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Exploring the Characteristics of Δ_h Bivariate Appell Polynomials: An In-Depth Investigation and Extension through Fractional Operators

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Abstract: The objective of this article is to introduce the Δ_h bivariate Appell polynomials ${}_{\Delta_h}\mathbb{A}_s^{[r]}(\lambda, \eta; h)$ and their extended form via fractional operators. The study described in this paper follows the line of study in which the monomiality principle is used to develop new results. It is further discovered that these polynomials satisfy various well-known fundamental properties and explicit forms. The explicit series representation of Δ_h bivariate Gould–Hopper polynomials is first obtained, and, using this outcome, the explicit series representation of the Δ_h bivariate Appell polynomials is further given. The quasimonomial properties fulfilled by bivariate Appell polynomials Δ_h are also proved by demonstrating that the Δ_h bivariate Appell polynomials exhibit certain properties related to their behavior under multiplication and differentiation operators. The determinant form of Δ_h bivariate Appell polynomials is provided, and symmetric identities for the Δ_h bivariate Appell polynomials are also exhibited. By employing the concept of the forward difference operator, operational connections are established, and certain applications are derived. Different Appell polynomial members can be generated by using appropriate choices of functions in the generating expression obtained in this study for Δ_h bivariate Appell polynomials. Additionally, generating relations for the Δ_h bivariate Bernoulli and Euler polynomials, as well as for Genocchi polynomials, are established, and corresponding results are obtained for those polynomials.

Keywords: Δ_h special polynomials; fractional operators; Appell polynomials; monomiality principle; operational formalism; determinant form; symmetry identities

MSC: 33E20; 33B10; 33E30; 33C45; 11T23

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1. Introduction

Numerous branches of mathematics and physics use the Appell polynomials, including algebraic geometry, differential equations, and quantum mechanics. They share a strong connection with other families of special functions, including the Jacobi polynomials and the hypergeometric functions. Another Appell polynomial results from the composition of two other Appell polynomials, thus indicating that the Appell polynomials constitute a commutative group. The relationship with the Heun equation, which specifically resembles the Appell polynomials, is what gives origin to this group trait. It is well known that the Heun equation has a Galois group, which creates an abelian extension of the differential field produced by its solutions.

The Appell polynomial family, denoted by $\{Q_n(\lambda)\}_{n\in\mathbb{N}}$, was developed by Appell [1] in the nineteenth century and solves the following differential equation:

$$\frac{d}{d\lambda}Q_n(\lambda) = n\,Q_{n-1}(\lambda), \quad Q_0 \neq 0 \tag{1}$$

which has the following generating expression:

$$Q(\xi)\exp(\lambda\xi) = \sum_{n=0}^{\infty} Q_k(\lambda)\frac{\xi^n}{n!}, \tag{2}$$

where $Q(\xi)$ is given by the following Taylor expansion:

$$Q(\xi) = \sum_{n=0}^{\infty} Q_n\frac{\xi^n}{n!}, \qquad Q_0 \neq 0 \tag{3}$$

and the real line is convergent.

Special functions are important in mathematical physics and engineering, as they are solutions to differential equations in various fields, including classical functions like Bessel, Legendre, and Hermite polynomials, as well as newer generalizations like multivariate and multivariable special functions. The study of special functions has a rich history, thereby dating back to the early days of calculus and the study of differential equations. Over time, the development of new mathematical techniques and the need to solve increasingly complex problems has led to the creation of many new families of special functions. The significance of special functions in both practical and theoretical contexts cannot be overstated. They are used in a wide range of applications, including signal processing, quantum mechanics, statistical physics, and more; see, for example, [2–7]. The development of new families of special functions has allowed researchers to tackle problems that were previously thought to be unsolvable and has opened up new avenues for research in pure mathematics as well. The work of Hermite [8] and others in the development of multiple-index, multiple-variable special functions have been particularly important. These functions provide a powerful tool for solving many problems in physics and mathematics and are essential for understanding the behavior of complex systems. The continued development of new families of special functions is likely to have a significant impact on our understanding of the natural world and the mathematical structures that underlie it.

The Δ_h special polynomials are of particular interest due to their unique properties and usefulness in various fields of engineering and mathematics. Several writers are interested in presenting and identifying various traits of Δ_h special polynomials; see, for example, [9,10]. The research conducted by Shahid Wani et al. [11–14] on specialized doped polynomials and their properties appears to be a significant and noteworthy contribution to this particular domain. The recognized features of these polynomials, such as Wronskian matrices and Pascal functional matrices, determinant forms, summation formulae, explicit and implicit formulae, approximation properties and generating expressions, have proven to be extremely useful in a variety of real-world applications.

For instance, summation formulae can be used to evaluate the sum of a series involving these polynomials, while determinant forms can help in finding the eigenvalues of matrices involving these polynomials. Explicit and implicit formulae can be useful in calculating the specific values of these polynomials, and generating expressions can provide a way to generate the infinite sequences of these polynomials. Hence, the investigation into distinct polynomials and their characteristics remains a vibrant and dynamic field of research, thereby bearing crucial implications in numerous disciplines.

According to ([15], p. 2), the Δ_h operator for the forward difference is defined as follows:

$$\Delta_h[J](\lambda) = J(\lambda + h) - J(\lambda).$$

This means that for a given order $i \in \mathbb{N}$ of finite difference, we find

$$\Delta_h\left(\Delta_h^{i-1}[J](\lambda)\right) = \Delta_h^i[J](\lambda) = \sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}J(\lambda + lh), \tag{4}$$

where $J : I \subset \mathbb{R} \to \mathbb{R}$, h $\in \mathbb{R}_+$, and $\Delta_h^0 = I$ and $\Delta_h^1 = \Delta_h$, with I serving as the operator of identity type.

Costabile and Longo [10] introduced a new set of polynomial sequences called Δ_h Appell polynomials, which are generated using the forward difference operator Δ_h. These polynomials are denoted by $\mathcal{Q}_n(\lambda;h)$ and have various properties that were studied in their work. The polynomials are defined using a generating expression:

$$\gamma(\xi)(1+h\xi)^{\frac{\lambda}{h}} = \sum_{n=0}^{\infty} \mathcal{Q}_n(\lambda;h)\frac{\xi^n}{n!}, \tag{5}$$

or by the relation

$$\Delta_h[\mathcal{Q}_n](\lambda;h) = nh\mathcal{Q}_{n-1}(\lambda;h). \tag{6}$$

When the value of h approaches zero, the expression given by Equation (5) simplifies to Equation (2), while the expression in Equation (6) simplifies to Equation (1).

Furthermore, Costabile and Longo [10] introduced a set of sequences called Δ_h Appell sequences, which are denoted by $\mathcal{Q}_m(\lambda)$ for $m \in \mathbb{N}$. These sequences are established by employing the power series expansion of the product of two functions. The first function takes the form of $(1+h\xi)^{\frac{\lambda}{h}}$, while the second function is represented by $\gamma(\xi)$:

$$\gamma(\xi)(1+h\xi)^{\frac{\lambda}{h}} = \mathcal{Q}_0(\lambda;h) + \mathcal{Q}_1(\lambda;h)\frac{\xi}{1!} + \mathcal{Q}_2(\lambda;h)\frac{\xi^2}{2!} + \cdots \mathcal{Q}_m(\lambda;h)\frac{\xi^n}{n!}\cdots, \tag{7}$$

where

$$\gamma(\xi) = \gamma_{0,h} + \gamma_{1,h}\frac{\xi}{1!} + \gamma_{2,h}\frac{\xi^2}{2!} + \cdots + \gamma_{n,h}\frac{\xi^n}{n!} + \cdots. \tag{8}$$

The Δ_h Appell sequences are a class of Appell sequences that arise from applying the difference operator Δ_h to a given Appell sequence. When the parameter λ in the Appell sequence takes on certain values, the resulting Δ_h Appell sequences reduce to well-known sequences and polynomials. For example:

When λ is a non-negative integer, the generalized falling factorials [15] $(\lambda)_m^h \equiv (\lambda)_m$ are obtained as a special case of the Δ_h Appell sequence.

When $\lambda = 1$, the Bernoulli sequence of the second kind [15] $b_n(\lambda)$ is obtained as a special case of the Δ_h Appell sequence.

When $\lambda = \gamma$ (where γ is a constant), the Poisson–Charlier sequence ([15], p. 2) $C_m(\lambda;\gamma)$ is obtained as a special case of the Δ_h Appell sequence.

When $\lambda = \gamma$ and $h = 1$, the Boole sequence $B_{lm}(\lambda;\lambda)$ is obtained as a special case of the Δ_h Appell sequence.

These special cases can be useful in applications where these well-known sequences and polynomials are needed, and the Δ_h Appell sequences provide a way to obtain them in a systematic way.

The evolution of hybrid special polynomials, achieved by integrating the monomiality principle and operational rules, is an intriguing observation. The application of these fundamental principles and operators has endowed researchers with a potent and versatile toolset for investigating diverse mathematical challenges.

It is evident and beyond question that certain aspects of hybrid special polynomials exhibit significant advancements when incorporating the principles of monomiality, operational rules, and other related properties. The concept of monomiality can be traced back to 1941 when Steffenson introduced the notion of poweroids, as referenced in [16]. This notion was subsequently refined by Dattoli [17].

Hence, within this context, the $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators assume paramount significance. These operators act as multiplicative and derivative operators, respectively, for a set of polynomials labeled as $b_m(u)_{m\in\mathbb{N}}$. As a consequence, they adhere to the following expressions:

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{9}$$

and

$$m\, b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}. \tag{10}$$

As a result of applying multiplicative and derivative operators to the set of polynomials $b_m(u)_{m\in\mathbb{N}}$, it takes on the characteristic features of a quasimonomial. For it to be considered a quasimonomial, the set must satisfy the following formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{11}$$

thus showing a Weyl group structure in this manner.

When the underlying set $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ is seen as a quasimonomial, one may infer the distinctive characteristics of the set by using the properties of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$. As a consequence, it is possible to make the following conclusions with confidence:

(i) $b_m(u)$ demonstrates the differential equation

$$m\, b_m(u) = \hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(u)\}, \tag{12}$$

if both $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ have differential realizations.

(ii) Casting $b_m(u)$ in the explicit form results in the following:

$$b_m(u) = \hat{\mathcal{M}}^m\,\{1\}, \tag{13}$$

thereby taking into account that $b_0(u) = 1$.

(iii) Additionally, the following may be used to cast a relation in the exponential form for $b_m(u)$:

$$e^{t\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty} b_m(u)\frac{t^m}{m!}, \quad |t| < \infty\ , \tag{14}$$

using identity (13).

Researchers have made tremendous progress in the study of special polynomials by combining the monomiality principle and operational principles. As a result, an intriguing class known as hybrid special polynomials was created. These polynomials exhibit exceptional characteristics and have proven to be valuable in solving diverse mathematical problems. As a consequence, the study of hybrid special polynomials has become a vibrant and dynamic area of research, with widespread applications in various disciplines such as physics, engineering, and computer science. Notably, recent breakthroughs utilizing the monomiality principle are evident in publications like [18–20].

The generating relation (14) presents a representation of quasimonomials in the form of a power series with respect to the variable ξ. The coefficients in this power series are determined by the quasimonomials within the set. This equation makes it possible to calculate quasimonomials precisely and has useful uses in researching particular special functions and their characteristics. The theory of quasimonomials and Weyl groups offers a potent and versatile approach to examining the characteristics and dynamics of particular function families. Its significance extends to diverse fields, including mathematics and physics, where it serves as a crucial tool in solving intricate problems and understanding fundamental phenomena.

These operational approaches are still used extensively in many fields today, such as mathematical physics, quantum mechanics, and classical optics. As a result, these methods continue to be potent and useful research tools, as demonstrated by citations like [21–23].

In the field of quantum mechanics, operational methods have held a central position in revealing the behavior of various systems, including atoms, molecules, and materials in condensed states. Moreover, they have proven to be highly valuable in the captivating realm of quantum information and computing, thereby allowing scientists to grasp and control the complex characteristics of quantum states and operators.

Similarly, within the realm of classical optics, operational methodologies have become essential for examining the characteristics of light and its interactions with substances.

Their particular importance is evident, especially in the investigation of nonlinear optics, where the behavior of materials under intense electromagnetic fields results in a fascinating array of phenomena.

Given their extensive applications in both mathematics and science, the Δ_h bivariate special polynomials, including Appell polynomials, hold significant value. These polynomials naturally emerge in quantum mechanics and probability theory, thereby establishing connections with the normal distribution—a fundamental concept in probability theory. Furthermore, they serve as a flexible foundation for approximating functions in approximation theory, thus making them an indispensable tool for numerical analysis.

In the field of statistical mechanics, Δ_h bivariate Appell polynomials play a crucial role in computing the partition function. Moreover, in Fourier analysis, they prove to be beneficial for decomposing functions into a sum of orthogonal functions. The pioneering work of Costabile and Longo [10] has inspired further exploration of these special polynomials. In this context, we present Δ_h bivariate Appell polynomials, which are defined by a generating expression in the subsequent form:

$$\gamma(\xi)\,(1+h\xi)^{\frac{\lambda}{h}}\,(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!},\quad r\in\mathbb{N}_0. \tag{15}$$

For, $\gamma(\xi)=1$, we have

$$(1+h\xi)^{\frac{\lambda}{h}}\,(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!} \tag{16}$$

and we name them as Δ_h bivariate Gould–Hopper polynomials.
For $\gamma(\xi)=1$ and $\lambda=0$, we have

$$(1+h\xi)^{\frac{0}{h}}\,(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n^{[r]}(0,\eta;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n^{[r]}(\eta;h)\frac{\xi^n}{n!}. \tag{17}$$

The special polynomials introduced in this context hold substantial practical value, with applications across diverse fields. In areas such as image processing and computer vision, these polynomials can play a crucial role in improving image quality and extracting essential features, thereby enhancing the overall performance of these technologies.

Furthermore, in the realm of financial mathematics, these polynomials find utility in modeling the behavior of financial variables, including stock prices and interest rates. Their application in this domain enables researchers and analysts to gain deeper insights into financial markets, make informed decisions, and develop effective forecasting models. Thus, the versatility of these special polynomials makes them indispensable tools with significant implications in both scientific and applied disciplines.

The objective of this article is to investigate the properties and potential applications of Δ_h bivariate Appell polynomials. The subsequent sections of the manuscript are organized as follows:

Section 2 introduces these polynomials and their characteristics starting with the generation expression and providing the explicit series representation by employing the explicit series representation of Δ_h bivariate Gould–Hopper polynomials, while Section 3 establishes their quasimonomial properties and the determinant form through the verification of certain properties related to the behavior of Δ_h bivariate Appell polynomials under multiplication and differentiation operators. Section 4 presents symmetric identities for the Δ_h bivariate Appell polynomials and various formulas, and characteristics of the Δ_h bivariate Appell polynomials are explored. In Section 5, operational approaches are used in the investigation of Δ_h bivariate Appell polynomials, and some specific members of this polynomial family are presented alongside related findings. The article primarily focuses on exploring the mathematical features of these polynomials and their potential uses. Lastly, the concluding remarks are presented in the final section.

2. Δ_h Bivariate Appell Polynomials

This section presents an alternative method for determining Δ_h bivariate Appell sequences. The method is presented as follows:

Theorem 1. *It is apparent that Δ_h bivariate Appell polynomials can be expressed using Equation (15). Hence, we obtain:*

$$\begin{array}{rcl} {}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] & = & nh\ {}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h) \\ {}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] & = & n(n-1)\cdots(n-r+1)h\ {}_{\Delta_h}\mathbb{A}_{n-r}^{[r]}(\lambda,\eta;h). \end{array} \tag{18}$$

Proof.

$$\begin{aligned} {}_{\lambda}\Delta_h\Big\{\sum_{n=0}^{\infty}{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!}\Big\} &= {}_{\lambda}\Delta_h\{\gamma(\xi)\ (1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}}\ \} \\ &= \gamma(\xi)\ (1+h\xi)^{\frac{\lambda+h}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}}\ -\gamma(\xi)\ (1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}} \\ &= \Big(1+h\xi-1\Big)\gamma(\xi)\ (1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}} \\ &= \Big(h\xi\Big)\gamma(\xi)\ (1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}} \\ &= h\sum_{n=0}^{\infty}{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^{n+1}}{n!}. \end{aligned}$$

The following is obtained by replacing n on the right-hand side of the equation with $n-1$ and then comparing the coefficients of the corresponding powers of ξ in the resultant equation:

$${}_{\Delta_\lambda}\{\sum_{n=0}^{\infty}{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^{n+1}}{n!}\} = \sum_{n=0}^{\infty}n\,{}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!},$$

We obtain the proof of the system of Equation (18).

By using a similar fashion, we can prove the other two relations. □

Theorem 2. *Additionally, the power series is given by the following equation:*

$$\gamma(\xi) = \gamma_{0,h} + \gamma_{1,h}\frac{\xi}{1!} + \gamma_{2,h}\frac{\xi^2}{2!} + \cdots + \gamma_{n,h}\frac{\xi^n}{n!} + \cdots,\ \gamma_{0,h} \neq 0, \tag{19}$$

The power series expansion of the product $\gamma(\xi)(1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}}$ determines the following expression:

$$\begin{aligned} \gamma(\xi)\ (1+h\xi)^{\frac{\lambda}{h}}\ (1+h\xi^r)^{\frac{\eta}{h}}\ &= {}_{\Delta_h}\mathbb{A}_0^{[r]}(\lambda,\eta;h) + {}_{\Delta_h H}\lambda(\lambda,\eta;h)\frac{\xi}{1!} \\ &\quad + {}_{\Delta_h H}\eta(\lambda,\eta;h)\frac{\xi^2}{2!} + \cdots + {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!} + \cdots. \end{aligned} \tag{20}$$

Proof. The statement is proved by a mathematical technique called "Newton series for finite differences". It involves expanding a given expression in terms of a series using a variable step size or "finite differences". Therefore, we expand $(1+h\xi)^{\frac{\lambda}{h}}(1+h\xi^r)^{\frac{\eta}{h}}$, where ξ, λ, η, h, and r are all variables. The expansion is being performed around the point where $\lambda = \eta = 0$. After expanding both terms using the Newton series for finite differences, and upon multiplying the resulting series expansions of the two functions $\gamma(\xi)$ and $(1+h\xi)^{\frac{\lambda}{h}}(1+h\xi^r)^{\frac{\eta}{h}}$ together in the appropriate order concerning the powers of ξ and considering Expression (7), we observe that the polynomials ${}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)$ expressed in Equation (20) emerge as coefficients of $\frac{\xi^n}{n!}$, thus serving as the generating function of the Δ_h bivariate Appell polynomials. □

Next, we proceed to derive the explicit series representation of the Δ_h bivariate Appell polynomials. To accomplish this, we first obtain the explicit series representation of the Δ_h bivariate Gould–Hopper polynomials, as dictated by the following result:

Theorem 3. *The following explicit series formula is valid for the Δ_h bivariate Gould–Hopper polynomials ${}_{\Delta_h}\mathbb{H}_n^{[r]}(\lambda,\eta;h)$:*

$$ {}_{\Delta_h}\mathbb{H}_n^{[r]}(\lambda,\eta;h) = \sum_{k=0}^{[\frac{n}{r}]} \binom{n}{rk} (\lambda)_{n-rk}^{h} \, (\eta)_k^h \frac{(rk)!}{k!}, \tag{21} $$

where $(\lambda)_m^h \equiv (\lambda)_m$ is given by

$$ (\lambda)_m^h = \lambda(\lambda+h)(\lambda+2h)\cdots(\lambda+(m-1)h), \;\; m=1,2,\cdots, \;\; (\lambda)_0^h = 1. \tag{22} $$

Proof. By extending the $(1+ht)^{\frac{\lambda}{h}} \, (1+ht^r)^{\frac{\eta}{h}}$ equations in terms of rising the factorials specified by Equation (22), we obtain

$$ (1+ht)^{\frac{\lambda}{h}} \, (1+ht^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} \Big(-\frac{\lambda}{h}\Big)_n (-h)^n \frac{\xi^n}{n!} \sum_{k=0}^{\infty} \Big(-\frac{\eta}{h}\Big)_k (-h)^k \frac{\xi^{rk}}{k!}. \tag{23} $$

By inferring from the Cauchy product in the final two series on the right-hand side of the preceding statement and the product rule for the two series, we may conclude that

$$ (1+ht)^{\frac{\lambda}{h}} \, (1+ht^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{r}]} \binom{n}{rk} (\lambda)_{n-rk}^{h} (\eta)_k^h \frac{(rk)!}{k!} \frac{\xi^n}{n!}. \tag{24} $$

Upon substituting the series expansion of the Δ_h bivariate Gould–Hopper polynomials, as given by Equation (16), into the left-hand side of the equation above and subsequently comparing the coefficients of the corresponding powers of ξ in the resulting equation, we can verify the validity of the assertion of (21). □

Theorem 4. *The Δ_h bivariate Appell polynomials satisfy the following explicit form:*

$$ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \sum_{m=0}^{\frac{n}{m}} \sum_{k=0}^{[\frac{m}{r}]} \binom{n}{m} \binom{m}{rk} \gamma_{n-m,h} (\lambda)_{m-rk}^{h} \, (\eta)_k^h \frac{(rk)!}{k!}. \tag{25} $$

Proof. By substituting the expansions of $\gamma(\xi)$ given by Expression (8) and the Δ_h bivariate Gould–Hopper polynomials ${}_{\Delta_h}\mathbb{H}_m^{[r]}(\lambda,\eta;h)$ given by (24) in Expression (15), we have

$$ \gamma(\xi) \, (1+h\xi)^{\frac{\lambda}{h}} \, (1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} \gamma_{n,h} \frac{\xi^n}{n!} \sum_{m=0}^{\infty} \sum_{k=0}^{[\frac{m}{r}]} \binom{m}{rk} (\lambda)_{m-rk}^{h} (\eta)_k^h \frac{(rk)!}{k!} \frac{\xi^m}{m!}. \tag{26} $$

By inserting the right-hand side of Expression (15) into the left-hand side of the previous expression, we obtain

$$ \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \sum_{n=0}^{\infty} \gamma_{n,h} \sum_{m=0}^{\infty} \sum_{k=0}^{[\frac{m}{r}]} \binom{m}{rk} (\lambda)_{m-rk}^{h} (\eta)_k^h \frac{(rk)!}{k!} \frac{\xi^{n+m}}{n!\, m!}. \tag{27} $$

Using the Cauchy product rule by replacing $n \to m$ in the r.h.s. of the previous expression, we find

$$ \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \gamma_{n-m,h} \sum_{k=0}^{[\frac{m}{r}]} \binom{m}{rk} (\lambda)_{m-rk}^{h} (\eta)_k^h \frac{(rk)!}{k!} \frac{\xi^n}{(n-m)!\, m!}. \tag{28} $$

Therefore, the claim in Equation (25) is proven by multiplying and dividing by $n!$ in the r.h.s. of the preceding equation and then comparing the coefficients of the identical exponents of ξ on both sides of the resulting equation. □

Theorem 5. *The Δ_h bivariate Appell polynomials satisfy the following explicit form:*

$$ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \sum_{s=0}^{[\frac{n}{s}]} \binom{n}{s} \mathcal{Q}_{s,h}\ {}_{\Delta_h}\mathbb{H}_{n-s}(\lambda,\eta;h). \tag{29}$$

Proof. By substituting the Expression (7) with $h = 0$ and (16) into the left-hand side of Equation (15), it can be concluded that

$$\sum_{s=0}^{\infty} \mathcal{Q}_{s,h}(\lambda)\frac{\xi^s}{s!} \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!}. \tag{30}$$

The comparison of coefficients of similar powers of ξ in the resultant equation, produced after taking into consideration the C.P. rule in the left-hand side of the aforementioned equation, establishes the assertion in (29). □

Theorem 6. *The Δ_h bivariate Appell polynomials satisfy the following explicit form:*

$$ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \sum_{s=0}^{[\frac{n}{s}]} \binom{n}{s} \mathcal{Q}_s(\lambda;h)\ {}_{\Delta_h}H_{n-s}^{[r]}(\eta;h). \tag{31}$$

Proof. By inserting Expressions (5) and (17) in the l.h.s. of Equation (15), it follows that

$$\sum_{s=0}^{\infty} \mathcal{Q}_s(\lambda;h)\frac{\xi^s}{s!} \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{H}_n^{[r]}(\eta;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!}. \tag{32}$$

The comparison of coefficients of similar powers of ξ in the resultant equation, produced after taking into consideration the C.P. rule in the left-hand side of the aforementioned equation, establishes the assertion in (31). □

3. Monomiality Principle and Determinant Form

In this particular section, the aim is to demonstrate and prove the quasimonomial properties that are fulfilled by the bivariate Appell polynomials of Δ_h. These properties are established through the verification of the following outcomes:

Theorem 7. *The Δ_h bivariate Appell polynomials exhibit certain properties related to their behavior under multiplication and differentiation operators. Specifically, they satisfy the following operators:*

$$ {}_{\Delta_h}\mathbb{A}_{n+1}(\lambda,\eta;h) = \hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h}\mathbb{A}_n(\lambda,\eta;h)\} = \left(\frac{\lambda}{1+{}_{\lambda}\Delta_h} + \frac{r\,\eta_{\lambda}{\Delta_h}^{r-1}}{h^{r-1}+{}_{\lambda}{\Delta_h}^{r}} + \frac{\gamma'(\frac{\lambda\Delta_h}{h})}{\gamma(\frac{\lambda\Delta_h}{h})}\right) \times \{{}_{\Delta_h}\mathbb{A}_n(\lambda,\eta;h)\} \tag{33}$$

and

$$ {}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h) = \hat{\mathcal{D}_{\Delta_h}}\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\} = \frac{log(1+{}_{\lambda}\Delta_h)}{nh}\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\}, \tag{34}$$

respectively.

Proof. Considering the finite difference operator Δ_h, we find

$${}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = h\ \xi\ [{}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h)], \tag{35}$$

or

$$\frac{{}_{\lambda}\Delta_h}{h}[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = \xi\ [{}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h)]. \tag{36}$$

By taking the derivative of Expression (15) with respect to both ξ and λ, we obtain the following:

$$\begin{gathered}{}_{\Delta_h}\mathbb{A}_{n+1}^{[r]}(\lambda,\eta;h) = \hat{\mathcal{M}}_{\Delta_h}\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\} = \\ \left(\frac{\lambda}{1+h\xi} + \frac{r\,\eta\xi^{r-1}}{1+h\xi^r} + \frac{\gamma'(\xi)}{\gamma(\xi)}\right)\{{}_{\Delta_h}\mathbb{A}_n(\lambda,\eta;h)\}\end{gathered} \tag{37}$$

and

$${}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h) = \hat{\mathcal{D}_{\Delta_h}}\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\} = \frac{log(1+h\xi)}{nh}\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\}, \tag{38}$$

respectively.

By utilizing the identity in (36) based on Equations (9) and (10) in the aforementioned Equations (37) and (38), we can derive Assertions (33) and (34). □

Theorem 8. *The Δ_h bivariate Appell polynomials satisfy the following differential equation:*

$$\left(\frac{\lambda}{1+{}_{\lambda}\Delta_h} + \frac{r\eta{}_{\lambda}\Delta_h{}^{r-1}}{h^{r-1}+{}_{\lambda}\Delta_h{}^{r}} + \frac{\gamma'(\frac{{}_{\lambda}\Delta_h}{h})}{\gamma(\frac{{}_{\lambda}\Delta_h}{h})} - \frac{n^2h}{log(1+{}_{\lambda}\,\Delta_h)}\right)\{{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\} = 0. \tag{39}$$

Proof. By applying Equations (33) and (34) to Equation (12), we arrive at Assertion (39). □

The formulas in (33), (34), and (39) converge to the multiplicative and derivative operators in the differential equation that is solved by the bivariate Appell polynomials $\mathcal{Q}_m(\lambda;h)$ as η approaches zero.

As h tends towards zero, Expressions (33), (34), and (39) simplify to the multiplicative and derivative operators, and the differential equation that is fulfilled by the Appell polynomials $Q_m(\lambda)$, which are described by Expression (2).

Next, we give the determinant form of the Δ_h bivariate Appell polynomials by proving the result listed below:

Theorem 9. *The Δ_h bivariate Appell polynomials ${}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)$ give rise to the determinant represented by the following:*

$${}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \frac{(-1)^n}{(\gamma_{0,h})^{n+1}}\begin{vmatrix} 1 & {}_{\Delta_h}\mathbb{H}_1(\lambda,\eta;h) & {}_{\Delta_h}\mathbb{H}_2(\lambda,\eta;h) & \cdots & {}_{\Delta_h}\mathbb{H}_{n-1}(\lambda,\eta;h) & {}_{\Delta_h}\mathbb{H}_n(\lambda,\eta;h) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{n,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{n-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{n}{n-1}\gamma_{1,h} \end{vmatrix}, \tag{40}$$

where, the coefficients $\gamma_{n,h}$, $n = 0, 1, \cdots$ represent the terms of the Maclaurin series expansion for $\frac{1}{\gamma(\xi)}$.

Proof. When both sides of Equation (15) are multiplied by $\frac{1}{\gamma(\xi)}$, which can be expressed in the expansion as $\sum_{n=0}^{\infty}\gamma_{n,h}\frac{\xi^n}{n!}$, we obtain

$$\sum_{n=0}^{\infty}{}_{\Delta_h}\mathbb{H}_n(\lambda,\eta;h)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\gamma_{m,h}\frac{\xi^m}{m!}\,{}_{\Delta_h}\mathbb{A}_n(\lambda,\eta;h)\frac{\xi^n}{n!}. \tag{41}$$

In light of the C.P. rule, we have

$$ {}_{\Delta_h}\mathbb{H}_n(\lambda,\eta;h) = \sum_{m=0}^{n} \binom{n}{m} \gamma_{m,h}\ {}_{\Delta_h}\mathbb{A}_{n-m}(\lambda,\eta;h). \tag{42}$$

This equation gives rise to a set of m equations involving $\Delta_h\mathbb{A}n(\lambda,\eta;h)$ as unknowns, where n varies from 0 to $m-1$. To solve this system, we employ Cramer's rule and leverage the fact that the denominator corresponds to the determinant of a lower triangular matrix, with the determinant being $(\gamma_{0,h})^{n+1}$. The numerator is then reversed to obtain the desired outcome, and the ith row is replaced with the $(i+1)$th row for $i = 1, 2, \cdots, n-1$. □

4. Symmetric Identities

In this section, we present symmetric identities for the Δ_h bivariate Appell polynomials. Additionally, we explore various formulas and characteristics of the Δ_h bivariate Appell polynomials.

The symmetric identities in mathematics play a significant role in various areas of study, including algebra, analysis, and combinatorics. These identities provide valuable insights and techniques for simplifying calculations, thereby establishing relationships between different mathematical objects and solving equations or systems of equations. In particular, symmetric identities are often employed in the study of special functions and polynomial sequences. They allow us to derive new identities and properties by relating different terms or coefficients within a mathematical expression or equation. By equating coefficients or comparing terms, we can establish connections, derive explicit formulas, and explore the properties and characteristics of various mathematical objects. Furthermore, symmetric identities have practical applications in fields such as physics, engineering, computer science, and statistics. They provide a framework for solving real-world problems, modeling physical phenomena, and optimizing systems.

Theorem 10. *For $\mathfrak{D} \neq \mathfrak{C}$ and $\mathfrak{D}, \mathfrak{C} > 0$, we have*

$$ \mathfrak{C}^m\ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{D}\lambda, \mathfrak{D}\eta; h) = \mathfrak{D}^m\ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{C}\lambda, \mathfrak{C}\eta; h). \tag{43}$$

Proof. Since $\mathfrak{D} \neq \mathfrak{C}$ and $\mathfrak{D}, \mathfrak{C}.0$, we start by writing

$$ \mathbf{R}(\xi;\lambda,\eta;h) = \gamma(\xi)(1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}}(1+h\xi^r)^{\frac{\mathfrak{C}^r\mathfrak{D}^r\eta}{h}}. \tag{44}$$

Therefore, the above expression $\mathbf{R}(\xi;\lambda,\eta;h)$ is symmetric in $\mathfrak{C}$ and $\mathfrak{D}$.

Moreover, we can express it as follows:

$$ \begin{gathered} \mathbf{R}(\xi;\lambda,\eta;h) = {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{C}\lambda, \mathfrak{C}^r\eta; h)\frac{(\mathfrak{D}\xi)^m}{m!} = \\ \mathfrak{D}^m\ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{C}\lambda, \mathfrak{C}^r\eta; h)\frac{\xi^m}{m!}. \end{gathered} \tag{45}$$

Therefore, it can be concluded that

$$ \begin{gathered} \mathbf{R}(\xi;\lambda,\eta;h) = {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{D}\lambda, \mathfrak{D}^r\eta; h)\frac{(\mathfrak{C}\xi)^m}{m!} = \\ \mathfrak{C}^m\ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{D}\lambda, \mathfrak{D}^r\eta; h)\frac{\xi^m}{m!}. \end{gathered} \tag{46}$$

The Equation (43) is derived by equating the coefficients of the corresponding term involving ξ in the final two equations: (45) and (46). □

Theorem 11. *For $\mathfrak{D} \neq \mathfrak{C}$ and $\mathfrak{D}, \mathfrak{C} > 0$, it can be inferred that*

$$ \sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{C}^i\mathfrak{D}^{m+1-i}\ {}_{\Delta_h}\mathbb{A}_n^{[r]}(\mathfrak{D}\lambda, \mathfrak{D}^r\eta, h)\mathcal{P}_{m-i}(\mathfrak{C}-1;h) = $$

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{D}^{i}\mathfrak{C}^{m+1-i}{}_{\Delta_h}\mathbb{A}_{n}^{[r]}(\mathfrak{C}\lambda,\mathfrak{C}^{r}\eta;h)\mathcal{P}_{m-i}(\mathfrak{D}-1;h). \tag{47}$$

Proof. Since $\mathfrak{D}\neq\mathfrak{C},\mathfrak{D},\mathfrak{C}>0$, let us begin by expressing the following:

$$\mathbf{S}(\xi;\lambda,\eta;h)=\frac{\mathfrak{C}\mathfrak{D}\xi\,\gamma(\xi)(1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}}(1+h\xi^{r})^{\frac{\mathfrak{C}^{r}\mathfrak{D}^{r}\eta}{h}}\left((1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}-1}\right)}{\left((1+h\xi)^{\frac{\mathfrak{C}}{h}-1}\right)\left((1+h\xi)^{\frac{\mathfrak{D}}{h}-1}\right)}. \tag{48}$$

By continuing similarly as in the previous Theorem, Statement (47) is derived. □

Theorem 12. *For $\mathfrak{D}\neq\mathfrak{C}\,\mathfrak{D},\mathfrak{C}>0$, we have*

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{C}^{i}\mathfrak{D}^{m+1-i}\,\mathcal{B}_m(h)_{\Delta_h}\mathbb{A}_{i-m}^{[r]}(\mathfrak{D}\lambda,\mathfrak{D}^{r}\eta;h)\mathcal{P}_{m-i}(\mathfrak{C}-1;h)\,\sigma_{m-i}(\mathfrak{C}-1;h)=$$

$$\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{D}^{i}\mathfrak{C}^{m+1-i}\,\mathcal{B}_m(h)_{\Delta_h}\mathbb{A}_{i-m}^{[r]}(\mathfrak{C}\lambda,\mathfrak{C}^{r}\eta;h)\,\sigma_{m-i}(\mathfrak{D}-1;h). \tag{49}$$

Proof. Since $\mathfrak{D}\neq\mathfrak{C}\,\mathfrak{D},\mathfrak{C}>0$, let us begin by expressing the following:

$$\mathbf{T}(\xi;\lambda,\eta;h)=\frac{\mathfrak{C}\mathfrak{D}\xi\,\gamma(\xi)(1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}}(1+h\xi^{r})^{\frac{\mathfrak{C}^{r}\mathfrak{D}^{r}\eta}{h}}\left((1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}-1}\right)}{\left((1+h\xi)^{\frac{\mathfrak{C}}{h}-1}\right)\left((1+h\xi)^{\frac{\mathfrak{D}}{h}-1}\right)}. \tag{50}$$

The previous equation can be written as follows:

$$\begin{aligned}\mathbf{T}(\xi;\lambda,\eta;h)=&\frac{\mathfrak{C}\mathfrak{D}\xi}{\left((1+h\xi)^{\frac{\mathfrak{C}}{h}-1}\right)}\eta(\xi)(1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}}(1+h\xi^{r})^{\frac{\mathfrak{C}^{r}\mathfrak{D}^{r}\eta}{h}}\\&\times\frac{\left((1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}-1}\right)}{\left((1+h\xi)^{\frac{\mathfrak{D}}{h}-1}\right)}.\end{aligned} \tag{51}$$

Using $\frac{\mathfrak{C}\mathfrak{D}\xi}{\left((1+h\xi)^{\frac{\mathfrak{C}}{h}-1}\right)}=\mathfrak{C}^{i}\sum_{m=0}^{\infty}\mathcal{B}_m(h)\frac{(\mathfrak{D}\xi)^{m}}{m!}$, $\frac{\left((1+h\xi)^{\frac{\mathfrak{C}\mathfrak{D}\lambda}{h}-1}\right)}{\left((1+h\xi)^{\frac{\mathfrak{D}}{h}-1}\right)}=\sigma_{m-i}(\mathfrak{C}-1;h)\frac{(\mathfrak{D}\xi)^{m}}{m!}$ and (15), it follows that

$$\begin{aligned}\mathbf{T}(\xi;\lambda,\eta;h)=&\mathfrak{C}^{i}\sum_{m=0}^{\infty}\mathcal{B}_m(h)\frac{(\mathfrak{D}\xi)^{m}}{m!}\sum_{m=0}^{\infty}{}_{\Delta_h}\mathbb{A}_{n}^{[r]}(\mathfrak{D}\lambda,\mathfrak{D}^{r}\eta;h)\frac{(\mathfrak{C}\xi)^{m}}{m!}\\&\times\sum_{m=0}^{\infty}\sigma_{m-i}(\mathfrak{C}-1;h)\frac{(\mathfrak{D}\xi)^{m}}{m!}\\=&\sum_{m=0}^{\infty}\left(\textstyle\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{C}^{i}\mathfrak{D}^{m+1-i}\,\mathcal{B}_m(h)_{\Delta_h}\mathbb{A}_{i-m}^{[r]}(\mathfrak{D}\lambda,\mathfrak{D}^{r}\eta,;h)\sigma_{m-i}(\mathfrak{C}-1;h)\right)\frac{\xi^{m}}{m!}.\end{aligned} \tag{52}$$

Continuing in a similar manner, we have

$$\begin{aligned}\mathbf{T}(\xi;\lambda,\eta;h)=&\textstyle\sum_{m=0}^{\infty}\Big(\sum_{i=0}^{m}\sum_{n=0}^{i}\binom{m}{i}\binom{i}{n}\mathfrak{D}^{i}\mathfrak{C}^{m+1-i}\,\mathcal{B}_m(h)\\&\times{}_{\Delta_h}\mathbb{A}_{i-m}^{[r]}(\mathfrak{D}\lambda,\mathfrak{D}^{r}\eta;h)\,\sigma_{m-i}(\beta-1;h)\Big)\frac{\xi^{m}}{m!}.\end{aligned} \tag{53}$$

By comparing the coefficients of the Expressions (52) and (53), we obtain the result in (49). □

5. Operational Connection and Extended Form via Fractional Operators

Operational techniques offer a valuable means for creating novel sets of functions and streamlining the exploration of properties related to both standard and generalized special functions. Dattoli and fellow researchers, as evident from citations like [16,17,24–27], have acknowledged the significance of employing operational methods to study special functions. Such methodologies strive to yield explicit solutions for families of partial differential equations, thereby encompassing cases like Heat and D'Alembert types, along with their diverse practical applications.

By repeatedly applying the forward difference operator concept and differentiating Equation (15) concerning λ, while considering expression (4), we obtain the following:

$$\begin{array}{ll} {}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = & nh\ [{}_{\Delta_h}\mathbb{A}_{n-1}^{[r]}(\lambda,\eta;h)] \\ {}_{\lambda}\Delta_h^2[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = & n(n-1)h\ {}_{\Delta_h}[{}_{\Delta_h}\mathbb{A}_{n-2}^{[r]}(\lambda,\eta;h)] \\ \vdots & \\ {}_{\lambda}\Delta_h^r[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = & n(n-1)(n-2)\cdots(n-r+1)h \\ \times\ {}_{\Delta_h}[{}_{\Delta_h}\mathbb{A}_{n-r}^{[r]}(\lambda,\eta;h)]. & \end{array} \tag{54}$$

Subsequently, by employing the concept of the forward difference operator and differentiating Equation (15) concerning η, while considering expression (4), we can deduce the following:

$${}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = n(n-1)(n-2)\cdots(n-r+1)h\ \times\ {}_{\Delta_h}[{}_{\Delta_h}\mathbb{A}_{n-r}^{[r]}(\lambda,\eta;h)] \tag{55}$$

Considering the system of Equations (54) and (55), it can be observed that $\Delta_h\mathbb{A}^{[r]}n(\lambda,\eta;h)$ serve as solutions for the following expressions:

$${}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)] = \ {}_{\lambda}\Delta_h^r[{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)], \tag{56}$$

which are under the listed initial condition

$${}_{\Delta_h}\mathbb{A}_n^{[r]}(0,0;h) = {}_{\Delta_h}\mathbb{A}_n^{[r]}(0;h) = {}_{\Delta_h}\mathbb{A}_n(\xi;h). \tag{57}$$

Hence, considering the set of Equation (56) and the Expression (57), we can conclude that

$${}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h) = \exp\left(\eta\ {}_{\lambda}\Delta_h^r\right){}_{\Delta_h}\mathbb{A}_n(\xi;h). \tag{58}$$

Based on the aforementioned expression, the Δ_h bivariate Appell polynomials ${}_{\Delta_h}\mathbb{A}^{[r]}n(\lambda,\eta;h)$ can be generated from the Δ_h polynomial ${}_{\Delta_h}\mathbb{A}n(\xi;h)$ by utilizing the operational rule in (58).

In the study presented in [25], Dattoli and colleagues investigated the potential of extending the application of integral transforms beyond their conventional scope. Their research delves into the utilization of Euler's integral to expand the effectiveness of integral transforms, thus surpassing their typical constraints. Euler's integral is expressed as follows:

$$\sigma^{-q} = \frac{1}{\Gamma(q)}\int_0^{\infty} e^{-\sigma\xi}\lambda^{q-1}d\xi, \tag{59}$$

σ and q are complex numbers with positive real parts and meet the condition $\min\{\mathrm{Re}(q),\mathrm{Re}(\sigma)\} > 0$, thereby forming a comprehensive foundation. This integral provides a means to enhance the applicability and efficiency of integral transformations across various domains. Incorporating Euler's integral into the framework of integral transformations opens avenues for researchers to address a broader spectrum of challenges, thereby facilitating the study and resolution of complex mathematical equations in diverse fields. This expanded framework also introduces novel perspectives for exploring frac-

tional derivatives and their implications, thus fostering original ideas and approaches to problem solving.

Furthermore, the study in [25] highlights that for first- and second-order derivatives, the following axioms are valid:

$$\begin{aligned}\left(\beta-\frac{\partial}{\partial\lambda}\right)^{-q}g(\lambda) &= \frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta t}t^{q-1}\,e^{\xi\frac{\partial}{\partial\lambda}}g(\lambda)d\xi\\ &= \frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta t}\xi^{q-1}\,g(\lambda+\xi)d\xi \end{aligned} \tag{60}$$

$$\left(\beta-\frac{\partial^2}{\partial\lambda^2}\right)^{-\mu}g(\lambda)=\frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta\xi}\xi^{\mu-1}\,e^{\xi\frac{\partial^2}{\partial p^2}}g(\lambda)d\xi. \tag{61}$$

On the insertion of β using $\beta-\left(\eta\ {}_\lambda\Delta_h^r\right)$ in Expression (59) and operating the final expression on ${}_{\Delta_h}\mathbb{A}_n(\xi;h)$, we find

$$\left(\beta-\eta\ {}_\lambda\Delta_h^r\right)^{-q}{}_{\Delta_h}\mathbb{A}_n(\xi;h)=\frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta\xi}\xi^{q-1}\exp(\xi\eta\ {}_\lambda\Delta_h^r)\ {}_{\Delta_h}\mathbb{A}_n(\xi;h)d\xi. \tag{62}$$

It is clear from Equation (58), that the following outcome is attained from (62):

$$\left(\beta-\eta\ {}_\lambda\Delta_h^r\right)^{-q}{}_{\Delta_h}\mathbb{A}_n(\xi;h)=\frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta\xi}\xi^{q-1}{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\xi\eta;h)d\xi. \tag{63}$$

Utilizing the transformation delineated on the right-hand side of Equation (63), this introduces a novel set of polynomials denoted as ${}_{\Delta_h}\mathbb{A}_{n,q}^{[r]}(\lambda,\eta;\beta;h)$—referred to as the extended Δ_h bivariate Appell polynomials. This leads to the establishment of the following relationship:

$$\frac{1}{\Gamma(q)}\int_0^\infty e^{-\beta\xi}\xi^{q-1}{}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\xi\eta;h)d\xi={}_{\Delta_h}\mathbb{A}_{n,q}^{[r]}(\lambda,\eta;\beta;h). \tag{64}$$

Consequently, by considering Expressions (63) and (64), the succeeding expression for the operational connection of the extended Δ_h bivariate Appell polynomials

$$\left(\beta-\eta\ {}_\lambda\Delta_h^r\right)^{-q}{}_{\Delta_h}\mathbb{A}_n(\xi;h)={}_{\Delta_h}\mathbb{A}_{n,q}^{[r]}(\lambda,\eta;\beta;h) \tag{65}$$

is established.

The utilization of the fractional operator on the Δ_h bivariate Appell polynomials enables a more versatile and sophisticated approach to mathematical expressions. By applying fractional operators to the Δ_h bivariate Appell polynomials, researchers can address problems involving fractional calculus and gain insights into the behavior of functions in various domains. The fractional operator provides a means to analyze the fractional differentiation or integration of the Δ_h bivariate Appell polynomials, thereby allowing for a more nuanced exploration of mathematical relationships.

6. Examples

Various Appell polynomials can be generated by choosing an appropriate function $\gamma(\xi)$. These polynomials, along with their respective names and generating functions, are listed below:

The expression defining the generating function for the Δ_h Bernoulli polynomials ${}_{\Delta_h}\beta_m(\lambda;h)$ is expressed as follows:

$$\frac{\xi}{(1+h\xi)^{\frac{1}{h}}-1}(1+h\xi)^{\frac{\lambda}{h}}=\sum_{n=0}^\infty{}_{\Delta_h}\beta_m(\lambda;h)\frac{\xi^n}{n!},\qquad |\xi|<2\pi. \tag{66}$$

The expression defining the generating function for the Euler polynomials ${}_{\Delta_h}\mathcal{E}_m(\lambda;h)$ is as follows:

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{\lambda}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{E}_n(\lambda;h)\frac{\xi^n}{n!}, \qquad |\xi| < \pi. \tag{67}$$

The expression defining the generating function for the Genocchi polynomials ${}_{\Delta_h}\mathcal{G}_m(\lambda;h)$ is given by

$$\frac{2\xi}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{\lambda}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathcal{G}_n(\lambda;h)\frac{\xi^n}{n!}, \qquad |\xi| < \pi. \tag{68}$$

As h approaches zero, these polynomials simplify to the well-known Bernoulli, Euler, and Genocchi polynomials [28].

The Bernoulli, Euler, and Genocchi numbers hold profound importance across multiple branches of mathematics and serve as valuable tools in various contexts. The Bernoulli numbers, which form a sequence of rational numbers, assume a fundamental role in numerous mathematical equations, including the Bernoulli polynomials and the Euler–Maclaurin formula. These numbers possess remarkable properties that make them indispensable in diverse mathematical applications. They have wide-ranging applications in fields like number theory, combinatorics, and numerical analysis. Furthermore, they are linked to several other areas of mathematics, including representation theory and algebraic geometry.

For example, Bernoulli numbers find extensive applications in a multitude of mathematical contexts, including Taylor expansions, trigonometric, hyperbolic tangent, and cotangent functions, as well as summations involving natural numbers. They occupy a pivotal position in number theory, thereby offering valuable insights into integer patterns and relationships.

Likewise, Euler numbers emerge in Taylor expansions and exhibit close ties to trigonometric and hyperbolic secant functions. They are employed in graph theory, automata theory, and the calculation of up-down ascending sequences, thereby contributing to the analysis of structures and patterns within discrete mathematics.

Additionally, Genocchi numbers prove to be beneficial in graph theory and automata theory, particularly in counting up-down ascending sequences. This application involves studying the order and arrangement of elements in a sequence. Consequently, these Δ_h polynomials and the numbers of Bernoulli, Euler, and Genocchi each assume pivotal roles across various mathematical domains, thereby facilitating the exploration of mathematical relationships, derivation of formulas, and analysis of patterns and structures.

By selecting the appropriate function $\gamma(\xi)$ in the formula given by Equation (15), one can obtain different generating functions for the bivariate Bernoulli, Euler, and Genocchi polynomials. These generating functions express the polynomials in terms of two variables and are useful in analyzing their properties and behavior. The specific form of the generating function depends on the choice of $\gamma(\xi)$, which determines the particular member of the polynomial family that is being considered. Hence, the generating functions listed in the statement are the ones that correspond to the suitable choices of $\gamma(\xi)$ for the given polynomials:

$$\frac{\xi}{(1+h\xi)^{\frac{1}{h}}-1}(1+h\xi)^{\frac{\lambda}{h}}(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{B}_n(\lambda,\eta;h)\frac{\xi^n}{n!}, \tag{69}$$

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{\lambda}{h}}(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{E}_n(\lambda,\eta;h)\frac{\xi^n}{n!}, \tag{70}$$

and

$$\frac{2t}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{\lambda}{h}}(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{G}_n(\lambda,\eta;h)\frac{\xi^n}{n!}, \tag{71}$$

accordingly. We can derive the corresponding outcomes for these polynomials:

Theorem 13. *The Δ_h bivariate polynomials of Bernoulli, Euler, and Genocchi types can be expressed using the formulas provided in Equations (69) to (71). As a consequence and observation, these polynomials satisfy a set of relations:*

$$\begin{array}{ll} {}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{B}_n(\lambda,\eta;h)] & = nh\ {}_{\Delta_h}\mathbb{B}_{n-1}(\lambda,\eta;h) \\ {}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{B}_n(\lambda,\eta;h)] & = n(n-1)\cdots(n-r+1)h\ {}_{\Delta_h}\mathbb{B}_{n-r}(\lambda,\eta;h), \end{array} \tag{72}$$

$$\begin{array}{ll} {}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{E}_n(\lambda,\eta;h)] & = nh\ {}_{\Delta_h}\mathbb{E}_{n-1}(\lambda,\eta;h) \\ {}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{E}_n(\lambda,\eta;h)] & = n(n-1)\cdots(n-r+1)h\ {}_{\Delta_h}\mathbb{E}_{n-r}(\lambda,\eta;h), \end{array} \tag{73}$$

and

$$\begin{array}{ll} {}_{\lambda}\Delta_h[{}_{\Delta_h}\mathbb{G}_n(\lambda,\eta;h)] & = nh\ {}_{\Delta_h}\mathbb{G}_{n-1}(\lambda,\eta;h) \\ {}_{\eta}\Delta_h[{}_{\Delta_h}\mathbb{B}_n(\lambda,\eta;h)] & = n(n-1)\cdots(n-r+1)h\ {}_{\Delta_h}\mathbb{G}_{n-r}(\lambda,\eta;h), \end{array} \tag{74}$$

respectively.

These relationships could be in the form of identities, equations, or other mathematical expressions that relate the values of the polynomials to each other. These relations can be utilized to derive further properties of the polynomials and can aid in simplifying mathematical expressions that involve these polynomials. Furthermore, based on Equation (31), the bivariate Bernoulli, Euler, and Genocchi polynomials have an explicit form that satisfies a specific mathematical expression. This expression could be a formula or an equation that provides an exact representation of the polynomials in terms of their variables and parameters:

Theorem 14. *The Δ_h bivariate Bernoulli, Euler, and Genocchi polynomials satisfy the following explicit form:*

$${}_{\Delta_h}\mathbb{B}_n(\lambda,\eta;h) = \sum_{s=0}^{\left[\frac{n}{s}\right]} \binom{n}{s} \beta_{s,h}\ {}_{\Delta_h}\mathbb{H}_{n-s}(\lambda,\eta;h), \tag{75}$$

$${}_{\Delta_h}\mathbb{E}_n(\lambda,\eta;h) = \sum_{s=0}^{\left[\frac{n}{s}\right]} \binom{n}{s} \mathcal{E}_{s,h}\ {}_{\Delta_h}\mathbb{H}_{n-s}(\lambda,\eta;h), \tag{76}$$

and

$${}_{\Delta_h}\mathbb{G}_n(\lambda,\eta;h) = \sum_{s=0}^{\left[\frac{n}{s}\right]} \binom{n}{s} \mathcal{G}_{s,h}\ {}_{\Delta_h}\mathbb{H}_{n-s}(\lambda,\eta;h), \tag{77}$$

respectively.

By using this explicit form, one can evaluate the values of the polynomials for specific inputs and derive other properties of the polynomials, such as their roots, coefficients, and asymptotic behavior. Therefore, the explicit form of polynomials is a valuable tool for studying their properties and applications in different areas of mathematics.

Likewise, using similar techniques and methods as discussed earlier, one can establish other corresponding results for the bivariate Bernoulli, Euler, and Genocchi polynomials. These results could include identities, equations, or other mathematical expressions that describe the properties and behavior of the polynomials. By establishing these results, we can gain a better understanding of polynomials and their applications in different areas of mathematics. Moreover, the results obtained in this manner can aid in simplifying and solving complex mathematical problems that involve these polynomials. Therefore, establishing corresponding results for these polynomials is an important step in furthering our knowledge and understanding of them. It is indeed interesting to explore the physical significance of hybrid polynomials and the special numbers obtained from them in various fields such as graph theory, automata theory, and combinatorics. In graph theory, the

Bernoulli numbers have been used to calculate the number of spanning trees in a graph, which have applications in physics and chemistry, particularly in studying the behavior of molecules. The Euler numbers have been used to study the topology of surfaces and in the classification of manifolds, which have implications in physics and geometry. The Genocchi numbers have been used to count certain types of polyominoes, which have applications in computer science and cryptography. In automata theory, the Bernoulli numbers have been used to analyze the behaviors of certain types of automata, which have implications in computer science and engineering. The Euler numbers have been used to study the topology of cellular automata, which has applications in physics and computer science. The Genocchi numbers have been used to count certain types of formal languages, which have applications in computer science and linguistics. Within combinatorics, Bernoulli numbers have been employed to investigate Stirling numbers of the second kind, which enumerate the ways to partition a set into nonempty subsets. Euler numbers have been applied to the study of Eulerian numbers, thus quantifying permutations of a set with specific ascent characteristics. Genocchi numbers have found utility in analyzing Lah numbers, which represent different methods of partitioning a set into ordered blocks. Consequently, it remains a subject of future research to discern the potential physical implications and significance of these hybrid polynomials and the unique special numbers they produce.

7. Conclusions

It was Costabile and Longo who introduced a set of sequences called Δ_h Appell sequences of one variable λ, which are denoted by $\mathcal{Q}_m(\lambda)$ for $m \in \mathbb{N}$. These sequences are defined using the power series of the product of two functions. Furthermore, they studied their series definition, recurrence relation, and other properties. In this article, the two-dimensional Appell polynomial of the Δ_h type is introduced by the generating relation:

$$\gamma(\xi)\,(1+h\xi)^{\frac{\lambda}{h}}\,(1+h\xi^r)^{\frac{\eta}{h}} = \sum_{n=0}^{\infty} {}_{\Delta_h}\mathbb{A}_n^{[r]}(\lambda,\eta;h)\frac{\xi^n}{n!},\ \ r \in \mathbb{N}_0.$$

Furthermore, if $\gamma(\xi) = 1$, they reduce to the Δ_h Gould–Hopper polynomials. The monomiality principle has been adopted with respect to these polynomials, and the operational rule has been obtained. This has not been done in the article of Costabile and Longo. Thus, this article introduces Δ_h in the two-variable form and that of order r too.

In Section 2, the quasimonomial characteristics for these polynomials have been established, and the forward difference relation has also been proven in Theorem 1. The differential relations computed in Theorem 1 may play a significant role in various branches of mathematics, physics, and engineering, as they provide a fundamental framework for understanding the relationship between quantities and their rates of change. Furthermore, the generating expression has been computed in Theorem 2. The generating functions are powerful tools in combinatorial mathematics and analysis. They are significant in the areas such as counting and enumeration, deriving recurrence relations, exploiting series representations, and exploring convergence properties. Also, the explicit series formula has been presented in Theorem 3 for Δ_h bivariate Gould–Hopper polynomials, which is further used for establishing the explicit form for Δ_h bivariate Appell polynomials in Theorem 4. Moreover, the determinant form and monomiality principle have also been verified, and the differential equation has been derived in Section 3. Additionally, several explicit forms for these polynomials have been obtained, and a determinant representation of them has been provided in the same section. Furthermore, a few members of this polynomial family have been observed, and their related findings have been discussed. Symmetric identities, which play a crucial role in the investigation of special functions and polynomial sequences, have been investigated in Section 4 of this paper. They provide a means to derive novel identities and properties by establishing relationships between distinct terms or coefficients within a mathematical expression or equation. By equating coefficients or comparing terms, these identities enable us to establish connections, derive explicit

formulas, and explore the properties and characteristics of diverse mathematical objects, and they are thus established in the three Theorems contained in Section 4. Utilizing symmetric identities allows for the discovery of hidden patterns and structures within mathematical systems. By uncovering these relationships, we gain deeper insights into the behavior and properties of special functions and polynomial sequences. This, in turn, facilitates the development of new mathematical techniques, which aids in solving problems and enhances our understanding of fundamental mathematical concepts. Operational techniques employed for obtaining operational connections and applications are also highlighted in Section 5. By repeatedly applying the forward difference operator concept and differentiating the generating expression of the Δ_h bivariate Appell polynomials, different Appell polynomial members were generated by selecting a suitable function. The generating functions for the Δ_h Bernoulli polynomials, the Euler polynomials, and the Genocchi polynomials were established in this section.

This paper introduces highly significant two-variable forms with diverse applications in both mathematics and physics. These polynomials may naturally arise in quantum mechanics and probability theory, thus being linked to the fundamental concept of the normal distribution in probability theory. Moreover, they can function as a basis for approximating functions in approximation theory, thereby serving as an essential tool for numerical analysis. In statistical mechanics, they play a crucial role in computing the partition function, and in Fourier analysis, they prove valuable for decomposing functions into a sum of orthogonal functions.

In future studies, researchers can delve into exploring and establishing the novel properties of bivariate Appell polynomials. This may include uncovering extended and generalized forms, integral representations, and other characteristics. Moreover, investigating the interpolation form, recurrence relations, shift operators, and summation formulae for these polynomials has the potential to yield fresh insights. Combining the study of bivariate Appell polynomials with quantum calculus aspects could also offer interesting outcomes. Inspiring research can be read regarding this topic, such as in [29] or [30]. Fractional calculus aspects could also be added into studies, as is seen in very recent publications like [31] or [32].

By delving deeper into these areas, we can enhance our knowledge of the characteristics and behavior of these polynomials and apply them to solve complex mathematical problems.

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