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The Existence of a Solution to a Class of Fractional Double Phase Problems

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Abstract: This paper focuses on the study of a class of fractional $p\&q$-Laplacian problems with unbalanced growth, which includes vanishing potential and a supercritical growth exponent. By employing the mountain pass theorem alongside the Truncation method, penalization method, and Moser iteration method, the main result establishes the existence of a nontrivial solution under conditions of low perturbations of supercritical nonlinearity. Furthermore, we derive $L^\infty(\mathbb{R}^N)$ estimates and the interior Hölder regularity of weak solutions in the context of supercritical growth.

Keywords: fractional double phase problem; supercritical exponent; penalization method; Moser iteration

MSC: 35R11; 47J30; 58E05

1. Introduction and Main Results

In this work, we address the existence of a non-negative solution to the following fractional $p\&q$-Laplacian problems:

$$\begin{cases} (-\Delta)_p^{s_1}u + (-\Delta)_q^{s_2}u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + \lambda|u|^{\gamma-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $0 < s_1 \leqslant s_2 < 1$ and $1 < p < q < \frac{N}{s_2}$, $\gamma \geqslant q_{s_2}^* = \frac{Nq}{N-s_2q}$ and λ is a non-negative parameter. The nonlocal operator $(-\Delta)_t^s$ denotes the fractional t-Laplacian operator, which is defined as follows:

$$(-\Delta)_t^s u(x) = 2\lim_{r\to 0}\int_{\mathbb{R}^N\setminus B_r(x)} \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}}dy \quad (u \in C_c^\infty(\mathbb{R}^N)).$$

The functions V, f are continuous and satisfy the following conditions:

(V) $V \in L_{loc}^\infty(\mathbb{R}^N)$ is non-negative and there exists $R > 0$ such that

$$\Lambda := \frac{1}{R^{(N-s_2q)(q_{s_2}^*-p)}} \inf_{|x|\geqslant R} |x|^{(N-s_2q)(q_{s_2}^*-p)}V(x) > 0.$$

It should be emphasized that $V(x) \to 0$ as $|x| \to \infty$ is permitted by the condition (V).

(f_1) $\lim_{|t|\to 0}\frac{f(t)}{|t|^{p-1}} = 0$;

(f_2) there exists $v \in (q, q_{s_2}^*)$ such that $\lim_{|t|\to\infty}\frac{f(t)}{|t|^{v-1}} = 0$;

(f_3) there exists $\theta \in (q, q_{s_2}^*)$ such that $0 < \theta F(t) := \theta\int_0^t f(\tau)d\tau \leqslant tf(t)$ for all $t > 0$;

(f_4) the map $\frac{f(t)}{|t|^{q-1}}$ is increasing for all $t > 0$.

In light of our objective to identify a solution to problem (1), we posit that $f(t) = 0$ for all $t \leqslant 0$.

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Problems of this nature arise when dealing with two distinct materials characterized by the power hardening exponents p and q. In this context, the fractional operator $(-\Delta)_t^s$ (with $t \in \{p, q\}$) serves to describe the geometric properties of a composite formed from these two materials.

This study focuses on Problem (1). Its key characteristics are as follows:

(i) The presence of two nonlocal operators with differing growth characteristics leads to the emergence of energy associated with a double phase.
(ii) We appropriately truncate the nonlinearity to address the lack of definition associated with the energy functional in the supercritical scenario.
(iii) Owing to the unboundedness of the $\mathbb{R}^N$, we address the challenge posed by the absence of compactness in $(PS)_c$ sequences by utilizing a penalization method.
(iv) The reaction integrates the various effects arising from the characteristic that potential can vanish at infinity, alongside the presence of both subcritical and supercritical nonlinearities.
(v) Our analysis combines the variational technique with the Moser iteration method.

In recent years, there has been a significant amount of research focused on nonlocal operators, particularly concerning fractional double phase patterns, owing to their compelling theoretical framework and diverse practical applications. These applications encompass thin obstacle problems, conservation laws, phase transitions, image processing, and anomalous diffusion, among others. Given that the focus of this paper is primarily on double phase problems, we will commence with a brief overview of the development of this research area. As far as we know, the earliest work on nonlinear problems related to unbalanced growth can be attributed to Ball [1], who examined nonlinear elasticity and its qualitative characteristics, including cavitation phenomena and discontinuous equilibrium solutions.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundaries. Define $u : \Omega \to \mathbb{R}^N$ as the displacement function, and let Du represent the $N \times N$ matrix corresponding to the deformation gradient. Therefore, the total energy can be expressed as an integral of the following form:

$$I(u) = \int_\Omega f(x, Du(x))dx, \tag{2}$$

where the energy function $f = f(x, \xi) : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}$ is quasiconvex in relation to the variable ξ. A straightforward illustration provided by Ball is given by functions f of the type $f(\xi) = g(\xi) + h(\det \xi)$. Here, $\det \xi$ denotes the determinant of the $N \times N$ matrix ξ, and g and h are non-negative convex functions that satisfy the following growth conditions:

$$g(\xi) \geqslant c_1|\xi|^p, \quad \lim_{t\to+\infty} h(t) = +\infty,$$

where c_1 is a positive constant, and $1 < p \leq N$. The condition $p \leq N$ was necessary for studying the existence of equilibrium solutions that incorporate cavities. The aforementioned patterns are closely associated with the examination of nonlinear issues and stationary wave phenomena in models pertinent to mathematical physics, including but not limited to composite materials, the stability of nonlinear damped Kirchhoff systems, fractional quantum mechanics in the context of particles interacting with stochastic fields, fractional super diffusion, and the fractional white-noise limit (refer to [2–4]). As a result, the investigation of double phase problems characterized by varying growth rates of the function f at both the origin and infinity has garnered significant scholarly interest.

When $s_1 = s_2 = 1,\ p = q = 2,\ \lambda = 0,$ problem (1) reduces to the following:

$$-\Delta u + V(x)u = f(u) \ \text{ in } \ \mathbb{R}^N. \tag{3}$$

Nonlinear elliptic equations of the form (3), which include a potential that vanishes at infinity, have been the subject of considerable investigation in the literature, as evidenced

in [5,6]. In particular, the study focused on solutions to a Schrödinger logarithmic equation characterized by a deepening potential well, as discussed in [7,8].

When $s_1 = s_2 = 1$, Equation (1) is transformed into a $p\&q$ elliptic problem of the following form:

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(u) + |u|^{\gamma-2}u \quad x \in \mathbb{R}^N. \tag{4}$$

As elucidated in [9], a primary motivation for investigating (4) is its relationship to more general reaction–diffusion systems:

$$u_t = \operatorname{div}(D(u)\nabla u) + c(x,u) \text{ and } D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

This phenomenon is observed in the fields of biophysics, plasma physics, and the design of chemical reactions. In these applications, the variable u represents the concentration, while the term $\operatorname{div}(D(u)\nabla u)$ denotes the diffusion with the diffusion coefficient $D(u)$. The reaction term $c(x,u)$ is associated with both the source and loss mechanisms. Generally, within chemical and biological frameworks, the reaction term $c(x,u)$ is typically expressed in polynomial form relative to the concentration u. As a result, quasilinear elliptic boundary value problems that incorporate this operator have been extensively examined in the existing literature, as evidenced by works such as [10,11].

In the nonlocal setting, its research has garnered significant attention because of two phenomena: the nonlinearity of the operator and its nonlocal nature. For instance, when $0 < s_1 = s_2 < 1$, $p = q = 2$, problem (1) simplifies to a fractional Schrödinger equation of the following form:

$$(-\Delta)^s u + V(x)u = f(u) + \lambda|u|^{\gamma-2} \text{ in } \mathbb{R}^N. \tag{5}$$

The motivation for problem (5) arises from the investigation of standing wave solutions within a specific class of fractional Schröinger equations, which effectively represent a variety of physical phenomena. Cardoso et al. [12] explored the existence of positive solutions to Equation (5) by employing variational techniques alongside the Moser iteration method. A notable aspect of their work is the simultaneous consideration of two scenarios, one in which $\gamma > 2_s^*$, and another where the potential can vanish at infinity. For further insights into the case when $\lambda = 0$, refer to [13,14]. Additionally, some researchers have concentrated on analyzing solutions to Equation (5) in which f exhibits critical or subcritical growth behavior.

A substantial body of literature has been dedicated to investigating the existence and regularity of solutions to the fractional p-Laplacian equation. For instance, in the case where $0 < s_1 = s_2 < 1$, $\lambda = 1$, the problem described in Equation (1) can be reformulated as a fractional $p\&q$-Laplacian equation:

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) + |u|^{\gamma-2}u \quad x \in \mathbb{R}^N, \tag{6}$$

In the research conducted by Zhang et al. [15], based on the assumptions $(f_1) - (f_4)$, they proved the existence of multiple positive solutions to problem (6) in the context of both critical and supercritical growth with respect to γ, where the potential function V attains its positive minimum and satisfies some suitable conditions. In addition, the authors in [16] employed the minimax theorem in conjunction with the Ljusternik–Schnirelmann theory to establish the existence of nontrivial solutions to Equation (6). This was achieved under the condition that the nonlinear term is a C^1 function exhibiting subcritical growth, notably without imposing the Ambrosetti–Rabinowitz (AR) condition, because this condition is very restrictive and eliminates many nonlinearities. Consequently, numerous researchers have sought to propose conditions that are less stringent than the (AR) condition. Several findings regarding fractional $p\&q$-Laplacian problems set in bounded domains or in the whole of $\mathbb{R}^N$ can be found in [17–19] and the references therein.

When $0 < s_2 \leqslant s_1 < 1$, Kumar and Sreenadh [20] studied the following unbalanced problem in the entire space $\mathbb{R}^N$:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)\frac{f(u)}{|x|^\beta} \quad x \in \mathbb{R}^N, \tag{7}$$

where $1 < q < p$, $ps_1 = N$ and $\beta \in [0, N)$, and the functions V, K, f are continuous and satisfy certain natural hypotheses. The authors established the existence of a nontrivial, nonnegative solution to (7) by combining a Moser–Trudinger-type inequality and variational method. Their findings pertain to scenarios in which the function f satisfies both subcritical and critical growth conditions concerning exponential nonlinearity. For further insights into $p\&q$-fractional elliptic problems with unbalanced growth, we refer to [21–23].

Inspired by the previously mentioned studies, the primary focus of our paper is the study of a class of fractional unbalanced double phase Problems (1). We point out that, to the best of our knowledge, there are only a few papers in the existing literature that address fractional $p\&q$-Laplacian problems with unbalanced growth. Furthermore, there are no results concerning the existence and regularity of solutions to Problem (1) in scenarios where the nonlinearity demonstrates both subcritical and supercritical growth and the potential may vanish at infinity. Consequently, the aim of this study is to provide an initial contribution in this area.

Our main result regarding the existence of solutions is as follows:

Theorem 1. *Assume that* (V) *and* $(f_1) - (f_4)$ *hold. Then, there exist* λ_0 *and* Λ_0 *such that for each* $\lambda \in [0, \lambda_0)$ *and* $\Lambda \geqslant \Lambda_0$, *Problem* (1) *has a bounded, nontrivial, non-negative solution.*

Remark 1. *It is worth mentioning that if condition V is replaced by*

$$\Lambda =: \liminf_{|x|\to\infty} |x|^{(N-s_2q)(q^*_{s_2}-p)} V(x) > 0,$$

the conclusion of Theorem 1 is still valid.

The proof of Theorem 1 is derived through the application of appropriate variational and topological arguments in [15,24]. First, it is important to note that Problem (1) contains a supercritical nonlinear term that renders the corresponding functional not well-defined within the space $\mathbb{X}_V$. To address this, we implement a suitable truncation of the nonlinearity on the right-hand side of Equation (1). Following this, to ensure the (PS) condition for the functional associated with the truncated equation and to derive uniform estimates for the solutions of this truncated equation, we employ the penalization method outlined in [25]. This approach involves an appropriate modification of the nonlinearity $f(u) + \lambda|u|^{\gamma-2}u$, which allows us to demonstrate the existence of a solution to the auxiliary problem. Ultimately, we show that, due to the significant impact of the supercritical term, the solution to the auxiliary problem also fulfills the the original Problem (1) for sufficiently small values of the positive parameter $\lambda > 0$.

Our main result on regularity is as follows:

Theorem 2. *Let* $\gamma \geqslant q^*_{s_2}$ *and u be a weak solution to problem* (1). *Then, there exists* $\alpha \in (0, 1)$ *such that* $u \in L^\infty(\mathbb{R}^N) \cap C^{0,\alpha}_{loc}(\mathbb{R}^N)$.

Remark 2. *Ambrosio et al. [19] studied the regularity results and proved the interior Hölder regularity of weak solutions to the following problem:*

$$(-\Delta)_p^{s} u + (-\Delta)_q^{s} u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad in \ \mathbb{R}^N,$$

where $V(x)$ *is a continuous potential that attains its positive minimum, and* f *is a continuous nonlinearity characterized by subcritical growth. In the present study, we do not impose a strict*

positivity condition on the potential $V(x)$ and permit $V(x) \to 0$ as $|x| \to \infty$, while the nonlinearity is characterized by supercritical growth. Additionally, in Theorem 2, we establish the conditions $0 < s_1 \leq s_2 < 1$ and $\lambda \neq 0$, in contrast to the conditions $0 < s_1 = s_2 < 1$ and $\lambda = 0$ presented in [19]. Consequently, Theorem 2 cannot be regarded as a merely trivial extension of the results found in [19].

We employ an appropriate Moser iteration method [26] to derive an $L^\infty(\mathbb{R}^N)$-estimate for the weak solution of the auxiliary problem and establish a interior Hölder regularity of weak solutions to problem (1). This result extends the interior regularity findings established in [27] for the fractional p-Laplacian to the fractional $p\&q$-case.

In this paper, the symbols C, C_1, C_2 denote positive constants, the specific values of which may vary from one instance to another throughout the text.

This paper is structured in the following manner: Section 2 establishes the mathematical framework necessary for the analysis of the solution to problem (1). In Section 3, we structure an auxiliary problem and demonstrate the existence of a solution for it. Section 4 presents an estimate pertaining to the $L^\infty(\mathbb{R}^N)$ norm of the solution to the auxiliary problem. Finally, Section 5 establishes the existence and regularity results for problem (1).

2. Preliminaries

Let $1 < p < \infty$ and $0 < s < 1$. Define $D^{s,p}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the following norm:

$$[u]_{s,p}^p := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy.$$

The fractional Sobolev space is defined as follows:

$$W^{s,p}(\mathbb{R}^N) := \{u : |u|_p < +\infty, [u]_{s,p} < +\infty\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := (|u|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where $|u|_p^p = \int_{\mathbb{R}^N} |u|^p dx$.

For $u, v \in W^{s,p}(\mathbb{R}^N)$, we define

$$\langle u, v\rangle_{s,p} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} dxdy.$$

Let us revisit the following essential embedding:

Theorem 3. *[28] Let $N > sp$. Then, there exists a constant $S_* := S_*(N, s, p) > 0$ such that*

$$|u|_{p_s^*}^p \leqslant S_*^{-1}[u]_{s,p}^p \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N).$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ for any $r \in [p, p_s^*]$ and compactly embedded in $L_{loc}^r(\mathbb{R}^N)$ for any $r \in [1, p_s^*)$.

Lemma 1. *Let $1 < p < q < \frac{N}{s_2}$, $0 < s_1 \leqslant s_2 < 1$. Then, there exists a constant $C > 0$ such that*

$$[u]_{s_1,p} \leqslant C[u]_{s_2,q} \quad \text{for all } u \in D^{s_2,q}(\mathbb{R}^N).$$

Proof. It is adequate to demonstrate the lemma for all functions $u \in C_c^\infty(\mathbb{R}^N)$. Consequently, let $u \in C_c^\infty(\mathbb{R}^N)$, from which it follows that

$$\begin{aligned}[u]_{s_1,p}^p &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy = \iint_{\mathbb{R}^{2N}} \frac{|u(x+h)-u(x)|^p}{|h|^{N+s_1p}}dxdh \\ &= \int_{|h|>1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{N+s_1p}}dxdh + \int_{|h|\leqslant 1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{N+s_1p}}dxdh \\ &:= A_1 + A_2.\end{aligned}$$

By utilizing the Hölder inequality, we derive the following result:

$$\begin{aligned}A_1 &= \int_{|h|>1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{N+s_1p}}dxdh \leqslant C\int_{|h|>1}\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|h|^{N+s_1p}}dxdh \\ &\leqslant C\int_{|h|>1} \frac{dh}{|h|^{N+s_1p}}\int_{\mathbb{R}^N}|u(x)|^p dx \leqslant C_1\int_{\mathbb{R}^N}|u(x)|^p dx = C_1|u|_p^p.\end{aligned}$$

Now, consider

$$A_2 = \int_{|h|\leqslant 1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{N+s_1p}}dxdh = \int_{|h|\leqslant 1} \frac{dh}{|h|^{N-(s_2-s_1)p}}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{s_2p}}dx.$$

Since $u \in C_c^\infty(\mathbb{R}^N)$, it follows that for every h with $|h| < 1$, the expression $u(x+h)-u(x)$ possesses compact support. Accordingly,

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^{s_2p}}dx &\leqslant C\int_{\mathbb{R}^N}\left(\frac{|u(x+h)-u(x)|^q}{|h|^{s_2q}}dx\right)^{\frac{p}{q}} \\ &\leqslant C\sup_{h>0}\int_{\mathbb{R}^N}\left(\frac{|u(x+h)-u(x)|^q}{|h|^{s_2q}}dx\right)^{\frac{p}{q}} \\ &\leqslant C_1(1-s_2)^{\frac{p}{q}}[u]_{s_2,q}^p,\end{aligned}$$

take note that Lemma A.1 in [29] confirms the validity of the final inequality. Therefore, we can conclude that

$$A_2 \leqslant C_1(1-s_2)^{\frac{p}{q}}[u]_{s_2,q}^p\int_{|h|\leqslant 1}\frac{dh}{|h|^{N-(s_2-s_1)p}} \leqslant C_2[u]_{s_2,q}^p.$$

Ultimately, we employ the Poincaré inequality to deduce that

$$[u]_{s_1,p}^p \leqslant C(|u|_p^p + [u]_{s_2,q}^p) \leqslant C_1[u]_{s_2,q}^p.$$

□

In order to address fractional $p\&q$-Laplacian problems, define

$$\mathbb{X} := W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N)$$

equipped with the norm

$$\|u\|_{\mathbb{X}} := \|u\|_{W^{s_1,p}(\mathbb{R}^N)} + \|u\|_{W^{s_2,q}(\mathbb{R}^N)}.$$

We also define the following Banach space:

$$\mathbb{X}_V := \left\{u \in \mathbb{X} : \int_{\mathbb{R}^N} V(x)(|u|^p + |u|^q)dx < \infty\right\}$$

endowed with the norm

$$\|u\|_{\mathbb{X}_V} := \|u\|_{V_{s_1,p}} + \|u\|_{V_{s_2,q}},$$

where

$$\|u\|_{V_{s_j,t}} := \left([u]^t_{s_j,t} + \int_{\mathbb{R}^N} V(x)|u|^t dx\right)^{\frac{1}{t}} \text{ for all } t > 1, \quad j \in \{1,2\}.$$

Definition 1. *A function $u \in \mathbb{X}_V$ is said to be a weak solution of problem (1) if*

$$\begin{aligned} &\langle u, \varphi\rangle_{s_1,p} + \langle u, \varphi\rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x)(|u|^{p-2}u + |u|^{q-2}u)\varphi dx \\ &= \int_{\mathbb{R}^N} f(u)\varphi dx + \lambda \int_{\mathbb{R}^N} |u|^{\gamma-1}u\varphi dx \qquad \text{for all } \varphi \in \mathbb{X}_V. \end{aligned}$$

It is commonly known that a weak solution to problem (1) serves as a critical point of the associated functional:

$$I_\lambda(u) := \frac{1}{p}\|u\|^p_{V_{s_1,p}} + \frac{1}{q}\|u\|^q_{V_{s_2,q}} - \int_{\mathbb{R}^N} F(u)dx - \frac{\lambda}{\gamma}\int_{\mathbb{R}^N} |u|^\gamma dx. \tag{8}$$

The assumptions (f_1) and (f_2), in conjunction with Theorem 3, indicate that the functional $I_\lambda(u)$ is well-defined in $\mathbb{X}_V$ if and only if $\gamma = q^*_{s_2}$. Consequently, the direct application of variational methods based on Equation (8) to analyze problem (1) is not feasible when $\gamma > q^*_{s_2}$. To address this challenge, we will propose a modification to the nonlinearity $f(u) + \lambda|u|^{\gamma-2}u$, which will be elaborated upon in the subsequent section.

3. Auxiliary Problems

Due to the presence of a supercritical nonlinear term in problem (1), we begin by appropriately truncating the nonlinearity $f(u) + \lambda|u|^{\gamma-2}u$. For an arbitrary integer $k \in \mathbb{N}$, we define the function $g_{\lambda,k} : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_{\lambda,k}(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ f(t) + \lambda t^{\gamma-1}, & \text{if } 0 \leqslant t \leqslant k, \\ f(t) + \lambda k^{\gamma-v}t^{v-1}, & \text{if } t \geqslant k. \end{cases} \tag{9}$$

It is evident that $g_{\lambda,k}$ adheres to the conditions $(f_1) - (f_4)$. Based on assumption (f_2) and (9), it can be inferred that

$$g_{\lambda,k}(t) \leqslant f(t) + \lambda k^{\gamma-v}t^{v-1} \leqslant C(1 + \lambda k^{\gamma-v})t^{v-1} \quad \text{for all } t \geqslant 0. \tag{10}$$

Additionally, let $G_{\lambda,k}(t) = \int_0^t g_{\lambda,k}(\tau)d\tau$, then

$$G_{\lambda,k}(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ F(t) + \frac{\lambda}{\gamma}t^\gamma, & \text{if } 0 \leqslant t \leqslant k, \\ F(t) + \frac{\lambda}{v}k^{\gamma-v}t^v + \lambda(\frac{1}{\gamma} - \frac{1}{v})k^\gamma, & \text{if } t \geqslant k \end{cases} \tag{11}$$

and

$$G_{\lambda,k}(t) \leqslant \frac{C}{v}(1 + \lambda k^{\gamma-v})t^v \quad \text{for all } t \geqslant 0. \tag{12}$$

In relation to the function $g_{\lambda,k}$, we examine the following auxiliary equation:

$$\begin{cases} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = g_{\lambda,k}(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{$A_{\lambda,k}$}$$

It is acknowledged that the embedding $\mathbb{X}_V \hookrightarrow L^r(\mathbb{R}^N)$ holds for any $r \in [q, q^*_{s_2}]$. Consequently, in view of (12), the energy functional $I_{\lambda,k}$ associated with problem $(A_{\lambda,k})$ is defined as follows:

$$I_{\lambda,k}(u) = \frac{1}{p}\|u\|^p_{V_{s_1,p}} + \frac{1}{q}\|u\|^q_{V_{s_2,q}} - \int_{\mathbb{R}^N} G_{\lambda,k}(u)dx$$

is well defined on $\mathbb{X}_V$. However, in addressing the auxiliary problem $(A_{\lambda,k})$, two primary challenges arise: the need to satisfy the (PS) condition for the functional $I_{\lambda,k}$ and the establishment of uniform estimates (independent of k and λ) for the solutions of $(A_{\lambda,k})$. Therefore, we adapted the penalization method to formulate a second auxiliary problem.

Denoting $\eta = \frac{q\theta}{\theta - q}$, for $(x,t) \in \mathbb{R}^N \times \mathbb{R}_+$, define

$$\tilde{g}_{\lambda,k}(x,t) = \begin{cases} g_{\lambda,k}(t), & \text{if } \eta\, g_{\lambda,k}(t) \leqslant V(x)(|t|^{p-2}t + |t|^{q-2}t), \\ \frac{1}{\eta}V(x)(|t|^{p-2}t + |t|^{q-2}t), & \text{if } \eta\, g_{\lambda,k}(t) > V(x)(|t|^{p-2}t + |t|^{q-2}t) \end{cases}$$

and $\tilde{g}_{\lambda,k}(x,t) = 0$ if $x \in \mathbb{R}^N$ and $t \leqslant 0$. Moreover, taking into account that $R > 1$, as stipulated in condition (V), we shall proceed to define the following:

$$h_{\lambda,k}(x,t) = \begin{cases} g_{\lambda,k}(t), & \text{if } |x| \leqslant R, \\ \tilde{g}_{\lambda,k}(x,t), & \text{if } |x| > R. \end{cases} \tag{13}$$

Then, it is clear that $h_{\lambda,k}$ satisfies the following assumptions:

- (h_1) for each $\lambda > 0$, $\lim_{|t|\to 0} \frac{h_{\lambda,k}(x,t)}{|t|^{p-1}} = 0$;
- (h_2) for each $\lambda > 0$, $h_{\lambda,k}(x,t) \leqslant g_{\lambda,k}(t)$ and $H_{\lambda,k}(x,t) \leqslant G_{\lambda,k}(t)$ for all $x \in \mathbb{R}^N$ and $t > 0$;
- (h_3) for each $\lambda > 0$, $0 < \theta H_{\lambda,k}(x,t) := \theta \int_0^t h_{\lambda,k}(x,\tau)d\tau \leqslant h_{\lambda,k}(x,t)t$ for all $|x| \leqslant R$ and $t > 0$;
- (h_4) for each $\lambda > 0$, $0 < pH_{\lambda,k}(x,t) \leqslant h_{\lambda,k}(x,t)t \leqslant \frac{1}{\eta}V(x)(t^p + t^q)$ for all $|x| > R$ and $t > 0$;
- (h_5) for each $\lambda > 0$ and $x \in \mathbb{R}^N$, the map $t \mapsto \frac{h_{\lambda,k}(t)}{t^{p-1}+t^{q-1}}$ is increasing for all $t > 0$.

Function (13) enables us to introduce a second auxiliary problem:

$$\begin{cases} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = h_{\lambda,k}(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{s_1,p}(\mathbb{R}^N) \cap W^{s_2,q}(\mathbb{R}^N),\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{$B_{\lambda,k}$}$$

The energy functional $J_{\lambda,k} : \mathbb{X}_V \to \mathbb{R}$ associated with problem $(B_{\lambda,k})$ is given as follows:

$$J_{\lambda,k}(u) = \frac{1}{p}\|u\|^p_{V_{s_1,p}} + \frac{1}{q}\|u\|^q_{V_{s_2,q}} - \int_{\mathbb{R}^N} H_{\lambda,k}(x,u)dx. \tag{14}$$

It is not difficult to verify that $J_{\lambda,k} \in C^1(\mathbb{X}_V, \mathbb{R})$ and its derivative can be expressed as follows:

$$\begin{aligned} \langle J'_{\lambda,k}(u), \varphi\rangle =& \langle u, \varphi\rangle_{s_1,p} + \langle u, \varphi\rangle_{s_2,q} + \int_{\mathbb{R}^N} V(x)(|u|^{p-2}u + |u|^{q-2}u)\varphi dx \\ & - \int_{\mathbb{R}^N} h_{\lambda,k}(x,u)\varphi dx \qquad \text{for all } u, \varphi \in \mathbb{X}_V. \end{aligned} \tag{15}$$

According to the definition of $h_{\lambda,k}$, the critical points of $J_{\lambda,k}$ correspond precisely to the weak solutions to the problem $(B_{\lambda,k})$. A key characteristic of the modified function is its adherence to a compactness condition. Subsequently, we demonstrate that $J_{\lambda,k}$ possesses mountain pass geometry.

Lemma 2. *The functional $J_{\lambda,k}$ satisfies the following conditions:*
(i) there exists $\alpha, \rho > 0$ such that $J_{\lambda,k}(u) \geqslant \alpha$ for $\|u\|_{\mathbb{X}_V} = \rho$;
(ii) there exists $e \in \mathbb{X}_V$ with $\|e\|_{\mathbb{X}_V} > \rho$ and $J_{\lambda,k}(e) < 0$.

Proof. (i) Based on (12) and (h_2), we have the following:

$$J_{\lambda,k}(u) \geqslant \frac{1}{p}\|u\|^p_{V_{s_1,p}} + \frac{1}{q}\|u\|^q_{V_{s_2,q}} - \frac{C}{v}(1+\lambda k^{\gamma-v})|u|^v_v.$$

Choosing $\|u\|_{\mathbb{X}_V} = \rho \in (0,1)$ and considering $1 < p < q$, we have $\|u\|_{V_{s_1,p}} < 1$ and $\|u\|^p_{V_{s_1,p}} \geqslant \|u\|^q_{V_{s_1,p}}$. This fact, combined with

$$a^t + b^t \geqslant C_t(a+b)^t \quad \forall a, b \geqslant 0 \quad \forall t > 1,$$

and Sobolev embedding, implies

$$J_{\lambda,k}(u) \geqslant \frac{1}{q}(\|u\|^q_{V_{s_1,p}} + \|u\|^q_{V_{s_2,q}}) - \frac{C}{v}(1+\lambda k^{\gamma-v})\|u\|^v_{\mathbb{X}_V} \geqslant \frac{C_t}{q}\|u\|^q_{\mathbb{X}_V} - C_1\|u\|^v_{\mathbb{X}_V},$$

for some constant $C_1 > 0$. Since $v > q$, there exists $\alpha > 0$ such that $J_{\lambda,k}(u) \geqslant \alpha$ for $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.

(ii) Take $u \in C_0^\infty(\mathbb{R}^N)$ such that $u \geqslant 0, u \neq 0$. Based on (11) and (f_3), it follows

$$G_{\lambda,k}(s) \geqslant F(s) \geqslant C_1 s^\theta - C_2,$$

for all $s \geqslant 0$. Then, we obtain

$$\begin{aligned} J_{\lambda,k}(tu) &\leqslant \frac{t^p}{p}\|u\|^p_{V_{s_1,p}} + \frac{t^q}{q}\|u\|^q_{V_{s_2,q}} - \int_{supp(u)} F(tu)dx \\ &\leqslant \frac{t^p}{p}\|u\|^p_{V_{s_1,p}} + \frac{t^q}{q}\|u\|^q_{V_{s_2,q}} - C_1 t^\theta \int_{supp(u)} u^\theta dx + C_2|supp(u)|, \end{aligned}$$

for $t \geqslant 0$ and some C_1, C_2. Therefore, one has $J_{\lambda,k}(tu) \to -\infty$ as $t \to +\infty$. By defining $e = tu$ with t being sufficiently large, condition (ii) is satisfied. □

A function $\psi \in C^1(X, \mathbb{R})$ (X is a Banach space) satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ ($(PS)_c$ condition for short) if any sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\psi(v_n) \to c \quad \text{and} \quad \|\psi'(v_n)\|_{X'} \to 0 \ \text{ as } \ n \to \infty$$

has a convergent subsequence.

Note that the sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$ such that $\psi(v_n) \to c$ and $\|\psi'(v_n)\|_{X'} \to 0$ as $n \to \infty$ is called the $(PS)_c$ sequence at level $c \in \mathbb{R}$.

We start by proving the boundedness of the Palais–Smale sequences.

Lemma 3. *If $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_V$ is a $(PS)_c$ sequence for $J_{\lambda,k}$, then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_V$.*

Proof. Using (h_3), (h_4) and $q > p$, for $n \in \mathbb{N}$ large enough, we observe

$$\begin{aligned}
C(1+\|u_n\|_{\mathbb{X}_V}) \geqslant & J_{\lambda,k}(u_n) - \frac{1}{\theta}\langle J'_{\lambda,k}(u_n), u_n\rangle \\
= & \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|^p_{V_{s_1,p}} + \left(\frac{1}{q}-\frac{1}{\theta}\right)\|u_n\|^q_{V_{s_2,q}} \\
& + \frac{1}{\theta}\int_{\{|x|\leqslant R\}}[h_{\lambda,k}(x,u_n)u_n - \theta H_{\lambda,k}(x,u_n)]dx \\
& + \frac{1}{\theta}\int_{\{|x|> R\}}[h_{\lambda,k}(x,u_n)u_n - \theta H_{\lambda,k}(x,u_n)]dx \\
\geqslant & \left(\frac{1}{q}-\frac{1}{\theta}\right)(\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}}) - \int_{\{|x|>R\}} H_{\lambda,k}(x,u_n)dx \\
\geqslant & \left(\frac{1}{q}-\frac{1}{\theta}\right)(\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}}) - \frac{1}{p\eta}\int_{\{|x|>R\}} V(x)(|u_n|^p + |u_n|^q)dx \\
\geqslant & \left[\left(\frac{1}{q}-\frac{1}{\theta}\right) - \frac{1}{p\eta}\right](\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}}) \\
= & \left(1-\frac{1}{p}\right)\left(\frac{1}{q}-\frac{1}{\theta}\right)(\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}}) \\
= & C_1(\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}}),
\end{aligned}$$

where $\eta = \frac{q\theta}{\theta - q}$ as defined above and $C_1 > 0$ because of $2 \leqslant p < q < \theta$.

Now, assume by contradiction that $\|u_n\|_{\mathbb{X}_V} \to \infty$. We have the following cases:

Case 1.$\|u_n\|_{V_{s_1,p}} \to \infty$ and $\|u_n\|_{V_{s_2,q}} \to \infty$.

Then, choose n to be sufficiently large such that $\|u_n\|^{q-p}_{V_{s_2,q}} \geqslant 1$. This indicates that $\|u_n\|^q_{V_{s_2,q}} \geqslant \|u_n\|^p_{V_{s_2,q}}$, which consequently implies that

$$C(1+\|u_n\|_{\mathbb{X}_V}) \geqslant C_1(\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^p_{V_{s_2,q}}) \geqslant C_2(\|u_n\|_{V_{s_1,p}} + \|u_n\|_{V_{s_2,q}})^p.$$

This creates a contradiction.

Case 2. $\|u_n\|_{V_{s_1,p}} \to \infty$ and $\|u_n\|_{V_{s_2,q}}$ is bounded.

In this case, we have

$$C(1+\|u_n\|_{\mathbb{X}_V}) = C(1+\|u_n\|_{V_{s_1,p}} + \|u_n\|_{V_{s_2,q}}) \geqslant C_1\|u_n\|^p_{V_{s_1,p}},$$

and thus

$$C\left(\frac{1}{\|u_n\|^p_{V_{s_1,p}}} + \frac{1}{\|u_n\|^{p-1}_{V_{s_1,p}}} + \frac{\|u_n\|_{V_{s_2,q}}}{\|u_n\|^p_{V_{s_1,p}}}\right) \geqslant C_1.$$

Since $p > 1$, we have $0 < C_1 \leqslant 0$ as $n \to \infty$. This is a contradiction.

Case 3. $\|u_n\|_{V_{s_2,q}} \to \infty$ and $\|u_n\|_{V_{s_1,p}}$ is bounded.

We can proceed as in Case 2.

As a consequence, the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_V$ is bounded. □

Next, we will demonstrate that the auxiliary function satisfies the Palais–Smale condition.

Lemma 4. *The functional $J_{\lambda,k}$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

Proof. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_V$ is a $(PS)_c$ sequence for the functional $J_{\lambda,k}$; that is,

$$J_{\lambda,k}(u_n) \to c \text{ and } J'_{\lambda,k}(u_n) \to 0 \text{ as } n \to \infty.$$

From Lemma 3, up to a subsequence, there exists $u \in \mathbb{X}_V$ such that $u_n \rightharpoonup u$ in $\mathbb{X}_V$. Since the Sobolev embedding $\mathbb{X}_V \hookrightarrow L^t_{loc}(\mathbb{R}^N)$ is compact for any $t \in [1, q^*_{s_2})$, combining (h_2), (10) and utilizing the Hölder inequality, we reach

$$\int_{|x|\leqslant r} h_{\lambda,k}(x,u_n)(u_n-u)dx = o_n(1) \quad \text{and} \quad \int_{|x|\leqslant r} h_{\lambda,k}(x,u)(u_n-u)dx = o_n(1), \tag{16}$$

for each $r > 0$. Moreover, by choosing a sufficiently large value for r and utilizing (h_2), (10), along with the the Hölder inequality, we obtain

$$\int_{|x|>r} h_{\lambda,k}(x,u_n)udx = o_n(1) \quad \text{and} \quad \int_{|x|>r} h_{\lambda,k}(x,u)(u_n-u)dx = o_n(1). \tag{17}$$

We will subsequently demonstrate that the weak limit u serves as a critical point of $J_{\lambda,k}$. Let us analyze the sequence

$$U_n(x,y) = \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1p}{p'}}}dxdy,$$

and let

$$U(x,y) = \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_1p}{p'}}}dxdy,$$

where $p' = \frac{p}{p-1}$. It can be readily established that the sequence U_n is bounded in $L^{p'}(\mathbb{R}^{2N})$ and $U_n \to U$ almost everywhere in $\mathbb{R}^{2N}$. Furthermore, there exists a subsequence, which we continue to denote as U_n, such that $U_n \rightharpoonup U$ in $L^{p'}(\mathbb{R}^{2N})$ because $L^{p'}(\mathbb{R}^{2N})$ is a reflexive space; that is,

$$\iint_{\mathbb{R}^{2N}} U_n(x,y)m(x,y)dxdy \to \iint_{\mathbb{R}^{2N}} U(x,y)m(x,y)dxdy \quad \forall m \in L^p(\mathbb{R}^{2N}).$$

Then, for any $\phi \in C^\infty_c(\mathbb{R}^N)$, taking

$$m(x,y) = \frac{\phi(x)-\phi(y)}{|x-y|^{\frac{N+s_1p}{p}}} \in L^p(\mathbb{R}^{2N}),$$

and then

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+s_1p}}dxdy \\ \to &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+s_1p}}dxdy. \end{aligned}$$

In a similar manner, we possess

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+s_2q}}dxdy \\ \to &\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+s_2q}}dxdy. \end{aligned}$$

Taking into account that

$$\int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\phi dx \to \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\phi dx,$$
$$\int_{\mathbb{R}^N} V(x)|u_n|^{q-2}u_n\phi dx \to \int_{\mathbb{R}^N} V(x)|u|^{q-2}u\phi dx,$$
$$\int_{\mathbb{R}^N} h_{\lambda,k}(x,u_n)\phi dx \to \int_{\mathbb{R}^N} h_{\lambda,k}(x,u)\phi dx.$$

Then, using the above limits and $\langle J'_{\lambda,k}(u_n),\phi\rangle = o_n(1)$, we obtain $\langle J'_{\lambda,k}(u),\phi\rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^N)$, which together with the density of $C_c^\infty(\mathbb{R}^N)$ in $\mathbb{X}_V$ implies that u is a critical point of $J_{\lambda,k}$.

Using $\langle J'_{\lambda,k}(u_n),u_n\rangle = o_n(1)$ and $\langle J'_{\lambda,k}(u),u\rangle = 0$, we know

$$\begin{aligned} o_n(1) =&(J'_{\lambda,k}(u_n) - J'_{\lambda,k}(u))\cdot(u_n-u)\\ =&\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}} - \|u\|^p_{V_{s_1,p}} - \|u\|^q_{V_{s_2,q}}\\ &- \int_{\mathbb{R}^N}[h_{\lambda,k}(x,u_n) - h_{\lambda,k}(x,u)](u_n-u)dx, \end{aligned}$$

and based on (16) and (17) and (h_4), for r values that are sufficiently large such that $r > R$, we obtain

$$\begin{aligned} &\|u_n\|^p_{V_{s_1,p}} + \|u_n\|^q_{V_{s_2,q}} - \|u\|^p_{V_{s_1,p}} - \|u\|^q_{V_{s_2,q}}\\ =&\int_{|x|>r} h_{\lambda,k}(x,u_n)u_n dx + o_n(1)\\ \leqslant&\frac{1}{\eta}\int_{|x|>r} V(x)(|u_n|^p + |u_n|^q) + o_n(1). \end{aligned} \tag{18}$$

Consequently, in order to ensure the completeness of the proof for Lemma 4, it is essential to demonstrate

$$\lim_{r\to\infty}\limsup_{n\to+\infty}\int_{|x|>r} V(x)(|u_n|^p + |u_n|^q) = 0. \tag{19}$$

Hence, fix $r > 2R$. Let $\psi_r \in C^\infty(\mathbb{R}^N)$ be such that

$$0 \leqslant \psi_r \leqslant 1,\ \ \psi_r = 0 \ \text{ if } \ |x| < \frac{r}{2},\ \ \psi_r = 1 \ \text{ if } \ |x| > r,\ \ |\nabla\psi_r(x)| \leqslant \frac{C}{r} \ \text{ for all } \ x \in \mathbb{R}^N,$$

for some constant $C > 0$.

Based on the boundedness of $\{\psi_r u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_V$, it can be inferred that $\langle J'_{\lambda,k}(u_n),\psi_r u_n\rangle = o_n(1)$. Therefore, for $n \in \mathbb{N}$ sufficiently large, we have

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))((\psi_r u_n)(x)-(\psi_r u_n)(y))}{|x-y|^{N+s_1p}}dxdy\\ &+\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))((\psi_r u_n)(x)-(\psi_r u_n)(y))}{|x-y|^{N+s_2q}}dxdy\\ &+\int_{\mathbb{R}^N} V(x)|u_n|^p\psi_r dx + \int_{\mathbb{R}^N} V(x)|u_n|^q\psi_r dx\\ =&o_n(1) + \int_{\mathbb{R}^N} h_{\lambda,k}(x,u_n)\psi_r u_n dx. \end{aligned}$$

According to (h_4), we obtain the following:

$$\begin{aligned}
&(1-\frac{1}{\eta})\int_{|x|>\frac{r}{2}} V(x)(|u_n|^p+|u_n|^q)\psi_r dx\\
=&o_n(1)-\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}\psi_r(x)dxdy-\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2q}}\psi_r(x)dxdy\\
&-\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi_r(x)-\psi_r(y))}{|x-y|^{N+s_1p}}u_n(y)dxdy\\
&-\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\psi_r(x)-\psi_r(y))}{|x-y|^{N+s_2q}}u_n(y)dxdy.
\end{aligned}\tag{20}$$

Utilizing the Hölder inequality, we can derive the following results:

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\psi_r(x)-\psi_r(y))}{|x-y|^{N+s_1p}}u_n(y)dxdy\right|\\
\leqslant&\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy\right)^{\frac{p-1}{p}}\left(\iint_{\mathbb{R}^{2N}}\frac{|\psi_r(x)-\psi_r(y)|^p}{|x-y|^{N+s_1p}}|u_n(y)|^pdxdy\right)^{\frac{1}{p}}.
\end{aligned}\tag{21}$$

From (20), (21) and the boundedness of $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{X}_V$, there exists $C>0$ such that

$$\begin{aligned}
&(1-\frac{1}{\eta})\int_{|x|>\frac{r}{2}} V(x)(|u_n|^p+|u_n|^q)\psi_r dx\\
\leqslant&C\left(\iint_{\mathbb{R}^{2N}}\frac{|\psi_r(x)-\psi_r(y)|^p}{|x-y|^{N+s_1p}}|u_n(y)|^pdxdy\right)^{\frac{1}{p}}\\
&+C\left(\iint_{\mathbb{R}^{2N}}\frac{|\psi_r(x)-\psi_r(y)|^q}{|x-y|^{N+s_2q}}|u_n(y)|^qdxdy\right)^{\frac{1}{q}}+o_n(1).
\end{aligned}\tag{22}$$

Based on the definition of ψ_r and polar coordinates, we have the following:

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}}\frac{|\psi_r(x)-\psi_r(y)|^p}{|x-y|^{N+s_1p}}|u_n(y)|^pdxdy\\
=&\int_{\mathbb{R}^N}\int_{|y-x|>r}\frac{|\psi_r(x)-\psi_r(y)|^p}{|x-y|^{N+s_1p}}|u_n(y)|^pdxdy+\int_{\mathbb{R}^N}\int_{|y-x|\leqslant r}\frac{|\psi_r(x)-\psi_r(y)|^p}{|x-y|^{N+s_1p}}|u_n(y)|^pdxdy\\
\leqslant&\int_{\mathbb{R}^N}|u_n(y)|^p\left(\int_{|y-x|>r}\frac{dx}{|x-y|^{N+s_1p}}\right)dy+\frac{C}{r^p}\int_{\mathbb{R}^N}|u_n(y)|^p\left(\int_{|y-x|\leqslant r}\frac{dx}{|x-y|^{N+s_1p-p}}\right)dy\\
\leqslant&\int_{\mathbb{R}^N}|u_n(y)|^p\left(\int_{|z|>r}\frac{dz}{|z|^{N+s_1p}}\right)dy+\frac{C}{r^p}\int_{\mathbb{R}^N}|u_n(y)|^p\left(\int_{|z|\leqslant r}\frac{dz}{|z|^{N+s_1p-p}}\right)dy\\
\leqslant&\frac{C}{r^{s_1p}}\int_{\mathbb{R}^N}|u_n(y)|^pdy+\frac{C}{r^p}r^{-s_1p+p}\int_{\mathbb{R}^N}|u_n(y)|^pdy\\
\leqslant&\frac{C}{r^{s_1p}}\int_{\mathbb{R}^N}|u_n(y)|^pdy\leqslant\frac{C}{r^{s_1p}}.
\end{aligned}\tag{23}$$

In a similar manner, we also possess the following:

$$\iint_{\mathbb{R}^{2N}}\frac{|\psi_r(x)-\psi_r(y)|^q}{|x-y|^{N+s_2q}}|u_n(y)|^qdxdy\leqslant\frac{C}{r^{s_2q}}.\tag{24}$$

Putting together (22)–(24), we can infer that (19) is verified. Furthermore, combining (18) with (19), we can derive the following:

$$\|u_n\|^p_{V_{s_1,p}}+\|u_n\|^q_{V_{s_2,q}}=\|u\|^p_{V_{s_1,p}}+\|u\|^q_{V_{s_2,q}}+o_n(1).$$

In view of the Brezis–Lieb Lemma in [30], we obtain the following:

$$\|u_n-u\|^p_{V_{s_1,p}}=\|u_n\|^p_{V_{s_1,p}}-\|u\|^p_{V_{s_1,p}}+o_n(1) \text{ and } \|u_n-u\|^q_{V_{s_2,q}}=\|u_n\|^q_{V_{s_2,q}}-\|u\|^q_{V_{s_2,q}}+o_n(1),$$

As a result

$$\|u_n-u\|^p_{V_{s_1,p}}+\|u_n-u\|^q_{V_{s_2,q}}=o_n(1),$$

which yields $u_n\to u$ in $\mathbb{X}_V$ as $n\to\infty$. This proof is now complete. □

In the subsequent phase, we are prepared to demonstrate the existence of positive solutions to the problem $(B_{\lambda,k})$. We will begin by establishing the minimax level

$$c_{\lambda,k}=\inf_{\tau\in\Gamma_{\lambda,k}}\max_{t\in[0,1]}J_{\lambda,k}(\tau(t)),$$

where $\Gamma_{\lambda,k}=\{\tau\in C([0,1],\mathbb{X}_V):\tau(0)=0 \text{ and } J_{\lambda,k}(\tau(1))<0\}$. Based on Lemma 2, we have $c_{\lambda,k}>0$.

Theorem 4. *Suppose that hypotheses $(f_1)-(f_4)$ and (V) are fulfilled. Then, the problem $(B_{\lambda,k})$ admits a nontrivial non-negative solution.*

Proof. In view of Lemmas 2–4, we can utilize the mountain pass theorem to deduce that there exists $u_{\lambda,k}\in\mathbb{X}_V$ such that $J_{\lambda,k}(u_{\lambda,k})=c_{\lambda,k}$ and $J'_{\lambda,k}(u_{\lambda,k})=0$. Moreover, $u_{\lambda,k}\geqslant 0$ in $\mathbb{R}^N$. Indeed, using $\langle J'_{\lambda,k}(u_{\lambda,k}),u^-_{\lambda,k}\rangle=0$, where $u^-_{\lambda,k}=\min\{u_{\lambda,k},0\}$ and $h_{\lambda,k}(\cdot,t)\leqslant 0$ for $t\leqslant 0$. Using the elementary inequality

$$|x-y|^{t-2}(x-y)(x^--y^-)\geqslant|x^--y^-|^t\quad\forall x,\,y\in\mathbb{R},\quad\forall t>1,$$

it is observable that

$$\|u^-_{\lambda,k}\|^p_{V_{s_1,p}}+\|u^-_{\lambda,k}\|^q_{V_{s_2,q}}\leqslant 0.$$

This indicates that $u^-_{\lambda,k}=0$, which leads to the conclusion that $u_{\lambda,k}\geqslant 0$ in $\mathbb{R}^N$. Finally, since $V\in L^\infty_{loc}(\mathbb{R}^N)$ and

$$(-\Delta)^{s_1}_p u_{\lambda,k}+(-\Delta)^{s_2}_q u_{\lambda,k}+V(x)|u_{\lambda,k}|^{p-2}u_{\lambda,k}+|u_{\lambda,k}|^{q-2}u_{\lambda,k})\geqslant 0 \text{ in } \mathbb{R}^N,$$

according to the strong maximum principle (Theorem 1.1 in [31]), it follows that $u_{\lambda,k}>0$ in $\mathbb{R}^N$. □

4. $L^\infty(\mathbb{R}^N)$–Estimate of the Solution $u_{\lambda,k}$ to $(B_{\lambda,k})$

This section begins with the demonstration of a uniform estimate for the $L^\infty(\mathbb{R}^N)$ norm of the critical point $u_{\lambda,k}$, as established in Theorem 4.

Lemma 5. *Let $u_{\lambda,k}$ be the critical point of $J_{\lambda,k}$. Then, there exists a constant $C>0$ such that*

$$[u_{\lambda,k}]^q_{s_2,q}\leqslant C,$$

where C is independent of λ, k and R .

Proof. Consider the functional $I_0:\mathbb{X}_V\mapsto\mathbb{R}$ defined by

$$I_0(u)=\frac{1}{p}\|u\|^p_{V_{s_1,p}}+\frac{1}{q}\|u\|^q_{V_{s_2,q}}-\int_{\mathbb{R}^N}F(u)dx.$$

It is easy to verify that I_0 satisfies the conditions (i) and (ii) of Lemma 2. Hence,

$$c_0 = \inf_{\tau\in\Gamma_0}\max_{t\in[0,1]} I_0(\tau(t)),$$

where $\Gamma_0 = \{\tau \in C([0,1], \mathbb{X}_V) : \tau(0) = 0 \text{ and } I_0(\tau(1)) < 0\}$. Based on (9) and (13), we obtain $J_{\lambda,k}(u) \leqslant I_0(u)$ for all $u \in \mathbb{X}_V$, and as a result, $c_{\lambda,k} \leqslant c_0$. Similar to the proof of Lemma 3, we obtain the following:

$$c_0 \geqslant c_{\lambda,k} = J_{\lambda,k}(u_{\lambda,k}) - \frac{1}{\theta}\langle J'_{\lambda,k}(u_{\lambda,k}), u_{\lambda,k}\rangle \geqslant \left(1-\frac{1}{p}\right)\left(\frac{1}{q}-\frac{1}{\theta}\right)[u_{\lambda,k}]^q_{s_2,q},$$

and this finalizes the proof. □

Lemma 6. *For each $\lambda > 0$ and $k \in \mathbb{N}$, $u_{\lambda,k} \in L^\infty(\mathbb{R}^N)$, there exists $C_1 > 0$ such that*

$$|u_{\lambda,k}|_\infty \leqslant C_1(1+\lambda k^{\gamma-v})^{\frac{1}{q(\beta_1-1)}}|u_{\lambda,k}|_{q^*_{s_2}},$$

where $\beta_1 = \frac{q^*_{s_2}-v+q}{q}$.

Proof. For simplicity, let us denote $u = u_{\lambda,k}$. For each $M > 0$, let $u_M := \min\{u, M\}$ and consider $\varphi(u) = uu_M^{q(\beta-1)}$ with $\beta > 1$ in (15). Then, we can infer

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y))}{|x-y|^{N+s_1p}}dxdy\\
&+\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y))}{|x-y|^{N+s_2q}}dxdy\\
&+\int_{\mathbb{R}^N} V(x)|u|^p u_M^{q(\beta-1)}dx + \int_{\mathbb{R}^N} V(x)|u|^q u_M^{q(\beta-1)}dx\\
=&\int_{\mathbb{R}^N} h_{\lambda,k}(x,u)uu_M^{q(\beta-1)}dx.
\end{aligned}$$

Combining (10), (h_2), and (h_4), we obtain the following:

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y))}{|x-y|^{N+s_1p}}dxdy\\
&+\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y))}{|x-y|^{N+s_2q}}dxdy\\
=&C\int_{\mathbb{R}^N}(1+\lambda k^{\gamma-v})u^v u_M^{q(\beta-1)}.
\end{aligned} \tag{25}$$

Define

$$\phi(t) := \frac{|t|^q}{q} \quad \text{and} \quad \Upsilon(t) := \int_0^t (\varphi'(\xi))^{\frac{1}{q}}d\xi.$$

Observe that φ is an increasing function. Thus,

$$(a-b)(\varphi(a)-\varphi(b)) \geqslant 0 \quad \text{for any} \quad a,b \in \mathbb{R}.$$

The Jensen inequality yields the following result:

$$\phi'(a-b)(\varphi(a)-\varphi(b)) \geqslant |\Upsilon(a)-\Upsilon(b)|^q \quad \text{for all } a,b \in \mathbb{R},$$

from which

$$|\Upsilon(u)(x)-\Upsilon(u)(y)|^q \leqslant |u(x)-u(y)|^{q-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y)).$$

It is important to observe that $\Upsilon(u) \geqslant \frac{1}{\beta}uu_M^{\beta-1}$. Consequently, by applying the Sobolev inequality, we can infer that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))((uu_M^{q(\beta-1)})(x)-(uu_M^{q(\beta-1)})(y))}{|x-y|^{N+s_2q}}dxdy\\ \geqslant&[\Upsilon(u)]^q_{s_2,q} \geqslant S_*|\Upsilon(u)|^q_{q^*_{s_2}} \geqslant \frac{S_*}{\beta^q}|uu_M^{\beta-1}|^q_{q^*_{s_2}},\end{aligned} \tag{26}$$

where S_* denotes the Sobolev constant of the embedding $W^{s_2,q}(\mathbb{R}^N) \hookrightarrow L^{q^*_{s_2}}(\mathbb{R}^N)$. Subsequently, by combining (25) and (26) in conjunction with the the Hölder inequality, we can deduce the following:

$$\begin{aligned}|uu_M^{\beta-1}|^q_{q^*_{s_2}} &\leqslant C\beta^q(1+\lambda k^{\gamma-v})\int_{\mathbb{R}^N} u^v u_M^{q(\beta-1)}dx\\ &= C\beta^q(1+\lambda k^{\gamma-v})\int_{\mathbb{R}^N} u^{v-q}u^q u_M^{q(\beta-1)}dx\\ &\leqslant C\beta^q(1+\lambda k^{\gamma-v})\left(\int_{\mathbb{R}^N} u^{q^*_s}dx\right)^{\frac{v-q}{q^*_{s_2}}}\left(\int_{\mathbb{R}^N}|uu_M^{\beta-1}|^{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}}dx\right)^{\frac{q^*_{s_2}-v+q}{qq^*_{s_2}}q}\\ &= C\beta^q(1+\lambda k^{\gamma-v})|u|^{v-q}_{q^*_{s_2}}\left|uu_M^{\beta-1}\right|^q_{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}}.\end{aligned} \tag{27}$$

Now, from Lemma 5, Theorem 3, and Relation (27), there exists $C>0$, independent of λ, such that

$$|uu_M^{\beta-1}|^q_{q^*_{s_2}} \leqslant C\beta^q(1+\lambda k^{\gamma-v})\left|uu_M^{\beta-1}\right|^q_{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}}. \tag{28}$$

If $u^\beta \in L^{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}}$, then, using the fact that $u_M \leqslant u$ and (28), it follows

$$|uu_M^{\beta-1}|^q_{q^*_{s_2}} \leqslant C\beta^q(1+\lambda k^{\gamma-v})|u|^{q\beta}_{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}\beta} < +\infty, \tag{29}$$

passing to the limit as $M \to +\infty$ in Relation (29) and using the Fatou Lemma, we have the following:

$$|u|_{q^*_{s_2}\beta} \leqslant C\beta^{\frac{1}{\beta}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta}}|u|_{\frac{qq^*_{s_2}}{q^*_{s_2}-v+q}\beta}. \tag{30}$$

In order to obtain the desired $L^\infty(\mathbb{R}^N)$ estimate, the next phase will involve an iterative procedure. For this, we will substitute $\beta=\beta_1 := \frac{q^*_{s_2}-v+q}{q}$ into (30), resulting in the following expression:

$$|u|_{q^*_{s_2}\beta_1} \leqslant C\beta_1^{\frac{1}{\beta_1}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta_1}}|u|_{q^*_{s_2}}.$$

Taking $\beta = \beta_2 := \beta_1^2$ in (30) and noting that $\frac{qq_{s_2}^*}{q_{s_2}^* - v + q}\beta_2 = q_{s_2}^*\beta_1$, based on the previous estimate, we obtain the following:

$$\begin{aligned}|u|_{q_{s_2}^*\beta_2} &\leqslant C\beta_2^{\frac{1}{\beta_2}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta_2}}|u|_{q_{s_2}^*\beta_1}\\ &\leqslant C\beta_1^{\frac{1}{\beta_1}}\beta_2^{\frac{1}{\beta_2}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta_1}+\frac{1}{q\beta_2}}|u|_{q_{s_2}^*}.\end{aligned}$$

Again, taking $\beta = \beta_3 := \beta_2\beta_1 = \beta_1^3$ in (30) and since $\frac{qq_{s_2}^*}{q_{s_2}^* - v + q}\beta_3 = q_{s_2}^*\beta_2$, we obtain

$$\begin{aligned}|u|_{q_{s_2}^*\beta_3} &\leqslant C\beta_3^{\frac{1}{\beta_3}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta_3}}|u|_{q_{s_2}^*\beta_2}\\ &\leqslant C\beta_1^{\frac{1}{\beta_1}}\beta_2^{\frac{1}{\beta_2}}\beta_3^{\frac{1}{\beta_3}}(1+\lambda k^{\gamma-v})^{\frac{1}{q\beta_1}+\frac{1}{q\beta_2}+\frac{1}{q\beta_3}}|u|_{q_{s_2}^*}.\end{aligned}$$

By implementing this iterative process m times for $m \geqslant 2$, with $\beta = \beta_m := \beta_{m-1}\beta_1 = \beta_1^m$ in (30) and along with $\frac{qq_{s_2}^*}{q_{s_2}^* - v + q}\beta_m = q_{s_2}^*\beta_{m-1}$, we can deduce the following:

$$|u|_{q_{s_2}^*\beta_m} \leqslant C\beta_1^{\frac{1}{\beta_1}}\beta_2^{\frac{1}{\beta_2}}\cdots\beta_m^{\frac{1}{\beta_m}}(1+\lambda k^{\gamma-v})^{\frac{1}{q}\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}+\cdots+\frac{1}{\beta_m}\right)}|u|_{q_{s_2}^*}. \tag{31}$$

This ensures that $u \in L^{q_{s_2}^*\beta_m}(\mathbb{R}^N)$ for all $m \geqslant 2$. Consequently, we have

$$\sum_{i=1}^{m}\frac{1}{\beta_i} \leqslant \sum_{i=1}^{\infty}\left(\frac{1}{\beta_1}\right)^i = \frac{1}{\beta_1 - 1} \quad \text{and} \quad \beta_1^{\frac{1}{\beta_1}}\beta_2^{\frac{1}{\beta_2}}\cdots\beta_m^{\frac{1}{\beta_m}} \leqslant \beta_1^{\sum_{i=1}^{\infty}\frac{i}{\beta_1^i}} = \beta_1^{\frac{\beta_1}{(\beta_1-1)^2}},$$

based on elementary calculus. Therefore, according to (31), we reach the following:

$$|u|_{q_{s_2}^*\beta_m} \leqslant C\beta_1^{\frac{\beta_1}{(\beta_1-1)^2}}(1+\lambda k^{\gamma-v})^{\frac{1}{q(\beta_1-1)}}|u|_{q_{s_2}^*} \quad \text{for all} \quad m \geqslant 2.$$

Since $\beta_m = \beta_1^m \to \infty$ as $m \to \infty$, by utilizing Lemma 5 and Theorem 3, we can conclude that $u \in L^{\infty}(\mathbb{R}^N)$ and

$$|u|_{\infty} \leqslant C_1(1+\lambda k^{\gamma-v})^{\frac{1}{q(\beta_1-1)}}|u|_{q_{s_2}^*},$$

where $C_1 = C\beta_1^{\frac{\beta_1}{(\beta_1-1)^2}}$. The proof is complete. □

5. Proof of the Main Results

Based on Lemma 6, for each $\lambda > 0$ and $k \in \mathbb{N}$, we have the following:

$$u_{\lambda,k}(x) \leqslant C_1(1+\lambda k^{\gamma-v})^{\frac{1}{q(\beta_1-1)}}|u_{\lambda,k}|_{q_{s_2}^*} \quad \text{a.e. in } x \in \mathbb{R}^N.$$

Based on Lemma 5 and Theorem 3, we obtain the following:

$$u_{\lambda,k}(x) \leqslant C(1+\lambda k^{\gamma-v})^{\frac{1}{q(\beta_1-1)}} \quad \text{a.e. in } x \in \mathbb{R}^N. \tag{32}$$

Now, fixing $k_0 > C$, it is known that $\lambda_0 > 0$ such that the $C(1+\lambda k_0^{\gamma-v})^{\frac{1}{q(\beta_1-1)}} < k_0$ holds for all $\lambda \in [0, \lambda_0)$. In particular, based on (32), we can infer

$$|u_{\lambda,k_0}|_{\infty} \leqslant k_0 \quad \forall\ \lambda \in [0, \lambda_0). \tag{33}$$

Consequently, based on the definition of g_{λ,k_0}, we arrive at the following:

$$g_{\lambda,k_0}(u_{\lambda,k_0}(x)) = f(u_{\lambda,k_0}(x)) + \lambda(u_{\lambda,k_0}(x))^{\gamma-1} \quad \text{a.e. in } x \in \mathbb{R}^N. \tag{34}$$

The following lemma is crucial for establishing the existence of the result.

Lemma 7. *For each $\lambda > 0$ and $k \in \mathbb{N}$, let $u_{\lambda,k}$ be the critical point of $J_{\lambda,k}$. Then,*

$$u_{\lambda,k}(x) \leqslant \frac{R^{N-s_1p}|u_{\lambda,k}|_\infty}{|x|^{N-s_1p}} \quad \forall\, |x| \geqslant R, \quad N > s_1p.$$

Proof. Define

$$\varpi_{j,t}(x) = \frac{R^{N-s_jt}|u_{\lambda,k}|_\infty}{|x|^{N-s_jt}} \quad x \neq 0.$$

Since $\frac{1}{|x|^{N-s_jt}}$ is (s_j,t)-harmonic, it follows that $(-\Delta)_p^{s_1}\varpi_{1,p} + (-\Delta)_q^{s_2}\varpi_{2,q} = 0$ weakly in $\mathbb{R}^N\backslash\{0\}$. Additionally, we have the following:

$$\begin{aligned}&(-\Delta)_p^{s_1}u_{\lambda,k} + (-\Delta)_q^{s_2}u_{\lambda,k}\\ =& -V(x)(|u_{\lambda,k}|^{p-2}u_{\lambda,k} + |u_{\lambda,k}|^{q-2}u_{\lambda,k}) + h_{\lambda,k}(x,u_{\lambda,k}) \qquad x \in \mathbb{R}^N.\end{aligned}$$

Therefore, based on condition (h_4), it follows that $(-\Delta)_p^{s_1}u_{\lambda,k} + (-\Delta)_q^{s_2}u_{\lambda,k} \leq 0$ for $|x| > R$. We will now introduce the function $\omega_{j,t} = u_{\lambda,k} - \varpi_{j,t}$. It is important to note that

$$u_{\lambda,k}(x) \leqslant |u_{\lambda,k}|_\infty \leqslant \frac{R^{N-s_jt}|u_{\lambda,k}|_\infty}{|x|^{N-s_jt}} = \varpi_{j,t}(x) \quad 0 < |x| \leqslant R.$$

As a consequence,

$$\begin{cases} (-\Delta)_p^{s_1}\omega_{1,p}(x) + (-\Delta)_q^{s_2}\omega_{2,q}(x) \leqslant 0, & |x| > R \\ \omega_{1,p}(x) \leqslant 0, \quad \omega_{2,q}(x) \leqslant 0, & 0 < |x| \leqslant R. \end{cases} \tag{35}$$

Since $u_{\lambda,k}$ is $q_{s_2}^*$-integrable and $u_{\lambda,k} \in C^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0,1)$, we have $u_{\lambda,k} \to 0$ as $|x| \to \infty$ (Theorem 3.4 in [32]). Hence,

$$\lim_{|x|\to\infty} \omega_{1,p}(x) = 0, \quad \lim_{|x|\to\infty} \omega_{2,q}(x) = 0. \tag{36}$$

Next, we shall proceed to show that $\omega_{1,p}(x) \leqslant 0$ for $|x| > R$. Since $\omega_{2,q}(x) \leqslant \omega_{1,p}(x)$ for $|x| > R$, suppose, by contradiction, there exists x_0 such that $\omega_{2,q}(x_0) > 0$ for $|x_0| > R$. Then, by virtue of $\omega_{2,q}(x) \leqslant 0$ for $0 < |x| \leqslant R$ and (36), there exists a point x_1 such that $\omega_{2,q}(x_1) = \max_{\mathbb{R}^N} \omega_{2,q} > 0$ for $|x_1| > R$. According to the definition of the fractional t-Laplacian operator, we have $(-\Delta)_p^{s_1}\omega_{1,p}(x_1) + (-\Delta)_q^{s_2}\omega_{2,q}(x_1) > 0$, which contradicts (35). As a result, $\omega_{1,p}(x) \leqslant 0$ for $|x| > R$. The proof of the lemma is complete. □

Proof of Theorem 1. Owing to $g_{\lambda,k_0}(t) \leqslant g_{\lambda_0,k_0}(t)$ for all $t \in R$ and for any $0 \leq \lambda < \lambda_0$, based on (10) and Lemma 7, we obtain the following:

$$\begin{aligned}\eta g_{\lambda,k_0}(u_{\lambda,k_0}(x)) &\leqslant \eta g_{\lambda_0,k_0}(u_{\lambda,k_0}(x))\\ &\leqslant \eta C(1+\lambda_0 k_0^{\gamma-v})\left(\frac{R^{N-s_2q}|u_{\lambda,k_0}|_\infty}{|x|^{N-s_2q}}\right)^{v-q} u_{\lambda,k_0}^{q-1}(x) \quad \forall\, |x| > R\end{aligned}$$

and

$$\eta g_{\lambda,k_0}(u_{\lambda,k_0}(x)) \leqslant \eta C(1+\lambda_0 k_0^{\gamma-v})\left(\frac{R^{N-s_1p}|u_{\lambda,k_0}|_\infty}{|x|^{N-s_1p}}\right)^{v-p} u_{\lambda,k_0}^{p-1}(x) \quad \forall\, |x|>R.$$

Since $v-q<v-p<q_{s_2}^*-p$, based on hypothesis (V) and (33), we have the following:

$$\begin{aligned}\eta g_{\lambda,k_0}(u_{\lambda,k_0}(x)) &\leqslant \frac{1}{2}\eta C(1+\lambda_0 k_0^{\gamma-v})\frac{V(x)}{\Lambda}(k_0^{v-p}u_{\lambda,k_0}^{p-1}(x)+k_0^{v-q}u_{\lambda,k_0}^{q-1}(x))\\ &\leqslant \frac{1}{2}\eta C(1+\lambda_0 k_0^{\gamma-v})k_0^{q_{s_2}^*-p}\frac{V(x)}{\Lambda}(u_{\lambda,k_0}^{p-1}(x)+u_{\lambda,k_0}^{q-1}(x)) \quad \forall\, |x|>R.\end{aligned}$$

For this reason, if $\Lambda \geqslant \Lambda_0 := \frac{1}{2}\eta C(1+\lambda_0 k_0^{\gamma-v})k_0^{q_{s_2}^*-p}$ and $\lambda \in [0,\lambda_0)$, then

$$\eta g_{\lambda,k_0}(u_{\lambda,k_0}(x)) \leqslant V(x)(u_{\lambda,k_0}^{p-1}(x)+u_{\lambda,k_0}^{q-1}(x)) \quad \forall\, |x|>R.$$

Consequently,

$$h_{\lambda,k_0}(x,u_{\lambda,k_0}(x)) = \tilde{g}_{\lambda,k_0}(x,u_{\lambda,k_0}(x)) = g_{\lambda,k_0}(u_{\lambda,k_0}(x)) \quad \forall\, |x|>R,$$

which together with (13) and (34) yields the following:

$$h_{\lambda,k_0}(x,u_{\lambda,k_0}(x)) = f(u_{\lambda,k_0}(x)) + \lambda(u_{\lambda,k_0}(x))^{\gamma-1} \quad \text{a.e. in } x\in\mathbb{R}^N,$$

for all $\lambda \in [0,\lambda_0)$ and $\Lambda \geqslant \Lambda_0$. Finally, utilizing the fact that u_{λ,k_0} is a critical point of J_{λ,k_0}, we have the following:

$$\begin{aligned}&\langle u_{\lambda,k_0},\varphi\rangle_{s_1,p}+\langle u_{\lambda,k_0},\varphi\rangle_{s_2,q}+\int_{\mathbb{R}^N} V(x)(|u_{\lambda,k_0}|^{p-2}u_{\lambda,k_0}+|u_{\lambda,k_0}|^{q-2}u_{\lambda,k_0})\varphi dx\\ =&\int_{\mathbb{R}^N} f(u_{\lambda,k_0}(x))\varphi dx+\lambda\int_{\mathbb{R}^N}(u_{\lambda,k_0}(x))^{\gamma-1}\varphi dx \quad \text{for all } \varphi\in\mathbb{X}_V.\end{aligned}$$

This indicates that u_{λ,k_0} constitutes a positive solution to the problem (1) for all $\lambda\in[0,\lambda_0)$ and $\Lambda\geqslant\Lambda_0$. This concludes the proof. □

Proof of Theorem 2. Lemma 6 has already established that each u (denote $u_{\lambda,k}=u$) belongs to $L^\infty(\mathbb{R}^N)$ and proves the desired estimate for the sup-norm. Now, we can focus on the interior Hölder regularity of the weak solution to problem (1).

Let $\Omega\subset\mathbb{R}^N$ be a bounded set. Define

$$\tilde{W}^{s,p}(\Omega)=\left\{u\in L^p_{\mathrm{loc}}(\mathbb{R}^N):\exists\Omega'\supset\supset\Omega\ s.t.\ \|u\|_{W^{s,p}(\Omega')}+\int_{\mathbb{R}^N}\frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}}dx<\infty\right\}.$$

Assume $2\leqslant p<q<\frac{N}{s_2}$, $\gamma\geqslant q_{s_2}^*$. Let u be a weak solution to Problem (1) and $R_0>0$ be such that $B_{R_0}:=B_{R_0}(0)\subset\Omega$. Since $u\in\mathbb{X}_V\cap L^\infty(\mathbb{R}^N)$, we have $u\in\tilde{W}^{s_1,p}(B_{R_0})\cap\tilde{W}^{s_2,q}(B_{R_0})$. By utilizing the growth assumptions on function f and the non-negativity of V, we can conclude the following:

$$|(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u|\leqslant|u|_\infty^{v-1}+\lambda|u|_\infty^{\gamma-1}:=K>0.$$

Therefore, according to Theorem 2.10 in [22] and standard covering arguments, there exists $\alpha\in(0,1)$ such that $u\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$. The proof of Theorem 2 is complete. □

Conclusions and Future Studies

In our study, we employ variational methods to establish the existence and regularity of solutions for a class of fractional double phase problems with unbalanced growth.

In the process of proving existence, influenced by the strong effects of the supercritical exponent and vanishing potential, we first modified Problem (1) using the truncation and penalization methods in two stages. We then applied the mountain pass theorem to establish the existence of a solution to the modified equation. Finally, we demonstrated the regularity of the solution using the Moser iteration method.

We expect our results to promote the development of research in the field of fractional $p\&q$-Laplacian problems. Next, we will develop and extend to double phase problems with Hardy singularity potential and logarithmic nonlinearities. At the same time, we will try our best to explore the practical application of the research questions.

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