|\mathfrak{K})$. Here, $n \in \mathbb{N}$ and the sequence $0 = v_0 = u_0 < v_1 < u_1 < v_2 < \cdots < v_n < u_n < v_{n+1} < \ldots$ consists of predetermined numbers. The functions Γ, ϱ, Υ, $\mathbf{\Delta}$, $\mathfrak{S}_i$, λ, and g operate within appropriate domains. Specifically, $\Gamma : [0,\xi] \times \mathcal{B} \to \mathcal{N}$, $\varrho : [0,\xi] \times \mathcal{B} \to (-\infty, \xi]$, $\mathbf{\Delta} : [0,\xi] \times \mathcal{B} \to \mathcal{N}$, $\mathfrak{S}_i : [0,\xi] \times \mathcal{B} \to \mathcal{N}$, $\lambda : \triangle \times \mathcal{B} \to \mathcal{L}_2^0(\mathfrak{K}, \mathcal{N})$, and $g : \triangle \times \mathcal{B} \to \mathcal{L}_2^0(\mathfrak{K}, \mathcal{N})$. Here, $\triangle = \{(v,u) : 0 \le u \le v \le \xi\}$.

In this work, we present the existence of solutions for system (1) using the resolvent operator and fixed-point (FP) methods, which are thoroughly examined throughout the manuscript. The initial result provides sufficient conditions to ensure both the existence and uniqueness of a mild solution for system (1). This is achieved by applying Banach's fixed-point theorem, particularly under the assumption that the nonlinear terms adhere to Lipschitz conditions. Additionally, we investigate the Ulam stability of the system.

2. Preliminaries

We consider the notation $(\mathcal{N}, \langle\cdot,\cdot\rangle_{\mathcal{N}}, \|\cdot\|_{\mathcal{N}})$ and $(\langle\cdot,\cdot\rangle_{\mathfrak{K}}, \|\cdot\|_{\mathfrak{K}})$ to represent two real, separable Hilbert spaces. Additionally, we define the complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\ge0}), \mathbb{P}$, which has a normal filtration $\{\mathfrak{F}_t\}_{t\ge0}$ that satisfies the standard properties: it is increasing, right continuous, and $\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets. Furthermore, $\{n(v) : v \ge 0\}$ represents a $\mathfrak{K}$-valued Wiener process defined on the probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\ge0})$, with an associated covariance operator $\mathcal{W}$.

$$\mathbb{E}(\langle n(v), x\rangle_{\mathfrak{K}} \langle n(u), y\rangle_{\mathfrak{K}}) = (v \wedge u)\langle \mathcal{W}x, y\rangle_{\mathfrak{K}},$$

for all $x, y \in \mathfrak{K}$, where $\mathcal{W}$ is a positive, self-adjoint operator of trace class acting on $\mathfrak{K}$. With regard to $\{\mathfrak{F}_t\}_{t\ge0}$, we specifically designate $n(v)$ as a $\mathfrak{K}$-valued $\mathcal{W}$-Wiener process. In particular, we introduce the subspace $\mathfrak{K}_0 = \mathcal{W}^{\frac{1}{2}}\mathfrak{K}$ of $\mathfrak{K}$ equipped with the inner product to delineate stochastic integrals regarding the $\mathcal{W}$-Wiener process $n(v)$.

$$\langle u, v\rangle_{\mathfrak{K}_0} = \langle \mathcal{W}^{\frac{-1}{2}}u, \mathcal{W}^{\frac{-1}{2}}v\rangle_{\mathfrak{K}}.$$

We assume the existence of a full orthonormal system, e_i, inside $\mathfrak{K}$ in the setting of a Hilbert space. Furthermore, for every $i = 1, 2 \ldots$, there is a bounded sequence of positive real numbers λ_i such that $\mathcal{W}e_i = \lambda_i e_i$. Moreover, we have a sequence of independent standard Brownian motions $n_i(v)_{i>1}$ such that

$$n(v) = \sum_{i=1}^{+\infty} \sqrt{\lambda_i}\; n_i(v)e_i \;\text{ for }\; v \ge 0 \;\text{ and }\; \mathfrak{F}_v = \mathfrak{F}_v^n,$$

where $\{n(u) : 0 \le u \le v\}$ generates the σ-algebra $\mathfrak{F}_v^n$. The set of all Hilbert–Schmidt operators mapping from $\mathfrak{K}_0$ to $\mathcal{N}$ is denoted as $L_2^0 := L_2(\mathfrak{K}_0, \mathcal{N})$. This space has a separable Hilbert structure, and, for any $v \in L_2^0$, it has the norm $\|v\|^2_{L_2^0} = \mathrm{tr}(v\mathcal{W}^{\frac{1}{2}})(v\mathcal{W}^{\frac{1}{2}})$. It is clear that this norm simplifies for any bounded operator $v \in L_2^0$ and can be given as

$$\|v\|^2_{L_2^0} = tr(v\mathcal{W}v^*) = \sum_{i=1}^{\infty} \left\| \sqrt{\lambda_i} v e_i \right\|^2.$$

Denote the Banach space of strongly measurable, square integrable random variables by $L_2(\Omega, \mathfrak{F}, \mathbb{P}; \mathcal{N})$, equipped with a norm:

$$\|v\|_{L_2(\Omega,\mathcal{N})} = (\mathbb{E}\|v\|^2_{\mathcal{N}})^{\frac{1}{2}}.$$

The phase space denoted as $\mathcal{B}$, as delineated by Hale and Kato in [23], is acknowledged to address the challenge posed by infinite delay. The phase space β is required to exhibit the attributes delineated in the lemma presented below.

Lemma 1 ([24]). *The stage of abstraction, denoted as $(\mathcal{B}, ||\cdot||)$, is a seminormed linear space with $\mathfrak{F}_0$ (measurable) functions contained within mapping $(-\infty, 0) \to \mathcal{N}$. It satisfies the following:*

1. *If $m : (-\infty, \xi) \to \mathcal{N}$ is a continuous function over the interval $[0, \xi]$, $m|_{[0,\xi]}$ belongs to the space of continuous functions $C([0, \xi], \mathcal{N})$, and m_0 belongs to $\mathcal{B}$, then the following conditions are satisfied:*

$$m_v \in \mathcal{B}, \quad |m(v)| \leq \mathfrak{K}|m_t|_{\mathcal{B}}$$

$$|m_v|_{\mathcal{B}} \leq N(v)|m_0|\mathcal{B} + N(v), \ \ \textit{where} \ \ \sup\{|m(u)| : 0 \leq u \leq v\}.$$

   In this case, $\mathfrak{K}$ is a positive constant, $N : [0, \infty) \to [0, \infty)$ is a continuous function, and $H : [0, \infty) \to [0, \infty)$ is locally bounded. Notably, the function $m(\cdot)$ is independent of H and N.
2. *$\mathcal{B}$ is a complete phase space.*
3. *On the set $\Re(-) = \{\rho(u, \psi) : (u, \psi) \in [0, \xi] \times \mathcal{B}\}$, extended to $\mathcal{B}$, the mapping $t \mapsto \phi_t$ is well defined. Moreover, the mapping $\mathcal{J}^\phi : \Re(-) \to (0, +\infty)$ is characterized by a continuous and bounded function such that, for every $v \in \Re(-)$, $|\phi_t|\mathcal{B} \leq \mathcal{J}^\phi(v)|\phi|\mathcal{B}$.*

Lemma 2 ([25]). *Consider $v : (-\infty, \xi] \to \mathcal{N}$ as a function satisfying $v_0 = \phi$ and $v|_{[0,\xi]} \in PC([0, \xi], \mathcal{N})$. Then*

$$\|v_t\|_{\mathcal{B}} \leq (\mathcal{N}_\xi + \mathfrak{J}^\phi)\|\phi\|_{\mathcal{B}} + N_\xi \sup_{v\in[0,\xi]} \{\|v(\sigma)\| : \sigma \in [0, \max\{0, v\}]\}, v \in \Re(-) \bigcup [0, \xi]$$

where

$$\mathfrak{J}^\phi = \sup_{v\in[0,\xi]} \mathfrak{J}^\phi(v), \quad \mathcal{H}_\xi = \sup_{v\in[0,\xi]} \mathcal{H}(v), \quad \mathcal{N}_\xi = \sup_{v\in[0,\xi]} \mathcal{N}(v)$$

Lemma 3 ([26]). *For any $q \geq 1$ and for any arbitrary predictable process $z(\cdot)$ taking values in $\mathcal{L}^0_2(K, \mathcal{N})$,*

$$\sup_{s\in[0,v]} \mathbb{E}\left\| \int_0^s z(r)dB(r) \right\|^{2q}_{\mathcal{N}} \leq C_q \left(\int_0^t \|z(r)\|^2_{\mathcal{L}^0_2} dr \right)^q, \ v \in [0, +\infty),$$

where $C_q = (q(2q-1))^q$.

Before delving deeper into the core discoveries of this investigation, let us offer a concise introduction to the concept of the Hausdorff measure of noncompactness from [27].

Definition 1. *Within the normed space Y, consider $\mathcal{W}$ as a bounded subset. The following defines the Hausdorff measure of noncompactness, or MNC for short:*

$$\varpi(\mathcal{W}) = \inf\{\epsilon \geq 0 : \mathcal{W} \ \textit{has a finite cover by balls of a radius less than} \ \epsilon\}.$$

Lemma 4 ([27]). *Let $\mathcal{W}$, $\mathcal{W}_1$, and $\mathcal{W}_2$ be bounded subsets of the normed space Y. Then, the Hausdorff measure of noncompactness satisfies the following properties:*

- $\mathcal{W}_1 \subseteq \mathcal{W}_2$ *then* $\varpi(\mathcal{W}_1) \subseteq \omega(\mathcal{W}_2)$ *(monotonicity);*
- $\varpi(\mathcal{W}) = \varpi(\overline{\mathcal{W}})$*;*
- $\varpi(\mathcal{W}_1 + \mathcal{W}_2) \leq \varpi(\mathcal{W}_1) + \varpi(\mathcal{W}_2)$*, where* $\mathcal{W}_1 + \mathcal{W}_2 = \{x + y : x \in \mathcal{W}_1, y \in \mathcal{W}_2\}$*;*
- $\varpi(\lambda\mathcal{W}) \leq |\lambda|\varpi(\mathcal{W})$ *for any* $\lambda \in \mathfrak{R}$*;*
- $\varpi(a + \mathcal{W}) \leq \varpi(\mathcal{W})$ *for every* $a \in Y$*;*
- $\varpi(\mathcal{W}) = 0$ *if* $\mathcal{W}$ *is relatively compact;*
- *With a Lipschitz constant* α*, we can obtain the following for any Lipschitz continuous mapping* $G : D(G) \subseteq Y \to Y$*:*

$$\varpi(G(\mathcal{W})) \leq \alpha\varpi(\mathcal{W}), \ for \ any \ subset \ \ \mathcal{W} \subseteq D(G).$$

Definition 2. *A mapping* $\mathfrak{K} : \mathcal{Q} \subset \mathcal{E} \to \mathcal{E}$ *is a* ϖ*-contraction if it is continuous and bounded, and, for any bounded and noncompact subset* $E \subset \mathfrak{U}$ *in a Banach space* $\mathcal{E}$*, the image* $\mathfrak{K}(E)$ *satisfies* $\varpi(\mathfrak{K}(E)) \leq \alpha\varpi(E)$ *for a constant* $0 < \alpha < 1$*.*

We also can see similar results in [28].

In order to investigate the existence of mild solutions for Equation (1), it is essential to introduce partial integro-differential equations and resolvent operators, which will serve as fundamental tools in establishing the main outcomes. Let us denote by $\mathfrak{X}$ and $\mathfrak{M}$ two Banach spaces. The space $\mathcal{E} := \mathcal{L}(\mathfrak{X}, \mathfrak{M})$ represents the Banach space from $\mathfrak{X}$ to $\mathfrak{M}$ of bounded linear operators equipped with the operator norm. In the case where $\mathfrak{X} = \mathfrak{M}$, we can use the abbreviated notation $\mathcal{L}(\mathfrak{X})$.

In the following discussion, we will consider a framework where $\mathfrak{X}_1$ is a Banach space. We define $\mathfrak{A}$ and $\mathfrak{B}$ as closed linear operators acting on $\mathfrak{X}_1$. Furthermore, we define $\mathfrak{X}2$ as the Banach space $D(\mathfrak{A})$ equipped with the graph norm $\|z\|_{\mathfrak{X}_2} = \|\mathfrak{A}z\| + \|z\|$ for $z \in \mathfrak{X}_2$. We also define $C(\mathbb{R}^+, \mathfrak{X}_2)$ as the space of continuous functions mapping from $\mathbb{R}^+$ to $\mathfrak{X}_2$. With these definitions in place, we can now proceed to analyze the following system:

$$\begin{cases} \acute{\varphi}(v) = \mathbb{A}\varphi(v) + \int_0^v \mathcal{G}(v-u)\varphi(u)du \ \ for \ \ v \in [0, \xi] \\ \varphi(0) = \varphi_0 \in \mathfrak{X}_1. \end{cases} \tag{2}$$

Definition 3 ([29]). *For system (2), the resolvent operator is a bounded linear function mapping* $\mathcal{R}(v) \in \mathcal{L}(\mathfrak{X}_1), v \geq 0$*, as long as it fulfills the requirements listed below:*

- $\mathcal{R}(0) = Id$ *and* $\|\mathcal{R}(v)\|_{\mathcal{L}(\mathfrak{X}_1)} \leq \mathbf{D}e^{\delta t}$ *for constants* $\mathbf{D}$ *and* δ*.*
- *For each* $x \in \mathfrak{X}_1$*,* $\mathcal{R}(v)x$ *is continuous for* $t \geq 0$*.*
- $\mathcal{R}(v) \in \mathcal{L}(\mathfrak{X}_2)$ *for* $t \geq 0$*. For* $x \in \mathfrak{X}_2$*,* $\mathcal{R}(.)x \in C^1(\mathbb{R}_+, \mathfrak{X}_1) \cap C(\mathbb{R}_+, \mathfrak{X}_2)$ *and*

$$\mathcal{R}^{'}(v)x = \mathbb{A}\mathcal{R}(v)x + \int_0^v \mathcal{G}(v-u)\mathcal{R}(u)xdu = \mathcal{R}(v)\mathbb{A}x + \int_0^v \mathcal{R}(v-u)\mathcal{G}(u)xdu, for \ v \geq 0.$$

The following presumptions are needed in the sequel:
(R_1) For a C_0 semigroup $(u(v))_{v\geq 0}$ on $\mathfrak{X}_1$, $\mathfrak{A}$ is its infinitesimal generator.
(R_2) A closed linear operator from D($\mathfrak{A}$) to $\mathfrak{X}_1$ and $\mathcal{G}(v) \in \mathcal{L}(\mathfrak{X}_2, \mathfrak{X}_1)$ for all $v \geq 0$ is $\mathcal{G}(v)$. The derivative $v \mapsto \mathcal{G}'(v)z$ remains bounded and uniformly continuous over the non-negative real numbers, while the mapping $v \mapsto \mathcal{G}(v)z$ for all $z \in \mathfrak{X}_2$ remains bounded and is differentiable.

Lemma 5 ([30])**.** *Assuming the truth of both statements* (R_1) *and* (R_2)*, we can conclude that* $\mathcal{R}(\cdot)$*, which is the corresponding resolvent operator, attains compactness if, for* $v > 0$*, the* C_0 *semigroup* $S(\cdot)$ *is compact.*

Building on the contributions of Grimmer [29], we propose the concept of mild solutions for the neutral stochastic integro-differential Equation (1) that incorporates $\mathcal{NII}$.

Lemma 6 ([12])**.** *If* $\int_0^\xi ||\zeta(u)||^2 du < \infty$*, then we obtain*

$$\mathbb{E}\left\| \int_0^\xi \zeta(u) dB^H(u) \right\|^2 \leq 2H\xi^{2H-1} \int_0^\xi \|\zeta(u)\|^2 du.$$

Definition 4. *We define a stochastic process* $\Psi : (-\infty, \xi] \to \mathcal{N}$ *as a mild solution to problem (1) if it satisfies the following conditions:*

1. *For any* $v \geq 0$*,* $\Psi(v)$ *is measurable and* $\mathfrak{F}_t$*-adapted.*
2. $\Psi(v)$ *has cdlag paths over* $v \in [0, \xi]$*. The initial condition* $\Psi = \phi_0 \in \mathcal{B}$ *is such that* $\Psi_0 \in L_2^0(\Omega, \mathcal{N})$*,* $v|_{[0,\xi]} \in PC([0, \xi], \mathcal{N})$*, and it adheres to the integral equation associated with the problem.*

$$\Psi(v) = \begin{cases} \eta(v), \;\; v \in (-\infty, 0], \\ \mathcal{R}(v)(\phi(0) - \Gamma(0, \phi(0))) + \sum_{i=0}^m \Gamma_i(v, \Psi_{\varrho(v,\varrho_v)}) + \int_0^v \mathcal{R}(v-u)\Delta(u, \Psi_{\varrho(u,\varrho_u)})du \\ + \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v, \Psi_{\varrho(v,\Psi)}, \int_0^v h(v, \varepsilon, \Psi_{\varrho(v,\Psi)})d\varepsilon) \right] du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u, \omega, \Psi_{\varrho(u,\varrho_u)})dB^H(\omega) \right] du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u, \varsigma, \Psi_{\varrho(u,\varrho_u)})dB(\varsigma) \right] du, \;\; v \in [0, v_1]; \\ \mathfrak{S}_i(v, \Psi_{\varrho(v,\varrho_v)}), \;\; v \in \bigcup_{i=1}^n (v_i, u_i]; \\ \mathcal{R}(v-u_i)\mathfrak{S}_i(u, \Psi_{\varrho(u,\varrho_u)}) + \sum_{i=0}^m \Gamma_i(v, \Psi_{\varrho(v,\varrho_v)}) \\ + \int_{u_i}^t \mathcal{R}(v-u)\Delta(u, \Psi_{\varrho(u,\varrho_u)})du + \int_{u_i}^v \mathcal{R}(v-u)\left[\Upsilon(v, \Psi_{\varrho(v,\Psi)}, \int_0^v h(v, \varepsilon, \Psi_{\varrho(v,\Psi)})d\varepsilon) \right] du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u, \omega, \Psi_{\varrho(u,\varrho_u)})dB^H(\omega) \right] du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v g(u, \varsigma, \Psi_{\varrho(u,\varrho_u)})dB(\varsigma) \right] du, \;\; v \in \bigcup_{i=1}^n (u_i, v_{i+1}]. \end{cases}$$

3. Main Results

Let $\mathcal{B}_\xi$ be the set consisting of all functions $\Psi : (-\infty, \xi] \to \mathcal{N}$ satisfying the following conditions: $\Psi_0 \in \mathcal{B}$ and $\Psi|_{[0,\xi]} \in PC([0, \xi], \mathcal{N})$. Here, $PC([0, \xi], \mathcal{N})$ denotes the set of functions with cdlag paths on the interval $[0, \xi]$ and $\mathcal{N}$ is a given space.

$$\|\Psi\|_\xi = \|\Psi_0\|_{\mathcal{B}} + \sup_{u \in [0,\xi]} (\mathbb{E}(\|\Psi\|^2))^{\frac{1}{2}}, \;\; \Psi \in \mathcal{B}_\xi.$$

Now consider the operator $\Theta : \mathcal{B}_\xi \to \mathcal{B}_\xi$ defined by

$$(\Theta\Psi)(v) = \begin{cases} \eta(v), \ \ v \in (-\infty, 0], \\ \mathcal{R}(v)[\phi(0) - \Gamma(0, \phi(0))] + \sum_{i=0}^{m} \Gamma_i(v, \Psi_{\varrho(v,\varrho_v)}) + \int_0^v \mathcal{R}(v-u)\mathbf{\Delta}(u, \Psi_{\varrho(u,\varrho_u)})du \\ + \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v, \Psi_{\varrho(v,\Psi)}, \int_0^v h(v, \varepsilon, \Psi_{\varrho(v,\Psi)})d\varepsilon)\right]du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u, \omega, \Psi_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u, \varsigma, \Psi_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du, \ \ v \in [0, v_1]; \\ \mathfrak{S}_i(v, \Psi_{\varrho(v,\varrho_v)}), \ \ v \in \bigcup_{i=1}^{n}(v_i, s_i]; \\ \mathcal{R}(v-u_i)\mathfrak{S}_i(u, \Psi_{\varrho(u,\varrho_u)}) + \sum_{i=0}^{m} \Gamma_i(v, \Psi_{\varrho(v,\varrho_v)}) \\ + \int_{s_i}^t \mathcal{R}(v-u)\mathbf{\Delta}(u, \Psi_{\varrho(u,\varrho_u)})du + \int_{u_i}^v \mathcal{R}(v-u)\left[\Upsilon(v, \Psi_{\varrho(v,\Psi)}, \int_0^v h(v, \varepsilon, \Psi_{\varrho(v,\Psi)})d\varepsilon)\right]du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u, \omega, \Psi_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v g(u, \varsigma, \Psi_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du, \ \ v \in \bigcup_{i=1}^{n}(u_i, v_{i+1}]. \end{cases} \tag{3}$$

Let $\overline{\eta}(.) : (-\infty, \xi] \to \mathcal{N}$ be the function defined by

$$\overline{\eta}(v) = \begin{cases} \eta(v) \ \ v \in (-\infty, 0]; \\ \mathcal{R}(v)\eta(0) \ \ v \in [0, \xi]. \end{cases}$$

Therefore, $\overline{\eta_0} = \eta$.

We are able to break down $\Psi(v) = \overline{\eta}(v) + y(v), v \in (-\infty, \xi]$. Clearly, Ψ satisfies Equation (3) if and only if $y_0 = 0$ and

$$(y)(v) = \begin{cases} -\mathcal{R}(v)\Gamma(0,\eta(0)) + \sum_{i=0}^{\infty}\Gamma_i\left(v, \overline{\eta}_{\varrho(v,\overline{\eta}_v+y_v)} + y_{\varrho(v,\overline{\eta}_v+y_v)}\right) \\ + \int_0^v \mathcal{R}(v-u)\mathbf{\Delta}\left(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}\right)du \\ + \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}, \int_0^v h(v,\varepsilon,\overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}d\varepsilon)\right]du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda\left(u,\omega,\overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)} + y_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)}\right)dB^H(\omega)\right]du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v g\left(u,\varsigma,\overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)} + y_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)}\right)dB(\varsigma)\right]du, \;\; v \in [0,v_1]; \\ \mathfrak{S}_i\left(v, \overline{\eta}_{\varrho(v,\overline{\eta}_v+y_v)} + y_{\varrho(v,\overline{\eta}_v+y_v)}\right), \;\; v \in \bigcup_{i=1}^n (v_i,s_i]; \\ \mathcal{R}(v-s_i)\mathfrak{S}_i(u, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}) \\ + \sum_{i=0}^{m}\Gamma_i\left(u_i, \overline{\eta}_{\varrho(u_i,\overline{\eta}_{s_i}+y_{s_i})} + y_{\varrho(u_i,\overline{\eta}_{s_i}+y_{s_i})}\right) \\ + \int_{u_i}^v \mathcal{R}(v-u)\mathbf{\Delta}\left(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}\right)du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\Upsilon(v_i, \overline{\eta}_{\varrho(v,\overline{\eta}_s+y_{u_i})} + y_{\varrho(u_i,\overline{\eta}_s+y_{u_i})}, \int_0^v h(v_i,\varepsilon,\overline{\eta}_{\varrho(u_i,\overline{\eta}_s+y_{u_i})} + y_{\varrho(u_i,\overline{\eta}_s+y_{u_i})}d\varepsilon)\right]du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v \lambda\left(u,\omega,\overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)} + y_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)}\right)dB^H(\omega)\right]du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)} + y_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)})dB(\varsigma)\right]du, \;\; v \in \bigcup_{i=1}^n (u_i,v_{i+1}]. \end{cases} \tag{4}$$

Define $\mathcal{B}_0^\xi = \{y \in \mathcal{B}_\xi : y_0 = 0 \in \mathcal{B}\}$**. For any** $y \in \mathcal{B}_0^\xi$**,**

$$\|y\|_\xi = \|y_0\|_{\mathcal{B}} + \sup_{s\in[0,\xi]} (E(\|y(u)\|^2))^{\frac{1}{2}} = \sup_{s\in[0,\xi]} (E(\|y(u)\|^2))^{\frac{1}{2}}, \;\; y \in \mathcal{B}_\xi.$$

The Banach space property of $(\mathcal{B}_0^\xi, \|\cdot\|_{\mathcal{B}_0^\xi})$ can be easily verified. For every set $r > 0$,

$$\mathcal{B}_r = \{y \in \mathcal{B}_\xi^0 : E||y||^2 \leq r\}.$$

In $\mathcal{B}_b^0$, for each r, $\mathcal{B}_r$ is a bounded, closed, and convex set. Lemma 2 is applied in this context.

$$\begin{aligned} &\left\|\bar{\eta}_{\varrho(v,\bar{\eta}_s+y_u)} + y_{\varrho(v,\bar{\eta}_s+y_u)}\right\| \\ &\leq 2\left(\|\bar{\eta}_s + y_u\|_{\mathcal{B}}^2 + \left\|y_{\varrho(v,\bar{\eta}_s+y_u)}\right\|_{\mathcal{B}}^2\right) \\ &\leq 4\left((\mathcal{N}_\xi + \mathcal{J}^\eta)^2\mathbb{E}\|\bar{\eta}_0\|_{\mathcal{B}}^2 + \mathcal{N}_\xi^2 \sup_{0\leq u\leq\xi} \mathbb{E}\|\bar{\eta}(u)\|^2 \left(\mathcal{N}_\xi + \mathcal{J}^\eta\right)^2\mathbb{E}\|y_0\|_{\mathcal{B}}^2 + \mathcal{N}_\xi^2 \sup \mathbb{E}\|y(u)\|^2\right) \\ &\leq 4\left(\mathcal{N}_\xi^2 \sup_{0\leq u\xi} \mathbb{E}\|y(u)\|^2 + \mathcal{N}_\xi^2\mathcal{M}^2\mathbb{E}\|\eta(0)\|^2 + (\mathcal{N}_\xi + \mathcal{J}^\eta)^2\|\eta\|_{\mathcal{B}}^2\right) \\ &\leq 4\left(\mathcal{N}_\xi^2 r + (\mathcal{N}_\xi^2\mathcal{M}^2\mathcal{K}^2 + \left(\mathcal{N}_\xi + \mathcal{J}^\eta)^2\right)\|\eta\|_{\mathcal{B}}^2\right) = \hat{r}, \;\; v \in [0,\xi]. \end{aligned}$$

Now we can define $\Pi : \mathcal{B}_0^{\xi} \to \mathcal{B}_0^{\xi}$ by

$$(\Pi y)(v) = \begin{cases} \eta(v), \quad v \in (-\infty, 0]; \\ -\mathcal{R}(v)\Gamma(0,\eta(0)) + \sum_{i=0}^{\infty} \Gamma_i\left(v, \overline{\eta}_{\varrho(v,\overline{\eta}_v + y_v)} + y_{\varrho(v,\overline{\eta}_v + y_v)}\right) \\ + \int_0^v \mathcal{R}(v-u)\mathbf{\Delta}\left(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s + y_u)} + y_{\varrho(u,\overline{\eta}_s + y_u)}\right) du \\ + \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s + y_u)} + y_{\varrho(u,\overline{\eta}_s + y_u)}, \int_0^v h(v,\varepsilon,\overline{\eta}_{\varrho(u,\overline{\eta}_s + y_u)} + y_{\varrho(u,\overline{\eta}_s + y_u)} d\varepsilon)\right] du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda\left(u,\omega,\overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega + y_\omega)} + y_{\varrho(\omega,\overline{\eta}_\omega + y_\omega)}\right) dB^H(\omega)\right] du \\ + \int_0^v \mathcal{R}(v-u)\left[\int_0^v g\left(u,\varsigma,\overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma + y_\varsigma)} + y_{\varrho(\varsigma,\overline{\eta}_\varsigma + y_\varsigma)}\right) dB(\varsigma)\right] du, \quad v \in [0, v_1]; \\ \mathfrak{S}_i\left(v, \overline{\eta}_{\varrho(v,\overline{\eta}_t + y_v)} + y_{\varrho(v,\overline{\eta}_t + y_v)}\right), \quad v \in \bigcup_{i=1}^n (v_i, s_i]; \\ \mathcal{R}(v-s_i)\mathfrak{S}_i(u, \overline{\eta}_{\varrho(u,\overline{\eta}_s + y_u)} + y_{\varrho(u,\overline{\eta}_s + y_u)}) \\ + \sum_{i=0}^{m} \Gamma_i\left(u_i, \overline{\eta}_{\varrho(u_i,\overline{\eta}_{s_i} + y_{s_i})} + y_{\varrho(u_i,\overline{\eta}_{s_i} + y_{s_i})}\right) \\ + \int_{u_i}^v \mathcal{R}(v-u)\mathbf{\Delta}\left(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s + y_u)} + y_{\varrho(u,\overline{\eta}_s + y_u)}\right) du \\ \int_{u_i}^v \mathcal{R}(v-u)\left[\Upsilon(v_i, \overline{\eta}_{\varrho(v,\overline{\eta}_s + y_{u_i})} + y_{\varrho(u_i,\overline{\eta}_s + y_{u_i})}, \int_0^v h(v_i,\varepsilon,\overline{\eta}_{\varrho(u_i,\overline{\eta}_s + y_{u_i})} + y_{\varrho(u_i,\overline{\eta}_s + y_{u_i})} d\varepsilon)\right] du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v \lambda\left(u,\omega,\overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega + y_\omega)} + y_{\varrho(\omega,\overline{\eta}_\omega + y_\omega)}\right) dB^H(\omega)\right] du \\ + \int_{u_i}^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma + y_\varsigma)} + y_{\varrho(\varsigma,\overline{\eta}_\varsigma + y_\varsigma)}) dB(\varsigma)\right] du, \quad v \in \bigcup_{i=1}^n (u_i, v_{i+1}]. \end{cases} \tag{5}$$

Here, $\tilde{M} = \sup_{v\in[0,\xi]} ||\mathcal{R}(v)||^2$.

3.1. Existence Under Lipschitz Conditions

The Banach FP theorem, which is applied in the demonstration, is based on the following assumptions:

(H_1) $\Gamma : [0,\xi] \times \mathcal{B} \to \mathcal{N}$, $\mathbf{\Delta} : [0,\xi] \times \mathcal{B} \to \mathcal{N}$, $g : \triangle \times \mathcal{B} \to \mathcal{L}_2^0(K,\mathcal{N})$, and $\lambda : \triangle \times \mathcal{B} \to \mathcal{L}_2^0(K,\mathcal{N})$ are all continuous. Additionally, there exist constants $M_1, M_2, L_\Upsilon, M_\Upsilon, L_h, M_3$, and M_4, all greater than zero, such that

$$\sum_{i=0}^{m} \mathbb{E}\|\Gamma_i(v,v_1) - \Gamma_i(v,v_2)\|_\xi^2 \le M_1\|v_1 - v_2\|_\mathcal{B}^2,$$
$$\mathbb{E}\|\mathbf{\Delta}(v,v_1) - \mathbf{\Delta}(v,v_2)\|_\xi^2 \le M_2\|v_1 - v_2\|_\mathcal{B}^2,$$
$$\mathbb{E}\|\Upsilon(v,v_1,v_2) - \Upsilon(u,u_{1,2})\| \le L_\Upsilon\|v_1 - u_1\| + M_\Upsilon\|v_2 - u_2\|$$
$$\mathbb{E}\|h(v,v_1,v_2) - h(u,u_1,u_2)\| \le L_h\|v_2 - u_2\|$$
$$\mathbb{E}\|g(v,v_1) - g(v,v_2)\|_{\mathcal{L}_0^2}^2 \le M_3\|v_1 - v_2\|_\mathcal{B}^2,$$
$$\mathbb{E}\|\lambda(v,v_1) - \lambda(v,v_2)\|_{\mathcal{L}_0^2}^2 \le M_4\|v_1 - v_2\|_\mathcal{B}^2,$$

for all $v \in [0,\xi]$, and $v_1, v_2 \in \mathcal{B}$.

(H_2) There exists $L_i > 0, i = 1, 2, \ldots, n$ such that the function $\mathfrak{S}_i : (v_i, u_i] \times \mathcal{B} \to \mathcal{N}, i = 1, 2 \ldots, n$ is continuous.

$$\mathbb{E}\|\mathfrak{S}_i(v, v_1) - \mathfrak{S}_i(v, v_2)\|_H^2 \leq L_i\|v_1 - v_2\|_{\mathcal{B}}^2$$

for all $v \in [0, \xi]$ and $v_1, v_2 \in \mathcal{B}$.

Theorem 1. *Given $\Psi_0 \in \mathcal{L}_2^0(\Omega, \mathcal{N})$ and assuming that $(R_1), (R_2), (H_1)$, and (H_2) hold true, a unique mild solution of (1) exists under the following condition:*

$$\mathcal{N}_\xi^2 \sup_{1 \leq i \leq n} 4\Big((1 + \tilde{M})^2 L_i + 2M_1 + \tilde{M}^2(M_2 + (L_\Upsilon + L_h M_\Upsilon \xi) + M_3 2H\xi^{2H-1} + M_4)\xi\Big) < 1. \tag{6}$$

Proof. To establish this result, it suffices to demonstrate the existence of a unique FP for the mapping Π. For any $y, z \in \mathcal{B}_\xi^0$ and $v \in [0; v_1]$, we obtain the following:

$$\begin{aligned}
&\mathbb{E}\|(\Pi y)(v) - (\Pi z)(v)\|_{\mathcal{N}}^2 \\
&\leq 3\mathbb{E}\Bigg\|\sum_{i=0}^{\infty}\Gamma_i\Big(v, \overline{\eta}_{\varrho(v,\overline{\eta}_t+y_v)} + y_{\varrho(v,\overline{\eta}_t+y_v)}\Big) - \sum_{i=0}^{\infty}\Gamma_i\Big(v, \overline{\eta}_{\varrho(v,\overline{\eta}_t+y_v)} + z_{\varrho(z,\overline{\eta}_z+y_z)}\Big)\Bigg\|_{\mathcal{N}}^2 \\
&+ 3\mathbb{E}\Bigg\|\int_0^v \mathcal{R}(v-u)\Big(\mathbf{\Delta}\Big(u, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}\Big) - \mathbf{\Delta}\Big(z, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + z_{\varrho(z,\overline{\eta}_z+y_z)}\Big)\Big)du\Bigg\|_{\mathcal{N}}^2 \\
&+ 3\mathbb{E}\Bigg\|\int_0^v \mathcal{R}(v-u)\Bigg(\Upsilon(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)}, \int_0^v h(v, \varepsilon, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + y_{\varrho(u,\overline{\eta}_s+y_u)})d\varepsilon)\Bigg)du \\
&- \int_0^v \mathcal{R}(v-u)\Bigg(\Upsilon(v, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + z_{\varrho(u,\overline{\eta}_s+z_u)}, \int_0^v h(v, \varepsilon, \overline{\eta}_{\varrho(u,\overline{\eta}_s+y_u)} + z_{\varrho(u,\overline{\eta}_s+z_u)})d\varepsilon)\Bigg)du\Bigg\|_{\mathcal{N}}^2 \\
&+ \Bigg\|\int_0^v \mathcal{R}(v-u)\Bigg(\int_0^s \lambda\Big(u, \omega, \overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)} + y_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)}\Big) - \lambda\Big(u, \omega, \overline{\eta}_{\varrho(\omega,\overline{\eta}_\omega+y_\omega)} + z_{\varrho(\omega,\overline{\eta}_\omega+z_\omega)}\Big)dB^H(\omega)\Bigg)du\Bigg\|_{\mathcal{N}}^2 \\
&+ \Bigg|\int_0^v \mathcal{R}(v-u)\Bigg(\int_0^s g\Big(u, \varsigma, \overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)} + y_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)}\Big) - g\Big(u, \varsigma, \overline{\eta}_{\varrho(\varsigma,\overline{\eta}_\varsigma+y_\varsigma)} + z_{\varrho(\varsigma,\overline{\eta}_\varsigma+z_\varsigma)}\Big)dB(\varsigma)\Bigg)du\Bigg\|_{\mathcal{N}}^2 \\
&\leq 3M_1\Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{N}}^2 + (3\tilde{M}^2 M_2)\int_0^u \Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{N}}^2 du \\
&+ \tilde{M}^2(L_\Upsilon + L_h M_\Upsilon v_1)\Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{N}}^2 du \\
&+ (3\tilde{M}^2 M_3 2Hv_1^{2H-1})\Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{N}}^2 + (3\tilde{M}^2 M_4)\int_0^u \Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{N}}^2 du \\
&\leq 3\Big(M_1 + \tilde{M}^2(M_2 + (L_\Upsilon + L_h M_\Upsilon v_1) + M_3 2Hv_1^{2H-1} + M_4 v_1)\Big) \times \mathcal{N}_\xi^2 \mathbb{E}\|y(v) - z(v)\|_{\mathcal{N}}^2 \\
&\leq 3\mathcal{N}_\xi^2[M_1 + (\tilde{M})^2(M_2 + (L_\Upsilon + L_h M_\Upsilon v_1) + M_3 2Hv_1^{2H-1} + M_4 v_1)]\mathbb{E}\|y(v) - z(v)\|_{\mathcal{N}}^2.
\end{aligned}$$

Considering $v \in \bigcup_{i=1}^n (v_i, s_i]$, we obtain

$$\begin{aligned}
&\mathbb{E}\|(\Pi y)(v) - (\Pi z)(v)\|_{\mathcal{N}}^2 \leq \Big\|\mathfrak{S}_i\Big(v, \overline{\eta}_{\varrho(v,\overline{\eta}_t+y_v)} + y_{\varrho(v,\overline{\eta}_t+y_v)}\Big) - \mathfrak{S}_i\Big(v, \overline{\eta}_{\varrho(v,\overline{\eta}_t+y_v)} + z_{\varrho(v,\overline{\eta}_t+z_v)}\Big)\Big\| \\
&\leq L_i\Big\|y_{\varrho(v,\bar{\eta}+y_v)} - z_{\varrho(v,\bar{\eta}+z_v)}\Big\|_{\mathcal{B}}^2 \\
&\leq L_i \mathcal{N}_\xi^2 \mathbb{E}\|y(v) - z(v)\|_{\mathcal{N}}^2.
\end{aligned}$$

In a similar manner, where $v \in \bigcup_{i=1}^{n}(u_i, v_{i+1}]$, we obtain

$$\mathbb{E}\|(\Pi y)(v) - (\Pi z)(v)\|$$
$$\leq 4\mathbb{N}_T^2\Big((\tilde{M})^2 L_i + 2M_1 + \tilde{M}^2\Big(M_2 + (L_{\Upsilon} + L_h M_{\Upsilon}(v_{i+1} - u_i)) + M_3 2H(v_{i+1} - u_i)^{2H-1}$$
$$+M_4(v_{i+1} - u_i))\Big)\mathbb{E}\|y(v) - z(v)\|_{\mathcal{N}}^2.$$

As a result, we obtain that, for every $v \in [0, \xi]$,

$$\mathbb{E}\|(\Pi y)(v) - (\Pi z)(v)\|$$
$$\leq 4\mathbb{N}_{\xi}^2\Big((1 + \tilde{M})^2 L_i + 2M_1 + \tilde{M}^2(M_2 + (L_{\Upsilon} + L_h M_{\Upsilon}\xi) + M_3 2H\xi^{2H-1} + M_4)\xi\Big)\mathbb{E}\|y(v) - z(v)\|_{\mathcal{N}}^2.$$

Because Π is a contraction mapping, as required by Theorem 1, there is only one FP, $y \in \mathcal{B}_{\xi}^0$. For any $v \in (-\infty, \xi]$, $m(v) = \bar{\eta}(v) + y(v)$. This suggests that only one mild solution to Equation (1) is v. □

3.2. Ulam–Hyers–Rassias Stability

Assume that $\psi \in PC([0, \xi], \Re)$ are non-decreasing. Let $\zeta \geq 0$ and $\varepsilon > 0$. Now, considering the subsequent inequality,

$$\mathbb{E}\Bigg\|d\Bigg(\tilde{\Psi}(v) - \sum_{i=1}^{n}\Gamma_i(v, \tilde{\Psi}_{\varrho(v,\Psi_v)})\Bigg) - \mathfrak{A}\tilde{\Psi}(v) - \int_0^v \mathcal{G}(v-u)\tilde{\Psi}(u)du - \mathbf{\Delta}(v, \tilde{\Psi}_{\varrho(v,\Psi_v)})$$
$$- \Upsilon(v, \tilde{\Psi}_{\varrho(v,\Psi)}, \int_0^v h(v, u, \tilde{\Psi}_{\varrho(v,\Psi)})du) - \int_0^v \lambda(v, s, \tilde{\Psi}_{\varrho(u,\Psi_)})dB^H(u)$$
$$- \int_0^v g(v, s, \tilde{\Psi}_{\varrho(u,\Psi_)})dB(u)\Bigg\|^2 \leq \zeta\psi, \;\; v \in \bigcup_{i=0}^{n}(u_k, v_{k+1}], \;\; n \in \mathbb{N},$$
$$\mathbb{E}\big\|\tilde{\Psi}(s) - \mathfrak{S}_k(\hat{v}, \Psi(\hat{v}))\big\|^2 \leq \varepsilon\zeta, \;\; v \in (v_k, u_k],$$
$$\mathbb{E}|\tilde{\Psi}(0) - \tilde{\Psi}_0|^2 \leq \varepsilon\zeta. \tag{7}$$

Construct a vector space

$$\chi = PC([0, \xi], \Re) \cap C((s_k, v_{k+1}], \Re).$$

Definition 5. *System (1) exhibits $\mathcal{UHR}$s concerning (ψ, ζ) if there exists a positive constant $C_{(\mathcal{M},\mathcal{L},\mathcal{P},\psi)}$ such that for every solution $\tilde{\Psi} \in \chi$ of inequality (7), there exists a function $\Psi \in PC([0, \xi], \Re)$ of system (1) with the property that*

$$\mathbb{E}|\tilde{\Psi}(s) - \Psi(s)||^2 \leq C_{(\mathcal{M},\mathcal{L},\mathfrak{p},\psi)}\varepsilon(\psi(\varepsilon) + \zeta), \;\; v \in [0, \xi].$$

Remark 1. *A function $\tilde{\Psi} \in \chi$ is deemed a solution of (7) if and only if there exists $\mathfrak{Q} \in \bigcap_{i=0}^{\mathcal{K}}((u_k, v_{k+1}], \Re]$ and $q \in \bigcap_{i=0}^{\mathcal{K}}((v_k, u_k], \Re)$ such that the following apply:*

1. $\mathbb{E}|\mathfrak{Q}(s)|^2 \leq \varepsilon\psi(s), v \in \bigcap_{i=0}^{\mathcal{K}}(u_k, v_{k+1}]; \mathbb{E}||q(s)||^2 \leq \varepsilon\zeta, v \in \bigcap_{i=0}^{\mathcal{K}}(u_k, v_k]$;
2. $d[\tilde{\Psi}(v) - \sum_{i=1}^{n}\Gamma_i(v, \tilde{\Psi}_{\varrho(v,\tilde{\Psi}_v)})] = \mathfrak{A}\tilde{\Psi}(v) + \int_0^v \mathcal{G}(v-u)\tilde{\Psi}(u)du + \mathbf{\Delta}(v, \tilde{\Psi}_{\varrho(v,\Psi_v)})$
   $+ \Upsilon(v, \tilde{\Psi}_{\varrho(v,\Psi)}, \int_0^v h(v, u, \tilde{\Psi}_{\varrho(v,\Psi)})du) + \int_0^v \lambda(v, s, \tilde{\Psi}_{\varrho(u,\Psi_)})dB^H(u)$
   $+ \int_0^v g(v, s, \tilde{\Psi}_{\varrho(u,\Psi_)})dB(u) + \mathfrak{Q}(s), \;\; v \in (u_k, v_{k+1}]$;
3. $\tilde{\Psi}(v) = \mathfrak{S}_k(\hat{v}, \tilde{\Psi}(\hat{v}))d\hat{v} + q(s), \;\; v \in (v_k, u_k]$.

By Remark 1, we have

$$\begin{aligned} d[\tilde{\Psi}(v)-\sum_{i=1}^{n}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})] =&\mathfrak{A}\tilde{\Psi}(v)+\int_0^v \mathcal{G}(v-u)\tilde{\Psi}(u)du+\mathbf{\Delta}(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})\\ &+\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,u,\tilde{\Psi}_{\varrho(v,\Psi)})du)+\int_0^v \lambda(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB^H(u)\\ &+\int_0^v g(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB(u)+\mathfrak{Q}(s),\ \ v\in(u_k,v_{k+1}],\\ \tilde{\Psi}(s) =&\mathfrak{S}_k(\hat{v},\tilde{\Psi}(\hat{v}))d\hat{v}+q(s),\ \ v\in(v_k,u_k],\\ I_{0^+}^{1-v}\tilde{\Psi}(v)/_{v=0} =&\tilde{\Psi}_0. \end{aligned}$$

Lemma 7. *If a function $\tilde{\Psi}\in\chi$ is a solution of (7), then the following apply:*

$$\begin{aligned} (i)\ \ \mathbb{E}\Bigg\|&\tilde{\Psi}(v)-\mathcal{R}(v)[\eta(0)-\Gamma(0,\eta(0))]-\sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)})-\int_0^v\mathcal{R}(v-u)\mathbf{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du\\ &-\int_0^v\mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du\\ &-\int_0^v\mathcal{R}(v-u)\left[\int_0^v\lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du\\ &-\int_0^v\mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\Bigg\|^2,\\ &\le\tilde{M}^2v_1\int_0^{v_1}\psi(v)dv,\ \ v\in[0,v_1]. \end{aligned}$$

$$\begin{aligned} (ii)\ \mathbb{E}\Bigg\|&\tilde{\Psi}(v)-\mathcal{R}(v-u_i)\mathfrak{S}_i(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})-\sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)})\\ &-\int_{u_i}^v\mathcal{R}(v-u)\mathbf{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du+\int_{u_i}^v\mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du\\ &-\int_{u_i}^v\mathcal{R}(v-u)\left[\int_0^v\lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du\\ &-\int_{u_i}^v\mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\Bigg\|^2,\\ &\le\tilde{M}^2\xi\int_0^v\psi(v)dv,\ \ v\in\bigcup_{i=1}^{n}(u_i,v_{i+1}].\ \ k=1,2,\dots,m. \end{aligned}$$

According to Remark 1, the following applies:
Case 1: For v within the interval $[0,v_1]$, it holds that

$$\begin{aligned} d[\tilde{\Psi}(v)-\sum_{i=1}^{n}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})] =&\ \mathfrak{A}\tilde{\Psi}(v)+\int_0^v \mathcal{G}(v-u)\tilde{\Psi}(u)du+\mathbf{\Delta}(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})\\ &+\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,u,\tilde{\Psi}_{\varrho(v,\Psi)})du)+\int_0^v \lambda(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB^H(u)\\ &+\int_0^v g(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB(u)+\mathfrak{Q}(s),\ \ v\in(u_k,v_{k+1}]. \end{aligned}$$

Then

$$\begin{aligned}\tilde{\Psi}(v) =&\mathcal{R}(v)[\eta(0)-\Gamma(0,\eta(0))]-\sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)})+\int_0^v \mathcal{R}(v-u)\mathbf{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du\\&+\int_0^v \mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du\\&+\int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du\\&+\int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du.\end{aligned}$$

From the above, we can obtain

$$\begin{aligned}&\mathbb{E}\Bigg\|\tilde{\Psi}(v)-\mathcal{R}(v)[\eta(0)-\Gamma(0,\eta(0))]-\sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)})-\int_0^v \mathcal{R}(v-u)\mathbf{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du\\&-\int_0^v \mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du\\&-\int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du\\&-\int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\Bigg\|^2\\&\le \mathbb{E}||\int_0^v \mathcal{R}(v-u)\mathcal{Q}(t)dt||^2\le \tilde{M}^2 v_1\int_0^v \psi(u)du.\end{aligned}$$

Case 2: For $v\in(v_k,u_k]$, we have

$$\mathbb{E}|\tilde{\Psi}(s)-\mathfrak{S}_k(v,\tilde{\Psi}(v))|^2\le \varepsilon\zeta.$$

Case 3: For $v\in(u_k,v_{k+1}]$, we have

$$\begin{aligned}d[\tilde{\Psi}(v)-\sum_{i=1}^{n}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})]=&\mathfrak{A}\tilde{\Psi}(v)+\int_0^v \mathcal{G}(v-u)\tilde{\Psi}(u)du+\mathbf{\Delta}(v,\tilde{\Psi}_{\varrho(v,\Psi_v)})\\&+\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,u,\tilde{\Psi}_{\varrho(v,\Psi)})du)+\int_0^v \lambda(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB^H(u)\\&+\int_0^v g(v,s,\tilde{\Psi}_{\varrho(u,\Psi_)})dB(u)+\mathcal{Q}(s).\end{aligned}$$

Then

$$\begin{aligned}\tilde{\Psi}(v)=&\mathcal{R}(v)[\eta(0)-\Gamma(0,\eta(0))]-\sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)})+\int_0^v \mathcal{R}(v-u)\mathbf{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du\\&+\int_0^v \mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)},\int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du\\&+\int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du\\&+\int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\end{aligned}$$

From the above, we can obtain

$$\mathbb{E}\left|\left|\tilde{\Psi}(v) - \mathcal{R}(v)[\eta(0) - \Gamma(0,\eta(0))] - \sum_{i=0}^{m} \Gamma_i(v, \tilde{\Psi}_{\varrho(v,\varrho_v)}) - \int_0^v \mathcal{R}(v-u)\mathbf{\Delta}(u, \tilde{\Psi}_{\varrho(u,\varrho_u)})du\right.\right.$$
$$- \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v, \tilde{\Psi}_{\varrho(v,\Psi)}, \int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du$$
$$- \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du$$
$$\left.\left.- \int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\right|\right|^2,$$
$$\leq \mathbb{E}||\int_0^v \mathcal{R}(v-u)\mathcal{Q}(t)dt||^2 \leq \tilde{M}^2\xi\int_0^v \psi(v)dv.$$

To establish stability, we predicate our analysis on the following assumptions:
(H_3) Positive constants M_k exist for $k = 1,2,\ldots,m$ such that

$$\mathbb{E}|\mathfrak{S}_k(s,y(s)) - \mathfrak{S}_k(s,z(s))|^2 \leq \sum_{k=1}^{m} M_k \mathbb{E}|y(s) - z(s)|^2 \forall \ s \ \in (s_k,t_k].$$

(H_4) Let $\psi \in C([0,\xi],\Re)$ be a non-decreasing function. There exists a constant $c_\psi > 0$ such that $\int_0^s \psi(v)\,dv < c_\psi \psi(v)$ for all $v \in [0,\xi]$.

Lemma 8 ([31]). *Consider the set $\Im_0 = \Im \cup \{0\}$, where $\Im = 1,2,\ldots,m$, and the subsequent inequality is satisfied.*

$$\Psi(s) \leq a(s) + \int_0^s b(\varepsilon)y(\eta)d\eta + \Sigma_{0<\eta_k<\eta}\alpha_k y(\eta_{\bar{k}}), \eta \geq 0.$$

Given that $x,a,b \in PC(\Re^+,\Re^+)$, where a is non-decreasing and $b(s) > 0$, and $\alpha_k > 0$ for $k \in \Im$, it follows that, for $s \in \Re^+$,

$$\Psi(s) \leq a(s)(1+\alpha)^k e\left(\int_0^s b(\varepsilon)d\varepsilon\right), s \in (s_k, s_{k+1}], \quad k \in \Im_0,$$

where $\alpha = \sup_{k\in\Im}\{\alpha_k\}$ and $s_0 = 0$.

Theorem 2. *Under the fulfillment of assumptions $(H_1),(H_3)$, and (H_4), system (1) attains $\mathcal{UHR}$s concerning (ψ,ζ).*

Proof. Suppose $\tilde{\Psi} \in \chi$ represents a solution of inequality (7), while φ denotes the unique solution of (1).

Case 1: For v within the interval $[0, v_1]$, it follows that

$$\begin{aligned}&\mathbb{E}\Bigg|\Bigg|\tilde{\Psi}(v) - \mathcal{R}(v)[\eta(0) - \Gamma(0,\eta(0))] - \sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)}) - \int_0^v \mathcal{R}(v-u)\boldsymbol{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du \\ &- \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)}, \int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du \\ &- \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du \\ &- \int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\Bigg|\Bigg|^2 \\ &\leq \mathbb{E}|| \int_0^v \mathcal{R}(v-u)\mathcal{Q}(t)dt||^2 \leq \tilde{M}^2 v_1 \int_0^v \psi(u)du \leq \tilde{M}^2 v_1 c_\psi \psi(\varepsilon) \\ &\leq c_p c_\psi \psi(s),\end{aligned}$$

where $c_p = \tilde{M}^2 v_1$.

Case 2: For $v \in (v_k, u_k]$, we have

$$\mathbb{E}|\tilde{\Psi}(s) - \mathfrak{S}_k(s,\tilde{\Psi}(s))|^2 \leq \varepsilon\zeta.$$

Case 3: For $v \in (u_k, v_{k+1}]$, we have

$$\begin{aligned}&\mathbb{E}\Bigg|\Bigg|\tilde{\Psi}(v) - \mathcal{R}(v)[\eta(0) - \Gamma(0,\eta(0))] - \sum_{i=0}^{m}\Gamma_i(v,\tilde{\Psi}_{\varrho(v,\varrho_v)}) - \int_0^v \mathcal{R}(v-u)\boldsymbol{\Delta}(u,\tilde{\Psi}_{\varrho(u,\varrho_u)})du \\ &- \int_0^v \mathcal{R}(v-u)\left[\Upsilon(v,\tilde{\Psi}_{\varrho(v,\Psi)}, \int_0^v h(v,\varepsilon,\tilde{\Psi}_{\varrho(v,\Psi)})d\varepsilon)\right]du \\ &- \int_0^v \mathcal{R}(v-u)\left[\int_0^v \lambda(u,\omega,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB^H(\omega)\right]du \\ &- \int_0^v \mathcal{R}(v-u)\left[\int_0^v g(u,\varsigma,\tilde{\Psi}_{\varrho(u,\varrho_u)})dB(\varsigma)\right]du\Bigg|\Bigg|^2, \\ &\leq \mathbb{E}|| \int_0^v \mathcal{R}(v-u)\mathcal{Q}(t)dt||^2 \leq \tilde{M}^2\xi \int_0^v \psi(v)dv \leq \tilde{M}^2\xi c_\psi \psi(\varepsilon) \leq c_p c_\psi \psi(\varepsilon).\end{aligned}$$

Hence, for $v \in [0, v_1]$, we have

$$\begin{aligned}&\mathbb{E}||\Psi(v) - \tilde{\Psi}(v)||^2 \\ &\quad\leq 4\left[c_p c_\psi \psi(\varepsilon) + \mathcal{N}_\xi^2\left\{M_1 + \tilde{M}^2(M_2 + (L_\Upsilon + L_h M_\Upsilon v_1) + M_3 2H v_1^{2H-1} + M_4 v_1)\right\}\right] \\ &\quad\times \int_0^v \mathbb{E}||\Psi(v) - \tilde{\Psi}(v)||^2 \\ &\quad\leq 4\left[c_p c_\psi \psi(\varepsilon) + \mathcal{N}_\xi^2\left\{M_1 + \tilde{M}^2(M_2 + (L_\Upsilon + L_h M_\Upsilon \xi) + M_3 2H \xi^{2H-1} + M_4 \xi)\right\}\right] \\ &\quad\times \int_0^v \mathbb{E}||\Psi(v) - \tilde{\Psi}(v)||^2.\end{aligned} \tag{8}$$

For $v \in (v_k, u_k]$, we have

$$\begin{aligned}\mathbb{E}||\Psi(v)-\tilde{\Psi}(s)||^2 =&\mathbb{E}||\Psi(v)-\mathfrak{S}_k(v_k,\tilde{\Psi}(v_k))||^2\\ =&\mathbb{E}||\Psi(v)-\mathfrak{S}_k(u_k,\Psi(u_k))+\mathfrak{S}_k(u_k,\Psi(u_k))-\mathfrak{S}_k(u_k,\tilde{\Psi}(u_k))||^2\\ \leq&2\{\mathbb{E}||\Psi(v)-\mathfrak{S}_k(u_k,\Psi(u_k))||^2+E||\mathfrak{S}_k(u_k,\Psi(u_k))-\mathfrak{S}_k(u_k,\tilde{\Psi}(u_k))||^2\}\\ \leq&2\{\varepsilon\zeta+\Sigma_{i=0}^{k}M_k||\Psi(v)-\tilde{\Psi}(v)||^2\}. \end{aligned} \tag{9}$$

For $v\in(u_k,v_{k+1}]$, k=1,2,...,m,

$$\leq 4\left[c_pc_\psi\psi(\varepsilon)+\mathcal{N}_\xi^2\left\{M_1+\tilde{M}^2(M_2+(L_\Upsilon+L_hM_\Upsilon\xi)+M_32H\xi^{2H-1}+M_4\xi)\right\}\right]$$
$$\times\int_{u_i}^{v}\mathbb{E}||\Psi(v)-\tilde{\Psi}(v)||^2+4\xi^{\nu-1}\sum_{i=1}^{k}M_k||\Psi(v)-\tilde{\Psi}(v)||^2\}. \tag{10}$$

By consolidating Equations (8) through (10), we can formulate an inequality resembling that described in Lemma 8. This inequality applies to $s\in\mathcal{W}$, as v falls within the interval $(v_k,s_{k+1}]$, with $k\in P_0$.

$$\begin{aligned}\mathbb{E}|\Psi(v)-\tilde{\Psi}(v)|^2\leq 4\Bigg[&c_pc_\psi\psi(\varepsilon)+\xi^{\nu-1}\sum_{i=1}^{k}M_k||\Psi(v)-\tilde{\Psi}(v)||^2\}\\ &c_p\mathcal{N}_\xi^2\left\{M_1+\tilde{M}^2(M_2+(L_\Upsilon+L_hM_\Upsilon\xi)+M_32H\xi^{2H-1}+M_4\xi)\right\}\Bigg]+2\varepsilon\phi\\ &\times\int_{u_i}^{v}\mathbb{E}||\Psi(v)-\tilde{\Psi}(v)||^2\\ &\leq 4c_p\varepsilon c_\psi(\psi(s)+\zeta)(1+M)^k e^{\int_0^s W ds},\end{aligned}$$

where $M=\sup\{\xi^{\nu-1}M_k\}, W=c_p\mathcal{N}_\xi^2\left\{M_1+\tilde{M}^2(M_2+(L_\Upsilon+L_hM_\Upsilon\xi)+M_32H\xi^{2H-1}+M_4\xi)\right\}$.

Then

$$\mathbb{E}||\Psi(v)-\tilde{\Psi}(v)||^2\leq 5C_{(\mathcal{M},\mathcal{L},\mathfrak{p},\psi)}\varepsilon(\psi(\varepsilon)+\zeta),\forall v\in[0,\xi].$$

In this instance, $C_{(\mathcal{M},\mathcal{L},\mathfrak{p},\psi)}$, depending on $\mathcal{M},\mathcal{L},\mathfrak{p},\psi$, is a constant. Thus, (1) is $\mathcal{UHR}s$ with reference to (ψ,ζ). □

4. Example

Let us now examine the following instance:

$$\begin{cases} \frac{\partial}{\partial t}\left[y(\xi,\eta) - \sum_{i=0}^{2}\mathfrak{P}_i(\xi, m-\xi, x, y(m, \varrho(||y(\xi)||,\eta)))du\right] = \frac{\partial^2}{\partial \eta^2}y(\xi,\eta) + \int_0^{\xi} a(\xi - m)\frac{\partial^2}{\partial \eta^2}y(\xi,\eta) \\ + \int_{-\infty}^{\xi}\mathfrak{P}_2(\xi, m-\xi, x, y(m, \varrho(||y(\xi)||, \int_0^{\xi} h(\xi y(m, \varrho(||y(\xi)||))d\xi))dm \\ + \int_{-\infty}^{\xi}\mathfrak{P}_3(\xi, m-\xi, x, y(m, \varrho(||y(\xi)||,\eta)))dm \\ + \int_0^{\xi}\int_{-\infty}^{m}\mathfrak{P}_4(m, z-m, x, y(z, \varrho(||y(\xi)||,\eta)))dB^H(z)dm \\ + \int_0^{\xi}\int_{-\infty}^{m}\mathfrak{P}_5(m, z-m, x, y(z, \varrho(||y(\xi)||,\eta)))dB(z)dm, \ \ (\xi,\eta) \in \bigcup_{i=0}^{n}(m_i, \xi_{i+1}] \times [0,\pi]; \\ y(\xi,\eta) = \int_{-\infty}^{\xi}\mathfrak{P}_5(\xi, m-\xi, x, y(m, \varrho(||y(\xi)||,\eta)))dm, \ \ (\xi,\eta) \in \bigcup_{i=0}^{n}(\xi_i, m_i] \times [0,\pi]; \\ y(\xi,0) = y(\xi,\pi) = 0, \ \ \xi \le 0 \\ y(\xi,\eta) = \eta(\xi,\eta), \xi \in (-\infty, 0], \end{cases} \tag{11}$$

where the fixed real numbers $\eta \in \mathcal{B}, 0 = \xi_0 = m_0 < \xi_1 < m_1 < \xi_2 < \dots \xi_n < m_n < \xi_{n+1} = 1$. A continuous function, $a : \Re^+ \to \Re^+$, is ϱ. Furthermore, the functions $\mathfrak{P}_i : \Re^4 \to \Re$, $i = 1,2,3,4$, are taken into consideration as continuous functions. Additionally, there is a continuous function $\varepsilon_i, \kappa_i : \Re \to \Re$, $i = 1,2,3,4$.
Determine the value of $\eta(\tau)(\eta) = \eta(\tau,\eta) \in \mathcal{B}$.

$$\begin{aligned} \Psi(\xi)(\eta) &= y(\xi,\eta), \\ \varrho(\xi,\eta)(\eta) &= \varrho(||\eta(0)||), \\ \sum_{i=0}^{2}\Gamma_i(\xi,v)(\eta) &= \sum_{i=0}^{2}\mathfrak{P}_1(\xi,m,x,v(m)(\eta))dm, \\ \Upsilon(\xi,v,w)(\eta) &= \int_{-\infty}^{0}\mathfrak{P}_2(\xi,m,x,v(m),\int_0^u(w,x,m,v(m))du(\eta))dm, \\ \mathbf{\Delta}(\xi,v)(\eta) &= \int_{-\infty}^{0}\mathfrak{P}_3(\xi,m,x,v(m)(\eta))dm, \\ \int_0^{\xi}\lambda(\xi,m,v)(\eta) &= \int_0^{\xi}\int_{-\infty}^{0}\mathfrak{P}_4(\xi,m,x,v(m)(\eta))dB^H(m)dm, \\ \int_0^{\xi}g(\xi,m,v)(\eta) &= \int_0^{\xi}\int_{-\infty}^{0}\mathfrak{P}_5(\xi,m,x,v(m)(\eta))dB(m)dm, \end{aligned}$$

Thus, it is necessary that we verify the nonlinear function hypotheses here.

$$\begin{aligned} \sum_{i=0}^{m}\mathbb{E}||\Gamma_i(v,v_1) - \Gamma_i(v,v_2)||_{\xi}^2 &\le \frac{1}{60}||v_1 - v_2||_{\mathcal{B}}^2, \\ \mathbb{E}||\mathbf{\Delta}(v,v_1) - \mathbf{\Delta}(v,v_2)||_{\xi}^2 &\le \frac{1}{10}||v_1 - v_2||_{\mathcal{B}}^2, \\ \mathbb{E}||\Upsilon(v,v_1,v_2) - \Upsilon(u,u_1,u_2)|| &\le \frac{1}{10e}||v_1 - u_1|| + \frac{1}{5}||v_2 - u_2|| \\ \mathbb{E}||h(v,v_1,v_2) - h(u,u_1,u_2)|| &\le \frac{1}{e^{10}}||v_2 - u_2|| \\ \mathbb{E}||g(v,v_1) - g(v,v_2)||_{\mathcal{L}_0^2}^2 &\le \frac{1}{e^2}||v_1 - v_2||_{\mathcal{B}}^2, \\ \mathbb{E}||\lambda(v,v_1) - \lambda(v,v_2)||_{\mathcal{L}_0^2}^2 &\le \frac{1}{5}||v_1 - v_2||_{\mathcal{B}}^2, \\ \mathbb{E}||\mathfrak{S}_i(v,v_1) - \mathfrak{S}_i(v,v_2)||_H^2 &\le \frac{1}{15}||v_1 - v_2||_{\mathcal{B}}^2. \end{aligned}$$

Here, $M_1 = 0.016$, $M_2 = 0.2$, $L_\Upsilon = 0.271$, $M_\Upsilon = 0.2$, $L_h = 0.000045$, $M_3 = 0.367$, $M_4 = 0.2$, $L_i = 0.067$, and $\tilde{M}^2 = 0.001$. Hence, from inequality (7), we obtain 0.250 < 1, and also

$$\mathbb{E}||\Psi(v) - \tilde{\Psi}(v)||^2 \leq 5C_{(\mathcal{M},\mathcal{L},\mathfrak{p},\psi)}\varepsilon(\psi(\varepsilon) + \zeta), \forall\ v \in [0,\xi], \leq 0.583.$$

All the conditions outlined in Theorem 1 are fulfilled. Therefore, Equation (11) has a unique solution and meets the requirements of Theorem 2 as well. Consequently, Equation (11) is $\mathcal{UHR}$ stable.

5. Conclusions

We have achieved diverse forms of existence outcomes for a category of stochastic integro-differential systems with neutral characteristics. These systems encompass SDDs and $\mathcal{NII}$ occurring within Hilbert spaces. Our approach to obtaining these results involves the application of functional analysis and stochastic analysis methodologies. Moving forward, our research aims to explore the concept of controllability within this framework. By doing so, we anticipate the ability to extend these processes to address more intricate scenarios, hence improving the system's performance in general. Additionally, we plan to integrate numerical techniques that incorporate various criteria to analyze and evaluate our findings. In summary, our future endeavors involve expanding upon our findings, investigating controllability, and incorporating numerical methodologies to provide a comprehensive treatment of the system's behavior.

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