



Article

On the Large- x Asymptotic of the Classical Solutions to the Non-Linear Benjamin Equation in Fractional Sobolev Spaces

Nabendra Parumasur * and Olabisi Aluko

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa; olabisialuko9.0a@gmail.com

* Correspondence: parumasurn1@ukzn.ac.za

Abstract: In this work, we study the large- x asymptotic of classical solutions to the non-linear Benjamin equation modeling propagation of small amplitude internal waves in a two fluid system. In our analysis, we extend known H^s -well-posedness results to the case of the variable-weight Sobolev spaces. The spaces provide a direct control over the asymptotics of classical solutions and their weak derivatives, and permit us to compute the bulk large- x asymptotic of classical solutions explicitly in terms of input data. The asymptotic formula provides a precise description of the qualitative behaviour of classical solutions in weighted spaces and yields a number of weighted persistence and continuation results automatically.

Keywords: non-linear Benjamin equation; fractional Sobolev spaces; large- x asymptotic; fourier analysis; well-posedness

1. Introduction

Studies of the uniqueness, existence, and qualitative properties of solutions to dispersive partial differential equations constitute a substantial body of modern research. In this paper, we concentrate on analysing a specific problem: the nonlinear Benjamin equation

$$u_t + \alpha u_x - \beta \mathcal{H}[u_{xx}] - \gamma u_{xxx} + \delta(u^2)_x = 0, \quad u(0) = u_0, \quad x \in \mathbb{R} = (-\infty, \infty), \quad (1)$$

where $\alpha, \beta, \gamma, \delta$ are real parameters and

$$\mathcal{H}[u](x) = \frac{\text{p.v.}}{\pi} \int_{\mathbb{R}} u(y) \frac{dy}{y-x},$$

is the standard Hilbert transform. The quasilinear PDE (1) was originally introduced by T.B. Benjamin in his studies of the propagation of internal waves in a two-fluid system [1,2]. We note that setting $\beta = 0$ in (1) provides the classical Korteweg–de Vries equation (KdV), while letting $\gamma = 0$ recovers the well-known Benjamin–Ono (BO) equation as a special case.

The analysis of (1), particularly the study of its low-regularity solutions, has garnered significant attention in the specialized literature. A variety of methods have been employed, ranging from bilinear estimates to the I -method (see for example, [3–6]). Historically, the first low-regularity results were obtained in [6], within the $L^2(\mathbb{R})$ settings. These results were subsequently extended to the $H^s(\mathbb{R})$, $-\frac{3}{4} < s \leq 0$ settings in [4]. Our work extends these findings by exploring well-posedness in the newly introduced $H_r^s(\mathbb{R})$ spaces, offering enhanced control over asymptotic behavior. The well-posedness analysis in [4,6] is local in time.

The first global well-posedness results in $H^s(\mathbb{R})$, for $-\frac{3}{4} < s < 0$, were obtained in [5]. The local analysis follows a similar approach to that in [4,6], but employs a refined bilinear estimate for a frequency cut-off operator I applied to the quadratic nonlinearity (uv) . In addition to demonstrating local solvability, this technique provides explicit estimates for



Citation: Parumasur, N.; Aluko, O.

On the Large- x Asymptotic of the Classical Solutions to the Non-Linear Benjamin Equation in Fractional Sobolev Spaces. *Fractal Fract.* **2024**, *8*, 635. <https://doi.org/10.3390/fractalfract8110635>

Academic Editors: Wei-Shih Du, Marko Kostić and Daniel Velinov

Received: 4 July 2024

Revised: 20 August 2024

Accepted: 25 October 2024

Published: 28 October 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

the growth rate of mild solutions and their intervals of existence. When combined with a continuation method, this approach establishes global well-posedness.

Furthermore, studies on variable dispersion, such as those examining special cases of the KdV and BO equations (see, for example, [7–11] and references therein), have shown how dispersion affects wave propagation and stability. Our research endeavours to consider the existing understanding of the impact of variable dispersion on the well-posedness and asymptotic properties of solutions to the nonlinear Benjamin equation within the framework of weighted Sobolev spaces. The study of low dispersion properties of a fractional variant of the Benjamin–Bona–Mahony (BBM) equation within $\mathcal{Z}_{s,r}$ -spaces for $\mathcal{Z}_{s,r}(\mathbb{R}) := H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$ has been carried out in [12], highlighting the importance of these spaces in controlling the asymptotic behaviour of solutions.

The local and global analyses in [4–6] do not extend to the critical case of $H^{-\frac{3}{4}}(\mathbb{R})$. The main obstacle is that the bilinear estimate used in [4,6], as well as its I analogue from [5], fails for $s \leq -\frac{3}{4}$. The critical case was recently addressed in [3], where the main tool used was the Besov–Bourgain spaces.

A distinctive feature of Equation (1) is its non-locality, primarily due to the presence of the global in space Hilbert transform $\mathcal{H}$. This non-locality presents significant challenges in predicting the long-term behaviour of internal waves in fluid dynamics. Furthermore, the Fourier symbol of the group generated by the linear part of (1) has finite regularity at the origin, complicating the analysis of wave propagation. By addressing these challenges, our study contributes to the accurate modeling of wave propagation, which is critical in both natural and engineered systems. As a consequence of the non-locality, one cannot expect super-algebraic spatial decay of solutions, even for the rapidly decreasing initial data u_0 from the Schwartz class $\mathcal{S}$. In context of the closely related Benjamin–Ono equation, this situation and its implications were thoroughly investigated in [13–17].

A systematic study of the interplay between asymptotic behavior, regularity, and the existence of global solutions to the Benjamin equation (1) was initiated in [18]. In particular, building on the ideas from [13–17], it is demonstrated in [18] that for initial data $u_0 \in \mathcal{Z}_{s,r} := H^s(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2r}dx)$, with $s > \frac{3}{2}$, $r \in (0, \frac{5}{2})$ and $r < \frac{5}{2}$, the Equation (1) is globally well-posed in $\mathcal{Z}_{s,r}$. The result indicates that, due to the jump discontinuity in the Fourier symbol of the operator $\mathcal{H}$, one cannot expect solutions to decay algebraically at infinity faster than x^{-3} . The situation improves slightly in the scale of weighted homogeneous Sobolev spaces $\dot{\mathcal{Z}}_{s,r} := \{u \in \mathcal{Z}_{s,r} \mid \hat{u}(0) = 0\}$, where the propagation of the discontinuity in the Fourier image of the solution can be properly controlled for $s \geq 2r$, $r \in [\frac{5}{2}, \frac{7}{2})$.

In this paper, we continue the investigation of the nonlinear and non-local dispersive model (1) in weighted Sobolev-like spaces, building on the work initiated in [18]. In contrast with the previously cited works, our analysis relies on variable-weight Sobolev spaces $H_r^s(\mathbb{R})$ (For the definition of the scale $H_r^s(\mathbb{R})$, see [19] and Section 2 below), which were introduced recently in [19]. This approach enables a more precise description of how solutions behave at large distances, which is crucial for understanding wave propagation in non-local dispersive systems, thereby revealing the intricate mechanisms underlying their behaviour. In the context of the Benjamin equation (and in similar dispersive models, such as the Benjamin–Ono equation), these spaces arise naturally. Unlike the weighted spaces $\mathcal{Z}_{s,r}$ and $\dot{\mathcal{Z}}_{s,r}$ from [13–18], they provide direct control over the large- x asymptotic behavior of classical solutions and their weak derivatives up to and including order s . This idea allows for a more refined analysis of the large- x asymptotic behavior of solutions.

Thanks to this property, we can demonstrate that for sufficiently regular input data, the jump discontinuities in the frequency space—caused by the finite regularity of the Fourier symbol associated with the linear part of (1)—cannot propagate beyond the origin. As a direct consequence, it follows that the large- x asymptotic behavior of classical solutions $u(t)$ is determined entirely by the behaviour of their Fourier images $\hat{u}(t)$ near the origin. Consequently, this asymptotic behavior can be computed explicitly in terms of quantities associated with the input data. Once the asymptotic identity is established, a range of qualitative results, similar to those in [18] for the Benjamin equation and [13–17]

for the Benjamin–Ono equation, automatically follows. Although our analysis focuses solely on the Benjamin equation (1), the approach is sufficiently general to be adapted for analyzing classical solutions to non-local quasi-linear dispersive models of Benjamin-type, where linear wave propagation operators exhibit finitely many jump discontinuities in the frequency domain.

The paper is organized as follows: Section 2 provides a brief review of the basic properties of the scale $H_r^s(\mathbb{R})$. In Section 3, we derive several technical estimates related to both the linear and nonlinear parts of (1). Section 4 discusses the model's well-posedness in weighted settings and its large- x asymptotic behavior. Section 6 concludes the paper.

Notation

- (i) x is the physical variable
- (ii) ξ is the frequency variable
- (iii) $*$ denotes Fourier convolution
- (iv) $\mathcal{F}[u](\xi)$ is the Fourier transform
- (v) $\mathcal{F}^{-1}[\hat{u}](x)$ is the inverse Fourier transform
- (vi) $L_w^p(\Omega)$ represent the weighted Lebesgue spaces.
- (vii) $\text{Re } \mathcal{S}'$ denotes the space of real valued tempered distributions
- (viii) c is a generic constant.

2. Function-Theoretic Framework

To begin, we define the normalized Fourier transform and its inverse

$$\begin{aligned}\mathcal{F}[u](\xi) &= \hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x) dx, \\ \mathcal{F}^{-1}[\hat{u}](x) &= u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{u}(\xi) d\xi,\end{aligned}$$

respectively.

In the forthcoming analysis, we employ the scale of the variable-weighted Sobolev spaces, introduced recently in [19]. For real valued functions, these spaces are defined as

$$H_r^s(\mathbb{R}) = \{u \in \text{Re } \mathcal{S}' \mid \|u\|_{H_r^s(\mathbb{R})} < \infty\}, \quad s > -\frac{1}{2}, \quad r \geq 0, \quad (2)$$

$$\langle u, v \rangle_{H_r^s(\mathbb{R})} = \frac{1}{2} \langle u, v \rangle_{L^2(\mathbb{R})} + \text{Re} \langle \mathcal{P}_\pm [\kappa_{r,k}^\pm u], \mathcal{P}_\pm [\kappa_{r,k}^\pm v] \rangle_{\dot{H}^s(\mathbb{R})}, \quad (3)$$

$$\|u\|_{H_r^s(\mathbb{R})}^2 = \langle u, u \rangle_{H_r^s(\mathbb{R})}, \quad (4)$$

where $\text{Re } \mathcal{S}'$ is the space of real valued tempered distributions, $\mathcal{P}_\pm = \frac{1}{2}(\mathcal{I} \pm i\mathcal{H})$ are Fourier multipliers (projectors) associated with the Heaviside functions $\hat{h}^\pm(\xi) = \frac{1 \pm \text{sgn}(\xi)}{2}$, $\kappa_{r,\ell}^\pm(x) = \frac{1}{\sqrt{2\pi}}(i\ell \mp x)^r$, and $\dot{H}^s(\mathbb{R})$ is the standard homogeneous Sobolev space of order s (see e.g., [20]). Note that the Fourier images of real valued distributions are Hermitian and, hence, the concrete choice of the sign (+ or −) in (3) is irrelevant and the symmetric, positive definite bilinear form $\langle \cdot, \cdot \rangle_{H_r^s(\mathbb{R})}$ defines a real Hilbert structure in $H_r^s(\mathbb{R})$. An interpolation argument, employed in Section 3, requires a complex version of $H_r^s(\mathbb{R})$. In a usual way, $H_r^s(\mathbb{R})$ is complexified by letting

$$\begin{aligned}\langle u, v \rangle_{H_r^s(\mathbb{R})} &= \langle \text{Re } u, \text{Re } v \rangle_{H_r^s(\mathbb{R})} + \langle \text{Im } u, \text{Im } v \rangle_{H_r^s(\mathbb{R})} \\ &\quad + i \langle \text{Im } u, \text{Re } v \rangle_{H_r^s(\mathbb{R})} - i \langle \text{Re } u, \text{Im } v \rangle_{H_r^s(\mathbb{R})}.\end{aligned} \quad (5)$$

Directly from (2)–(5), it follows that for $s \geq 0$, $H_0^s(\mathbb{R})$, and $H^s(\mathbb{R})$ are equal as Banach spaces. However, when $s, r > 0$, the former class carries information about the large x asymptotic of all weak derivatives up to and including order s , and, hence, is more suitable for the analysis of the fine interplay between regularity and the large x behaviour of solutions of (1).

Basic properties of $H_r^s(\mathbb{R})$ are established in [19]. In particular, we have

$$H_{r_0}^{s_0}(\mathbb{R}) \hookrightarrow H_{r_1}^{s_1}(\mathbb{R}), \quad -\frac{1}{2} < s_1 \leq s_0 \leq s_1 + r_0 - r_1, \quad 0 \leq r_1 \leq r_0, \quad (6)$$

$$[H_{r_0}^{s_0}(\mathbb{R}), H_{r_1}^{s_1}(\mathbb{R})]_\theta = H_{r_0(1-\theta)+r_1\theta}^{s_0(1-\theta)+s_1\theta}(\mathbb{R}), \quad s_0, s_1 > -\frac{1}{2}, \quad r_0, r_1 \geq 0, \quad \theta \in (0, 1), \quad (7)$$

$$\|fg\|_{H_r^s(\mathbb{R})} \leq c\|f\|_{H_r^s(\mathbb{R})}\|g\|_{H_r^s(\mathbb{R})}, \quad s > \frac{1}{2}, \quad r \geq 0, \quad (8)$$

where $[\cdot, \cdot]_\theta$ is the standard complex interpolation functor (see [20]). We note that the embeddings (6) are dense, as the Schwartz class $\mathcal{S}$ of smooth rapidly decreasing test functions is dense in each of $H_r^s(\mathbb{R})$, $s > -\frac{1}{2}$, and $r \geq 0$.

In addition to (6)–(8), we need the following technical

Lemma 1. *The embedding*

$$H_{r_0}^{s_0}(\mathbb{R}) \hookrightarrow H^{s_1}(\mathbb{R}), \quad 0 \leq s_1 < s_0 < r_0, \quad (9)$$

is dense and compact.

Proof. As $\|\cdot\|_{H_r^s(\mathbb{R})}^2 = \|\operatorname{Re} \cdot\|_{H_r^s(\mathbb{R})}^2 + \|\operatorname{Im} \cdot\|_{H_r^s(\mathbb{R})}^2$, it is sufficient to verify the real valued version of (9) only (for details see [21]).

(a) The denseness and continuity of (9) follows from (6). To show compactness, consider $u \in H_{r_0}^{s_0}(\mathbb{R})$, with $\|u\|_{H_{r_0}^{s_0}(\mathbb{R})} \leq 1$. In view of the assumption $s_1 < s_0$, we have

$$\begin{aligned} \int_{|\zeta| > |\xi|} (1 + |\zeta|^2)^{s_1} |\hat{u}(\zeta)|^2 d\zeta &\leq \sup_{|\zeta| > |\xi|} (1 + |\zeta|^2)^{s_1 - s_0} \|u\|_{H^{s_0}(\mathbb{R})}^2 \\ &\leq c(1 + |\xi|^2)^{s_1 - s_0}, \quad \xi \in \mathbb{R}, \end{aligned}$$

uniformly for $\|u\|_{H_{r_0}^{s_0}(\mathbb{R})} \leq 1$.

(b) As u is assumed to be real valued, its Fourier image $\hat{u}$ is Hermitian (i.e., $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $\xi \in \mathbb{R}$). Therefore, for $0 \leq h \leq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} |\hat{u}(\xi + h) - \hat{u}(\xi)|^2 d\xi &\leq 2c \int_0^h (1 + |\xi|^2)^{s_1} |\hat{u}(\xi)|^2 d\xi \\ &\quad + c \int_{\mathbb{R}_+} (1 + |\xi|^2)^{s_1} |\hat{u}(\xi + h) - \hat{u}(\xi)|^2 d\xi \\ &=: c(2I_0 + I_1), \end{aligned}$$

with $c = (1 + \sup_{\xi \in \mathbb{R}_+} \frac{(1 + |\xi + 1|^2)^{s_1}}{(1 + |\xi|^2)^{s_1}}) < \infty$. To bound I_0 , as in [19], we let

$$\mathcal{J}_{\mp}^{-r}[\cdot](\xi) := \sqrt{2\pi}(\hat{\kappa}_{-r,\ell}^{\pm} * \cdot)(\xi), \quad \xi \in \mathbb{R}_{\pm}.$$

By the definition of $\|\cdot\|_{H_{r_0}^{s_0}(\mathbb{R})}$, for $\xi \in \mathbb{R}_+$, we have $\hat{u} = \mathcal{J}_{-}^{-r_0}[\hat{v}_{r_0}]$, with $\hat{v}_{r_0} \in L_{s_0}^2(\mathbb{R}_+)$. This identity, combined with the explicit formula

$$\hat{\kappa}_{-r,\ell}^{\pm}(\xi) = \frac{e^{-i\pi r/2} \sqrt{2\pi}}{\Gamma(r)} (\mp \xi)^{r-1} e^{\pm \ell \xi} \chi_{\mathbb{R}_{\mp}}(\xi), \quad \xi \in \mathbb{R},$$

the assumption $s_0 < r_0$ and the Cauchy–Schwarz inequality, indicates that

$$|\hat{u}(\xi)| \leq \left(\frac{2\pi \Gamma(2(r_0 - s_0))}{\ell^{2(r_0 - s_0)} \Gamma(r_0)} \right)^{\frac{1}{2}} \|\hat{v}_{r_0}\|_{L_{s_0}^2(\mathbb{R}_+)} \leq c\|u\|_{H_{r_0}^{s_0}(\mathbb{R})},$$

when $\xi \in \mathbb{R} \setminus \{0\}$. Hence, $I_0 \leq ch$, uniformly for $\|u\|_{H_{r_0}^{s_0}(\mathbb{R})} \leq 1$.

To estimate I_1 , we let $\hat{v}_{r'} = \mathcal{J}_-^{r'-r_0}[\hat{v}_{r_0}]$, $0 \leq r' \leq r_0$, $\xi \in \mathbb{R}_+$. The analysis presented in [22] indicates that

$$\hat{u} = \mathcal{J}_-^{r'}[\hat{v}_{r'}] \in L_{s'(\mathbb{R}_+)}^2, \quad -\frac{1}{2} < s' \leq s_0 \leq s' + r_0 - r', \quad 0 \leq r' \leq r_0, \quad (10)$$

$$\|\hat{v}_{r'}\|_{L_{s'}^2(\mathbb{R}_+)} \leq c \|\hat{v}_{r_0}\|_{L_{s_0}^2(\mathbb{R}_+)}, \quad -\frac{1}{2} < s' \leq s_0 \leq s' + r_0 - r', \quad 0 \leq r' \leq r_0. \quad (11)$$

Using these facts, the inclusion $\text{supp } \hat{\kappa}_{-r',\ell}^+ \in \mathbb{R}_-$, and the standard Minkowski inequality, we obtain initially

$$\begin{aligned} I_1 &= 2\pi \int_{\mathbb{R}_+} (1 + |\xi|^2)^{s_1} |(\hat{\kappa}_{-r',\ell}^+ * \hat{v}_{r'})(\xi + h) - (\hat{\kappa}_{-r',\ell}^+ * \hat{v}_{r'})(\xi)|^2 d\xi \\ &\leq 2\pi \left[\int_{\mathbb{R}} |\hat{\kappa}_{-r',\ell}^+(\xi + h) - \hat{\kappa}_{-r',\ell}^+(\xi)| d\xi \left(\int_{\mathbb{R}_+} |\hat{v}_{r'}|^2(\xi) (1 + |\xi - \zeta|^2)^{s_1} d\xi \right)^{\frac{1}{2}} \right]^2 \end{aligned}$$

then, using the elementary inequality $(1 + |\xi - \zeta|^2)^{s_1} \leq c[(1 + |\xi|^2)^{s_1} + |\xi|^{2s_1}]$ (which holds uniformly for $\xi, \zeta \in \mathbb{R}$, with some constant $c > 0$) and Formulas (3), (10) and (11),

$$\begin{aligned} I_1 &\leq c \left[\int_{\mathbb{R}_-} (1 + |\xi|^2)^{\frac{s_1}{2}} |\hat{\kappa}_{-r',\ell}^+(\xi - h) - \hat{\kappa}_{-r',\ell}^+(\xi)| d\xi \right]^2 \\ &\quad (\|\hat{v}_{r'}\|_{L^2(\mathbb{R}_+)}^2 + \|\hat{v}_{r'}\|_{L_{s_1}^2(\mathbb{R}_+)}^2) \\ &\leq c \left[\int_{\mathbb{R}_-} (1 + |\xi|^2)^{\frac{s_1}{2}} |\hat{\kappa}_{-r',\ell}^+(\xi - h) - \hat{\kappa}_{-r',\ell}^+(\xi)| d\xi \right]^2 \|u\|_{H_{r_0}^{s_0}(\mathbb{R})}^2 \end{aligned} \quad (12)$$

provided $0 \leq s_0 \leq r_0 - r'$ and $-\frac{1}{2} < s_1 \leq s_0 \leq s_1 + r_0 - r'$. We note that the right hand side of (12) tends to zero, as $h \rightarrow 0$, as $\hat{\kappa}_{-r',\ell}^+ \in L^1(\mathbb{R}_+, |\xi|^s d\xi)$, for all $s \geq 0$, provided $r' > 0$.

(c) From parts (a) and (b) of the proof, it follows that the unit ball of $H_{r_0}^{s_0}(\mathbb{R})$ is equibounded, equitight, and equicontinuous in $H^{s_1}(\mathbb{R})$, provided $s_1 < s_0$ and $0 \leq s_0 \leq r_0 - r'$, $-\frac{1}{2} < s_1 \leq s_0 \leq s_1 + r_0 - r'$, for some $0 < r' \leq r$. Hence, on account of the Kolmogorov–Riesz theorem, the embedding (9) is indeed compact if $0 \leq s_1 < s_0 < r_0$. The proof is complete. $\square$

3. Technical Estimates

The analysis of (1) relies on two technical estimates that allow us to control the linear and nonlinear quantities appearing in the Benjamin equation. We begin with the linear part of (1).

Let

$$\mathcal{A} = -\alpha \partial_x + \beta \mathcal{H} \partial_{xx} + \gamma \partial_{xxx}$$

and let H be a Hilbert space over $\mathbb{C}$, obtained as the completion of the Schwartz class $\mathcal{S}$ with respect to the inner product $\langle \cdot, \cdot \rangle_H$. To the couple $\mathcal{A}$ and H , we associate the bilinear form

$$\mathcal{Q}_H(u, v) = \langle \mathcal{A}u, v \rangle_H, \quad u, v \in \mathcal{S}.$$

Passing to the Fourier images and using the Cauchy–Schwarz inequality, it is not difficult to verify that

$$\mathcal{Q}_{H^{s_0}(\mathbb{R})}(u, v) = -\overline{\mathcal{Q}_{H^{s_0}(\mathbb{R})}(v, u)}, \quad (13)$$

$$|\mathcal{Q}_{H^{s_0}(\mathbb{R})}(u, v)| \leq \|u\|_{H^{s_1}(\mathbb{R})} \|v\|_{H^{2s_0+3-s_1}(\mathbb{R})}, \quad s_0, s_1 \in \mathbb{R}. \quad (14)$$

Furthermore, we have

Lemma 2. For $s > -\frac{1}{2}$, $r > 0$, the bilinear form $\mathcal{Q}_{H_r^s(\mathbb{R})}(\cdot, \cdot)$ satisfies

$$|\mathcal{Q}_{H_r^s(\mathbb{R})}(u, u)| \leq r(1 + r^2)c_s(\|u\|_{H_r^s(\mathbb{R})}^2 + \|u\|_{H^{s+2r}(\mathbb{R})}^2), \quad (15)$$

with $c_s > 0$ controlled by $s > -\frac{1}{2}$ and the coefficients of $\mathcal{A}$ only.

Proof. As in Lemma 1, we prove (15) for $u \in \text{Re } \mathcal{S}$ only. The real version of (15) then follows by a standard density argument, and the complex version follows from (5).

Assume $u \in \text{Re } \mathcal{S}$, then (3) and (13) yield

$$|\mathcal{Q}_{H_r^s(\mathbb{R})}(u, u)| = \left| \langle \mathcal{P}_+ [[\kappa_{r,\ell}^+, \mathcal{A}]u], \mathcal{P}_+ [\kappa_{r,\ell}^+ u] \rangle_{\dot{H}^s(\mathbb{R})} \right|,$$

where $[\kappa_{r,\ell}^+, \mathcal{A}] = \kappa_{r,\ell}^+ \mathcal{A} - \mathcal{A} \kappa_{r,\ell}^+$ is the commutator of $\mathcal{A}$ and $\kappa_{r,\ell}^+$. On account of the identity $\mathcal{P}_\pm \mathcal{H} = \mp i \mathcal{P}_\pm$ and the inclusion $\text{supp } \hat{\kappa}_{-r,\ell}^\pm \subset \mathbb{R}_\mp$, we have

$$\mathcal{P}_+ [[\kappa_{r,\ell}^+, \mathcal{A}]u] = \mathcal{P}_+ \left[u \cdot (\mathcal{A}_0 \kappa_{r,\ell}^+) + \partial_x [u \cdot (\mathcal{A}_1 \kappa_{r,\ell}^+)] + \partial_{xx} [u \cdot (\mathcal{A}_2 \kappa_{r,\ell}^+)] \right],$$

with

$$\mathcal{A}_0 = \alpha \partial_x - i \partial_{xx} - \gamma \partial_{xxx}, \quad \mathcal{A}_1 = 2i\beta \partial_x + 3\gamma \partial_{xx}, \quad \mathcal{A}_2 = -3\gamma \partial_x.$$

The last formula, combined with (11) and the interpolation inequality. (The inequality follows directly from (7), as the complex interpolation functor $[\cdot, \cdot]_\theta$ is exact of exponent $\theta \in (0, 1)$, see, e.g., [20] (Chapter 4).

$$\|u\|_{H_{r_0(1-\theta)+s_1\theta}^{s_0(1-\theta)+s_1\theta}(\mathbb{R})} \leq \|u\|_{H_0^{s_0}(\mathbb{R})}^{1-\theta} \|u\|_{H_{r_1}^{s_1}(\mathbb{R})}^\theta, \quad s_0, s_1 > -\frac{1}{2}, \quad r_0, r_1 \geq 0, \quad \theta \in (0, 1),$$

yields

$$\begin{aligned} \left\| \mathcal{P}_+ [[\kappa_{r,\ell}^+, \mathcal{A}]u] \right\|_{\dot{H}^s(\mathbb{R})} &\leq r(1 + r^2)c'_s \sum_{j=0}^2 \left\| \mathcal{P}_+ [\kappa_{r-1,\ell}^+ u] \right\|_{\dot{H}^{s+j}(\mathbb{R})} \\ &\leq r(1 + r^2)c_s \left(\left\| \mathcal{P}_+ [\kappa_{r,\ell}^+ u] \right\|_{\dot{H}^s(\mathbb{R})} + \left\| \mathcal{P}_+ [\kappa_{r-1,\ell}^+ u] \right\|_{\dot{H}^{s+2}(\mathbb{R})} \right) \\ &\leq r(1 + r^2)c_s (\|u\|_{H_r^s(\mathbb{R})} + \|u\|_{H^{s+2r}(\mathbb{R})}), \end{aligned}$$

where the generic constants $c_s, c'_s > 0$ depend on $s > -\frac{1}{2}$ and the coefficients α, β and γ of the operator $\mathcal{A}$ only. The last bound, together with the standard Cauchy–Schwarz inequality, completes the proof. $\square$

Let H be a Hilbert space employed in the definition of $\mathcal{Q}_H(\cdot, \cdot)$. To control the quadratic nonlinearity $(u^2)_x$, we define

$$\mathcal{T}_H(u, v, w) = \frac{1}{2} \langle u, \overline{wv_x} \rangle_H + \frac{1}{2} \langle v, \overline{wu_x} \rangle_H, \quad u, v, w \in \mathcal{S}.$$

Partial integration, combined with the standard Gagliardo–Nirenberg inequality, indicates that

$$|\mathcal{T}_{L^2(\mathbb{R})}(u, v, w)| \leq \|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} \|w\|_{H^s(\mathbb{R})}, \quad s > \frac{3}{2}. \quad (16)$$

In addition to (16), we have

Lemma 3. For $s > \frac{1}{2}$, $r \geq 0$, $\mathcal{T}_{H_r^s(\mathbb{R})}(\cdot, \cdot, \cdot)$ extends to a bounded trilinear form in $H_r^s(\mathbb{R})^2 \times H^{s+1}(\mathbb{R})$. In particular, we have

$$|\mathcal{T}_{H_r^s(\mathbb{R})}(u, v, w)| \leq c_{s,r} \|u\|_{H_r^s(\mathbb{R})} \|v\|_{H_r^s(\mathbb{R})} \|w\|_{H^{s+1}(\mathbb{R})}, \quad (17)$$

with $c_{s,r} > 0$ depending on s and r only.

Proof. (a) As in the proof of Lemma 1, we consider real valued functions only. Assume initially that $r = 0$ and $u, v, w \in \text{Re } \mathcal{S}$. Direct calculations give

$$\begin{aligned} |\mathcal{T}_{H^s(\mathbb{R})}(u, v, w)| &\leq \frac{1}{2} |\langle uv, w_x \rangle_{L^2(\mathbb{R})}| \\ &\quad + \frac{1}{4} \int_{\mathbb{R}} |\hat{w}(\xi)| d\xi \int_{\mathbb{R}} (|\xi|^{2s} - |\xi - \zeta|^{2s}) \cdot |\zeta| \\ &\quad \cdot [|\hat{u}(\zeta)\hat{v}(\xi - \zeta)| + |\hat{v}(\zeta)\hat{u}(\xi - \zeta)|] d\zeta \\ &\quad + \frac{1}{4} \int_{\mathbb{R}} |\xi \hat{w}(\xi)| d\xi \int_{\mathbb{R}} (|\xi|^{2s} + |\xi - \zeta|^{2s}) \\ &\quad \cdot [|\hat{u}(\zeta)\hat{v}(\xi - \zeta)| + |\hat{v}(\zeta)\hat{u}(\xi - \zeta)|] d\zeta. \end{aligned}$$

Using the elementary estimate $|\xi_1^{2s} - \xi_2^{2s}| \leq 2s|\xi_1 - \xi_2|(\xi_1^{2s-1} + \xi_2^{2s-1})$ (which holds uniformly for all $\xi_1, \xi_2 \in \mathbb{R}_+$ and $s > \frac{1}{2}$), followed by Young's inequality (with exponents $2s > 1$, $\frac{2s}{2s-1} > 1$) and changing the order of integration, we infer

$$\begin{aligned} |\mathcal{T}_{H^s(\mathbb{R})}(u, v, w)| &\leq \frac{1}{2} |\langle uv, w_x \rangle_{L^2(\mathbb{R})}| \\ &\quad + \frac{8s+1}{4} \int_{\mathbb{R}} |\hat{u}(\xi)| |\xi|^{2s} d\xi \int_{\mathbb{R}} |\zeta \hat{w}(\zeta)| \cdot |\hat{v}(\xi - \zeta)| d\zeta \\ &\quad + \frac{8s+1}{4} \int_{\mathbb{R}} |\hat{v}(\xi)| |\xi|^{2s} d\xi \int_{\mathbb{R}} |\zeta \hat{w}(\zeta)| \cdot |\hat{u}(\xi - \zeta)| d\zeta. \end{aligned}$$

This bound, together with the elementary inequality $(|\xi_1| + |\xi_2|)^s \leq 2^s(|\xi_1|^s + |\xi_2|^s)$, $s \geq 0$, yields

$$\begin{aligned} |\mathcal{T}_{H^s(\mathbb{R})}(u, v, w)| &\leq \frac{1}{2} |\langle uv, w_x \rangle_{L^2(\mathbb{R})}| \\ &\quad + (8s+1)2^{s-1} \int_{\mathbb{R}} |\xi \hat{w}(\xi)| d\xi \int_{\mathbb{R}} |\zeta|^s |\hat{u}(\zeta)| \cdot |\xi - \zeta|^s |\hat{v}(\xi - \zeta)| d\zeta \\ &\quad + (8s+1)2^{s-2} \int_{\mathbb{R}} |\hat{u}(\xi)| d\xi \int_{\mathbb{R}} |\zeta|^{s+1} |\hat{w}(\zeta)| \cdot |\xi - \zeta|^s |\hat{v}(\xi - \zeta)| d\zeta \\ &\quad + (8s+1)2^{s-2} \int_{\mathbb{R}} |\hat{v}(\xi)| d\xi \int_{\mathbb{R}} |\zeta|^{s+1} |\hat{w}(\zeta)| \cdot |\xi - \zeta|^s |\hat{u}(\xi - \zeta)| d\zeta. \end{aligned}$$

On account of the embedding $H^s(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, the last estimate implies

$$|\mathcal{T}_{H^s(\mathbb{R})}(u, v, w)| \leq c_{s,0} \|u\|_{H^s(\mathbb{R})} \|v\|_{H^s(\mathbb{R})} \|w\|_{H^{s+1}(\mathbb{R})}, \quad s > \frac{1}{2}, \quad (18)$$

where the generic constant $c_{s,0} > 0$ depends on $s > \frac{1}{2}$ only. By virtue of (18), the trilinear form $\mathcal{T}_{H^s(\mathbb{R})}(u, v, w)$ extends continuously to $H^s(\mathbb{R})^2 \times H^{s+1}(\mathbb{R})$ and the case of $r = 0$ is settled.

(b) Next, we let $s = 1$ and $r \in \mathbb{N}$ and for the sake of brevity, denote $u^\pm = \mathcal{P}_\pm[u]$. The identities

$$\begin{aligned} (uv)^+ &= u^+v^+ + (u^+v^- + u^-v^+)^+, \\ \langle u^\pm, v^\mp \rangle_{\dot{H}^s(\mathbb{R})} &= 0, \quad u, v \in \dot{H}^s(\mathbb{R}), \quad s \in \mathbb{R}, \end{aligned}$$

and the commutativity of Fourier multipliers $\mathcal{P}_\pm$ and ∂_x , combined together, give

$$\begin{aligned} \mathcal{T}_{H_r^1(\mathbb{R})}(u, v, w) &= -\frac{1}{2} \langle w_x, uv \rangle_{L^2(\mathbb{R})} \\ &\quad + \operatorname{Re} \left[\langle (\kappa_{r,\ell}^+ u)^+, w(\kappa_{r,\ell}^+ v)_x^+ \rangle_{\dot{H}^1(\mathbb{R})} + \langle (\kappa_{r,\ell}^+ v)^+, w(\kappa_{r,\ell}^+ u)_x^+ \rangle_{\dot{H}^1(\mathbb{R})} \right] \\ &\quad - r \operatorname{Re} \left[\langle (\kappa_{r,\ell}^+ u)^+, w(\kappa_{r-1,\ell}^+ v)^+ \rangle_{\dot{H}^1(\mathbb{R})} + \langle (\kappa_{r,\ell}^+ v)^+, w(\kappa_{r-1,\ell}^+ u)^+ \rangle_{\dot{H}^1(\mathbb{R})} \right] \\ &\quad + \operatorname{Re} \left[\langle (\kappa_{r,\ell}^+ u)^+, w^+(\kappa_{r,\ell}^+ v)_x^- \rangle_{\dot{H}^1(\mathbb{R})} + \langle (\kappa_{r,\ell}^+ v)^+, w^+(\kappa_{r,\ell}^+ u)_x^- \rangle_{\dot{H}^1(\mathbb{R})} \right] \\ &\quad - r \operatorname{Re} \left[\langle (\kappa_{r,\ell}^+ u)^+, w^+(\kappa_{r-1,\ell}^+ v)^- \rangle_{\dot{H}^1(\mathbb{R})} + \langle (\kappa_{r,\ell}^+ v)^+, w^+(\kappa_{r-1,\ell}^+ u)^- \rangle_{\dot{H}^1(\mathbb{R})} \right] \\ &:= I_1 + I_2 - rI_3 + I_4 - rI_5. \end{aligned}$$

We estimate each term separately.

For $s > \frac{1}{2}$, $r > 0$, the embedding $H_r^s(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$ provides

$$|I_1| \leq c \|u\|_{H_r^1(\mathbb{R})} \|v\|_{H_r^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})},$$

with $c > 0$ depending on $r > 0$ only. Partial integration yields the bound

$$|I_2| \leq c \|u\|_{H_r^1(\mathbb{R})} \|v\|_{H_r^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})},$$

with an absolute constant $c > 0$. To estimate I_3 , we pass to the frequency space and change the order of summation. On account of the Cauchy–Schwarz inequality and Formulas (2)–(6), we obtain

$$\begin{aligned} |I_3| &\leq c \|u\|_{H_r^1(\mathbb{R})} (\|(\kappa_{r-1,\ell}^+ v)^+\|_{\dot{H}^1(\mathbb{R})} \|w\|_{H^1(\mathbb{R})} + \|(\kappa_{r-1,\ell}^+ v)^+\|_{L^2(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}) \\ &\quad + c \|v\|_{H_r^1(\mathbb{R})} (\|(\kappa_{r-1,\ell}^+ u)^+\|_{\dot{H}^1(\mathbb{R})} \|w\|_{H^1(\mathbb{R})} + \|(\kappa_{r-1,\ell}^+ u)^+\|_{L^2(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}) \\ &\leq c_r \|u\|_{H_r^1(\mathbb{R})} \|v\|_{H_r^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}, \end{aligned}$$

with a constant $c_r > 0$ that depends on $r \in \mathbb{N}$ only.

In order to estimate I_4 and I_5 , we note that $\operatorname{supp} \hat{u}^\pm \subset \overline{\mathbb{R}_\pm}$ and, therefore,

$$\begin{aligned} |\langle u^+, w^+ v^- \rangle_{\dot{H}^1(\mathbb{R})}| &\leq \int_{\mathbb{R}_+} |\xi \hat{w}^+(\xi)| d\xi \int_0^\xi |(\xi - \zeta) \hat{v}^-(\xi - \zeta)| \cdot |\zeta \hat{u}^+(\zeta)| d\zeta \\ &\leq c \|u\|_{\dot{H}^1(\mathbb{R})} \|v\|_{\dot{H}^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}, \end{aligned}$$

with an absolute constant $c > 0$, for any $u, v, w \in \operatorname{Re} \mathcal{S}$. Using this fact, the elementary identity $\kappa_r^+ = (-1)^r \sum_{k=0}^r \binom{k}{r} (-2i\ell)^{r-k} \kappa_k^-$ and the embedding (6), we infer

$$|I_4| + r|I_5| \leq c'_r \|u\|_{H_r^1(\mathbb{R})} \|v\|_{H_r^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})},$$

with a constant $c'_r > 0$ that depends on $r \in \mathbb{N}$ only. Hence, our estimates, combined with the standard density argument, give

$$|\mathcal{T}_{H_r^1(\mathbb{R})}(u, v, w)| \leq c_{1,r} \|u\|_{H_r^1(\mathbb{R})} \|v\|_{H_r^1(\mathbb{R})} \|w\|_{H^2(\mathbb{R})}, \quad r \geq 0, \quad (19)$$

for all $u, v \in H_r^1(\mathbb{R})$, $r \in \mathbb{N}$, and $w \in H^2(\mathbb{R})$, with some $c_{1,r} > 0$.

(c) The proof of the general case $s > \frac{1}{2}$, $r \geq 0$, relies on the following straightforward modification of the complex interpolation argument:

Given a regular compatible interpolation pair (H_0, H_1) of complex Hilbert spaces with $H := H_0 \cap H_1$ being dense in H_i , $i = 0, 1$; a regular compatible interpolation pair of

complex Banach spaces (B_0, B_1) , with H being dense in $B := B_0 \cap B_1$; and the family of trilinear forms

$$\begin{aligned}\mathcal{T}_\theta(u, v, w) &= \langle u, \overline{T(v, w)} \rangle_{[H_0, H_1]_\theta} + \langle v, \overline{T(u, w)} \rangle_{[H_0, H_1]_\theta}, \quad \theta \in (0, 1), \\ T &\in \mathcal{L}(H_i \times B_i, H_i), \quad i = 0, 1,\end{aligned}$$

defined initially in H^3 and satisfying

$$|\mathcal{T}_i(u, v, w)| \leq c_i \|u\|_{H_i} \|v\|_{H_i} \|w\|_{B_i}, \quad i = 0, 1.$$

We claim that $\mathcal{T}_\theta$ extends to a bounded trilinear form in $[H_0, H_1]_\theta^2 \times [B_0, B_1]_\theta$ and

$$|\mathcal{T}_\theta(u, v, w)| \leq c_0^{1-\theta} c_1^\theta \|u\|_{[H_0, H_1]_\theta} \|v\|_{[H_0, H_1]_\theta} \|w\|_{[B_0, B_1]_\theta}, \quad \theta \in (0, 1). \quad (20)$$

To see that (20) holds, it is sufficient to note that H_i , $i = 0, 1$, can be realized as completions of H with respect to the inner products $\langle \cdot, \cdot \rangle_{H_0} = \langle \mathcal{M} \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{H_1} = \langle \mathcal{I} - \mathcal{M} \cdot, \cdot \rangle_H$, where $\mathcal{M}$, $\mathcal{I} - \mathcal{M}$ are positive definite bounded symmetric linear maps in H . As the complex interpolation functor $[\cdot, \cdot]_\theta$ is the exact of exponent $\theta \in (0, 1)$, the theory of exact interpolation of Hilbert spaces (see the classical paper [23]) implies that $\langle \cdot, \cdot \rangle_{[H_0, H_1]_\theta} = \langle \mathcal{M}^{1-\theta}(\mathcal{I} - \mathcal{M})^\theta \cdot, \cdot \rangle_H$, $\theta \in (0, 1)$.

Further, the map

$$z \mapsto \mathcal{M}^{1-z}(\mathcal{I} - \mathcal{M})^z u,$$

$u \in H$, is analytic in the strip $S := \{0 < \operatorname{Re} z < 1\}$ and is continuous and bounded in its closure $\bar{S}$ (see, e.g., [24] for the standard results on spectral resolution of bounded normal operators and the associated functional calculus). Hence, for $f_u(z), f_v(z) \in \mathcal{F}(H_0, H_1)$ and $f_w(z) \in \mathcal{F}(B_0, B_1)$ (The notation $\mathcal{F}(H_0, H_1)$ and $\mathcal{F}(B_0, B_1)$ for the spaces of vector valued analytic functions in S , with boundary values in H_i and B_i , $i = 0, 1$, respectively, is standard, see, e.g., [20]), with $f_u(\theta) = u$, $f_v(\theta) = v$ and $f_w(\theta) = w$, the function

$$\begin{aligned}g(z) &= c_0^{z-1} c_1^{-z} \langle \mathcal{M}^{1-z}(\mathcal{I} - \mathcal{M})^z f_u(z), \overline{T(f_v(z), f_w(z))} \rangle_H \\ &\quad + c_0^{z-1} c_1^{-z} \langle \mathcal{M}^{1-z}(\mathcal{I} - \mathcal{M})^z f_v(z), \overline{T(f_u(z), f_w(z))} \rangle_H\end{aligned}$$

is well defined, analytic in S and continuous and bounded in $\bar{S}$. The last assertion allows us to repeat verbatim the standard interpolation and density arguments (see e.g., [20]) (Theorem 4.1.2, p. 88, or 4.4.2, p. 97) and thus (20) is settled.

(d) To complete the proof of (17), we note that the scale of Hilbert spaces $H_r^s(\mathbb{R})$ and the family $\mathcal{T}_{H_r^s(\mathbb{R})}(\cdot, \cdot, \cdot)$, $s > \frac{1}{2}$, $r \geq 0$, of the trilinear forms fall in the scope of the interpolation argument discussed above.

Indeed, every pair $(1, r)$ of parameters is a convex combination of points $(1, 0)$ and $(1, n)$, $r \leq n \in \mathbb{N}$. Further, on account of (6), the endpoint spaces $H^1(\mathbb{R})$ and $H_n^1(\mathbb{R})$ satisfy all of the density constraints imposed in item (c) above. Hence, direct application of the interpolation identities (3) and (20) to the bounds (18) and (19) settles the claim for $s = 1$ and $r \geq 0$. In general, every pair $(s, r) \in (\frac{1}{2}, \infty) \times \mathbb{R}_+$ can be realized as a convex combination of two points laying in the lines $\ell_0 = (\frac{1}{2}, \infty) \times \{0\}$ and $\ell_1 = \{1\} \times \mathbb{R}_+$, respectively. Hence, using (3) and (20) one more time, we obtain (17) for all $s > \frac{1}{2}$ and $r \geq 0$. $\square$

4. Large- x Asymptotic of Classical Solutions

Technical estimates of Sections 2 and 3 allow us to deduce the global weighted solvability of (1) directly from the well-known unweighted well-posedness results, see, e.g., the discussion in [4–6,18], the classical papers [25,26], and note that (1) is the low order perturbation of the Korteweg–de Vries equation.

Theorem 1. Assume that $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, with $\frac{1}{2} < s < r$ and $s + 2r \geq 3$. Then, for any finite value of $0 < T < \infty$, the unique global classical solution u to (1) satisfies,

$$u \in L^\infty([0, T]; H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})). \quad (21)$$

Proof. (a) It is well known, that for the initial data $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, the Benjamin equation is classically globally well-posed (see the remark preceding Theorem 1). In particular, for such input data and any finite value of $0 < T < \infty$, we have $u \in L^\infty([0, T], H^s(\mathbb{R}))$. Furthermore, for any $v \in L^2(\mathbb{R})$

$$\langle u_t, v \rangle_{L^2} = \mathcal{Q}_{L^2(\mathbb{R})}(u, v) - 2\delta \langle uu_x, v \rangle_{L^2(\mathbb{R})}, \quad t \in (0, T], \quad (22)$$

$$u(0) = u_0, \quad n \geq 0. \quad (23)$$

Hence, to complete the proof of Theorem 1, we need show that under the assumption $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, the inclusion (21) holds.

(b) For the sake of brevity, for $s > -\frac{1}{2}$, $r \geq 0$, we let

$$W_r^s(\mathbb{R}) = H_r^s(\mathbb{R}) \cap H^{s+2r}(\mathbb{R}), \quad \langle \cdot, \cdot \rangle_{W_r^s(\mathbb{R})} := \langle \cdot, \cdot \rangle_{H_r^s(\mathbb{R})} + \langle \cdot, \cdot \rangle_{H^{s+2r}(\mathbb{R})}.$$

On account of Lemma 1, the embedding

$$W_r^s(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}), \quad 0 < s < r, \quad (24)$$

is dense and compact. Hence, for $0 < s < r$, $L^2(\mathbb{R})$ can be realized as a completion of $W_r^s(\mathbb{R})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})} = \langle \mathcal{M} \cdot, \cdot \rangle_{W_r^s(\mathbb{R})}$, with some positive definite bounded symmetric map $\mathcal{M} \in \mathcal{L}(W_r^s(\mathbb{R}))$. It is not difficult to verify that $\mathcal{M}$ extends uniquely to a symmetric positive definite bounded linear map $\mathcal{M}_r^s \in \mathcal{L}(L^2(\mathbb{R}))$ and that $\mathcal{M}_r^s L^2(\mathbb{R}) \subset W_r^s(\mathbb{R})$. By virtue of (24), the last inclusion indicates that $\mathcal{M}_r^s$, with $0 < s < r$, is compact.

(c) From part (a) of the proof, the spectrum of $\mathcal{M}_r^s$ is discrete, while the collection of associated eigenfunctions $\{\varphi_k\}_{k \geq 0}$ is a complete orthogonal basis in $L^2(\mathbb{R})$ and in $W_r^s(\mathbb{R})$, $0 < s < r$.

We denote $\mathbb{P}_n = \text{span}\{\varphi_k\}_{k=0}^n$ and let $\mathcal{P}_n : L^2(\mathbb{R}) \rightarrow \mathbb{P}_n$ be the associated orthogonal projector. From the definition of $W_r^s(\mathbb{R})$, for $s + 2r \geq \frac{3}{2}$, we have $\{\varphi_k\}_{k \geq 0} \subset H^{\frac{3}{2}}(\mathbb{R})$. Hence, in view of Lemma 2 and part (a) of the proof, the sequence of linear Galerkin approximations

$$\langle u_{nt}, v \rangle_{L^2(\mathbb{R})} = \mathcal{Q}_{L^2(\mathbb{R})}(u_n, v) - 2\delta \langle uu_{nx}, v \rangle_{L^2(\mathbb{R})}, \quad v \in \mathbb{P}_n, \quad (25)$$

$$u_n(0) = \mathcal{P}_n[u_0], \quad n \geq 0, \quad (26)$$

is well defined.

The special choice of the orthogonal basis indicates that

$$\langle u_{nt}, v \rangle_{W_r^s(\mathbb{R})} = \mathcal{Q}_{W_r^s(\mathbb{R})}(u_n, v) - 2\delta \langle uu_{nx}, v \rangle_{W_r^s(\mathbb{R})}, \quad v \in \mathbb{P}_n. \quad (27)$$

Letting $v = u_n$ in (27) and using Lemmas 2 and 3, for $\frac{1}{2} < s < r$, we obtain

$$\frac{d}{dt} \|u_n\|_{W_r^s(\mathbb{R})}^2 \leq 2[r(1+r^2)c_s + 2\delta c_{s,r} \|u\|_{H^{s+2r+1}(\mathbb{R})}] \|u_n\|_{W_r^s(\mathbb{R})}^2. \quad (28)$$

Bound (27), combined with the standard Gronwall's inequality, indicates that

$$\{u_n\}_{n \geq 0} \in L^\infty([0, T]; W_r^s(\mathbb{R})), \quad \frac{1}{2} < s < r, \quad s + 2r \geq \frac{3}{2}, \quad (29)$$

uniformly in $n \geq 0$, for every finite fixed value of $0 < T < \infty$. In turn, for $s + 2r \geq 3$, using (14), (16), (24) and (29), integrating over the interval $[0, T]$ and using the Cauchy–Schwarz inequality, we obtain

$$\|u_{nt}\|_{L^2([0,T] \times \mathbb{R})} \leq cT^{\frac{1}{2}} [1 + \|u\|_{L^\infty([0,T]; H^{s+2r+1}(\mathbb{R}))}] \|u_n\|_{L^\infty([0,T]; H^{s+2r}(\mathbb{R}))},$$

where the generic constant $c > 0$ depends on the coefficients α, β, γ , and δ of (1) only. From the last bound, it follows that

$$\{u_{nt}\}_{n \geq 0} \in L^2([0, T] \times \mathbb{R}), \quad (30)$$

uniformly in $n \geq 0$.

(d) The rest of the proof is standard. Using the uniform inclusions (29) and (30) and passing to subsequences, we conclude that: (i) $\{u_n\}_{n \geq 0}$ and $\{u_{nt}\}_{n \geq 0}$ converge weak-* to some w and w_t in $L^\infty([0, T]; W_r^s(\mathbb{R}))$ and $L^2([0, T] \times \mathbb{R})$, respectively; (ii) $u_n(0)$ converges weakly to $w(0)$ (and by construction, strongly to u_0) in $L^2(\mathbb{R})$; and (iii) in view of (24), $\{w_n\}_{n \geq 0}$, converges strongly in $L^2([0, T] \times \mathbb{R})$. Passing to the weak limit in (25) and (26), we see that for $s + 2r \geq 3$,

$$\langle w_t, v \rangle_{L^2(\mathbb{R})} = \mathcal{Q}_{L^2(\mathbb{R})}(w, v) - 2\delta \langle uw_x, v \rangle_{L^2(\mathbb{R})}, \quad v \in L^2(\mathbb{R}), \quad (31)$$

$$w(0) = u_0, \quad (32)$$

Subtracting (22), (23), (31) and (32) and letting $v = w - u$, in view of (13) and (16), we obtain

$$\frac{d}{dt} \|u - w\|_{L^2(\mathbb{R})}^2 \leq 2\delta \|u\|_{H^{s+2r+1}(\mathbb{R})} \|u - w\|_{L^2(\mathbb{R})}^2.$$

As $\|u(0) - w(0)\|_{L^2(\mathbb{R})} = 0$, Gronwall's inequality gives $\|u - w\|_{L^\infty([0,T]; L^2(\mathbb{R}))} = 0$, and as $w \in L^\infty([0, T]; W_r^s(\mathbb{R}))$, (21) follows. The proof is complete. $\square$

By virtue of Theorem 1, Fourier images of classical solutions with input data u_0 from $H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, $\frac{1}{2} < s < r$, $s + 2r \geq 3$ are regular away from the origin. Hence, the large- x asymptotic of $u(t)$ is controlled solely by the small ξ behavior of its Fourier image $\hat{u}(t)$. Passing to the frequency domain in (1), we have

$$\hat{u}(t) = e^{t\hat{\sigma}} \hat{u}_0 - \frac{i\tilde{\zeta}\delta}{\sqrt{2\pi}} \int_0^t e^{(t-s)\hat{\sigma}} \hat{u}^{*2}(s) ds, \quad t \in \mathbb{R}, \quad (33)$$

$$\hat{\sigma}(\tilde{\zeta}) = -i\alpha\tilde{\zeta} + i\beta\tilde{\zeta}|\tilde{\zeta}| - i\gamma\tilde{\zeta}^3, \quad (34)$$

where $\hat{\sigma}$ is the Fourier symbol of operator $\mathcal{A}$ and $*2$ denotes the Fourier convolution square. The large- x asymptotic of $e^{At}u_0 = \mathcal{F}^{-1}[e^{t\hat{\sigma}}\hat{u}_0]$ is described by

Lemma 4. Assume $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, $s + 2r \geq 3$. If $\frac{1}{2} < s < r - p - \frac{1}{2}$, for some positive integer p , then

$$e^{At}u_0(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{p-1} \frac{1}{(x-\alpha t)^{k+1}} \left[\langle \mathcal{P}_+[u_0], c_k^+(t) \rangle_{L^2(\mathbb{R})} + \langle \mathcal{P}_-[u_0], c_k^-(t) \rangle_{L^2(\mathbb{R})} \right] + o(|x|^{-p}), \quad |x| \rightarrow \infty, \quad (35)$$

$$c_k^\pm(x, t) = (\mp i)^{k+1} \frac{d^k}{d\tilde{\zeta}^k} [e^{\pm i\tilde{\zeta}(x+t\tilde{\zeta}(\gamma\tilde{\zeta}-\beta))}]_{\tilde{\zeta}=0}, \quad 0 \leq k \leq p, \quad (36)$$

uniformly in bounded time intervals.

Proof. (a) We begin with some basic estimates. As $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, we have $\mathcal{P}_\pm[u_0] = 2\pi\mathcal{P}_\pm[\kappa_{-r,\ell}^\pm v_r^\pm]$, with $v_r^\pm \in \dot{H}^s(\mathbb{R})$. Consequently,

$$\hat{u}_0(\xi) = \sqrt{2\pi}[\hat{\kappa}_{-r,\ell}^\pm * \hat{v}_r^\pm](\xi), \quad \xi \in \mathbb{R}_\pm, \quad (37)$$

$$\hat{\kappa}_{-r,\ell}^\pm = \frac{e^{-i\pi r/2}}{\Gamma(r)} e^{\pm \ell \xi} (\mp \xi)^{r-1} \chi_{\mathbb{R}_\mp}, \quad \xi \in \mathbb{R}, \quad (38)$$

where $\hat{v}_r^\pm \in L_s^2(\mathbb{R}_\mp)$. The Cauchy–Schwarz inequality, (37) and (38) and assumption $\frac{1}{2} < s < r - p - \frac{1}{2}$, combined together, indicate that

$$|[\hat{\kappa}_{-r',\ell}^\pm * \hat{v}_r^\pm](0)| \leq c \|u_0\|_{H_r^s(\mathbb{R})}, \quad r - p \leq r', \quad (39)$$

with $c > 0$, depending on s, r and r' only. Further, on the account of (6), we have

$$\begin{aligned} \|u_0\|_{H_p^0(\mathbb{R})} + \|u_0\|_{H_p^s(\mathbb{R})} &\leq c \|u_0\|_{H_r^s(\mathbb{R})}, \\ \|u_0\|_{H^{2p}(\mathbb{R})} + \|u_0\|_{H^{s+2p}(\mathbb{R})} &\leq \|u_0\|_{H^{s+2r+1}(\mathbb{R})}, \\ \|u_0\|_{L^2(\mathbb{R})} + \|u_0\|_{H^s(\mathbb{R})} &\leq \|u_0\|_{H^{s+2r+1}(\mathbb{R})}. \end{aligned}$$

Consequently, for any $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$, with $\theta_1 + \theta_2 + \theta_3 = 1$, from the interpolation identity (7), it follows that

$$\|u_0\|_{H_{\theta_1 \cdot p}^{\theta_2 \cdot 2p}(\mathbb{R})} + \|u_0\|_{H_{\theta_1 \cdot p}^{s+\theta_2 \cdot 2p}(\mathbb{R})} \leq c \|u_0\|_{H_r^s(\mathbb{R})}^{\theta_1} \|u_0\|_{H^{s+2r+1}(\mathbb{R})}^{\theta_2 + \theta_3},$$

for some $c > 0$. This estimate, combined with the definition (2)–(5) of $H_r^s(\mathbb{R})$ -norm, implies that

$$(\pm \xi)^{s'} [\hat{\kappa}_{-r',\ell}^\pm * \hat{v}_r^\pm](\xi) \in L^2(\mathbb{R}_\pm) \cap L_s^2(\mathbb{R}_\pm), \quad 0 \leq r' \leq p, \quad 0 \leq s' \leq 2(p - r').$$

From our assumption, $\frac{1}{2} < s$. Hence,

$$(\pm \xi)^{s'} [\hat{\kappa}_{-r',\ell}^\pm * \hat{v}_r^\pm](\xi) \in L^1(\mathbb{R}_\pm), \quad 0 \leq r' \leq p, \quad 0 \leq s' \leq 2(p - r'). \quad (40)$$

(b) We write

$$\hat{\sigma}(\xi) = -i\xi\alpha - i\hat{\sigma}^+(\xi) - i\hat{\sigma}^-(\xi), \quad \hat{\sigma}^\pm(\xi) = \xi^2(\gamma\xi \mp \beta)\chi_{\mathbb{R}_\pm}, \quad \xi \in \mathbb{R}.$$

The inclusion $u_0 \in H^{s+2r+1}(\mathbb{R})$ indicates that $\hat{u}_0 \in L^1(\mathbb{R})$. Therefore, (35) and (36), combined with the Fubini–Tonelli theorem, provides

$$\begin{aligned} \mathcal{P}_\pm[e^{\mathcal{A}t}u_0](x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_\pm} e^{i(x-\alpha t)\xi} e^{-it\hat{\sigma}^\pm(\xi)} \hat{u}_0(\xi) d\xi \\ &= \frac{e^{-i\pi r/2}}{\Gamma(r)} \int_{\mathbb{R}_+} e^{-\ell\xi} \hat{v}_r^\pm(\pm\xi) d\xi \int_0^\xi e^{\pm i(x-\alpha t)\zeta} e^{\ell\zeta - it\hat{\sigma}^\pm(\pm\zeta)} (\xi - \zeta)^{r-1} d\zeta \\ &= \frac{e^{-i\pi r/2}}{\Gamma(r)} \int_{\mathbb{R}_+} e^{-\ell\xi} \hat{v}_r^\pm(\pm\xi) K^\pm(x, t, \xi) d\xi. \end{aligned}$$

Partial integration gives

$$\begin{aligned} K^\pm(x, t, \xi) &= \sum_{k=0}^{p-1} \frac{(\pm i)^{k+1}}{(x-\alpha t)^{k+1}} \frac{d^k}{d\xi^k} \left[(\xi - \zeta)^{r-1} e^{\ell\zeta - it\hat{\sigma}^\pm(\pm\zeta)} \right]_{\zeta=0} \\ &\quad + \frac{(\pm i)^p}{(x-\alpha t)^p} \int_0^\xi e^{\pm i(x-\alpha t)\zeta} \frac{d^p}{d\xi^p} \left[(\xi - \zeta)^{r-1} e^{\ell\zeta - it\hat{\sigma}^\pm(\pm\zeta)} \right] d\zeta. \end{aligned}$$

Hence, from the last two formulas, we infer

$$\mathcal{P}_\pm[e^{At}u_0](x) = \sum_{k=0}^{p-1} \frac{(\pm i)^{k+1}}{(x-\alpha t)^{k+1}} \langle \hat{\sigma}_r^\pm, \hat{c}_{k,r}^\pm \rangle_{L^2(\mathbb{R}_\pm)} + R_p^\pm(x, t), \quad (41)$$

$$R_p^\pm(x, t) = \frac{e^{-i\pi r/2}}{\Gamma(r)} \frac{(\pm i)^p}{(x-\alpha t)^p} \int_{\mathbb{R}_+} e^{\pm i(x-\alpha t)\xi} d\xi \int_{\xi}^{\infty} e^{-\ell\xi} \frac{d^p}{d\xi^p} \left[(\xi - \zeta)^{r-1} e^{\ell\xi - it\hat{\sigma}^\pm(\pm\xi)} \right] \hat{\sigma}_r^\pm(\pm\xi) d\xi, \quad (42)$$

$$\hat{c}_{k,r}^\pm(\xi, t) = \frac{e^{-i\pi r/2}}{\Gamma(r)} e^{\mp \ell\xi} \frac{d^k}{d\xi^k} \left[(\pm\xi - \zeta)^{r-1} e^{\ell\xi - it\hat{\sigma}^\pm(\pm\xi)} \right]_{\zeta=0}, \quad 0 \leq k \leq p. \quad (43)$$

It is not difficult to verify that $\hat{c}_{k,r}^\pm(\xi, t)$ is a linear combination of the powers $(\pm\xi)^{r-1-j}$, $0 \leq j \leq k$. In view of (39), this observation allows us to conclude that each inner product appearing in (41) is finite. Differentiating and using the Faa di Bruno formula, we have

$$\frac{d^k}{d\xi^k} \left[e^{\ell\xi - it\hat{\sigma}^\pm(\pm\xi)} \right] = e^{\ell\xi - it\hat{\sigma}^\pm(\pm\xi)} p_{2k}^\pm(t, \xi),$$

where $p_{2k}^\pm$ are polynomials of degree $2k$ in ξ . Hence, the inner integral in (42) can be written as the sum of terms appearing in (40) and, consequently, is absolutely integrable in $\mathbb{R}_+$. On the account of classical Lebesgue–Riemann lemma, we conclude that

$$R_p^\pm(x, t) = o(|x|^{-p}), \quad |x| \rightarrow \infty,$$

uniformly in bounded time intervals. Finally, using (41)–(43) and the elementary identities

$$e^{-i\pi r/2} \langle \hat{\sigma}_r^\pm, e^{-i\pi(r-k)/2} \hat{\kappa}_{k-r,\ell}^\mp \rangle_{L^2(\mathbb{R}_\pm)} = \frac{i^k}{\sqrt{2\pi}} \langle \mathcal{P}_\pm[u_0], \kappa_{k,\ell}^\mp \rangle_{L^2(\mathbb{R})}, \quad 0 \leq k \leq p,$$

after some simplifications, we arrive at (35) and (36). $\square$

As $H_r^s(\mathbb{R})$ and $H^{2s+r+1}(\mathbb{R})$, with $s > \frac{1}{2}$, $r \geq 0$ are Banach algebras (see [19]), it follows that $u(t)^2 \in H_r^s(\mathbb{R}) \cap H^{2s+r+1}(\mathbb{R})$. Further, on the account of (2)–(5) and (6)–(8), we have $(u(t)^2)_x \in H_r^{s-1}(\mathbb{R}) \cap H^{s+2r}(\mathbb{R})$. Hence, for $\frac{3}{2} < s < r - p + \frac{1}{2}$, repeating verbatim arguments of Lemma 4, we obtain

$$\begin{aligned} e^{As}(u(t-s)^2)_x &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{p-1} \frac{1}{(x-\alpha s)^{k+1}} \left[\langle \mathcal{P}_+[(u(t-s)^2)_x], c_k^+(s) \rangle_{L^2(\mathbb{R})} \right. \\ &\quad \left. + \langle \mathcal{P}_-[(u(t-s)^2)_x], c_k^-(s) \rangle_{L^2(\mathbb{R})} \right] \\ &\quad + o(|x|^{-p}), \quad |x| \rightarrow \infty, \end{aligned} \quad (44)$$

uniformly in bounded time intervals.

For the sake of brevity, we let

$$\begin{aligned} [u]_k &= i \langle \mathcal{P}_+[u] - \mathcal{P}_-[u], x^k \rangle_{L^2(\mathbb{R})} = - \int_{\mathbb{R}} \mathcal{H}[u](x) x^k dx, \quad k \geq 0, \\ \{u\}_k &= \langle \mathcal{P}_+[u] + \mathcal{P}_-[u], x^k \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} u(x) x^k dx, \quad k \geq 0. \end{aligned}$$

The bulk asymptotic of the classical solutions to (1) is given by the following

Theorem 2. Assume $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, with $\frac{3}{2} < s < r - \frac{9}{2}$. Then,

$$\begin{aligned} u(x, t) &= \frac{[u_0]_0}{2\pi(x-\alpha t)} + \frac{[u_0]_1}{2\pi(x-\alpha t)^2} + \frac{\pi[u_0]_2 + t(2\pi\beta\{u_0\}_0 + \delta[u_0]_0^2)}{2\pi^2(x-\alpha t)^3} \\ &\quad + \frac{\pi[u_0]_3 - 3t(2\pi\gamma[u_0]_0 - 2\pi\beta\{u_0\}_1 + \delta[u_0]_0[u_0]_1) + 3t^2(\pi\delta\beta\|u_0\|_{L^2(\mathbb{R})}^2 - \delta\alpha[u_0]_0^2)}{2\pi^2(x-\alpha t)^4} \\ &\quad + o(|x|^{-4}), \quad |x| \rightarrow \infty, \end{aligned} \quad (45)$$

uniformly in bounded time intervals.

Proof. (a) Directly from (35), (36) and (44), it follows that

$$\begin{aligned} e^{At}u_0(x) &= \frac{[u_0]_0}{2\pi(x-\alpha t)} + \frac{[u_0]_1}{2\pi(x-\alpha t)^2} + \frac{[u_0]_2 + 2t\beta\{u_0\}_0}{2\pi(x-\alpha t)^3} \\ &\quad + \frac{[u_0]_3 - 6t\gamma[u_0]_0 + 6t\beta\{u_0\}_1}{2\pi(x-\alpha t)^4} + o(|x|^{-4}), \\ e^{As}(u(x, t-s)^2)_x &= \frac{[(u(t-s)^2)_x]_0}{2\pi(x-\alpha s)} + \frac{[(u(t-s)^2)_x]_1}{2\pi(x-\alpha s)^2} + \frac{[(u(t-s)^2)_x]_2 + 2s\beta\{(u(t-s)^2)_x\}_0}{2\pi(x-\alpha s)^3} \\ &\quad + \frac{[(u(t-s)^2)_x]_3 - 6s\gamma[(u(t-s)^2)_x]_0 + 6s\beta\{(u(t-s)^2)_x\}_1}{2\pi(x-\alpha s)^4} + o(|x|^{-4}), \end{aligned}$$

for large values of x and uniformly in bounded time intervals. We still need to compute moments, associated with $(u(t)^2)_x$, and the Hilbert transform.

(b) Elementary calculations and conservation of $L^2(\mathbb{R})$ -norm along classical trajectories of (1) indicate that

$$\{(u^2)_x\}_0(t) = 0, \quad \{(u^2)_x\}_1(t) = -\|u(t)\|_{L^2(\mathbb{R})}^2 = -\|u_0\|_{L^2(\mathbb{R})}^2.$$

From the remark preceding formula (44), $u(t) \in H_r^{s-1}(\mathbb{R}) \cap H^{s+2r}(\mathbb{R})$ while by our assumption $r > 6$. Hence, the restrictions $i\zeta[\hat{u}(t)^{*2}](\zeta)|_{\mathbb{R}_\pm}$ as well as their derivatives of orders $0 \leq s < \frac{11}{2}$ extend continuously to $\overline{\mathbb{R}_\pm}$. Using this fact, it is not difficult to verify that

$$i^{k-1}[(u(t)^2)_x]_k = \lim_{\xi \rightarrow 0^+} \frac{d^k}{d\xi^k} [\zeta[\hat{u}(t)^{*2}](\zeta)] - \lim_{\xi \rightarrow 0^-} \frac{d^k}{d\xi^k} [\zeta[\hat{u}(t)^{*2}](\zeta)], \quad k \geq 0.$$

Direct calculations yield

$$\begin{aligned} [(u(t)^2)_x]_0 &= [(u(t)^2)_x]_1 = 0, \quad [(u(t)^2)_x]_2 = -\frac{1}{\pi}[u(t)]_0^2, \\ [(u(t)^2)_x]_3 &= \frac{3}{\pi}[u(t)]_0[u(t)]_1. \end{aligned}$$

In connection with the last formula, we note that the quantity $[u(t)]_0$ is preserved along the classical trajectories of (1) and, hence, $[u(t)]_0 = [u_0]_0 = 0$. Applying the functional $[\cdot]_1$ to the both sides of (1), using partial integration and the identity $[(u^2)_x]_1 = 0$, we arrive at

$$\frac{d}{dt}[u]_1 = \alpha[u]_0,$$

so that $[u(t)]_1 = [u_0]_1 + \alpha t[u_0]_0$. Combining all the formulas together and integrating $e^{As}u(x, t-s)$ with respect to s over $[0, t]$, after some simplifications, (45) follows. $\square$

The proof of Theorem 2 relies on Lemma 4 with $p = 4$. The cases of $p = 1, 2, 3$ can be treated using analogy, and the resulting asymptotic formulas are obtained by truncating (45) at the p -th power of $(x - \alpha t)^{-1}$.

As an immediate consequence of Theorem 2, we have the following analogue of qualitative results obtained in [18] for the Benjamin equation (see also [13–17] for the closely related Benjamin–Ono equation).

Corollary 1. Assume $u_0 \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$, with $s + 2r \geq 3$.

- (i) If $\frac{3}{2} < s < r - \frac{1}{2}$, then $u \in C([0, T], L_{s''}^2(\mathbb{R}))$, for each $0 \leq s'' < \frac{1}{2}$.
- (ii) If $\frac{3}{2} < s < r - \frac{3}{2}$, $u_0 \in L_{s'}^2(\mathbb{R})$ and $\frac{1}{2} < s' \leq \frac{3}{2}$, then $u \in C([0, T], L_{s''}^2(\mathbb{R}))$, for each $0 \leq s'' < \frac{3}{2}$.
- (iii) If $\frac{3}{2} < s < r - \frac{5}{2}$, $u_0 \in L_{s'}^2(\mathbb{R})$ and $\frac{3}{2} < s' \leq \frac{5}{2}$, then $u \in C([0, T], L_{s''}^2(\mathbb{R}))$, for each $0 \leq s'' < \frac{5}{2}$.
- (iv) If $\frac{3}{2} < s < r - \frac{7}{2}$, $\{u_0\}_0 = 0$, $u_0 \in L_{s'}^2(\mathbb{R})$ and $\frac{5}{2} < s' \leq \frac{7}{2}$, then $u \in C([0, T], L_{s''}^2(\mathbb{R}))$, for each $0 \leq s'' < \frac{7}{2}$.
- (v) If $\frac{3}{2} < s < r - \frac{9}{2}$, $u(t_i) \in L_{s'}^2(\mathbb{R})$, at three distinct time points t_i , $i = 0, 1, 2$ and $s' \geq \frac{7}{2}$, then $u(t) = 0$, for all $t \in \mathbb{R}$.
- (vi) If $\frac{3}{2} < s < r - \frac{9}{2}$, $\{u_0\}_0 = 0$ and the associated nontrivial classical solution $u(t)$ satisfies $u(t_0) \in L_{s'}^2(\mathbb{R})$, $s' \geq \frac{7}{2}$, at time $t_0 \in \mathbb{R}$, then $u(t_1) \in L_{s'}^2(\mathbb{R})$, at $t_1 = t_0 - \frac{2\{u(t_0)\}_1}{\delta\|u_0\|_{L^2(\mathbb{R})}^2}$. In particular, there exists a classical solution to (1) with $t_0 \neq t_1$.
- (vii) There exists two distinct and nontrivial classical solutions u_1 and u_2 of (1), satisfying $u_1, u_2 \in L^\infty([0, T]; H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R}))$ and $u_1 - u_2 \in L^\infty([0, T]; L_{s''}^2(\mathbb{R}))$, for each $0 \leq s'' < \frac{9}{2}$ and any finite value of $0 < T < \infty$.

Proof. (a) Items (i)–(iv) follow from a direct evaluation of the coefficients appearing in (45). In particular, in (i) $[u_0]_0 \neq 0$, from assumptions (ii), (iii), and (iv), we have $[u_0]_0 = 0$, $[u_0]_0 = [u_0]_1 = 0$ and $[u_0]_0 = [u_0]_1 = [u_0]_2 = \{u_0\}_0 = 0$, respectively.

(b) If (v) holds, then the quadratic polynomial $q(t) = [u_0]_3 + 6t(\beta\{u_0\}_1 - \gamma[u_0]_0) + 6t^2\delta\beta\|u_0\|_{L^2(\mathbb{R})}^2$ has three distinct real roots and, hence, vanishes identically. In particular, this implies that $\|u_0\|_{L^2(\mathbb{R})}^2 = 0$, and as the last quantity is invariant under the Benjamin flow, $\|u(t)\|_{L^2(\mathbb{R})}^2 = 0$ for all $t \in \mathbb{R}$.

(c) Under the assumptions of (vi), at time t_0 , the principal asymptotic terms in (45) vanish identically. This implies, in particular, that

$$\begin{aligned} [u(t_0)]_0 &= [u(t_0)]_1 = [u(t_0)]_2 = [u(t_0)]_3 = \{u(t_0)\}_0 = 0, \\ 2(t - t_0) \frac{\{u(t_0)\}_1}{\delta\|u_0\|_{L^2(\mathbb{R})}^2} + (t - t_0)^2 &= 0. \end{aligned}$$

Hence, $u(t_1) \in L_{s'}^2(\mathbb{R})$, at $t_1 = t_0 - \frac{2\{u(t_0)\}_1}{\delta\|u_0\|_{L^2(\mathbb{R})}^2}$. Further, for $u_0 \in \mathcal{S}$, with $\frac{d}{d\xi}\hat{u}_0(0) \neq 0$, we have $t_0 = 0$ and $\{u_0\}_1 \neq 0$. As a consequence, the associated classical solution satisfies (vi) with $t_0 \neq t_1$.

(d) Finally, choosing the initial data $u_{0,i} \in \mathcal{S}$, $i = 1, 2$, so that $\frac{d^k}{d\xi^k}\hat{u}_{0,i}(0) = 0$, $k = 0, 1, 2, 3$, $i = 0, 1$, and $\|u_{0,1}\|_{L^2(\mathbb{R})} = \|u_{0,2}\|_{L^2(\mathbb{R})}$, it is not difficult to verify that

$$[u_{0,i}]_0 = [u_{0,i}]_1 = [u_{0,i}]_2 = [u_{0,i}]_3 = \{u_{0,i}\}_0 = \{u_{0,i}\}_1 = 0, \quad i = 1, 2.$$

Further, for the data from the Schwartz class $\mathcal{S}$, the inclusion $u_{0,i} \in H_r^s(\mathbb{R}) \cap H^{s+2r+1}(\mathbb{R})$ holds for any $s, r \geq 0$. Hence, repeating the calculations of Lemma 4 and Theorem 2 verbatim, but with $p = 5$, we obtain

$$u_i(x, t) = \frac{3t^2\delta\beta\|u_{0,i}\|_{L^2(\mathbb{R})}^2}{2\pi(x - \alpha t)^4} + \mathcal{O}(|x|^{-5}), \quad |x| \rightarrow \infty, \quad i = 1, 2.$$

The last identity yields (vii). $\square$

5. Numerical Illustration

We illustrate the main results contained in Theorem 1 and the asymptotic formula (45) given in Theorem 2. In the Figure 1, in the left panel, we notice the asymptotic behaviour of the solution using (45). In the right panel, we illustrate the numerical solution obtained

via the standard Fourier pseudospectral method. Although the initial condition has an exponential decay rate at infinity, the full solution does not sustain this feature and validates the result presented in Theorem 1. The figures in the bottom panel represent the 3D plot and contour plot of the solution in the complex plane obtained using formula (45). The asymptotic behaviour of the solution (dark green) is clearly noticed away from the poles.

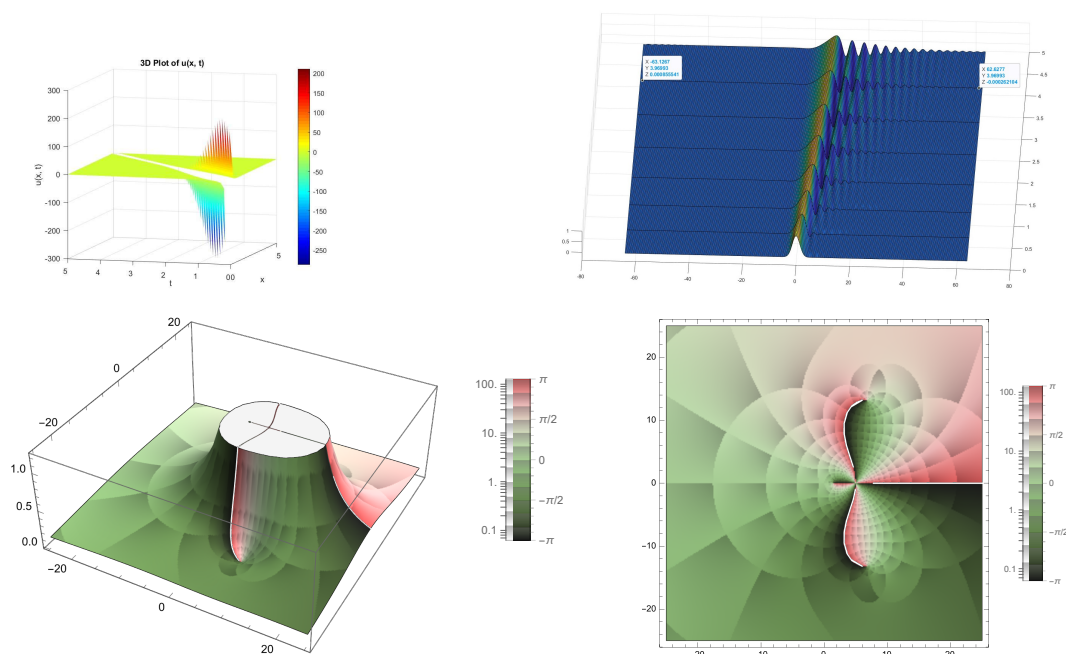


Figure 1. (Top): Solution using asymptotic formula (45) (left); Numerical Solution (right). (Bottom): 3D Complex Plot (left); Contour Plot in the complex plane (right).

6. Conclusions

In this paper, we have shown that, for suitably chosen initial data, the Fourier transforms of classical solutions to (1) remain regular away from the origin. As a result, for such data, the large- x asymptotic of $u(t)$ is controlled solely by the behavior of its Fourier image $\hat{u}(t)$ near the origin, and its bulk asymptotic can be computed explicitly in terms of the input data. The analysis of the paper and, in particular, the explicit large- x asymptotic formula (45) yields a number of qualitative results, similar to those established earlier in [15–18], automatically. Finally, theoretical analysis of the paper lays the foundation for building robust numerical schemes for the Benjamin equation posed in the real line.

Author Contributions: Writing—original draft preparation, N.P. and O.A.; writing—review and editing, N.P. and O.A.; formal analysis, N.P.; visualization, N.P.; software, O.A.; supervision, N.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Benjamin, T. A new kind of solitary waves. *Fluid Mech.* **1992**, *245*, 401–411. [\[CrossRef\]](#)
2. Benjamin, T. Solitary and periodic waves of a new kind. *Philos. Trans. R. Soc. Lond. Ser. A* **1996**, *354*, 1775–1806.
3. Chen, W.; Guo, Z.; Xiao, J. Sharp well-posedness for the Benjamin equation. *Nonlinear Anal.* **2011**, *74*, 6209–6230. [\[CrossRef\]](#)
4. Kozono, H.; Ogawa, T.; Tanisaka, H. Well-posedness for the Benjamin equations. *J. Korean Math. Soc.* **2001**, *38*, 1205–1234.
5. Li, Y.; Wu, Y. Global well-posedness for the Benjamin equation in low regularity. *Nonlinear Anal.* **2010**, *73*, 1610–1625. [\[CrossRef\]](#)
6. Linares, F. L^2 global well-posedness of the initial value problem associated to the Benjamin equation. *J. Differ. Equ.* **1999**, *152*, 377–393. [\[CrossRef\]](#)

7. Cunha, O.R.A. The generalized fractional KdV equation in weighted Sobolev spaces. *arXiv* **2022**, arXiv:2212.11160. [[CrossRef](#)]
8. Kenig, C.E.; Pilod, D.; Ponce, G.; Vega, L. On the unique continuation of solutions to non-local non-linear dispersive equations. *Comm. Partial. Differ. Equ.* **2020**, *45*, 872–886. [[CrossRef](#)]
9. Mendez, A.J. On the propagation of regularity for solutions of the dispersion generalized Benjamin-Ono equation. *Anal. PDE* **2020**, *13*, 2399–2440. [[CrossRef](#)]
10. Riano, O. On persistence properties in weighted spaces for solutions of the fractional Korteweg-de Vries equation. *Nonlinearity* **2021**, *34*, 4604–4660. [[CrossRef](#)]
11. Flamarion, M.V.; Pelinovsky, E. Interaction of Interfacial Waves with an External Force: The Benjamin-Ono Equation Framework. *Symmetry* **2023**, *15*, 1478. [[CrossRef](#)]
12. Fonseca, G.; Riano, O.; Rodriguez-Blanco, G. On The Persistence for the Fractionary BBM equation with low dispersion in weighted Sobolev spaces. *arXiv* **2024**, arXiv:2405.19613v1. [[CrossRef](#)]
13. Iorio, R. On the Cauchy problem for the Benjamin-Ono equation. *Commun. Partial. Differ. Equ.* **1986**, *11*, 1031–1087.
14. Iorio, R. Unique continuation principle for the Benjamin-Ono equation. *Differ. Integral Equ.* **2003**, *16*, 1281–1291. [[CrossRef](#)]
15. Fonseca, G.; Ponce, G. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces. *J. Funct. Anal.* **2011**, *260*, 436–459. [[CrossRef](#)]
16. Fonseca, G.; Linares, F.; Ponce, G. The IVP for the Benjamin-Ono equation in weighted Sobolev spaces II. *J. Funct. Anal.* **2012**, *262*, 2031–2049. [[CrossRef](#)]
17. Fonseca, G.; Linares, F.; Ponce, G. The IVP for the dispersion generalized Benjamin-Ono equation in weighted Sobolev spaces. *Ann. Inst. Henri Poincaré C Anal.* **2013**, *30*, 763–790. [[CrossRef](#)]
18. Urrea, J. The Cauchy problem associated to the Benjamin equation in weighted Sobolev spaces. *J. Differ. Equ.* **2013**, *254*, 1863–1892. [[CrossRef](#)]
19. Shindin, S.; Parumasur, N.; Aluko, O. Analysis of Malmquist-Takenaka-Christov rational approximations with applications to the nonlinear Benjamin equation. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *94*, 105571. [[CrossRef](#)]
20. Bergh, J.; Löfström, J. *Interpolation Spaces: An Introduction*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1976.
21. Aluko, O.B. Analysis of Nonlinear Benjamin Equation Posed on a Real Line. Ph.D. Thesis, University of KwaZulu-Natal, Durban, South Africa, 2022.
22. Banasiak, J.; Parumasur, N.; Poka, W.; Shindin, S. Pseudospectral Laguerre approximation of transport-fragmentation equations. *Appl. Math. Comput.* **2014**, *239*, 107–125. [[CrossRef](#)]
23. Donoghue, W. The interpolation of quadratic norms. *Acta Math.* **1967**, *118*, 251–270. [[CrossRef](#)]
24. Rudin, W. *Functional Analysis*; McGraw-Hill: New York, NY, USA, 1991.
25. Bona, J.; Smith, R. The initial-value Problem for the Korteweg-de Vries equation. *Phil. Trans. R. Soc. Lond.* **1975**, *278*, 555–601. [[CrossRef](#)]
26. Kato, T. On the Korteweg-de Vries equation. *Manuscripta Math.* **1979**, *28*, 89–99. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.