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Existence and Stability Results for Piecewise Caputo–Fabrizio Fractional Differential Equations with Mixed Delays

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Abstract: In this article, by using the differential Caputo–Fabrizio operator, we suggest a novel family of piecewise differential equations (DEs). The issue under study contains a mixed delay period under the criteria of anti-periodic boundaries. It is possible to utilize the piecewise derivative to describe a variety of complex, multi-step, real-world situations that arise from nature. Using fixed point (FP) techniques, like Banach’s FP theorem, Schauder’s FP theorem, and Arzelá Ascoli’s FP theorem, the Hyer–Ulam (HU) stability and the existence theorem conclusions are investigated for the considered problem. Eventually, a supportive example is given to demonstrate the applicability and efficacy of the applied concept.

Keywords: fixed point technique; Caputo–Fabrizio operator; delay term; boundary condition; stability analysis

MSC: 47H09; 26A33; 34A08; 93A30



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1. Introduction

Many mathematicians and writers have begun to take fractional differential and integral equations into consideration, where simulation and planning for the future can utilize these equations. Also, the fractional derivatives and integrals are handled by several types of fractional operators, which were presented in [1–3] and have been clearly applied as powerful mathematical tools for predicting many natural phenomena. The authors of [4] listed numerous practical and scientific uses for fractional calculus.

The classical theory of fractional calculus has long been a necessary part of the main curricula in most branches of science. Numerous investigations have been carried out in the literature on the existence of problems involving fractional differential equations (FDEs) and their numerical solutions; see, for instance, [5–7]. Both FDEs with the power law kernel and classical DEs have been studied and investigated. The requirements for these equations with power-law and classical fractional differentiation have been secured in a number of ways. The prerequisites for the existence of solutions to such DE issues are helpful within the context of pure mathematics. However, these criteria are ineffective when dealing with issues in the real world.

Caputo and Fabrizio recently presented new non-singular kernels for FDEs [8]. We direct readers to [9,10] for information on the characteristics of these differential operators with non-singular kernels. Researchers have been interested in this idea, which is applied in a variety of science, engineering, and technological sectors. A non-integer order derivative with a non-singular kernel is a mathematical notion that was introduced by Caputo et al. [11], and its applications to the hysteresis phenomenon were shown. By

incorporating the latest developments in the fractional differential operator, numerous additional theoretical conclusions and numerical techniques have been produced.

The proportional delay differential equations (DDEs) give rise to a class of DEs that are frequently used in the modeling of real-world problems where the status at time s is a time function that depends on previous or past times. In situations involving decision-making, these equations are essential. The applications of DDEs can be seen in fields such as population studies, medicine, physiology, number theory, physics, electrodynamics, and other related fields. These applications have received the attention of many writers; for more misconceptions about the importance of these applications, see [12–18].

Stability theory is frequently required for dynamical problems. Lyapunov and Mittag-Leffler stabilities have been created extremely effectively for common traditional fractional calculus issues as well as exponential types of stabilities. HU stability [19] has recently received the necessary attention. In [20], some stability and existence results from the FP technique were investigated. Also, the existence and stability analysis of FDEs have been studied [21–23].

It is important to note that bodily problems frequently exhibit a multi-step tendency in their nature and are prone to rapid alterations. To put it another way, these issues lack unique behaviors and maintain multiplicity in their dynamical behaviors. The aforementioned differential operators cannot accurately represent such problems. So, Atangana and Araz [24] have presented piecewise differential and integral operators to overcome such a situation. The ability of these operators to handle issues involving crossover and impulsive conduct is well known. Common examples of impulsive situations that can benefit from the application of piecewise differential operators include earthquakes, economic fluctuations, and gaseous dynamics with related tendencies. Shah et al. [25] have investigated a dynamical problem of the Cauchy type with piecewise derivative (PD) using an FP technique. A non-singular type of derivative is included in [26] as a generalization of the PD.

Inspired by the above applications, in this study, we propose the following piecewise Caputo–Fabrizio DE (PCFDE, for short) problem with mixed time delay:

$${}^{PCF}D^\rho \zeta(s) = \begin{cases} Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))), & \sigma \in (0, 1), \rho \in (0, 1], \eta(s) \geq 0, s \in [0, \xi], \\ \zeta(0) = -\zeta(\xi), \end{cases} \tag{1}$$

where $Z : [0, \xi] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ denotes a piecewise continuous function and ${}^{PCF}D^\rho$ represents a PCF derivative of order ρ . Our investigation of the considered problem involves the study of the HU stability and existence theory. The left-hand side of the issue (1) can be written as:

$${}^{PCF}D^\rho \zeta(s) = \begin{cases} \frac{d\zeta}{ds}, & \text{if } s \in [0, s_1], \\ {}^{CF}D^\rho \zeta(s), & \text{if } s \in [s_1, \xi]. \end{cases} \tag{2}$$

The outline and moral of the manuscript are arranged as follows: Section 2 defines PCF operators and provides the FP theorem’s assertions. The requirements for the unique solution (US) to the considered problem are developed in Section 3. Section 4 presents the HU stability analysis for the relevant problem. An illustrative example is provided in Section 5 to test the validity of our results. Ultimately, Section 6 provides the conclusion and suggests open problems.

2. Preliminaries

The newly established definitions of piecewise integrals and derivatives are provided in this section of the study. Here, we also provide certain lemmas that are essential for reaching our main results.

Definition 1 ([24]). For a continuous function ζ with a fractional order $\rho \in (0, 1]$,

(i) the PCF integral is described as

$${}^{PCF}I^\rho \zeta(s) = \begin{cases} \int_0^{s_1} \zeta(s) ds, & s \in [0, s_1], \\ \frac{1-\rho}{\Lambda(\rho)} \zeta(s) + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s \zeta(s) ds, & \text{if } s \in [s_1, \xi], \end{cases}$$

where $\Lambda(\rho)$ is a normalization function satisfying $\Lambda(0) = \Lambda(1) = 1$.

(ii) The PCF derivative is defined by

$${}^{PCF}D^\rho \zeta(s) = \begin{cases} \frac{d\zeta}{ds}, & \text{if } s \in [0, s_1], \\ {}^{CF}D^\rho \zeta(s), & \text{if } s \in [s_1, \xi], \end{cases}$$

where ${}^{CF}D^\rho$ is the Caputo–Fabrizio derivative operator [24].

Lemma 1 ([10]). For chosen constants $\lambda_j \in \mathbb{R}, j = 0, 1, 2, \dots, u - 1$, where $u = [\rho] + 1$ and $[\alpha]$ refer to the integer part of α , the relation below holds:

$$I^\rho [{}^{CF}D^\rho \zeta(s)] = \zeta(s) + \lambda_0 + \lambda_1 s + \dots + \lambda_{u-1} s^{u-1}.$$

Theorem 1 ([26]). (Banach FP theorem) Assume that $\Pi \neq \emptyset$ is a closed subset of a Banach space \mathcal{U} , and \mathfrak{S} is a contraction self-mapping on Π ; then it owns a unique FP.

Theorem 2 ([26]). (Schauder’s FP theorem) Suppose that $\overline{\Pi}$ is a non-empty subset of of a Banach space $\overline{\mathcal{U}}$ and $\overline{\mathfrak{S}} : \overline{\Pi} \rightarrow \overline{\Pi}$ is a continuous and compact mapping; then there is at least one FP of $\overline{\mathfrak{S}}$.

3. Main Consequences

Describe a Banach space as

$$\mathcal{U} = \{ \zeta : [0, \xi] \rightarrow \mathbb{R}, \zeta \in C([0, \xi]) \}$$

equipped with the norm $\|\zeta\| = \sup_{s \in [0, \xi]} \{ |\zeta(s)| \}$, where $C([0, \xi])$ is the space of all continuous functions on $[0, \xi]$.

Lemma 2. The problem

$$\begin{cases} {}^{PCF}D^\rho \zeta(s) = \chi(s), & \rho \in (0, 1], \\ \zeta(0) = -\zeta(\xi), \end{cases}$$

has the solution below

$$\zeta(s) = \begin{cases} -\zeta(\xi) + \int_0^{s_1} \chi(\mu) d\mu, & \text{if } s \in [0, s_1], \\ \zeta(s_1) + \frac{1-\rho}{\Lambda(\rho)} \chi(s) + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s \chi(\mu) d\mu, & \text{if } s \in [s_1, \xi]. \end{cases} \tag{3}$$

Proof. We skip the proof because it is simple to understand. \square

Corollary 1. In light of Lemma 2, the solution of the problem (1) is provided by

$$\zeta(s) = \begin{cases} -\zeta(\xi) + \int_0^{s_1} Z(\mu, \zeta(\mu), \zeta(\sigma\mu), \zeta(\mu - \eta(\mu))) d\mu, & \text{if } s \in [0, s_1], \\ \zeta(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(\mu, \zeta(\mu), \zeta(\sigma\mu), \zeta(\mu - \eta(\mu))) \\ + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(\mu, \zeta(\mu), \zeta(\sigma\mu), \zeta(\mu - \eta(\mu))) d\mu, & \text{if } s \in [s_1, \xi]. \end{cases} \tag{4}$$

To reach our desired goal here, we need the following hypotheses:

(H_i) For $Z : [0, \xi] \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $\zeta, \tilde{\zeta} \in \mathcal{U}$, there exists a constant Y_Z so that

$$\begin{aligned} & \left| Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) - Z\left(s, \tilde{\zeta}(s), \tilde{\zeta}(\sigma s), \tilde{\zeta}(s - \eta(s))\right) \right| \\ & \leq Y_Z \left\{ \left| \zeta(s) - \tilde{\zeta}(s) \right| + \left| \zeta(\sigma s) - \tilde{\zeta}(\sigma s) \right| + \left| \zeta(s - \eta(s)) - \tilde{\zeta}(s - \eta(s)) \right| \right\}. \end{aligned}$$

(H_{ii}) There are functions $\ell_1, \ell_2 \in \mathcal{U}$ such that

$$|Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s)))| \leq \ell_1(s) + \ell_2(s)(|\zeta(s)| + |\zeta(\sigma s)| + |\zeta(s - \eta(s))|),$$

with $\ell_1^* = \sup_{s \in [0, \xi]} \{\ell_1(s)\}$ and $\ell_2^* = \sup_{s \in [0, \xi]} \{\ell_2(s)\}$.

Further, define an operator $\mathfrak{S} : \mathcal{U} \rightarrow \mathcal{U}$ by

$$\mathfrak{S}(\zeta) = \begin{cases} -\zeta(\xi) + \int_0^{s_1} Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) ds, & \text{if } s \in [0, s_1], \\ \zeta(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(\mu, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) \\ + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) d\mu, & \text{if } s \in [s_1, \xi]. \end{cases}$$

Now, we present the theorem related to the existence of the solution to the proposed problem (1).

Theorem 3. *The considered problem (1) has a US provided that $\max\{W_1, W_2\} < 1$, where $W_1 = 3Y_Z s_1$ and $W_2 = 3\left(\frac{1-\rho+\xi\rho}{\Lambda(\rho)}\right)Y_Z$.*

Proof. Let $\zeta, \tilde{\zeta} \in \mathcal{U}$; to organize the proof, we have the following two cases:

(I) If $s \in [0, s_1]$, one has

$$\begin{aligned} & \left| \mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta}) \right| \\ & \leq \int_0^{s_1} \left| Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) - Z\left(s, \tilde{\zeta}(s), \tilde{\zeta}(\sigma s), \tilde{\zeta}(s - \eta(s))\right) \right| ds \\ & \leq Y_Z \int_0^{s_1} \left\{ \left| \zeta(s) - \tilde{\zeta}(s) \right| + \left| \zeta(\sigma s) - \tilde{\zeta}(\sigma s) \right| + \left| \zeta(s - \eta(s)) - \tilde{\zeta}(s - \eta(s)) \right| \right\} ds \\ & \leq 3Y_Z s_1 \left| \zeta(s) - \tilde{\zeta}(s) \right|. \end{aligned}$$

Set $3Y_Z s_1 = W_1$, and taking the suprimum, we get

$$\left\| \mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta}) \right\| \leq W_1 \left\| \zeta - \tilde{\zeta} \right\|. \tag{5}$$

(II) If $s \in [s_1, \xi]$, one can write

$$\begin{aligned} & \left| \mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta}) \right| \\ & \leq \frac{1-\rho}{\Lambda(\rho)} \left| Z(\mu, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) - Z\left(s, \tilde{\zeta}(s), \tilde{\zeta}(\sigma s), \tilde{\zeta}(s - \eta(s))\right) \right| \\ & \quad + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^{\xi} \left| Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) - Z\left(s, \tilde{\zeta}(s), \tilde{\zeta}(\sigma s), \tilde{\zeta}(s - \eta(s))\right) \right| ds. \end{aligned}$$

Using Hypothesis (H_i) and the norm on both sides, we can write

$$\begin{aligned} \|\mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta})\| &\leq \frac{3(1-\rho)Y_Z}{\Lambda(\rho)}\|\zeta - \tilde{\zeta}\| + \frac{3\rho Y_Z}{\Lambda(\rho)}(\xi - s_1)\|\zeta - \tilde{\zeta}\| \\ &\leq 3\left(\frac{1-\rho + \xi\rho}{\Lambda(\rho)}\right)Y_Z\|\zeta - \tilde{\zeta}\|. \end{aligned}$$

Put $3\left(\frac{1-\rho + \xi\rho}{\Lambda(\rho)}\right)Y_Z = W_2$, we have

$$\|\mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta})\| \leq W_2\|\zeta - \tilde{\zeta}\|. \tag{6}$$

It follows from (5) and (6) that

$$\|\mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta})\| \leq \begin{cases} W_1\|\zeta - \tilde{\zeta}\|, & \text{if } s \in [0, s_1], \\ W_2\|\zeta - \tilde{\zeta}\|, & \text{if } s \in [s_1, \xi]. \end{cases}$$

Ultimately, for $s \in [0, \xi]$, we obtain

$$\|\mathfrak{S}(\zeta) - \mathfrak{S}(\tilde{\zeta})\| \leq \max\{W_1, W_2\}\|\zeta - \tilde{\zeta}\|.$$

Consequently, \mathfrak{S} is a contraction; hence, the problem (1) has a US according to Banach FP theorem. \square

Theorem 4. *If the hypotheses (H_i) and (H_{ii}) hold, then there exists at least one solution to the supposed problem (1).*

Proof. Define a closed convex subset Ξ of \mathcal{U} by $\Xi = \{\zeta \in \mathcal{U} : \|\zeta\| \leq M\}$ with $\max\{\frac{\alpha + \ell_1^* s_1}{1 - \ell_2^* Q s_1}, \frac{P + \theta \ell_1^*}{1 - Q \ell_2^*}\} \leq M$, where $\alpha = |\zeta(\xi)|$ and $P = |\zeta(s_1)|$. Also, describe the mapping $\mathcal{T} : \Xi \rightarrow \Xi$ as

$$\mathcal{T}(\zeta) = \begin{cases} -\zeta(\xi) + \int_0^{s_1} Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) ds, & \text{if } s \in [0, s_1], \\ \zeta(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(\mu, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) \\ + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) d\mu, & \text{if } s \in [s_1, \xi]. \end{cases}$$

We split the proof into the following main steps:

Step (A): Prove that \mathcal{T} is continuous.

Case₁. Let $\{\zeta_u\} \in \Xi$, which converges to $\zeta \in \Xi$; then, by the condition (H_i) , for $s \in [0, s_1]$, one has

$$\begin{aligned} |\mathcal{T}\zeta_u(s) - \mathcal{T}\zeta(s)| &\leq \left| \int_0^{s_1} Z(s, \zeta_u(s), \zeta_u(\sigma s), \zeta_u(s - \eta(s))) - Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) ds \right| \\ &\leq 3s_1 Y_Z |\zeta_u(s) - \zeta(s)|. \end{aligned}$$

Since $\zeta_u \rightarrow \zeta$ as $u \rightarrow \infty$, then $|\mathcal{T}\zeta_u(s) - \mathcal{T}\zeta(s)| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $\|\mathcal{T}\zeta_n - \mathcal{T}\zeta\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that \mathcal{T} is continuous.

Case₂. Based on the same assumptions of Case₁, for $s \in [s_1, \xi]$, one can write

$$\begin{aligned}
 |\Upsilon \zeta_u(s) - \Upsilon \zeta(s)| &\leq \frac{1-\rho}{\Lambda(\rho)} (Z(s, \zeta_u(s), \zeta_u(\sigma s), \zeta_u(s-\eta(s))) - Z(s, \zeta(s), \zeta(\sigma s), \zeta(s-\eta(s)))) \\
 &\quad \left| \int_{s_1}^s Z(s, \zeta_u(s), \zeta_u(\sigma s), \zeta_u(s-\eta(s))) - Z(s, \zeta(s), \zeta(\sigma s), \zeta(s-\eta(s))) \right| \\
 &\leq 3 \left(\frac{1-\rho + \xi \rho}{\Lambda(\rho)} \right) Y_Z |\zeta_u(s) - \zeta(s)|.
 \end{aligned}$$

Because $\zeta_u \rightarrow \zeta$ as $u \rightarrow \infty$, we conclude that $\|\Upsilon \zeta_n - \Upsilon \zeta\| \rightarrow 0$ as $n \rightarrow \infty$, which leads to Υ being continuous.

Step (B): Show that Υ is bounded and $\Upsilon(\Xi) \subset \Xi$.

Case₃. Let $\zeta \in \Xi$; then, for $s \in [0, s_1]$, one has

$$\begin{aligned}
 |\Upsilon(\zeta)| &\leq |\zeta(\xi)| + \int_0^{s_1} |Z(s, \zeta(s), \zeta(\sigma s), \zeta(s-\eta(s)))| ds \\
 &\leq |\zeta(\xi)| + s_1 [\ell_1(s) + \ell_2(s)(|\zeta(s)| + |\zeta(\sigma s)| + |\zeta(s-\eta(s))|)].
 \end{aligned}$$

Put $|\zeta(s)| + |\zeta(\sigma s)| + |\zeta(s-\eta(s))| = Q|\zeta(s)|$, $|\zeta(\xi)| = \alpha$, where $Q > 0$, and taking the supremum, we have

$$\|\Upsilon\| \leq \alpha + (\ell_1^* + \ell_2^* Q \|\zeta\|) s_1 \leq M,$$

where

$$M \geq \frac{\alpha + \ell_1^* s_1}{1 - \ell_2^* Q s_1}.$$

Therefore, $\|\Upsilon\| \leq M$, that is, Υ is bounded and $\Upsilon(\zeta) \in \Xi$, as a result, $\Upsilon(\Xi) \subset \Xi$.

Case₄. According to the same conditions of Case₂, for $s \in [s_1, \xi]$, we get

$$\begin{aligned}
 \|\Upsilon(\zeta)\| &\leq \sup_{s \in [s_1, \xi]} \left\{ |\zeta(s_1)| + \frac{1-\rho}{\Lambda(\rho)} |Z(\mu, \zeta(s), \zeta(\sigma s), \zeta(s-\eta(s)))| \right. \\
 &\quad \left. + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s |Z(s, \zeta(s), \zeta(\sigma s), \zeta(s-\eta(s)))| d\mu \right\} \\
 &\leq \sup_{s \in [s_1, \xi]} \left\{ |\zeta(s_1)| + \frac{1-\rho}{\Lambda(\rho)} [\ell_1(s) + Q\ell_2(s)|\zeta(s)|] \right. \\
 &\quad \left. + \frac{\rho}{\Lambda(\rho)} (s-s_1)[\ell_1(s) + Q\ell_2(s)|\zeta(s)|] \right\} \\
 &\leq |\zeta(s_1)| + \frac{1-\rho}{\Lambda(\rho)} (Q\ell_2^* \|\zeta\| + \ell_1^*) + \frac{\rho}{\Lambda(\rho)} (Q\ell_2^* \|\zeta\| + \ell_1^*) (s-s_1) \\
 &\leq |\zeta(s_1)| + Q\ell_2^* \left(\frac{1-\rho + s\rho}{\Lambda(\rho)} \right) M + \left(\frac{1-\rho + s\rho}{\Lambda(\rho)} \right) \ell_1^* \leq M.
 \end{aligned}$$

For simplicity, set $|\zeta(s_1)| = P$ and $\left(\frac{1-\rho + s\rho}{\Lambda(\rho)} \right) = \theta$; then, we have

$$\|\Upsilon(\zeta)\| \leq P + Q\ell_2^* \theta + \theta \ell_1^* \leq M,$$

where $M \geq \frac{P + \theta \ell_1^*}{1 - Q\ell_2^* \theta}$. If $\max\left\{ \frac{\alpha + \ell_1^* s_1}{1 - \ell_2^* Q s_1}, \frac{P + \theta \ell_1^*}{1 - Q\ell_2^* \theta} \right\} \leq M$, we conclude that $\|\Upsilon(\zeta)\| \leq M$. Hence, Υ is bounded in both cases and $\Upsilon(\Xi) \subset \Xi$.

Step. (C): Claim that Υ is equi-continuous.

Case₅. Assume that $s_v < s_w \in [0, s_1]$ and, similar to Step (C), suppose Ξ is a bounded set of \mathcal{U} , one has

$$\begin{aligned} |\mathcal{T}\zeta(s_w) - \mathcal{T}\zeta(s_v)| &\leq \left| \int_{s_v}^{s_w} Z(s_w, \zeta(s_w), \zeta(\sigma s_w), \zeta(s_w - \eta(s_w))) - Z(s_v, \zeta(s_v), \zeta(\sigma s_v), \zeta(s_v - \eta(s_v))) \right| ds \\ &\leq 3(s_w - s_v)Y_Z |\zeta(s_w) - \zeta(s_v)| \rightarrow 0 \text{ as } s_v \rightarrow s_w. \end{aligned}$$

As \mathcal{T} is bounded, then \mathcal{T} is uniformly continuous.

Case₆. Assume that $s_v < s_w \in [s_1, \xi]$, one can write

$$\begin{aligned} &|\mathcal{T}\zeta(s_w) - \mathcal{T}\zeta(s_v)| \\ &\leq \frac{1-\rho}{\Lambda(\rho)} |Z(s_w, \zeta(s_w), \zeta(\sigma s_w), \zeta(s_w - \eta(s_w))) - Z(s_v, \zeta(s_v), \zeta(\sigma s_v), \zeta(s_v - \eta(s_v)))| \\ &\quad + \frac{\rho}{\Lambda(\rho)} \int_{s_v}^{s_w} |Z(s_w, \zeta(s_w), \zeta(\sigma s_w), \zeta(s_w - \eta(s_w))) - Z(s_v, \zeta(s_v), \zeta(\sigma s_v), \zeta(s_v - \eta(s_v)))| \\ &\quad \frac{3(1-\rho)Y_Z}{\Lambda(\rho)} |\zeta(s_w) - \zeta(s_v)| + \frac{\rho(s_w - s_v)}{\Lambda(\rho)} |\zeta(s_w) - \zeta(s_v)|. \end{aligned}$$

Because \mathcal{T} is bounded and continuous on $[0, \xi]$, then it is uniformly continuous. Taking $s_v \rightarrow s_w$, then $|\mathcal{T}\zeta(s_w) - \mathcal{T}\zeta(s_v)| \rightarrow 0$. From Ca₅ and Ca₆, we conclude that $\|\mathcal{T}\zeta(s_w) - \mathcal{T}\zeta(s_v)\| \rightarrow 0$ as $s_v \rightarrow s_w$. Hence, \mathcal{T} is equi-continuous. Clearly, all requirements of relative compactness are satisfied. Since \mathcal{T} has at least one FP, the Schauder FP theorem can be applied. This supports at least one solution to Problem (1). \square

4. Stability Results

This part is devoted to studying HU and generalizing HU (GHU) stability for the US of the problem (1). In this regard, let $\zeta \in C([0, \xi], \mathbb{R}_+)$ and assume the inequality below is true for $s \in [0, \xi]$ and $\nu > 0$:

$$\left| {}^{PCF}D^\rho \widehat{\zeta}(s) - Z\left(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))\right) \right| \leq \nu, \quad \sigma \in (0, 1), \rho \in (0, 1], \eta(s) \geq 0. \tag{7}$$

We point out that the definition of HU stability below was taken directly from [19].

Definition 2. The solution to the considered problem (1) is called HU stable if there is a constant $\wp > 0$ such that, for each solution $\widehat{\zeta} \in \mathcal{U}$ of the problem (7), there is a US $\zeta \in \mathcal{U}$ of (1) such that the inequality below is true:

$$|\zeta(s) - \widehat{\zeta}(s)| \leq \wp \nu.$$

Moreover, if there exists a non-decreasing function $\vartheta \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\vartheta(0) = 0$ such that

$$|\zeta(s) - \widehat{\zeta}(s)| \leq \vartheta(\nu),$$

then the solution of (1) is called a GHU stable.

Remark 1. Assume that V is an independent function of $\widehat{\zeta}$ such that $V(0) = 0$, then

- (a) $|V(s)| \leq \nu, s \in [0, \xi]$;
- (b) for each $s \in [0, \xi]$,

$$\begin{cases} {}^{PCF}D^\rho \widehat{\zeta}(s) = Z\left(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))\right) + V(s), \\ \widehat{\zeta}(0) = -\widehat{\zeta}(\xi), s \in [0, \xi]. \end{cases} \tag{8}$$

Lemma 3. The solution of (8) fulfills the relation below.

$$\left\{ \begin{aligned} & \left| \zeta(s) - \left(-\widehat{\zeta}(s) + \int_0^{s_1} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds \right) \right| \leq s_1 \nu, \quad s \in [0, s_1], \\ & \left| \zeta(s_1) - \left(-\widehat{\zeta}(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) \right. \right. \\ & \left. \left. + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds \right) \right| \leq \left(\frac{1-\rho+s\rho}{\Lambda(\rho)} \right) \nu, \quad s \in [s_1, \xi]. \end{aligned} \right. \tag{9}$$

Proof. Problem (8) has the following solution in light of Corollary 1:

$$\widehat{\zeta}(s) = \begin{cases} -\widehat{\zeta}(s) + \int_0^{s_1} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds + \int_0^{s_1} V(s) ds, & s \in [0, s_1], \\ -\widehat{\zeta}(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) \\ + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds \\ + \frac{1-\rho}{\Lambda(\rho)} V(s) + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s V(s) ds, & s \in [s_1, \xi]. \end{cases} \tag{10}$$

Using (10), we get

$$\left\{ \begin{aligned} & \left| \zeta(s) - \left(-\widehat{\zeta}(s) + \int_0^{s_1} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds \right) \right| \leq s_1 \nu, \quad s \in [0, s_1], \\ & \left| \zeta(s_1) - \left(-\widehat{\zeta}(s_1) + \frac{1-\rho}{\Lambda(\rho)} Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) \right. \right. \\ & \left. \left. + \frac{\rho}{\Lambda(\rho)} \int_{s_1}^s Z(s, \widehat{\zeta}(s), \widehat{\zeta}(\sigma s), \widehat{\zeta}(s - \eta(s))) ds \right) \right| \leq \left(\frac{1-\rho+s\rho}{\Lambda(\rho)} \right) \nu, \quad s \in [s_1, \xi], \end{aligned} \right.$$

which fulfills (9). □

Theorem 5. In light of Hypothesis H_i and Lemma 3, the solution of the system (1) is HU stable, provided that $W_1, W_2 < 1$, where $W_1 = 3Y_Z s_1$ and $W_2 = 3\left(\frac{1-\rho+\xi\rho}{\Lambda(\rho)}\right) Y_Z$.

Proof. Assume that $\zeta \in \mathcal{U}$ is a US of (1) and $\widehat{\zeta} \in \mathcal{U}$ is any solution to (7). Then, for $s \in [0, s_1]$, one has

$$\|\zeta - \widehat{\zeta}\| \leq s_1 \nu + 3s_1 Y_Z \|\zeta - \widehat{\zeta}\|. \tag{11}$$

Similar to Theorem 3, put $W_1 = 3Y_Z s_1$, then (11) can be written as

$$\|\zeta - \widehat{\zeta}\| \leq \frac{s_1}{1 - W_1} \nu. \tag{12}$$

Analogously, for $s \in [s_1, \xi]$, one gets

$$\|\zeta - \widehat{\zeta}\| \leq s_1 \nu + \frac{3(1-\rho)}{\Lambda(\rho)} Y_Z \|\zeta - \widehat{\zeta}\| + \frac{\rho \xi}{\Lambda(\rho)} Y_Z \|\zeta - \widehat{\zeta}\|. \tag{13}$$

Similar to Theorem 3, set $W_2 = 3\left(\frac{1-\rho+\xi\rho}{\Lambda(\rho)}\right) Y_Z$, then (13) takes the form

$$\|\zeta - \widehat{\zeta}\| \leq \frac{s_1}{1 - W_2} \nu. \tag{14}$$

If $\max\{\frac{s_1}{1-W_1}, \frac{s_1}{1-W_2}\} = \wp$, it follows from (12) and (14) that

$$\|\zeta - \widehat{\zeta}\| \leq \wp \nu.$$

Hence, the US of (1) is HU stable. Further, define the non-decreasing function ϑ by $\vartheta(s) = \frac{s}{3}$. Clearly, $\vartheta(0) = 0$ and

$$\|\zeta - \tilde{\zeta}\| \leq \wp \vartheta(v).$$

Therefore, the system (1) is GHU-stable. \square

5. Supportive Example

Example 1. Consider the following problem:

$$\begin{cases} {}^{PCF}D^{\frac{1}{3}}\zeta(s) = e^{-s} \left(\frac{\zeta(s)}{50+|\zeta(s)|} + \frac{\zeta(\frac{1}{3}s)}{50+|\zeta(\frac{1}{3}s)|} + \frac{\zeta(s-0.35)}{50+|\zeta(s-0.35)|} \right), s \in [0, 1], \\ \zeta(0) = -\zeta(1). \end{cases}$$

Clearly, $\xi = 1, \rho = \sigma = \frac{1}{3}, s_1 = 0, \eta(s) = 0.35 > 0$ and

$$Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) = e^{-s} \left(\frac{\zeta(s)}{50 + |\zeta(s)|} + \frac{\zeta(\frac{1}{3}s)}{50 + |\zeta(\frac{1}{3}s)|} + \frac{\zeta(s - 0.35)}{50 + |\zeta(s - 0.35)|} \right).$$

Assume the normalization function $\Lambda(\rho) = \rho^2 - \rho + 1$, then $\Lambda(\frac{1}{3}) = \frac{7}{9}$.

Now, for $s \in [0, 1]$, and $\zeta, \tilde{\zeta} \in \mathbb{R}$, we have

$$\begin{aligned} & \left| Z(s, \zeta(s), \zeta(\sigma s), \zeta(s - \eta(s))) - Z(s, \tilde{\zeta}(s), \tilde{\zeta}(\sigma s), \tilde{\zeta}(s - \eta(s))) \right| \\ & \leq \frac{3}{50} \left| \zeta(s) - \tilde{\zeta}(s) \right| + \left| \zeta(\sigma s) - \tilde{\zeta}(\sigma s) \right| + \left| \zeta(s - \eta(s)) - \tilde{\zeta}(s - \eta(s)) \right|, \end{aligned}$$

which implies that, $Y_z = \frac{1}{50}$. If $s_1 = 0.5$, then $W_1 = 3Y_z s_1 = \frac{3}{100} = 0.03 < 1$ and $W_2 = 3 \left(\frac{1-\rho+\xi\rho}{\Lambda(\rho)} \right) Y_z = \frac{3}{50} \left(\frac{9}{7} \right) = \frac{27}{350} \approx 0.08 < 1$. Hence, $\max\{W_1, W_2\} = 0.08 < 1$. Therefore, all requirements of Theorem 3 are fulfilled; then, there is a US to Problem (1). Moreover, $\max\left\{ \frac{s_1}{1-W_1}, \frac{s_1}{1-W_2} \right\} \approx \max\{0.515, 0.543\} = 0.543 = \wp < 1$. Hence, all assumptions of Theorem 5 hold; then, the US of (1) is HU-stable.

6. Conclusions and Future Work

To a class of Caputo–Fabrizio derivative-enhanced DDES with non-integer order proportionality, we have extended the idea of the PD in this publication. There are numerous physical conditions that naturally display crossover and multi-step behavior, which the normal and other fractional differential operators fail to adequately characterize. These conditions can be effectively modeled using this sort of derivative. In the primary findings, we used FP techniques to extract the requirements for the existence and stability of the presented problem. In order to verify and demonstrate the application of the results, an illustrative example was used. Future work could include applying the idea of piecewise derivatives to additional fractional variable orders as function classes of DEs. It will be fascinating to investigate these further differential operators with variable kernels in piecewise form. This idea can also be used to solve issues that have been studied in abstract spaces.

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