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Multiple Solutions for a Critical Steklov Kirchhoff Equation

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Abstract: In the present work, we study some existing results related to a new class of Steklov $p(x)$-Kirchhoff problems with critical exponents. More precisely, we propose and prove some properties of the associated energy functional. In the first existence result, we use the mountain pass theorem to prove that the energy functional admits a critical point, which is a weak solution for such a problem. In the second main result, we use a symmetric version of the mountain pass theorem to prove that the investigated problem has an infinite number of solutions. Finally, in the third existence result, we use a critical point theorem proposed by Kajikiya to prove the existence of a sequence of solutions that tend to zero.

Keywords: variational methods; ψ-Hilfer fractional derivative; critical exponent; concentration-compactness principle; fractional Kirchhoff equation; Steklov problem

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1. Introduction

In this study, we examine a specific class of fractional Kirchhoff-type problem with a $p(x)$-Laplacian operator, which is expressed as follows:

$$\begin{cases} M\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\mathbf{L}_{p(x)}^{\varpi,\nu;\psi}u = |u|^{r(x)-2}u + \mathfrak{g}(x,u), \ in\ \Lambda = (0,T)\times(0,T), \\ \left|\mathbb{D}_{0+}^{\varpi,\nu;\mu}u\right|^{p(x)-2}\frac{\partial u}{\partial v} = |u|^{s(x)-2}u \quad \text{on } \partial\Lambda, \end{cases} \tag{1}$$

where

$$\mathbf{L}_{p(x)}^{\varpi,\nu;\psi}u = {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}\left(\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)-2}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right), \tag{2}$$

where ${}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(\cdot)$ and ${}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(\cdot)$ are ψ-Hilfer fractional partial derivatives of order $\frac{1}{p^+} < \varpi < 1$ and type $0 \le \nu \le 1$. $\frac{\partial u}{\partial v}$ is the outer-unit normal derivative. p, r, and $s \in C(\overline{\Lambda})$ are such that

$$1 < p^- = \inf_\Lambda p(x) \le p^+ = \sup_\Lambda p(x) < 2,$$

$$p^+ < r(x) \le p^*(x), \text{ for all } x \in \Lambda,$$

and

$$p^+ < s(x) < p_*(x) \text{ for all } x \in \partial\Lambda,$$

where $p^*(x)$ is the critical exponent, which is defined as follows:

$$p^*(x) = \begin{cases} \frac{p(x)}{2-p(x)} \text{ if } p(x) < 2, \\ \infty \text{ if } p(x) \ge 2. \end{cases}$$

The Kirchhoff function (M) is continuous, and the nonlinearity $(\mathfrak{g})$ is of the Carathéodory type.

In recent years, considerable attention has been paid to problems involving the ψ-Hilferfractional derivative with a p-Laplacian operator [1–8]. Problems that involve the growth of the critical exponent are more complicated, since the embedding is continuous and not compact. To solve this lack of compactness, Lions [9] used the concentration-compactness principle. Recently, several problems have arisen that involve critical growth behavior. For a more thorough exploration of this subject, we recommended that interested readers consult references [10–13] and other related sources.

The investigation of problems characterized by variable exponents, critical growth, and those involving the ψ-Hilfer fractional derivative with a p-Laplacian operator has garnered considerable interest in recent years. These issues are not only intriguing but also relevant to various applications, including the modeling of electrorheological fluids [14–16], image processing [17], medicine [18], economics and finance [19,20], physics [21,22], and biology [23]. Moreover, they represent difficult mathematical problems that require a detailed examination.

The authors of [24] explored an equation with critical variable exponents.

$$\begin{cases} (-\Delta)_{p(x)}u = |u|^{r(x)-2}u + a(x)|u|^{q(x)-2}u \quad \text{in } \Lambda, \\ u = 0 \quad \text{on } \partial\Lambda, \end{cases}$$

where p, r, and q are such that

$$1 < p(x) < q(x) \leq r(x) \leq p^*(x), \forall\, x \in \Lambda.$$

It is assumed that the set $A = \{x \in \Lambda : r(x) = p^*(x)\}$ is not empty, which suggests the appearance of critical growth behavior. To demonstrate the existence of solutions, the authors utilized variational methods alongside the mountain pass theorem. These approaches enabled them to formulate an appropriate functional and apply critical point theory to identify non-trivial solutions to the problem.

The problems related to Kirchhoff-type problems with variable exponents are attracting significant attention and becoming increasingly important to various research groups due to a range of theoretical and practical inquiries [25–29]. Furthermore, it is important to highlight the growing interest in Kirchhoff problems involving fractional operators, which has seen a remarkable increase over the years [30–32]. The $p(x)$-Laplacian introduces more intricate nonlinearity, leading to several fundamental challenges. For approximation, the authors of [4] explored the Kirchhoff fractional $p(\xi)$-Laplacian problem without a critical exponent.

$$\begin{cases} \mathfrak{S}\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\mathbf{L}_{p(x)}^{\varpi,\nu;\psi}u = \mathfrak{g}(x,u), \; in \; \Lambda = [0,T]\times[0,T], \\ u = 0, \; on \; \partial\Lambda \end{cases} \qquad (P_2)$$

The authors utilized variational techniques, along with the mountain pass theorem and the fountain theorem, to demonstrate both the existence and the multiplicity of solutions for the problem (P_2).

We note that the novelty in our study compared with reference [24] is that the perturbed nonlinearity can be critical, which implies more difficulties. This is due to the embedding, which is only continuous when the exponent is critical, and to solve this we use a concentration-compactness principle. Moreover, Kirchhoff-type problems are important, since they arise in the description of nonlinear vibrations of an elastic string.

Inspired by the findings reported in reference [4], our article seeks to advance the discussion by examining the critical case of the problem mentioned above. To achieve this, we apply a recent concentration-compactness principle for ψ-Hilfer spaces with variable exponents to analyze the weighted Kirchhoff problem (1). Our research offers a generalization, enhancement, and extension of the earlier references, incorporating additional suitable conditions. As a result, this project is of considerable significance and provides important insights.

In conclusion, our study makes a meaningful addition to the current body of literature by investigating the critical case of the Kirchhoff problem under Steklov boundary conditions. By employing robust mathematical methods and leveraging recent concentration-compactness principles, we demonstrate the existence and multiplicity of solutions for problem (1), thereby deepening the understanding of this significant area of research.

2. Preliminaries of Variational Spaces

In this section, we present the spaces related to the variational formulation of the main problem. Concerning the properties of the Lebegue spaces and the Hilfer fractional derivative ($\mathbb{D}_{0+}^{\varpi,\nu;\mu}$), we refer the reader to the lemmas presented in [4,5,33,34]. Next, in order to define the working space, we begin by introducing the following set:

$$C_+\left(\overline{\Lambda}\right) = \left\{p \in C\left(\overline{\Lambda}\right), p(\xi) > 1, \forall \xi \in \overline{\Lambda}\right\}.$$

For all $p \in C_+\left(\overline{\Lambda}\right)$, consider

$$p^- = \inf_{\overline{\Lambda}} p(\xi), p^+ = \sup_{\overline{\Lambda}} p(\xi).$$

Additionally, we define

$$\mathcal{L}^{p(\xi)}(\Lambda) = \left\{u : \Lambda \to \mathbb{R}, \text{ measurable} : \int_\Lambda |u(\xi)|^{p(\xi)} d\xi < \infty\right\},$$

with the norm on $\mathcal{L}^{p(\xi)}(\Lambda)$ defined as

$$|u|_{\mathcal{L}^{p(\xi)}(\Lambda)} = \inf\left\{\mu > 0 : \int_\Lambda \left|\frac{u(\xi)}{\mu}\right|^{p(\xi)} d\xi \leq 1\right\}.$$

We also define

$$\mathcal{L}^{p(\xi)}(\partial\Lambda) = \left\{u : \Lambda \to \mathbb{R}, \text{ measurable} : \int_{\partial\Lambda} |u(\xi)|^{p(\xi)} d\sigma < \infty\right\},$$

with the norm on $\mathcal{L}^{p(\xi)}(\partial\Lambda)$ defined as

$$|u|_{\mathcal{L}^{p(\xi)}(\partial\Lambda)} = \inf\left\{\mu > 0 : \int_{\partial\Lambda} \left|\frac{u(\xi)}{\mu}\right|^{p(\xi)} d\sigma \leq 1\right\}.$$

Spaces $\left(\mathcal{L}^{p(\xi)}(\Lambda), |\cdot|_{\mathcal{L}^{p(\xi)}(\Lambda)}\right)$ and $\left(\mathcal{L}^{p(\xi)}(\partial\Lambda), |\cdot|_{\mathcal{L}^{p(\xi)}(\partial\Lambda)}\right)$ are Banach spaces, which we refer to as variable-exponent Lebesgue spaces.

The μ-fractional space is expressed as [4,5]

$$\mathcal{H}_{p(\xi)}^{\varpi,\nu;\mu}(\Lambda) = \left\{u \in L^{p(\xi)}(\Lambda) : \left|\mathbb{D}_{0+}^{\varpi,\nu;\mu} u\right| \in L^{p(\xi)}(\Lambda)\right\}$$

with the norm

$$\| u \| = \| u \|_{\mathcal{H}_{p(\xi)}^{\varpi,\nu;\mu}(\Lambda)} = \| u \|_{L^{p(\xi)}(\Lambda)} + \| \mathbb{D}_{0+}^{\varpi,\nu;\mu} u \|_{L^{p(\xi)}(\Lambda)}.$$

$\mathcal{H}^{\varpi,\nu;\mu}_{p(\xi),0}(\Lambda)$ denotes the closure of $C_0^\infty(\Lambda)$ in $\mathcal{H}^{\varpi,\nu;\mu}_{p(\xi)}(\Lambda)$. We recall that Banach spaces $\mathcal{L}^{p(\xi)}(\Lambda)$ and $\mathcal{H}^{\varpi,\nu;\mu}_{p(\xi)}(\Lambda)$ are also separable and reflexive. Moreover, if $p_1, p_2 \in C_+(\overline{\Lambda})$ such that $p_1(\xi) \leq p_2(\xi)$ for all $\xi \in \overline{\Lambda}$, then embedding $\mathcal{L}^{p_2(\xi)}(\Lambda) \hookrightarrow \mathcal{L}^{p_1(\xi)}(\Lambda)$ is continuous.

The following result is important in our proofs.

Theorem 1 (Concentration-compactness principle (see [35]))**.** *Let $p(x)$ and $r(x)$ be two continuous functions such that*

$$p^- = \inf_{\overline{\Omega}} p(x) \leq p^+ = \sup_{\overline{\Omega}} p(x) < n and 1 < r(x) \leq p^*(x) \text{in} \Omega \subset \mathbb{R}^n.$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $\mathcal{H}^{\varpi,\nu;\psi}_{p(\xi)}(\Lambda)$ with a weak limit (u) such that

- $|u_n|^{r(x)} \rightharpoonup \nu$ *weakly in the sense of measures;*
- $|{}^H\mathbb{D}^{\varpi,\nu;\psi}_{0+} u_n|^{p(x)} \rightharpoonup \psi$ *weakly in the sense of measures.*

Also assume that $A = \{x \in \Omega : r(x) = p^*(x)\}$ is nonempty. Then, for some countable index set (I), we have

$$\nu = |u|^{r(x)} + \sum_{i\in I} \nu_i \delta_{x_i}, \nu_i > 0.$$

$$\psi \geq |{}^{\mathrm{H}}\mathbb{D}^{\varpi,\nu;\psi}_{0+} u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \mu_i > 0.$$

$$S \nu_i^{\frac{1}{p^*(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}.$$

where $\{x_i\}_{i\in I} \subset A$ and S is the best constant in the Gagliardo–Nirenberg–Sobolev inequality for variable exponents, namely

$$S = S_r(\Omega) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{|| |{}^{\mathrm{H}}\mathbb{D}^{\varpi,\nu;\psi}_{0+} \phi| ||_{\mathcal{L}^{p(x)}}}{||\phi||_{\mathcal{L}^{r(x)}}}.$$

Finally, we note that in our proofs, we use both the mountain pass and symmetric mountain pass theorems, which can be found in [36]; moreover, we use the following theorem:

Theorem 2 (Kajikiya theorem; see [37])**.** *Let E be an infinite, dimensional, real Banach space. Let $\mathfrak{J} \in C^1(E, \mathbb{R})$, satisfying the following conditions:*

(c_1) *$\mathfrak{J}$ is an even functional such that $\mathfrak{J}(0) = 0$;*
(c_2) *$\mathfrak{J}$ satisfies the (PS) condition;*
(c_3) *For any $k \in \mathbb{N}$, there exist a k-dimensional subspace of E (E_k) and a number ($r_k > 0$) such that $\sup_{E_k \cap S_{r_k}} \mathfrak{J}(u) < 0$, where $S_{r_k} = \{u \in E, ||u|| = r_k\}$.*
Then, the functional $\mathfrak{J}$ has a sequence of critical points ($\{u_k\}_{k\in\mathbb{N}}$) satisfying $||u_k|| \to 0$ as $k \to 0$.

3. Main Results

In this part, we present the principal results of this work. To do this, we make the following assumptions.

($\mathbf{H_1}$) The function $g(x, u)$ can be expressed as $a(x)h(u)$, where a and h are measurable functions satisfying the following condition: there exists $c_1 > 0$, $P, q \in C_+(\overline{\Lambda})$ such that for all $(x, u) \in \Lambda \times \mathbb{R}$, we have

$$a(x) \in \mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}(\Lambda),\ \ h(u) \leq c_1 |u|^{q(x)-1}$$

and

$$p^+ < q(x) < P(x) < p^*(x). \tag{3}$$

($\mathbf{H_2}$) There exists $t_0 > 0$ such that the Kirchhoff function (M) satisfies $M(t) \geq t_0$.
($\mathbf{H_3}$) There exists $\omega \in (0,1)$ with $1-\omega \geq \frac{1}{p^+}$ such that

$$\hat{M}(t) \geq (1-\omega)M(t)t$$

where $\hat{M}(t) = \int_0^t M(\xi)\,d\xi$.
($\mathbf{H_4}$) There exist $M_1 > 0$ and $\frac{p^+}{1-\omega} < \theta < \min(r^-, s^-)$ such that for all $x \in \Lambda$ and $|u| \geq M_1$, we have

$$0 < \theta a(x)H(u) \leq a(x)h(u)u,$$

where $H(t) = \int_0^t h(\xi)\,d\xi$.
($\mathbf{H_5}$) For all $x \in \overline{\Lambda}$, we have $h(-u) = -h(u)$.
($\mathbf{H_6}$) There exists a nonempty open ball ($B \subset \Omega$) such that

$$\lim_{t\to 0} \frac{a(x)H(t)}{|t|^{p_B^-}} = \infty uniformly for a.a. x \in B,$$

where $p_B^- = \inf_{x\in B} p(x)$.

Subsequently, we define a weak solution for the problem (1) in the following way.

Definition 1. *We assume that $u \in X$ is a weak solution for Equation (1) if, for any $v \in X$, we have*

$$M\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\int_\Lambda \left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\,{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}v$$
$$-\int_\Lambda |u|^{r(x)-2}uvdx - \int_\Lambda a(x)h(u)vdx - \int_{\partial\Lambda}|u|^{s(x)-2}uvdx = 0.$$

We are now prepared to present and demonstrate the first key results.

Theorem 3. *Under Hypotheses ($\mathbf{H_1}$)–($\mathbf{H_4}$), problem (1) has a non-trivial weak solution.*

Theorem 4. *Under Hypotheses ($\mathbf{H_1}$)–($\mathbf{H_5}$), problem (1) has an infinite number of solutions.*

Theorem 5. *Under Hypotheses ($\mathbf{H_1}$)–($\mathbf{H_6}$), problem (1) has an infinite number of solutions ($\{v_k\}_{k\in\mathbb{N}}$) satisfying $||v_k|| \to 0$ as $k \to 0$.*

Now, we introduce the functional ($\mathfrak{J}(u)$) associated with problem (1), which characterizes the critical points and plays a key role in the existence of solutions.

$$\begin{aligned}\mathfrak{J}(u) &= \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right) - \int_\Lambda \frac{|u|^{r(x)}}{r(x)}dx - \int_\Lambda a(x)H(u)dx - \int_{\partial\Lambda}\frac{|u|^{s(x)}}{s(x)}dx,\\ &= L(u) - I(u) - J(u) - T(u).\end{aligned}$$

where $\hat{M}(t) = \int_0^t M(s)ds$ and

$$L(u) = \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right), I(u) = \int_\Lambda \frac{|u|^{r(x)}}{r(x)}dx, J(u) = \int_\Lambda a(x)H(u)dx and$$

$$T(u) = \int_{\partial\Lambda}\frac{|u|^{s(x)}}{s(x)}dx.$$

We recall from [25] that $L \in C^1(X,\mathbb{R})$. Moreover, for all $u, v \in X$, we have

$$< L'(u), v >= M\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\,{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}v dx.$$

We also recall the following proposition.

Proposition 1 (see [25]). *Let* $\mathfrak{L} : \mathcal{H}_{p(x)}^{\varpi,\nu;\psi}(\Lambda) \to \left(\mathcal{H}_{p(x)}^{\varpi,\nu;\psi}(\Lambda)\right)^*$ *be a functional defined as follows:*

$$\mathfrak{L}(u) = \int_\Lambda \frac{|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u|^{p(x)}}{p(x)}dx.$$

Then the following statements hold:

1. *For all* $u, v \in \mathcal{H}_{p(x)}^{\varpi,\nu;\psi}(\Lambda)$, *we have*

$$< \mathfrak{L}'(u), v >= \int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\,{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}v dx.$$

2. *Operator* $\mathfrak{L}'$, *from X to its dual* X^*, *is continuous and bounded; moreover, it is strictly monotone.*
3. $\mathfrak{L}'$ *is a mapping of the* (S_+) *type, that is, if* $u_n \rightharpoonup u$ *in X and* $\limsup_{n\to\infty} < \mathfrak{L}'(u_n) - \mathfrak{L}'(u), u_n - u > \leq 0$, *then* $u_n \to u$ *strongly in X.*

Remark 1. *Using Hypothesis* $(\boldsymbol{H_1})$ *and the Hölder inequality,* $J \in C^1(X, \mathbb{R})$. *Therefore, for all* $u, v \in X$, *we obtain*

$$< J'(u), v >= \int_\Lambda a(x)h(u(x))v(x)dx.$$

From Proposition 1 and Remark 1, it follows that $\mathfrak{J} \in C^1(X, \mathbb{R})$. Then, for all $u, v \in X$, we have

$$\begin{aligned} < \mathfrak{J}'(u), v > \; &= \; M\left(\int_\Lambda \tfrac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\,{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}v dx \\ &- \; \int_\Lambda |u|^{r(x)-2}uv dx - \int_{\partial\Lambda}|u|^{s(x)-2}uv dx - \int_\Lambda a(x)h(u(x))v(x)dx. \end{aligned}$$

Therefore, the weak solutions of the problem (1) are associated with the critical points of functional $\mathfrak{J}$.

Now, we establish a key result that offers a lower bound for functional $\mathfrak{J}(u)$ associated with the problem (1) under the assumption of $(\mathbf{H_1})$.

Lemma 1. *Assume that* $(\boldsymbol{H_1})$*–*$(\boldsymbol{H_3})$ *are satisfied. Then, there exist* $m, \eta > 0$ *such that, for* $u \in X$,

$$if \quad \| u \| = \rho, \; then, \; \mathfrak{J}(u) \geq d.$$

Proof. Let $u \in X$, with $\| u \| < 1$. Under Hypothesis $(\mathbf{H_1})$, for all $x \in \Lambda$, we have

$$H(u) \leq \frac{c_1}{q(x)}|u|^{q(x)}. \tag{4}$$

$1 < P(x) < p^*(x),\ 1 < r(x) \leq p^*(x), 1 < s(x) < p_*^{\varpi}(x)$ proves the existence of $c_3,\ c_4,$ $c_5 > 0$, such that

$$|u|_{\mathcal{L}^{P(x)}(\Lambda)} \leq c_3\| u \|,\ |u|_{\mathcal{L}^{r(x)}(\Lambda)} \leq c_4\| u \|, |u|_{\mathcal{L}^{s(x)}(\partial\Lambda)} \leq c_5\| u \|. \tag{5}$$

Now, under Hypotheses $(\mathbf{H_2})$–$(\mathbf{H_3})$, we obtain

$$
\begin{aligned}
L(u) &= \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right) \\
&\geq (1-\omega)M\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx \\
&\geq \frac{(1-\omega)t_0}{p^+}\int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx \geq \frac{(1-\omega)t_0}{p^+}||u||^{p^+}.
\end{aligned} \tag{6}
$$

Next, according to (4)–(6), we have

$$
\begin{aligned}
\mathfrak{J}(u) &= \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right) - \int_\Lambda a(x)H(u)dx - \int_\Lambda \frac{|u|^{r(x)}}{r(x)}dx, \\
&= L(u) - J(u) - I(u) - T(u) \\
&\geq \frac{(1-\omega)t_0}{p^+}||u||^{p^+} - \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}(\Lambda)}||u||^{q^-} - \frac{c_4}{r^-}||u||^{r^-} - \frac{c_5}{s^-}||u||^{s^-} \\
&\geq ||u||^{p^+}\left(\frac{(1-\omega)t_0}{p^+} - \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}(\Lambda)}||u||^{q^- - p^+} - \frac{c_4}{r^-}||u||^{r^- - p^+} - \frac{c_5}{s^-}||u||^{s^- - p^+}\right) \\
&\geq ||u||^{p^+}\left(\frac{(1-\omega)t_0}{p^+} - t||u||^{\min(q^- - p^+, r^- - p^+, s^- - p^+)}\right),
\end{aligned}
$$

where

$$
t = \frac{c_3}{q^-}|a|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}(\Lambda)} + \frac{c_4}{r^-} + \frac{c_5}{s^-}.
$$

Since q^-, s^- and r^- are both greater than p^+, we can set $\| u \| = \rho$ to be sufficiently small such that

$$
\frac{(1-\omega)t_0}{p^+} - t\rho^{\min(q^- - p^+, r^- - p^+, s^- - p^+)} > 0.
$$

Therefore, we obtain

$$
\mathfrak{J}(u) \geq \rho^{p^+}\left(\frac{(1-\omega)t_0}{p^+} - t\,\rho^{\min(q^- - p^+, r^- - p^+, s^- - p^+)}\right) := d > 0.
$$

□

In the upcoming lemma, we demonstrate a result related to the boundedness of a Palais–Smale sequence within a space (X).

Lemma 2. *Suppose that conditions $(\boldsymbol{H_2})$–$(\boldsymbol{H_4})$ are satisfied. Let $\{u_n\}$ be a Palais–Smale sequence in X. Then, $\{u_n\}$ is bounded in X.*

Proof. Let $\{u_n\}$ be a sequence in X such that

$$
\mathfrak{J}(u_n) \to c, \; and \; \mathfrak{J}'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,
$$

where c is a positive constant.

Since $\mathfrak{J}(u_n) \to c$, there exists $T_1 > 0$, such that

$$
|\mathfrak{J}(u_n)| \leq T_1. \tag{7}
$$

On the other hand, since we have $\mathfrak{J}'(u_n) \to 0$ in X^*, $< \mathfrak{J}'(u_n), u_n >$ converges to zero. Therefore, it is bounded. Hence, there exists $T_2 > 0$, such that

$$
|< \mathfrak{J}'(u_n), u_n >| \leq T_2. \tag{8}
$$

Next, we prove, by contradiction, that $\{u_n\}$ is bounded. We consider a subsequence such that $\| u_n \| \geq 1$, and $\| u_n \| \to \infty$.

According to (7), (6) and using $p^+ < \theta < \min(r^-, s^-)$, we obtain

$$
\begin{aligned}
T_1 \geq \mathfrak{J}(u_n) &= L(u_n) - I(u_n) - J(u_n) - T(u_n) \\
&\geq \frac{(1-\omega)t_0}{p^+} \int_\Lambda \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} dx - \frac{1}{r^-} \int_\Lambda |u_n|^{r(x)} dx - \frac{1}{s^-} \int_{\partial\Lambda} |u_n|^{s(x)} dx - J(u_n) \\
&\geq \frac{(1-\omega)t_0}{p^+} \int_\Lambda \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} dx - \frac{1}{\theta} \int_\Lambda |u_n|^{r(x)} dx - \frac{1}{\theta} \int_{\partial\Lambda} |u_n|^{s(x)} dx - J(u_n).
\end{aligned} \tag{9}
$$

Now, according to (8) and ($\mathbf{H_2}$), we obtain

$$
\begin{aligned}
T_2 &\geq -< \mathfrak{J}'(u_n), u_n > \\
&= -M\left(\int_\Lambda \frac{1}{p(x)} \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u \right|^{p(x)} dx \right) \int_\Lambda \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} dx + \int_\Lambda |u_n|^{r(x)} dx \\
&+ \int_{\partial\Lambda} |u_n|^{s(x)} dx + < J'(u_n), (u_n) > \\
&\geq -t_0 \int_\Lambda \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} dx + \int_\Lambda |u_n|^{r(x)} dx + \int_{\partial\Lambda} |u_n|^{s(x)} dx + < J'(u_n), (u_n) > .
\end{aligned}
$$

By merging the inequality mentioned above with (9), we obtain

$$
\begin{aligned}
\theta T_1 + T_2 &\geq \left(\frac{(1-\omega)\theta}{p^+} - 1 \right) t_0 \int_\Lambda \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} dx + < J'(u_n), (u_n) > -\theta J(u_n) \\
&\geq \left(\frac{(1-\omega)\theta}{p^+} - 1 \right) t_0 \| u_n \|^{p^-} + \int_\Lambda a(x)(h(u_n)u_n - \theta H(u_n)) dx.
\end{aligned}
$$

Using ($\mathbf{H_4}$), we obtain

$$
\theta T_1 + T_2 \geq \left(\frac{(1-\omega)\theta}{p^+} - 1 \right) t_0 \| u_n \|^{p^-}. \tag{10}
$$

According to ($\mathbf{H_4}$) and ($\mathbf{H_2}$), $\frac{p^+}{1-\omega} < \theta$ and $t_0 > 0$; then, we have $\left(\frac{(1-\omega)\theta}{p^+} - 1 \right) t_0 > 0$, so

$$
\left(\frac{(1-\omega)\theta}{p^+} - 1 \right) t_0 \| u_n \|^{p^-} \to \infty
$$

According to (10), this is absurd. Therefore, $\{u_n\}$ is bounded in X. □

Next, we introduce a set (A) defined by $A = \{x \in \Omega : r(x) = p^*(x)\}$ as nonempty set. We also define a set expressed as $A_\delta = \{x \in \Omega : dist((x, A) < \delta\}$ for $\delta > 0$. We note that $r_\delta^- = \inf_{\overline{A_\delta}} r(x)$ and $r_A^- = \inf_{\overline{A}} r(x)$.

If $\{u_n\}$ is a Palais–Smale sequence with an energy level of c, then, as stated in Theorem 1, we obtain the following convergence results:

$$
|u_n|^{r(x)} \rightharpoonup \nu = |u|^{r(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \nu_i > 0. \tag{11}
$$

$$
\left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n \right|^{p(x)} \rightharpoonup \phi \geq \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u \right|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}, \mu_i > 0. \tag{12}
$$

$$
S \nu_i^{\frac{1}{p^*(x_i)}} \leq \mu_i^{\frac{1}{p(x_i)}}. \tag{13}
$$

If $I = \varnothing$, then $u_n \to u$ in $\mathcal{L}^{r(x)}(\Omega)$. It should be noted that $\{x_i\}_{i \in A} \subset A$. We aim to demonstrate that if $c < \left(\frac{1}{p^+} - \frac{1}{r_A^-} \right) S^n$, then $I = \varnothing$, where S is defined in Theorem 1.

Lemma 3. *If conditions ($\mathbf{H_1}$)–($\mathbf{H_4}$) are satisfied, let $\{u_n\}$ be a Palais–Smale sequence in X with a energy level of c. If $c < \left(\frac{1}{p^+} - \frac{1}{r_A^-} \right) S^n$, then the index set (I) is empty.*

Proof. Suppose that $I \neq \varnothing$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(0) \neq 0$. Now, we consider the functions $\varphi_{i,\epsilon}(x) = \varphi\Big(\frac{x-x_i}{\epsilon}\Big)$. We have $< \mathfrak{J}'(u_n), \varphi_{i,\epsilon} u_n > \to 0$. Thus,

$$\begin{aligned} & < \mathfrak{J}'(u_n), \varphi_{i,\epsilon} u_n > \\ = \; & M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Omega |{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{(p(x)-2)} {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(\varphi_{i,\epsilon} u_n\Big) dx \\ - \; & \int_\Omega |u_n|^{r(x)} \varphi_{i,\epsilon} dx - \int_\Omega a(x) h(u_n(x)) \varphi_{i,\epsilon} u_n dx - \int_{\partial\Omega} |u_n|^{s(x)} \varphi_{i,\epsilon} dx. \end{aligned}$$

Then, we obtain

$$\begin{aligned} 0 \; = \; & \lim_{n\to\infty}\Big(M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Omega \Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{(p(x)-2)} {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(\varphi_{i,\epsilon}) u_n dx \\ + \; & \int_\Omega \varphi_{i,\epsilon} d\mu - \int_\Omega \varphi_{i,\epsilon} d\nu - \int_\Omega a(x) h(u_n(x)) \varphi_{i,\epsilon} u_n dx - \int_{\partial\Omega} |u_n|^{s(x)} \varphi_{i,\epsilon} dx\Big). \end{aligned}$$

According to Hölder's inequality and using Hypothesis $(\mathbf{C_2})$, we have

$$\lim_{n\to\infty} \int_\Omega \Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{(p(x)-2)} {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(\varphi_{i,\epsilon}) u_n dx = 0.$$

Therefore, we obtain

$$\lim_{\epsilon\to 0} \int_\Omega a(x) h(u_n(x)) \varphi_{i,\epsilon} u_n dx = 0, \lim_{\epsilon\to 0} \int_{\partial\Omega} |u_n|^{s(x)} \varphi_{i,\epsilon} dx = 0$$

$$\lim_{\epsilon\to 0} \int_\Omega \varphi_{i,\epsilon} d\mu = \mu_i \varphi(0), \lim_{\epsilon\to 0} \int_\Omega \varphi_{i,\epsilon} d\nu = \nu_i \varphi(0).$$

Then, $(\mu_i - \nu_i)\varphi(0) = 0$, which implies $\mu_i = \nu_i$. Consequently,

$$S \nu_i^{\frac{1}{p^*(x_i)}} \leq \nu_i^{\frac{1}{p(x_i)}},$$

Thus, we conclude that $\nu_i = 0$ or $S^n \leq \nu_i$.
Now, since $r(x), s(x), \theta > p^+ > 1 - \omega$ and using Hypothesis $(\mathbf{H_4})$, we have

$$\begin{aligned} c \; = \; & \lim_{n\to\infty} \mathfrak{J}(u_n) = \lim_{n\to\infty}\Big(\mathfrak{J}(u_n) - \tfrac{1}{p^+} < \mathfrak{J}'(u_n), u_n >\Big) \\ = \; & \lim_{n\to\infty}\Big(\hat{M}\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u\Big|^{p(x)} dx\Big) - \int_\Omega \tfrac{|u_n|^{r(x)}}{r(x)} dx - \int_\Omega a(x) H(u_n) dx - \int_\Omega \tfrac{|u_n|^{s(x)}}{s(x)} dx \\ - \; & \tfrac{1}{p^+} M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx + \tfrac{1}{p^+}\int_\Omega a(x) h(u_n) u_n dx \\ + \; & \tfrac{1}{p^+}\int_\Omega |u_n|^{r(x)} dx + \tfrac{1}{p^+}\int_{\partial\Omega} |u_n|^{s(x)} dx\Big) \\ \geq \; & \lim_{n\to\infty}\Big((1-\omega) M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} \\ - \; & \tfrac{1}{p^+} M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx \\ + \; & \int_\Omega\Big(\tfrac{1}{p(x)} - \tfrac{1}{r(x)}\Big)|u_n|^{r(x)} dx + \int_{\partial\Omega}\Big(\tfrac{1}{p(x)} - \tfrac{1}{s(x)}\Big)|u_n|^{s(x)} dx \\ + \; & \tfrac{1}{p^+}\int_\Omega a(x) h_1(u_n) u_n dx - \int_\Omega a(x) H(u_n) dx\Big) \\ \geq \; & \lim_{n\to\infty}\Big(\Big(1 - \omega - \tfrac{1}{p^+}\Big) M\Big(\int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u_n\Big|^{p(x)} dx\Big) \int_\Lambda \tfrac{1}{p(x)}\Big|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u\Big|^{p(x)} dx \\ + \; & \int_\Omega\Big(\tfrac{1}{p^+} - \tfrac{1}{r(x)}\Big)|u_n|^{r(x)} dx + \int_{\partial\Omega}\Big(\tfrac{1}{p^+} - \tfrac{1}{s(x)}\Big)|u_n|^{s(x)} dx \\ + \; & \tfrac{1}{\theta}\int_\Omega a(x) h(u_n) u_n dx - \int_\Omega a(x) H(u_n) dx \\ \geq \; & \lim_{n\to\infty} \int_\Omega\Big(\tfrac{1}{p^+} - \tfrac{1}{r(x)}\Big)|u_n|^{r(x)} dx \\ \geq \; & \lim_{n\to\infty} \int_{A_\delta}\Big(\tfrac{1}{p^+} - \tfrac{1}{r^-_{A_\delta}}\Big)|u_n|^{r(x)} dx. \end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\int_{A_\delta}\left(\frac{1}{p^+}-\frac{1}{r^-_{A_\delta}}\right)|u_n|^{r(x)}dx &= \left(\frac{1}{p^+}-\frac{1}{r^-_{A_\delta}}\right)(\int_{A_\delta}|u|^{r(x)}+\sum_{i\in I}\nu_i)\\ &\geq \left(\frac{1}{p^+}-\frac{1}{r^-_{A_\delta}}\right)\nu_i\\ &\geq \left(\frac{1}{p^+}-\frac{1}{r^-_{A_\delta}}\right)S^n.\end{aligned} \tag{14}$$

Therefore, since δ is positive and arbitrary and r is continuous, we have

$$c\geq\left(\frac{1}{p^+}-\frac{1}{r^-_A}\right)S^n.$$

Then, if $c<\left(\frac{1}{p^+}-\frac{1}{r^-_A}\right)S^n$, the index set (I) is empty. □

We now introduce the subsequent lemma that demonstrates a significant convergence result.

Lemma 4. *Assume that conditions* $(\boldsymbol{H_1})$–$(\boldsymbol{H_4})$ *are satisfied and let* $\{u_n\}$ *be a Palais–Smale sequence in* X *with an energy level of* c*. Then, there exists a subsequence of* $\{u_n\}$ *that converges strongly in* X*.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence in X. Then, according to Lemma 2, we know that the sequence $\{u_n\}$ is bounded in X. Therefore, there exists a subsequence still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in X.

On the other hand, according to Lemma 3 and the compact embedding, we obtain

$$\begin{cases} u_n\to u,\ stronglyin\ \mathcal{L}^{P(x)}(\Lambda),\\ u_n\to u,\ stronglyin\ \mathcal{L}^{r(x)}(\Lambda),\\ u_n\to u,\ stronglyin\ \mathcal{L}^{s(x)}(\partial\Lambda).\end{cases}$$

Next, we prove that $u_n\to u$ strongly in X. To this end, we begin by remarking that

$$\begin{aligned}<\mathfrak{J}'(u_n),u_n-u> &= <L'(u_n),u_n-u>-\int_\Lambda|u_n|^{r(x)-2}u_n(u_n-u)dx\\ &- \int_\Lambda a(x)h(u_n)(u_n-u)dx-\int_{\partial\Lambda}|u_n|^{s(x)-2}u_n(u_n-u)dx.\end{aligned}$$

Therefore, by applying Hölder's inequality, we obtain

$$\begin{aligned}\int_\Lambda|u_n|^{r(x)-1}|u_n-u|dx &\leq |u_n-u|_{\mathcal{L}^{r(x)}}||u|^{r(x)-1}|_{\mathcal{L}^{\frac{r(x)}{r(x)-1}}}\\ &\leq |u_n-u|_{\mathcal{L}^{r(x)}}\max\left(|u_n|^{r^+-1}_{\mathcal{L}^{r(x)}},|u_n|^{r^--1}_{\mathcal{L}^{r(x)}}\right)\\ &\leq c_1|u_n-u|_{\mathcal{L}^{r(x)}}\max(||u_n||^{r^+-1},||u_n||^{r^--1}).\end{aligned}$$

This yields

$$\lim_{n\to\infty}\int_\Lambda|u_n|^{r(x)-2}u_n(u_n-u)dx=0. \tag{15}$$

Similarly, we can obtain

$$\lim_{n\to\infty}\int_{\partial\Lambda}|u_n|^{s(x)-2}u_n(u_n-u)dx=0. \tag{16}$$

Now, by using Hypothesis $(\mathbf{H_1})$ and Hölder's inequality, one has

$$\begin{aligned}\int_\Lambda a(x)h(u_n)(u_n-u)dx &\leq \int_\Lambda c_1|a(x)||u_n|^{q(x)-1}|u_n-u|dx \\ &\leq c_1|u_n-u|_{\mathcal{L}^{P(x)}}|a(x)|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}}||u_n|^{q(x)-1}|_{\mathcal{L}^{\frac{P(x)}{q(x)-1}}} \\ &\leq c_1|u_n-u|_{\mathcal{L}^{P(x)}}|a(x)|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}}\max(|u_n|^{q^+-1}|_{\mathcal{L}^{P(x)}},|u_n|^{q^--1}|_{\mathcal{L}^{P(x)}}) \\ &\leq c_1|u_n-u|_{\mathcal{L}^{P(x)}}|a(x)|_{\mathcal{L}^{\frac{P(x)}{P(x)-q(x)}}}\max(||u_n||^{q^+-1}|,||u_n||^{q^--1}|).\end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty}\int_\Lambda a(x)h(u_n)(u_n-u)dx=0. \tag{17}$$

On the other hand, by combining Equations (15) and (16) with Equation (17) and using the fact that $<\mathfrak{J}'(u_n),u_n-u>\to 0$, we conclude that

$$<L'(u_n),u_n-u>=M\left(\int_\Lambda\frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u_n\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u_n{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(u_n-u)dx\to 0.$$

Since Hypothesis $(\mathbf{C_2})$ implies that $M\left(\int_\Lambda\frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u\right|^{p(x)}dx\right)\neq 0$, we obtain

$$<\mathfrak{L}'(u_n),u_n-u>=\int_\Lambda\left|{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u_n\right|^{(p(x)-2)}{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}u_n{}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi}(u_n-u)dx\to 0.$$

Also, the fact that $u_n\rightharpoonup u$ implies that $<\mathfrak{L}'(u),u_n-u>\to 0$.
Hence, based on the above information, we deduce that

$$\lim_{n\to\infty}<\mathfrak{L}'(u_n)-\mathfrak{L}'(u),u_n-u>=0.$$

Finally, based on the fact that the functional $\mathfrak{L}'$ is of the (S_+) type, we conclude that $u_n\to u$. □

To go deeper into the characteristics of the $\mathfrak{J}$ functional and its critical points, we present the following lemma.

Lemma 5. *If conditions* $(\mathbf{H_2})$–$(\mathbf{H_4})$ *hold, then there exists* $u_0\in X$ *such that*

$$||u_0||>\rho,\ \textit{and}\ \mathfrak{J}(u_0)<0$$

where η *is defined in Lemma 1.*

Proof. According to $(\mathbf{H_4})$,

$$G(x,t)\geq\xi|t|^\theta,|t|\geq M_1 andx\in\Lambda. \tag{18}$$

According to Hypotheses $(\mathbf{H_2})$ and $(\mathbf{H_3})$, the function $t\longmapsto\frac{\hat{M}(t)}{t^{1/\varpi-1}}$ is decreasing. Therefore, for all $t_0>0$, when $t>t_0$, $\frac{\hat{M}(t)}{t^{\frac{1}{\varpi-1}}}\leq\frac{\hat{M}(t_0)}{t_0^{\frac{1}{\varpi-1}}}$; then,

$$\hat{M}(t)\leq\hat{M}(t_0)\left(\frac{t}{t_0}\right)^{\frac{1}{1-\omega}}\leq ct^{\frac{1}{1-\omega}},\ for\ t>t_0. \tag{19}$$

Let $t>1$ be sufficiently large and let $u\in X$ be such that $\int_\Lambda|u|^\theta\neq 0$. Then, according to (18) and (19),

$$\begin{aligned}\mathfrak{J}(tu) &\leq \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|t^{\mathrm{H}}\mathbb{D}_{0+}^{\omega,\nu;\psi}tu\right|^{p(x)}dx\right) - \int_\Lambda a(x)H(tu)dx \\ &\leq C\left(\int_\Lambda \left|^{\mathrm{H}}\mathbb{D}_{0+}^{\omega,\nu;\psi}u\right|^{p(x)}dx\right)^{1/1-\omega} t^{\frac{p^+}{1-\omega}} - c\xi t^\theta \int_\Lambda |u|^\theta dx.\end{aligned}$$

Since $\theta > \frac{p^+}{1-\omega}$, it follows that

$$\mathfrak{J}(tu) \to -\infty, \quad \text{as} \quad t \to \infty.$$

Therefore, we can set $t_0 > 0$ and set $u_0 = t_0 e$ such that $||u_0|| > \rho$ and $\mathfrak{J}(u_0) < 0$. This completes the proof. □

We now present a lemma that offers an important conclusion about the boundlessness of a set given specific conditions.

Lemma 6. *Under Hypotheses* (***H₁***)–(***H₄***)*, if F is a finite-dimensional subspace of X, then the set*

$$T = \{u \in F, such that \mathfrak{J}(u) \geq 0\},$$

is bounded in X.

Proof. Let $u \in T$. Then, according to (18) and (19), we have

$$\begin{aligned}\mathfrak{J}(u) &\leq \hat{M}\left(\int_\Lambda \frac{1}{p(x)}\left|t^{\mathrm{H}}\mathbb{D}_{0+}^{\omega,\nu;\psi}u\right|^{p(x)}dx\right) - \int_\Lambda a(x)H(u)dx \\ &\leq C\left(\int_\Lambda \left|^{\mathrm{H}}\mathbb{D}_{0+}^{\omega,\nu;\psi}u\right|^{p(x)}dx\right)^{1/1-\omega} - \xi \int_\Lambda |u|^\theta dx \\ &\leq C(||u||^{\frac{p^+}{1-\omega}} + ||u||^{\frac{p^-}{1-\omega}}) - \xi|u|^\theta_{\mathcal{L}^\theta},\end{aligned}$$

where $|.|_{\mathcal{L}^\theta}$ and $||.||$ are equivalent norms in F. Thus, there exists a positive constant (k) such that

$$||u||^\theta \leq k|u|^\theta_{\mathcal{L}^\theta}.$$

Therefore, we have

$$\mathfrak{J}(u) \leq \frac{p^+}{1-\omega}(||u||^{\frac{p^+}{1-\omega}} + ||u||^{\frac{p^-}{1-\omega}}) - \frac{\xi}{k}||u||^\theta.$$

Hence, the fact that $\frac{p^-}{1-\omega} < \frac{p^+}{1-\omega} < \theta$ implies that $\mathfrak{J}$ is bounded in X. □

Proof of Theorem 3. By combining Lemmas 1 and 4 with Lemma 5, we deduce that all conditions of the mountain pass theorem are satisfied. Therefore, this theorem ensures the existence of a critical point of the energy functional, which implies the existence of a non-trivial solution to problem (1). □

Proof of Theorem 4. First of all, we begin by remarking that $\mathfrak{J}(0) = 0$; moreover, according to condition (**H₅**), $\mathfrak{J}$ is an even functional. Therefore, combining the last information with Lemmas 1, 4, and 6, we can see that all the conditions of the symmetric mountain pass theorem hold. Therefore, this theorem ensures the existence of an unbounded sequence of critical points of the functional energy, which results in the existence of infinitely many non-trivial solutions to the problem (1). □

Proof of Theorem 5. Let $k \in \mathbb{N}$ and $E_k = span\{\phi_1, \phi_2, ..., \phi_k\}$, where ϕ_i is an eigenfunction corresponding to the i-th eigenvalue of the problem expressed as $-\Delta u = \lambda u$ in B, $u = 0$ on ∂B, and it is extended on Ω by setting $\phi_i = 0$ for $x \in \Omega \setminus B$, where B is given by Hypothesis

$(\mathbf{H_6})$. Since E_k is a finitely dimensional space, there exists a positive constant (μ_k) such that for all $u \in E_k$, we have

$$\| u \| \leq \mu_k \| u \|_{p_B^-}. \quad (20)$$

On the other hand, according to Hypothesis $(\mathbf{H_6})$, we can choose

$$M_k > \frac{M_0 \mu^{p_B^-}}{p^-} \quad (21)$$

such that for a.a. $x \in B$ and for all $|t| < \varepsilon_k$, we have

$$a(x)H(t) \geq M_K |t|^{p_B^-}. \quad (22)$$

Let $r_k \in \{0, \min(1, \varepsilon_k)\}$ and $S_{r_k} = \{u \in E, ||u|| = r_k\}$. Then, according to (H_2), $\hat{M}(t) \leq M_0 t$ for all $t \geq 0$. Therefore, using Equation (22), if u is a function such that $supp(u) \subset B$, then we obtain

$$\begin{aligned} \mathfrak{J}(u) &\leq \hat{M}\left(\int_\Lambda \frac{1}{p(x)} \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u \right|^{p(x)} dx \right) - \int_\Lambda a(x) H(u) dx, \\ &\leq M_0 \int_\Lambda \frac{1}{p(x)} \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u \right|^{p(x)} dx - \int_\Lambda a(x) H(u) dx, \\ &\leq \frac{M_0}{p^-} \int_B \left| {}^{\mathrm{H}}\mathbb{D}_{0+}^{\varpi,\nu;\psi} u \right|^{p(x)} dx - M_K \int_B |u|^{p_B^-} dx. \end{aligned}$$

Now, if $u \in E_k \cap S_{r_k}$, then according to Equation (20), we have

$$\begin{aligned} \mathfrak{J}(u) &\leq \frac{M_0}{p^-} ||u||^{p_B^-} - M_K ||u||_{p_B^-}^{p_B^-} \\ &\leq \frac{M_0}{p^-} ||u||^{p_B^-} - M_K (\mu_k^{-1} ||u||)^{p_B^-} \\ &\leq \left(\frac{M_0}{p^-} - M_K \mu_k^{-p_B^-} \right) r_k^{p_B^-}. \end{aligned}$$

Finally, according to (21), $\sup\limits_{E_k \cap S_{r_k}} \mathfrak{J}(u) < 0$. Hence, $\mathfrak{J}$ satisfies condition (c_3) of Theorem 2. On the other hand, if $\mathfrak{J}(0) = 0$, the $\mathfrak{J}$ functional is even. Additionally, Lemmas 1 and 4 confirm that all the requirements outlined in Theorem 2 are satisfied. Therefore, we can conclude that problem (1) has a sequence of critical points ($\{u_k\}_{k \in \mathbb{N}}$) satisfying $||u_k|| \to 0$ as $k \to 0$. Thus, the proof of Theorem 5 is complete. □

4. Conclusions and Discussion

In this paper, we have proven two existing results for a Kirchhoff-type problem involving a fractional ψ-Hilfer derivative with variable exponents and critical nonlinearity. In the first main result, we studied the existence of solutions by proving that the associated energy functional satisfies the geometry of the mountain pass theorem. In the second main result, we investigated the existence an infinitely number of solutions for such a problem by using the symmetric version of the mountain pass theorem. It is noted that to manipulate the embedding for the critical exponent, we used a concentration-compactness principle.

We hope to develop other works by considering double-phase problems with singular nonlinearity.

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