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Solvability of a ϱ-Hilfer Fractional Snap Dynamic System on Unbounded Domains

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Abstract: This paper is devoted to studying the ϱ-Hilfer fractional snap dynamic system under the ϱ-Riemann–Liouville fractional integral conditions on unbounded domains $[a,\infty), a \geq 0$, for the first time. The results concerning the existence and uniqueness, along with the Ulam–Hyers, Ulam–Hyers–Rassias, and semi-Ulam–Hyers–Rassias stabilities, are established in an appropriate special Banach space according to fractional calculus, fixed point theory, and nonlinear analysis. At the end, a numerical example is presented for the interpretation of the main results.

Keywords: ϱ-Hilfer fractional derivatives; snap system; fixed point theorems; Ulam–Hyers–Rassias stability

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1. Introduction

The study of fractional calculus has recently gained momentum and has become an important research area. Fractional derivatives provide excellent tools for characterizing the memory and genetic properties of various materials, processes, and dynamic systems [1–9]. In fractional calculus, many operators have appeared; one of these is called the Hilfer fractional derivative with respect to another non-negative increasing function ϱ (ϱ-Hilfer fractional derivatives), and this has unified several operators that exist in the literature [10]. So, this definition has attracted the attention of many researchers and has been used to cover a variety of fractional differential problems; for example, see [11–19] and the references cited therein.

Fixed point theory is a mathematical discipline that explores the existence and behavior of fixed points of mappings. The study of fixed points has many applications in various fields, such as analysis, geometry, topology, physics, economics, and engineering. For instance, fixed point theory can be used to assess the stability of dynamic systems in physics. In general, fixed point theory provides a powerful tool for finding solutions to problems involving nonlinear equations and systems; we refer the readers to some works that have used fixed point techniques [20–33].

In 2021, Rezapour et al. focused on the existence of solutions and established four classes of the Hyres–Ulam stability of a generalized FBVP with a p-Laplacian operator and with three-point integral boundary conditions given by

$$\begin{cases} {}^{C}\mathfrak{D}_{a^+}^{\nu_1,\xi_1}\left(\varphi_p\left({}^{C}\mathfrak{D}_{a^+}^{\nu_2,\xi_2}\Psi(u)\right)\right) = \mathcal{W}(u,\Psi(u)), & u\in[u_0,K],\ u_0\geq 0,\\ \Psi(u_0)+\mu_1\Psi(K)=\sigma_1\int_0^K \hbar_1(r)\,\mathrm{d}r, & \mu_1\neq -1,\\ \Psi'(u_0)+\mu_2\Psi(K)=\sigma_2\int_0^K \hbar_2(r)\,\mathrm{d}r, & \mu_2\neq -\left(\frac{u_0}{K}\right)^{\xi_2-1},\\ {}^{C}\mathfrak{D}_{a^+}^{\nu_2,\xi_2}\Psi(u_0)=0,\ {}^{C}\mathfrak{D}_{a^+}^{\nu_2,\xi_2}\Psi(K)=\gamma\,{}^{C}\mathfrak{D}_{a^+}^{\nu_2,\xi_2}\Psi(\eta), & \eta\in(u_0,K), \end{cases} \tag{1}$$

so that ${}^{C}\mathfrak{D}_{a^+}^{\nu_i,\xi_i}$ is a generalized derivative in the sense of Caputo and is of the order $0<\nu_i<1$ and $\xi_i>0$ for $i=1,2$ [34]. The authors of [35] investigated a snap dynamic system via simulations, analysis, and a real circuit in terms of integer derivatives:

$$\begin{cases} \mathfrak{D}^1\Psi_1=\Psi_2(u),\ \mathfrak{D}^1\Psi_2=\Psi_3(u),\ \mathfrak{D}^1\Psi_3=\Psi_4(u),\\ \mathfrak{D}^1\Psi_4=\mathcal{G}(\Psi_1,\Psi_2,\Psi_3,\Psi_4), \end{cases} \tag{2}$$

where $\mathcal{G}(\Psi_1,\Psi_2,\Psi_3,\Psi_4)=-a\Psi_1-\Psi_2-\Psi_4+b\Psi_1\Psi_3$, which can be reformulated as

$$\mathfrak{D}^4\Psi_1=\mathcal{G}\left(\Psi_1,\mathfrak{D}^1\Psi_1,\mathfrak{D}^2\Psi_1,\mathfrak{D}^3\Psi_1\right). \tag{3}$$

In 2021, Mahmoud et al. ([36]) examined the qualitative properties of the following Caputo fractional snap dynamic system:

$$\begin{cases} {}^{C}\mathfrak{D}^p\Psi_1=\Psi_2(u),\ {}^{C}\mathfrak{D}^p\Psi_2=\Psi_3(u),\ {}^{C}\mathfrak{D}^p\Psi_3=\Psi_4(u),\\ {}^{C}\mathfrak{D}^p\Psi_4=-2\Psi_1-\Psi_2-\Psi_4+\Psi_1\Psi_3, \end{cases} \tag{4}$$

where $p=0.95$. Recently, in 2021, Samei et al. [37] studied a new structure of the ϱ-Caputo fractional snap dynamic system with initial value boundary conditions on bounded domains $[a,b]$:

$$\begin{cases} {}^{C}\mathfrak{D}^{p_1;\varrho}\Psi_1(u)=\Psi_2(u), & \Psi_1(a)=\lambda_1, \quad u\in[a,b],\\ {}^{C}\mathfrak{D}^{p_2;\varrho}\Psi_2(u)=\Psi_3(u), & \Psi_2(a)=\lambda_2,\\ {}^{C}\mathfrak{D}^{p_3;\varrho}\Psi_3(u)=\Psi_4(u), & \Psi_3(a)=\lambda_3,\\ {}^{C}\mathfrak{D}^{p_4;\varrho}\Psi_4(u)=\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4), & \Psi_4(a)=\lambda_4, \end{cases} \tag{5}$$

where ${}^{C}\mathfrak{D}^{q;\varrho}$ is the ϱ-Caputo fractional derivative of the order $q\in(0,1]$ and $q\in\{p_1,p_2,p_3,p_4\}$. Very recently, in 2022 [38], some authors established qualitative theories, such as those of existence, uniqueness, and Ulam–Hyers stability, for a nonlinear fractional snap dynamic system in the sense of ϱ-Caputo fractional derivatives under non-periodic boundary conditions on bounded domains $[a,b]$:

$$\begin{cases} {}^{C}\mathfrak{D}^{p_1;\varrho}\Psi_1(u)=\Psi_2(u), & \Psi_1(a)=\lambda_1\Psi_1(b),\\ {}^{C}\mathfrak{D}^{p_2;\varrho}\Psi_2(u)=\Psi_3(u), & \Psi_2(a)=\lambda_2\Psi_2(b),\\ {}^{C}\mathfrak{D}^{p_3;\varrho}\Psi_3(u)=\Psi_4(u), & \Psi_3(a)=\lambda_3\Psi_3(b),\\ {}^{C}\mathfrak{D}^{p_4;\varrho}\Psi_4(u)=\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4), & \Psi_4(a)=\lambda_4\Psi_4(b). \end{cases} \tag{6}$$

Motivated by these articles and by a generalization of a recently published articles [39,40], our focus is centered on studying the existence and uniqueness, as well as

various types of Ulam–Hyers stability, of solutions for the following ϱ-Hilfer fractional snap dynamic system on unbounded domains $[a,\infty)$:

$$\begin{cases} {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) = \Psi_2(u), \\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Psi_2(u) = \Psi_3(u), \\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Psi_3(u) = \Psi_4(u), \\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_4,\beta_4;\varrho}\Psi_4(u) = \mathcal{G}(u,\Psi_1(u),\Psi_2(u),\Psi_3(u),\Psi_4(u)), \end{cases} \tag{7}$$

for $u \in J := [a,\infty)$, $a \geq 0$, under the ϱ-Riemann–Liouville (R-L) fractional integral conditions:

$$\begin{cases} \mathfrak{I}_{a^+}^{1-\gamma_1;\varrho}\Psi_1(a) = \lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1), \quad \delta_1 \in J,\ \theta_1 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_2;\varrho}\Psi_2(a) = \lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2), \quad \delta_2 \in J,\ \theta_2 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_3;\varrho}\Psi_3(a) = \lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3), \quad \delta_3 \in J,\ \theta_3 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_4;\varrho}\Psi_4(a) = \lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4), \quad \delta_4 \in J,\ \theta_4 > 0, \end{cases} \tag{8}$$

where $\lambda_i \in \mathbb{R}, i = 1,\ldots,4$, ${}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha,\beta;\varrho}$ is the ϱ-Hilfer fractional derivative of order $\alpha \in (0,1]$ and type $\beta \in [0,1]$, where $\alpha \in \{\alpha_i\}$ and $\beta \in \{\beta_i\}$, respectively, $\mathfrak{I}_{a^+}^{\theta;\varrho}$ is the ϱ-R-L fractional integral of order $\theta = \{1-\gamma_i,\theta_i\}$, where $\alpha_i \leq \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, and $\mathcal{G} : J \times \Sigma^4 \to \Sigma$ and $\omega_i : J \to \Sigma$ are continuous functions in the real Banach space Σ.

It is worth mentioning that the analysis space and approach in this paper are different from those in [36–38]. Additionally, the novelty and contributions of this work lie in the study of a more general ϱ-Hilfer fractional snap dynamic system under the ϱ-R-L fractional integral boundary conditions on unbounded domains $[a,\infty)$, $a \geq 0$ in a specially applicable Σ. In particular, it reduces to snap problem (3) when $\varrho(u) = 1$, $\alpha_i = 1$, $\beta_i \to 1$, and it reduces to snap problem (4) when $\varrho(u) = 1, \alpha_i = p, \beta_i \to 1$; it returns to snap problem (5) with boundary conditions at $\beta_i \to 1$, $\theta_i \to 0$, $\omega_i(\delta_i) = 1$, and it reduces to snap problem (6) with boundary conditions for $\beta_i \to 1$, $\theta_i \to 0$, $\omega_i(\delta_i) = \Psi_i(b)$. Moreover, we can conclude that there are many new problems that are special cases of our problem.

This paper is arranged as follows: In Section 2, we introduce various definitions and preliminaries. In Section 3, we establish the existence and uniqueness of the solution of the fractional snap system (4). In Section 4, we discuss several kinds of stability, such as Ulam–Hyers ($\mathcal{U}.\mathcal{H}$), Ulam–Hyers–Rassias ($\mathcal{U}.\mathcal{H}.\mathcal{R}$), and semi-Ulam–Hyers–Rassias stability (s-$\mathcal{U}.\mathcal{H}.\mathcal{R}$). Finally, an example is given to illustrate the main results in Section 5.

2. Preliminaries

In this section, we recall some important preliminaries that are related to our analysis. The R-L fractional integral of an integrable function g on $[a,b]$ of the order $\vartheta_1 > 0$ and the R-L fractional derivative of a function g of the order $\vartheta_1 \in (n-1,n]$ with respect to another function ϱ are given by

$$\mathfrak{I}_{a^+}^{\vartheta_1;\varrho}g(u) = \frac{1}{\Gamma(\vartheta_1)}\int_a^u \varrho'(v)(\varrho(u)-\varrho(v))^{\vartheta_1-1}g(v)dv, \qquad u > a \in \mathbb{R},$$

where $\varrho \in C^1([a,b])$ is an increasing function and $\Gamma(\cdot)$ is the following Euler Gamma function:

$${}^{\mathrm{R}}\mathfrak{D}_{a^+}^{\vartheta_1;\varrho}g(u) = \left(\frac{1}{\varrho'(u)}\frac{\mathrm{d}}{\mathrm{d}u}\right)^n \int_a^u \frac{(\varrho(u)-\varrho(v))^{n-\vartheta_1-1}}{\Gamma(n-\vartheta_1)}\varrho'(v)g(v)\,\mathrm{d}v, \qquad u > a,$$

where $\varrho \in C^n([a,b])$ is an increasing and integrable function with $\varrho'(u) \neq 0$ for each $u \in [a,b]$, and $n = [\vartheta_1] + 1$ such that $[\vartheta_1]$ is an integer part of ϑ_1 [2].

Lemma 1 ([2]). *Let $\vartheta_1, \vartheta_2, \zeta > 0$. Then,*

(1) $\mathfrak{I}_{a^+}^{\vartheta_1;\varrho}\left(\mathfrak{I}_{a^+}^{\vartheta_2;\varrho}\varrho(u)\right) = \mathfrak{I}_{a^+}^{\vartheta_1+\vartheta_2;\varrho}g(u)$;

(2) ${}^{\mathrm{R}}\mathfrak{D}_{a^+}^{\vartheta_1;\varrho}\left(\mathfrak{I}_{a^+}^{\vartheta_1;\varrho}g(u)\right) = g(u)$;

(3) $\left[\mathfrak{I}_{a^+}^{\vartheta_1;\varrho}(\varrho(u)-\varrho(a))^{\zeta-1}\right](u) = \dfrac{\Gamma(\zeta)}{\Gamma(\zeta+\vartheta_1)}(\varrho(u)-\varrho(a))^{\zeta+\vartheta_1-1}$;

(4) $\left[{}^{\mathrm{R}}\mathfrak{D}_{a^+}^{\vartheta;\varrho}(\varrho(u)-\varrho(a))^{\zeta-1}\right](u) = \dfrac{\Gamma(\zeta)}{\Gamma(\zeta-\vartheta)}(\varrho(u)-\varrho(a))^{\zeta-\vartheta-1}, 0 < \vartheta \le 1$.

Definition 1 ([10]). *Let $\varrho \in C^n([a,b])$ be an increasing and integrable function with $\varrho'(u) \neq 0$ for $u \in [a,b]$. Then, the ϱ-Hilfer fractional derivative of a function g of the order $\vartheta_1 \in (n-1,n]$ and type $\vartheta_2 \in [0,1]$ with respect to another function ϱ is defined by*

$$ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\vartheta_1,\vartheta_2;\varrho}g(u) = \mathfrak{I}_{a^+}^{\vartheta_2(n-\vartheta_1);\varrho}\left(\left(\frac{1}{\varrho'(u)}\frac{\mathrm{d}}{\mathrm{d}u}\right)^n \mathfrak{I}_{a^+}^{(1-\vartheta_2)(n-\vartheta_1);\varrho}g(u)\right). $$

Moreover, the operator ${}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\vartheta_1,\vartheta_2;\varrho}$ can be written as

$$ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\vartheta_1,\vartheta_2;\varrho} = \mathfrak{I}_{a^+}^{\vartheta_2(n-\vartheta_1);\varrho}\,{}^{R}\mathfrak{D}_{a^+}^{\vartheta;\varrho}, \qquad \vartheta = \vartheta_1 + n\vartheta_2 - \vartheta_1\vartheta_2. \tag{9} $$

Lemma 2 ([2,10]). *If $\vartheta_1 \in (0,1]$, $\vartheta_2 \in [0,1]$, $0 \le \vartheta < 1$, $\vartheta = \vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2$, and $g^1 \in C([a,b])$, $\mathfrak{I}_{a^+}^{1-\vartheta;\varrho}g \in C^1([a,b])$, then*

$$ \begin{aligned} \mathfrak{I}_{a^+}^{\vartheta_1;\varrho}\,{}^{H}\left(\mathfrak{D}_{a^+}^{\vartheta_1,\vartheta_2;\varrho}g(u)\right) &= \mathfrak{I}_{a^+}^{\vartheta;\varrho}\,{}^{R}\left(\mathfrak{D}_{a^+}^{\vartheta;\varrho}g(u)\right) \\ &= g(u) - \frac{(\varrho(u)-\varrho(a))^{\vartheta-1}}{\Gamma(\vartheta)}\mathfrak{I}_{a^+}^{1-\vartheta;\varrho}g(a), \qquad \forall\, u \in [a,b]. \end{aligned} $$

Theorem 1 ([41]). *Assume that the generalized complete metric space is denoted by $(\mathfrak{G}, \tilde{d})$ and let the operator $\Lambda : \mathfrak{G} \to \mathfrak{G}$ be contractive with the Lipschitz constant $\ell < 1$. If there is a positive integer r, where $\tilde{d}(\Lambda^{r+1}u, \Lambda^r u) < \infty$, for some $u \in \mathfrak{G}$, then the following holds.*

(i) *The sequence $\{\Lambda^r\}$ converges to a fixed point $u_0 \in \mathfrak{G}$;*

(ii) *u_0 is the unique fixed point of Λ in $\mathfrak{G}^* = \{v \in \mathfrak{G} : \tilde{d}(\Lambda^r u, v) < \infty\}$;*

(iii) *If $v \in \mathfrak{G}^*$, then $\tilde{d}(v, u_0) \le \frac{1}{1-\ell}\tilde{d}(\Lambda v, v)$.*

Clearly, the ϱ-Hilfer fractional snap dynamic system in (7) and (8) can be rewritten as follows:

$$ \begin{cases} {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_4,\beta_4;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\right) = \breve{\mathcal{G}}_{\Psi_1}(u), \\ \mathfrak{I}_{a^+}^{1-\gamma_1;\varrho}\Psi_1(a) = \lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1), & \delta_1 \in J,\ \theta_1 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\right) = \lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2), & \delta_2 \in J,\ \theta_2 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\right)\right) = \lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3), & \delta_3 \in J,\ \theta_3 > 0, \\ \mathfrak{I}_{a^+}^{1-\gamma_4;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\right)\right)\right) = \lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4), & \delta_4 \in J,\ \theta_4 > 0, \end{cases} \tag{10} $$

for $u \in J = [a,\infty)$, $a \ge 0$, where

$$ \begin{aligned} \breve{\mathcal{G}}_{\Psi_1}(u) = \mathcal{G}\Big(u, \Psi_1(u), {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u), {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right), \\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\Big). \end{aligned} \tag{11} $$

Now, we consider the Banach space $\mathcal{C}(J,\Sigma)$, which is endowed by the norm $\|\Psi_1\|_\Sigma = \sup_{u\in J}\|\Psi_1(u)\|$. For the compatibility of our work and due to inspiration from [42–45], we introduce the following suitable special Banach space $(\Omega, \|\cdot\|_\Omega)$:

$$\begin{aligned}\Omega = \Big\{\Psi_1 \,\Big|\, &\Psi_1(u)\in\mathcal{C}(J,\Sigma),\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u),\ {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big),\\ &{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\in\mathcal{C}^1(J,\Sigma),\\ &\sup_{u\in J}\frac{\|\Psi_1(u)\|}{\xi(u)}<\infty,\ \sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big\|<\infty,\\ &\sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big\|<\infty,\\ &\sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\Big\|<\infty\Big\},\end{aligned}$$

which has the supremum norm

$$\begin{aligned}\|\Psi_1\|_\Omega = \max\Big\{&\sup_{u\in J}\frac{\|\Psi_1(u)\|}{\xi(u)},\ \sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big\|,\\ &\sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big\|,\\ &\sup_{u\in J}\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\Big\|\Big\},\end{aligned}$$

where the function $\xi: J\to(0,\infty)$ is non-decreasing and continuous.

3. The Existence and Uniqueness Results

Firstly, we introduce an important corresponding fractional integral equation to the ϱ-Hilfer fractional snap dynamic system (10), as given in the next lemma.

Lemma 3. *Consider a continuous function Ψ_1 that is differentiable. Then, the solution of the ϱ-Hilfer fractional snap dynamic system (10) is given in the following fractional integral form:*

$$\begin{aligned}\Psi_1(u) = &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2)\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}\tag{12}$$

Proof. In view of Lemma 2, and by applying $\mathfrak{I}_{a^+}^{\alpha_4;\varrho}$ on both sides of (10), we have

$$\begin{aligned}&{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\\ &\quad=\frac{(\varrho(u)-\varrho(a))^{\gamma_4-1}}{\Gamma(\gamma_4)}\mathfrak{I}_{a^+}^{1-\gamma_4;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\Big)\Big)\Big)\\ &\qquad+\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}$$

By using the boundary condition

$$\mathfrak{I}_{a^+}^{1-\gamma_4;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\Big)\Big)\Big)=\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4),$$

we get

$$\begin{aligned}&{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\\&\quad=\frac{(\varrho(u)-\varrho(a))^{\gamma_4-1}}{\Gamma(\gamma_4)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}\tag{13}$$

Now, by applying $\mathfrak{I}_{a^+}^{\alpha_3;\varrho}$ on both sides of (13) and by using Lemma 1 with the substitution of the boundary condition

$$\mathfrak{I}_{a^+}^{1-\gamma_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\right)\right)=\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3),$$

we find

$$\begin{aligned}{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)&=\frac{(\varrho(u)-\varrho(a))^{\gamma_3-1}}{\Gamma(\gamma_3)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3-1}}{\Gamma(\gamma_4+\alpha_3)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}\tag{14}$$

Again, by applying $\mathfrak{I}_{a^+}^{\alpha_2;\varrho}$ on both sides of (14) and by substituting the boundary condition

$$\mathfrak{I}_{a^+}^{1-\gamma_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(a)\right)=\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2),$$

we obtain

$$\begin{aligned}{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)&=\frac{(\varrho(u)-\varrho(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2)\\&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2-1}}{\Gamma(\gamma_3+\alpha_2)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}\tag{15}$$

Similarly, with the boundary condition $\mathfrak{I}_{a^+}^{1-\gamma_1;\varrho}\Psi_1(a)=\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)$ and by applying $\mathfrak{I}_{a^+}^{\alpha_1;\varrho}$ on both sides of (15), we have

$$\begin{aligned}\Psi_1(u)&=\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2)\\&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u).\end{aligned}$$

Then, the proof is complete. □

Before establishing the existence and uniqueness results, we need to introduce the following hypotheses.

Hypothesis 1. *Let $\mathcal{G}: I\times\Omega^4\to\Omega$ be a continuously differentiable function, and we assume that $\sigma_i(\cdot)\geq 0,(i=1,\ldots,4)$ are continuous functions satisfying*

$$\left\|\mathcal{G}\left(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4\right)-\mathcal{G}\left(u,\bar{\Psi}_1,\bar{\Psi}_2,\bar{\Psi}_3,\bar{\Psi}_4\right)\right\|\leq\sum_{i=1}^{4}\sigma_i(u)\frac{\left\|\Psi_i(u)-\bar{\Psi}_i(u)\right\|}{\xi(u)},$$

for all $\Psi_i,\bar{\Psi}_i\in\Omega,(i=1,2,3,4)$ and $u\in J$.

Hypothesis 2. *There are constants* $\breve{L}, \breve{Q} > 0$, *where* $\breve{Q} \in (0,1)$, *and these satisfy the following restrictions:*

$$\sup_{u\in J}\left\{\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha;\varrho}[\sigma_1(u)+\sigma_2(u)+\sigma_3(u)+\sigma_4(u)]\right\} \leq \breve{Q} < 1, \tag{16}$$

$$\begin{aligned}\sup_{u\in J}\Bigg\{\frac{1}{\xi(u)}\Bigg(&\frac{(\varrho(u)-\varrho(a))^{\eta_1-1}}{\Gamma(\eta_1)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\ &+\frac{(\varrho(u)-\varrho(a))^{\eta_2-1}}{\Gamma(\eta_2)}\left(\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\right)\\ &+\frac{(\varrho(u)-\varrho(a))^{\eta_3-1}}{\Gamma(\eta_3)}\left(\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\right)\\ &+\frac{(\varrho(u)-\varrho(a))^{\eta_4-1}}{\Gamma(\eta_4)}\left(\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\right)\\ &+\mathfrak{I}_{a^+}^{\alpha;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\Bigg)\Bigg\} \leq \breve{L} < \infty,\end{aligned} \tag{17}$$

where $\omega_i^* = \sup_{u\in[a,\delta_i]}\|\omega_i(\delta_i)\|$, $(i = 1,\ldots,4)$, *and*

$$\begin{aligned}\alpha &= \alpha_4+\alpha_3+\alpha_2+\alpha_1,\ \text{or}\ \alpha_4+\alpha_3+\alpha_2,\ \text{or}\ \alpha_4+\alpha_3,\ \text{or}\ \alpha_4;\\ \eta_1 &= \gamma_1\ \text{or}\ \gamma_1-\alpha_1,\ \text{or}\ \gamma_1-\alpha_2-\alpha_1,\ \text{or}\ \gamma_1-\alpha_3-\alpha_2-\alpha_1;\\ \eta_2 &= \gamma_2+\alpha_1,\ \text{or}\ \gamma_2,\ \text{or}\ \gamma_2-\alpha_2,\ \text{or}\ \gamma_2-\alpha_3-\alpha_2;\\ \eta_3 &= \gamma_3+\alpha_2+\alpha_1,\ \text{or}\ \gamma_3+\alpha_2,\ \text{or}\ \gamma_3,\ \text{or}\ \gamma_3-\alpha_3;\\ \eta_4 &= \gamma_4+\alpha_3+\alpha_2+\alpha_1,\ \text{or}\ \gamma_4+\alpha_3+\alpha_2,\ \text{or}\ \gamma_4+\alpha_3,\ \text{or}\ \gamma_4.\end{aligned} \tag{18}$$

The following lemma is for the study of the existence and uniqueness of solutions for the ϱ-Hilfer fractional snap dynamic system (10), and it is used by employing the well-known Banach fixed point theorem.

Theorem 2. *If the hypotheses* (HP_1) *and* (HP_2) *hold, then, the* ϱ*-Hilfer fractional snap dynamic system* (10) *has one solution on the unbounded domain* J.

Proof. Based on Lemma 3, let the mapping $T : \Omega \to \Omega$ be defined as follows:

$$\begin{aligned}(T\Psi_1)(u) = &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2)\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)+\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u),\end{aligned} \tag{19}$$

where $\breve{\mathcal{G}}_{\Psi_1}(u)$ is defined in Equation (11). For simplicity, let $\breve{\mathcal{G}}_{\Psi_1}(0) = \mathcal{G}(u,0,0,0,0)$. Indeed, T maps Ω into Ω, that is, for any $\Psi_1 \in \Omega$, by using (HP_1) and (HP_2), we get

$$\begin{aligned}\frac{\|(T\Psi_1)(u)\|}{\xi(u)} \leq &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)\xi(u)}|\lambda_1|\mathfrak{I}_{a^+}^{\theta_1;\varrho}\|\omega_1(\delta_1)\|\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)\xi(u)}|\lambda_2|\mathfrak{I}_{a^+}^{\theta_2;\varrho}\|\omega_2(\delta_2)\|\\ &+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)\xi(u)}|\lambda_3|\mathfrak{I}_{a^+}^{\theta_3;\varrho}\|\omega_3(\delta_3)\|\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)\xi(u)}|\lambda_4|\mathfrak{I}_{a^+}^{\theta_4;\varrho}\|\omega_4(\delta_4)\|\\
&+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(u)\|\\
\leq\ &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)\xi(u)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)\xi(u)}\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)\xi(u)}\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)\xi(u)}\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\\
&+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\Bigg[\sigma_1(u)\frac{\|\Psi_1(u)\|}{\xi(u)}+\sigma_2(u)\frac{\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right\|}{\xi(u)}\\
&+\sigma_3(u)\frac{\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right\|}{\xi(u)}\\
&+\sigma_4(u)\frac{\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\right\|}{\xi(u)}\Bigg]\\
&+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\\
\leq\ &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)\xi(u)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)\xi(u)}\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)\xi(u)}\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)\xi(u)}\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\\
&+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\\
&+\frac{\|\Psi_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq\breve{L}+\breve{\mathcal{Q}}\|\Psi_1\|_\Omega,
\end{aligned}
$$

for $\breve{\mathcal{Q}}\in(0,1)$. In addition, due to Equation (9), we have

$$
\begin{aligned}
\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right\|\leq\ &\frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_1)\xi(u)}|\lambda_1|\mathfrak{I}_{a^+}^{\theta_1;\varrho}\|\omega_1(\delta_1)\|\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_2-1}}{\Gamma(\gamma_2)\xi(u)}|\lambda_2|\mathfrak{I}_{a^+}^{\theta_2;\varrho}\|\omega_2(\delta_2)\|\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2-1}}{\Gamma(\gamma_3+\alpha_2)\xi(u)}|\lambda_3|\mathfrak{I}_{a^+}^{\theta_3;\varrho}\|\omega_3(\delta_3)\|\\
&+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2)\xi(u)}|\lambda_4|\mathfrak{I}_{a^+}^{\theta_4;\varrho}\|\omega_4(\delta_4)\|\\
&+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(u)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_1)\xi(u)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\
&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_2-1}}{\Gamma(\gamma_2)\xi(u)}\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\\
&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2-1}}{\Gamma(\gamma_3+\alpha_2)\xi(u)}\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\\
&\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2)\xi(u)}\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\\
&\quad+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\\
&\quad+\frac{\|\Psi_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{L}+\breve{Q}\|\Psi_1\|_\Omega,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right\|\\
&\quad\leq \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_2-\alpha_1)\xi(u)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_2)\xi(u)}\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_3-1}}{\Gamma(\gamma_3)\xi(u)}\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3-1}}{\Gamma(\gamma_4+\alpha_3)\xi(u)}\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\\
&\quad\quad+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\\
&\quad\quad+\frac{\|\Psi_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{L}+\breve{Q}\|\Psi_1\|_\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right)\right\|\\
&\quad\leq \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_3-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_3-\alpha_2-\alpha_1)\xi(u)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_3-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_3-\alpha_2)\xi(u)}\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_3-\alpha_3-1}}{\Gamma(\gamma_3-\alpha_3)\xi(u)}\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\\
&\quad\quad+\frac{(\varrho(u)-\varrho(a))^{\gamma_4-1}}{\Gamma(\gamma_4)\xi(u)}\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\\
&\quad\quad+\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|+\frac{\|\Psi_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{L}+\breve{Q}\|\Psi_1\|_\Omega,
\end{aligned}
$$

for $\breve{Q}\in(0,1)$. Hence, $\|\Psi_1\|_\Omega\leq \breve{L}+\breve{Q}\|\Psi_1\|_\Omega<\infty$, which means that $T:\Omega\to\Omega$. Now, we will prove that T is contractive on Ω. By using (HP_1) and (HP_2), for any $\Psi,\bar{\Psi}_1\in\Omega$, we find

$$
\frac{\|(T\Psi_1)(u)-(T\bar{\Psi}_1)(u)\|}{\xi(u)}\leq\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\left\|\breve{\mathcal{G}}_{\Psi_1}(u)-\breve{\mathcal{G}}_{\bar{\Psi}_1}(u)\right\|
$$

$$
\begin{aligned}
&\leq \frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\Bigg[\sigma_1(u)\frac{\|\Psi_1(u)-\bar{\Psi}_1(u)\|}{\xi(u)}\\
&+\sigma_2(u)\frac{\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\right\|}{\xi(u)}\\
&+\frac{\sigma_3(u)}{\xi(u)}\Big[\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\\
&-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big\|\Big]\\
&+\frac{\sigma_4(u)}{\xi(u)}\Big[\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\\
&-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big)\Big\|\Big]\Bigg]\\
&\leq \frac{\|\Psi_1-\bar{\Psi}_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}\|\Psi_1\|_\Omega.
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\right\|\\
&\quad\leq \frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(u)-\breve{\mathcal{G}}_{\bar{\Psi}_1}(u)\|\\
&\quad\leq \frac{\|\Psi_1-\bar{\Psi}_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}\|\Psi_1\|_\Omega,\\
&\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1)(u)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1)(u)\right\|\\
&\quad\leq \frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(u)-\breve{\mathcal{G}}_{\bar{\Psi}_1}(u)\|\\
&\quad\leq \frac{\|\Psi_1-\bar{\Psi}_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}\|\Psi_1\|_\Omega,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\Big)\Big)\\
&\qquad-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\Big)\Big)\Big\|\\
&\quad\leq \frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(u)-\breve{\mathcal{G}}_{\bar{\Psi}_1}(u)\|\\
&\quad\leq \frac{\|\Psi_1-\bar{\Psi}_1\|_\Omega}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}\|\Psi_1\|_\Omega.
\end{aligned}
$$

Thus, we conclude that

$$\|T\Psi_1-T\bar{\Psi}_1\|_\Omega\leq \breve{Q}\|\Psi_1-\bar{\Psi}_1\|_\Omega.$$

Since $\breve{Q}\in(0,1)$, T is a contractive operator. According to the Banach fixed point theorem, the operator T possesses one fixed point Ψ_0 in Ω, i.e., $T\Psi_0=\Psi_0$. Hence, the ϱ-Hilfer fractional snap dynamic system (10) has one solution on the unbounded domain J. □

4. The Stability

Throughout this section, we will study different types of stability, such as $\mathcal{U}.\mathcal{H}, \mathcal{U}.\mathcal{H}.\mathcal{R}$, and s-$\mathcal{U}.\mathcal{H}.\mathcal{R}$. For definitions thereof and more details, we refer readers to [46]. The aim in this section is to define the appropriate metrics $\mathrm{d}_1(\cdot)$ and $\mathrm{d}_2(\cdot)$ on the Banach space Ω. The metric $\mathrm{d}_1(\cdot)$ is defined by

$$\begin{aligned}\mathrm{d}_1(\Psi_1, \bar{\Psi}_1) = \inf_{u\in J}\Big\{ N \in J \,\Big|\, & \frac{\|\Psi_1(u) - \bar{\Psi}_1(u)\|}{\xi(u)} \le N v(u), \\ & \frac{1}{\xi(u)}\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big\| \le N v(u), \\ & \frac{1}{\xi(u)}\Big\| {}^{H}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big) \\ & - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big\| \le N v(u), \\ & \frac{1}{\xi(u)}\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big) \\ & - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big)\Big\| \le N v(u)\Big\},\end{aligned}$$

for any non-negative, non-decreasing continuous function $v(u)$ on J, and the metric $\mathrm{d}_2(\cdot)$ is defined for any non-negative, non-increasing continuous function $v(u)$ on J as follows:

$$\begin{aligned}\mathrm{d}_2(\Psi_1, \bar{\Psi}_1) = \sup_{u\in J}\Big\{ N \in J \,\Big|\, & \frac{\|\Psi_1(u) - \bar{\Psi}_1(u)\|}{v(u)\xi(u)} \le N, \\ & \frac{\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big\|}{v(u)\xi(u)} \le N, \\ & \frac{\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big\|}{v(u)\xi(u)} \le N, \\ & \frac{1}{v(u)\xi(u)}\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big) \\ & - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\bar{\Psi}_1(u)\Big)\Big)\Big\| \le N\Big\}.\end{aligned}$$

According to [47], it is not difficult to prove that $\mathrm{d}_1(\cdot)$ and $\mathrm{d}_2(\cdot)$ are metrics on a special Banach space Ω. Next, we state and prove the $\mathcal{U}.\mathcal{H}, \mathcal{U}.\mathcal{H}.\mathcal{R}$, and s-$\mathcal{U}.\mathcal{H}.\mathcal{R}$ stability theorems on the unbounded domains J.

Theorem 3. *Assume that* (HP$_1$) *and* (HP$_2$) *hold and that* $\Psi_1 : J \to \Omega$ *is a continuously differentiable function, and let* $v(u)$ *be a non-negative, non-decreasing continuous function on* J *that satisfies*

$$\begin{aligned}\Big\| \Psi_1(u) - & \frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}\lambda_1 \mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1) \\ & + \frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)}\lambda_2 \mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2) \\ & + \frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)}\lambda_3 \mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3) \\ & + \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)}\lambda_4 \mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4) \\ & + \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\mathcal{G}_{\Psi_1}(u)\Big\| \le \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho} v(u), \qquad u \in J, \end{aligned} \tag{20}$$

$$\begin{aligned}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) &- \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2-1}}{\Gamma(\gamma_3+\alpha_2)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4)\\ &- \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u)\Big\| \le \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}v(u),\end{aligned} \tag{21}$$

$$\begin{aligned}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big) &- \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_2-\alpha_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_2)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2) - \frac{(\varrho(u)-\varrho(a))^{\gamma_3-1}}{\Gamma(\gamma_3)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3-1}}{\Gamma(\gamma_4+\alpha_3)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4) - \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u)\Big\| \le \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}v(u),\end{aligned} \tag{22}$$

$$\begin{aligned}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big) &- \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_3-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_3-\alpha_2-\alpha_1)}\lambda_1\mathfrak{I}_{a^+}^{\theta_1;\varrho}\omega_1(\delta_1)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_3-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_3-\alpha_2)}\lambda_2\mathfrak{I}_{a^+}^{\theta_2;\varrho}\omega_2(\delta_2) - \frac{(\varrho(u)-\varrho(a))^{\gamma_3-\alpha_3-1}}{\Gamma(\gamma_3-\alpha_3)}\lambda_3\mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3)\\ &- \frac{(\varrho(u)-\varrho(a))^{\gamma_4-1}}{\Gamma(\gamma_4)}\lambda_4\mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4) - \mathfrak{I}_{a^+}^{\alpha_4;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u)\Big\| \le \mathfrak{I}_{a^+}^{\alpha_4;\varrho}v(u),\end{aligned} \tag{23}$$

for $u \in J$. Then, there is one solution $\Psi_0 \in \Omega$ for all $u \in J$ and $0 < \breve{Q} < 1$, which satisfies

$$\begin{aligned}&\frac{1}{\xi(u)}\|\Psi_1(u)-\Psi_0(u)\| \le \frac{M}{1-\breve{Q}}v(u),\\ &\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big\| \le \frac{M}{1-\breve{Q}}v(u),\\ &\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\\ &\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big)\Big\| \le \frac{M}{1-\breve{Q}}v(u),\\ &\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big)\\ &\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big)\Big)\Big\| \le \frac{M}{1-\breve{Q}}v(u),\end{aligned} \tag{24}$$

where

$$\sup_{u\in J}\frac{(\varrho(u)-\varrho(a))^{\alpha}}{\Gamma(\alpha+1)\xi(u)} \le M < \infty, \tag{25}$$

which implies that the solution of the ϱ-Hilfer fractional snap dynamic system (10) is $\mathcal{U}.\mathcal{H}.\mathcal{R}$ stable and, accordingly, is $\mathcal{U}.\mathcal{H}$ stable.

Proof. At the beginning, $T : \Omega \to \Omega$ was as defined in (19). By the metric $\mathrm{d}_1(\cdot)$ and hypotheses (HP_1) and (HP_2), for any $\Psi_1, \bar{\Psi}_1 \in \Omega$, we have

$$\frac{1}{\xi(u)}\|(T\Psi_1)(u)-(T\bar{\Psi}_1)(u)\| \le \frac{Nv(u)}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right] \le \breve{Q}Nv(u),$$

$$\frac{1}{\xi(u)}\left\|\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1\right)(u)-\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1\right)(u)\right\|$$
$$\leq \frac{Nv(u)}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}Nv(u),$$
$$\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\right)\right\|$$
$$\leq \frac{Nv(u)}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}Nv(u),$$

and

$$\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right)\right.$$
$$\left.-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\right)\right)\right\|$$
$$\leq \frac{Nv(u)}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha_4;\varrho}\left[\sum_{i=1}^{4}\sigma_i(u)\right]\leq \breve{Q}Nv(u),$$

for $\breve{Q}\in(0,1)$. Hence, we find

$$\mathrm{d}_1(T\Psi_1,T\bar{\Psi}_1)\leq \breve{Q}N=\breve{Q}\mathrm{d}_1(\Psi_1,\bar{\Psi}_1),\qquad 0<\breve{Q}<1.$$

Next, due to inequalities (20)–(23), we get

$$\frac{\|(\Psi_1)(u)-(T\Psi_1)(u)\|}{\xi(u)}\leq \sup_{u\in J}\frac{(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)\xi(u)}v(u)=Mv(u),\tag{26}$$

$$\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right\|$$
$$\leq \sup_{u\in J}\frac{v(u)(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3+\alpha_2}}{\Gamma(\alpha_4+\alpha_3+\alpha_2+1)\xi(u)}=Mv(u),\tag{27}$$

$$\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right\|$$
$$\leq \sup_{u\in J}\frac{(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3}}{\Gamma(\alpha_4+\alpha_3+1)\xi(u)}v(u)=Mv(u),\tag{28}$$

and

$$\frac{1}{\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\right.$$
$$\left.-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right)\right\|$$
$$\leq \sup_{u\in J}\frac{(\varrho(u)-\varrho(a))^{\alpha_4}}{\Gamma(\alpha_4+1)\xi(u)}v(u)=Mv(u).\tag{29}$$

Therefore, inequalities (26)–(29) imply that

$$\mathrm{d}_1(\Psi_1,T\Psi_1)\leq M<\infty.$$

Furthermore, according to Theorem 1 (parts (i) and (ii)), T has one fixed point Ψ_0, i.e., $T\Psi_0 = \Psi_0$. Moreover, from Theorem 1 (part (iii)), we have

$$\mathrm{d}_1(\Psi_1, \Psi_0) \leq \frac{1}{1-\breve{Q}} \mathrm{d}_1(T\Psi_1, \Psi_1) \leq \frac{M}{1-\breve{Q}}, \qquad 0 < \breve{Q} < 1.$$

Thus, the solution of the ϱ-Hilfer fractional snap dynamic system (10) is $\mathcal{U}.\mathcal{H}.\mathcal{R}$ stable. Along with this, if $v(u) = 1$, then the solution of the ϱ-Hilfer fractional snap dynamic system (10) is $\mathcal{U}.\mathcal{H}$ stable. □

Theorem 4. *Let* (HP$_1$) *and* (HP$_2$) *hold, let* $\Psi_1 : J \to \Omega$ *be a continuously differentiable function, and let* $v(u)$ *be a non-negative, non-increasing continuous function on* J*, all of which satisfy*

$$\begin{aligned}
&\Big\| \Psi_1(u) - \frac{(\varrho(u)-\varrho(a))^{\gamma_1-1}}{\Gamma(\gamma_1)} \lambda_1 \mathfrak{I}_{a^+}^{\theta_1;\varrho} \omega_1(\delta_1) \\
&\quad + \frac{(\varrho(u)-\varrho(a))^{\gamma_2+\alpha_1-1}}{\Gamma(\gamma_2+\alpha_1)} \lambda_2 \mathfrak{I}_{a^+}^{\theta_2;\varrho} \omega_2(\delta_2) \\
&\quad + \frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_3+\alpha_2+\alpha_1)} \lambda_3 \mathfrak{I}_{a^+}^{\theta_3;\varrho} \omega_3(\delta_3) \\
&\quad + \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2+\alpha_1-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2+\alpha_1)} \lambda_4 \mathfrak{I}_{a^+}^{\theta_4;\varrho} \omega_4(\delta_4) \\
&\quad + \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho} \breve{\mathcal{G}}_{\Psi_1}(u) \Big\| \leq \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2+\alpha_1;\varrho} \rho,
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho} \Psi_1(u) - \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_1)} \lambda_1 \mathfrak{I}_{a^+}^{\theta_1;\varrho} \omega_1(\delta_1) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_2-1}}{\Gamma(\gamma_2)} \lambda_2 \mathfrak{I}_{a^+}^{\theta_2;\varrho} \omega_2(\delta_2) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_3+\alpha_2-1}}{\Gamma(\gamma_3+\alpha_2)} \lambda_3 \mathfrak{I}_{a^+}^{\theta_3;\varrho} \omega_3(\delta_3) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3+\alpha_2-1}}{\Gamma(\gamma_4+\alpha_3+\alpha_2)} \lambda_4 \mathfrak{I}_{a^+}^{\theta_4;\varrho} \omega_4(\delta_4) \\
&\quad - \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho} \breve{\mathcal{G}}_{\Psi_1}(u) \Big\| \leq \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3+\alpha_2;\varrho} \rho,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho} \Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho} \Psi_1(u) \Big) - \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_2-\alpha_1)} \lambda_1 \mathfrak{I}_{a^+}^{\theta_1;\varrho} \omega_1(\delta_1) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_2)} \lambda_2 \mathfrak{I}_{a^+}^{\theta_2;\varrho} \omega_2(\delta_2) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_3-1}}{\Gamma(\gamma_3)} \lambda_3 \mathfrak{I}_{a^+}^{\theta_3;\varrho} \omega_3(\delta_3) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_4+\alpha_3-1}}{\Gamma(\gamma_4+\alpha_3)} \lambda_4 \mathfrak{I}_{a^+}^{\theta_4;\varrho} \omega_4(\delta_4) - \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho} \breve{\mathcal{G}}_{\Psi_1}(u) \Big\| \leq \mathfrak{I}_{a^+}^{\alpha_4+\alpha_3;\varrho} \rho,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\Big\| {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho} \Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho} \Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho} \Psi_1(u) \Big) \Big) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_1-\alpha_3-\alpha_2-\alpha_1-1}}{\Gamma(\gamma_1-\alpha_3-\alpha_2-\alpha_1)} \lambda_1 \mathfrak{I}_{a^+}^{\theta_1;\varrho} \omega_1(\delta_1) \\
&\quad - \frac{(\varrho(u)-\varrho(a))^{\gamma_2-\alpha_3-\alpha_2-1}}{\Gamma(\gamma_2-\alpha_3-\alpha_2)} \lambda_2 \mathfrak{I}_{a^+}^{\theta_2;\varrho} \omega_2(\delta_2)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\varrho(u)-\varrho(a))^{\gamma_3-\alpha_3-1}}{\Gamma(\gamma_3-\alpha_3)}\lambda_3 \mathfrak{I}_{a^+}^{\theta_3;\varrho}\omega_3(\delta_3) \\
&- \frac{(\varrho(u)-\varrho(a))^{\gamma_4-1}}{\Gamma(\gamma_4)}\lambda_4 \mathfrak{I}_{a^+}^{\theta_4;\varrho}\omega_4(\delta_4) - \mathfrak{I}_{a^+}^{\alpha_4;\varrho}\breve{\mathcal{G}}_{\Psi_1}(u)\Big\| \leq \mathfrak{I}_{a^+}^{\alpha_4;\varrho}\rho, \qquad (33)
\end{aligned}
$$

for $u \in J$, where $\rho > 0$. Then, there is one solution $\Psi_0 \in \Omega$ and a constant $\lambda > 0$ for all $u \in J$ and $0 < \breve{Q} < 1$ satisfying

$$
\begin{aligned}
&\frac{1}{\xi(u)}\|\Psi_1(u)-\Psi_0(u)\| \leq \frac{\lambda\,\rho\,M}{1-\breve{Q}}v(u), \\
&\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big\| \leq \frac{\lambda\,\rho\,M}{1-\breve{Q}}v(u), \\
&\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big) \\
&\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big)\Big\| \leq \frac{\lambda\,\rho\,M}{1-\breve{Q}}v(u), \\
&\frac{1}{\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\Big)\Big) \\
&\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_0(u)\Big)\Big)\Big\| \leq \frac{\lambda\,\rho\,M}{1-\breve{Q}}v(u),
\end{aligned}
$$

where

$$
\sup_{u\in J}\frac{(\varrho(u)-\varrho(a))^{\alpha}}{\Gamma(\alpha+1)\xi(u)} \leq M < \infty,
$$

which implies that the solution of the ϱ-Hilfer fractional snap dynamic system (10) is s-$\mathcal{U}.\mathcal{H}.\mathcal{R}$ stable.

Proof. Let the operator $T : \Omega \to \Omega$ be defined as in (19); as in Theorem 3, the metric $\mathrm{d}_2(\cdot)$ and hypotheses 1 and 2 imply that

$$
\begin{aligned}
&\frac{1}{v(u)\xi(u)}\|(T\Psi_1)(u)-(T\bar{\Psi}_1)(u)\| \leq \breve{Q}N, \\
&\frac{1}{v(u)\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u) - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\Big\| \leq \breve{Q}N, \\
&\frac{1}{v(u)\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\Big) \\
&\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\Big)\Big\| \leq \breve{Q}N,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{v(u)\xi(u)}\Big\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\Big)\Big) \\
&\qquad - {}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\Big({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\bar{\Psi}_1(u)\Big)\Big)\Big\| \leq \breve{Q}N,
\end{aligned}
$$

for $\breve{Q} \in (0,1)$. Hence, we conclude that

$$
\mathrm{d}_2(T\Psi_1, T\bar{\Psi}_1) \leq \breve{Q}N = \breve{Q}\mathrm{d}_2(\Psi_1,\bar{\Psi}_1), 0 < \breve{Q} < 1.
$$

Now, in view of the continuity and positiveness of the decreasing function $v(u), \forall u \in J$, we have $\frac{1}{v(u)} \leq \lambda, \forall u \in J, \lambda > 0$. In addition, from inequalities (30)–(33), we find

$$
\frac{\|(\Psi_1)(u)-(T\Psi_1)(u)\|}{v(u)\xi(u)} \leq \sup_{u\in J}\frac{\rho(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3+\alpha_2+\alpha_1}}{\Gamma(\alpha_4+\alpha_3+\alpha_2+\alpha_1+1)v(u)\xi(u)} = \lambda\,\rho\,M, \qquad (34)
$$

$$\frac{1}{v(u)\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right\| \leq \sup_{u\in J}\frac{\rho(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3+\alpha_2}}{\Gamma(\alpha_4+\alpha_3+\alpha_2+1)v(u)\xi(u)}=\lambda\,\rho\,M, \tag{35}$$

$$\frac{1}{v(u)\xi(u)}\left\|{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right\| \leq \sup_{u\in J}\frac{\rho(\varrho(u)-\varrho(a))^{\alpha_4+\alpha_3}}{\Gamma(\alpha_4+\alpha_3+1)v(u)\xi(u)}=\lambda\,\rho\,M, \tag{36}$$

and

$$\begin{aligned}\frac{1}{v(u)\xi(u)}\Big\|&{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}\Psi_1(u)\right)\right)\\&-{}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_3,\beta_3;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\varrho}\left({}^{\mathrm{H}}\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\varrho}T\Psi_1(u)\right)\right)\Big\|\\&\leq \sup_{u\in J}\frac{\rho(\varrho(u)-\varrho(a))^{\alpha_4}}{\Gamma(\alpha_4+1)v(u)\xi(u)}=\lambda\,\rho\,M.\end{aligned} \tag{37}$$

Thus, by (34)–(37), we obtain $\mathrm{d}_2(\Psi_1,T\Psi_1)\leq\lambda\,\rho\,M<\infty$. Based on Theorem 1 (parts (i) and (ii)), there is one fixed point Ψ_0 i.e., $T\Psi_0=\Psi_0$. Furthermore, according to Theorem 1 (part (iii)), we conclude that

$$\mathrm{d}_2(\Psi_1,\Psi_0)\leq\frac{1}{1-\breve{Q}}\mathrm{d}_2(T\Psi_1,\Psi_1)\leq\frac{\lambda\,\rho\,M}{1-\breve{Q}},\qquad 0<\breve{Q}<1.$$

Hence, from the above conclusions, the solution of the ϱ-Hilfer fractional snap dynamic system (10) is s-$\mathcal{U}.\mathcal{H}.\mathcal{R}$ stable, and the proof is complete. □

5. Examples

Herein, we give some examples of the ϱ-Hilfer fractional snap dynamic system on unbounded domains $[a,\infty)$ based on numerical simulations to analyze their solutions. In these examples, we consider different cases of the function ϱ to cover the Caputo, Caputo–Hadamard, and Katugampola versions.

Example 1. *Based on system (7), we consider a ϱ-Hilfer fractional snap dynamic system on the unbounded domains $J=[1.05,\infty)$ as*

$$\begin{cases}{}^{\mathrm{H}}\mathfrak{D}_{1.05}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(u)=\Psi_2(u),\\ {}^{\mathrm{H}}\mathfrak{D}_{1.05}^{\frac{43}{80},\frac{23}{25};\varrho}\Psi_2(u)=\Psi_3(u),\\ {}^{\mathrm{H}}\mathfrak{D}_{1.05}^{\frac{2}{15},\frac{11}{20};\varrho}\Psi_3(u)=\Psi_4(u),\\ {}^{\mathrm{H}}\mathfrak{D}_{1.05}^{\frac{7}{20},\frac{13}{50};\varrho}\Psi_4(u)=\mathcal{G}(u,\Psi_1(u),\Psi_2(u),\Psi_3(u),\Psi_4(u)),\end{cases} \tag{38}$$

with

$$\begin{aligned}\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4)=&\frac{\sqrt{u^2+1}}{47(2+\sqrt{u^2+1})}+\frac{1}{\exp(u^2+1)}\left(\frac{2\sqrt[3]{u+12}|\Psi_1(u)|}{47(0.5+|\Psi_1(u)|)}\right.\\&\left.+\frac{3|u|\sin^{-1}(\Psi_2(u))}{94}+\frac{u}{47}\left(\frac{\tan^{-1}(\Psi_3(u))}{\sqrt{5}+\tan^{-1}(\Psi_3(u))}\right)\right.\end{aligned}$$

$$+\frac{2u}{47}\left(\frac{\sin(\Psi_4(u))}{8+\sin(\Psi_4(u))}\right)\Bigg), \tag{39}$$

for $u \in J$, $a = 1.05 \geq 0$, under the ϱ-Riemann–Liouville (R-L) fractional integral conditions:

$$\begin{cases} \mathfrak{I}_{a^+}^{\frac{167}{961};\varrho}\Psi_1(a) = -\dfrac{121}{16}\,\mathfrak{I}_{a^+}^{\frac{7}{8};\varrho}\omega_1(3.76), \\ \mathfrak{I}_{a^+}^{\frac{37}{1000};\varrho}\Psi_2(a) = -\dfrac{72}{39}\,\mathfrak{I}_{a^+}^{\frac{3}{10};\varrho}\omega_2(2.41), \\ \mathfrak{I}_{a^+}^{\frac{39}{100};\varrho}\Psi_3(a) = \dfrac{215}{108}\,\mathfrak{I}_{a^+}^{\frac{9}{13};\varrho}\omega_3(4.58), \\ \mathfrak{I}_{a^+}^{\frac{481}{1000};\varrho}\Psi_4(a) = -\dfrac{83}{105}\,\mathfrak{I}_{a^+}^{\frac{3}{7};\varrho}\omega_4(5.93), \end{cases} \tag{40}$$

where $\omega_i(u) = \frac{\sqrt{2}u^2}{5i(u^2+i)}$, $i = 1,2,3,4$. One can check that

$$\alpha_i = \left\{\frac{67}{90}, \frac{43}{80}, \frac{2}{15}, \frac{7}{20}\right\} \subset (0,1], \qquad \beta_i = \left\{\frac{8}{25}, \frac{23}{25}, \frac{11}{20}, \frac{13}{50}\right\} \subset [0,1],$$

$$\alpha_i \leq \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i = \begin{cases} \frac{794}{961}, & i = 1, \\ \frac{963}{1000}, & i = 2, \\ \frac{61}{100}, & i = 3, \\ \frac{519}{1000}, & i = 4, \end{cases} \qquad 1 - \gamma_i = \begin{cases} \frac{167}{961}, & i = 1, \\ \frac{37}{1000}, & i = 2, \\ \frac{39}{100}, & i = 3, \\ \frac{481}{1000}, & i = 4, \end{cases}$$

$$\lambda_i = \left\{-\frac{121}{16}, -\frac{72}{39}, \frac{215}{108}, -\frac{83}{105}\right\} \subset \mathbb{R}, \qquad \theta_i = \left\{\frac{7}{8}, \frac{3}{10}, \frac{9}{13}, \frac{3}{7}\right\} \subset (0,\infty),$$

and

$$\delta = \{3.76,\ 2.41,\ 4.58,\ 5.93\} \subset J.$$

Thus, the ϱ-Hilfer fractional snap dynamic system (38)–(40) can be rewritten as follows:

$$\begin{cases} {}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{7}{20},\frac{13}{50};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{2}{15},\frac{13}{50};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{43}{80},\frac{23}{25};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{8}{25};\varrho}\Psi_1(u)\right)\right)\right) = \breve{\mathcal{G}}_{\Psi_1}(u), \\ \mathfrak{I}_{1.05}^{\frac{167}{961};\varrho}\Psi_1(1.05) = -\dfrac{121}{16}\mathfrak{I}_{1.05}^{\frac{7}{8};\varrho}\omega_1(3.76), \\ \mathfrak{I}_{1.05}^{\frac{37}{1000};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(1.05)\right) = -\frac{72}{39}\,\mathfrak{I}_{1.05}^{\frac{3}{10};\varrho}\omega_2(2.41), \\ \mathfrak{I}_{1.05}^{\frac{39}{100};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{43}{80},\frac{23}{25};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(1.05)\right)\right) = \dfrac{215}{108}\,\mathfrak{I}_{1.05}^{\frac{9}{13};\varrho}\omega_3(4.58), \\ \mathfrak{I}_{1.05}^{\frac{481}{1000};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{2}{15},\frac{13}{50};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{43}{80},\frac{23}{25};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(1.05)\right)\right)\right) \\ \qquad = -\dfrac{83}{105}\mathfrak{I}_{1.05}^{\frac{3}{7};\varrho}\omega_4(5.93), \end{cases} \tag{41}$$

where

$$\breve{\mathcal{G}}_{\Psi_1}(u) = \mathcal{G}\Big(u, \Psi_1(u), {}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(u), {}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{43}{80},\frac{23}{25};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(u)\right), {}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{2}{15},\frac{13}{50};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{43}{80},\frac{23}{25};\varrho}\left({}_{1.05}\mathrm{H}\mathfrak{D}^{\frac{67}{90},\frac{8}{25};\varrho}\Psi_1(u)\right)\right)\Big). \tag{42}$$

Now, we have

$$\begin{aligned} &\left|\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4) - \mathcal{G}(u,\Psi_1^*,\Psi_2^*,\Psi_3^*,\Psi_4^*)\right| \\ &\quad = \left|\frac{\sqrt{u^2+1}}{47(2+\sqrt{u^2+1})} + \frac{1}{\exp(u^2+1)}\left(\frac{2\sqrt[3]{u+12}|\Psi_1(u)|}{47(0.5+|\Psi_1(u)|)}\right.\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{3|u|\sin^{-1}(\Psi_2(u))}{94}+\frac{u}{47}\frac{\tan^{-1}(\Psi_3(u))}{\sqrt{5}+\tan^{-1}(\Psi_3(u))}+\frac{2u}{47}\frac{\sin(\Psi_4(u))}{8+\sin(\Psi_4(u))}\Bigg)\\
&-\left[\frac{\sqrt{u^2+1}}{47(2+\sqrt{u^2+1})}+\frac{1}{\exp(u^2+1)}\left(\frac{2\sqrt[3]{u+12}|\Psi_1^*(u)|}{47(0.5+|\Psi_1^*(u)|)}\right.\right.\\
&\left.\left.+\frac{3|u|\sin^{-1}(\Psi_2^*(u))}{94}+\frac{u}{47}\frac{\tan^{-1}(\Psi_3^*(u))}{\sqrt{5}+\tan^{-1}(\Psi_3^*(u))}+\frac{2u}{47}\frac{\sin(\Psi_4^*(u))}{8+\sin(\Psi_4^*(u))}\right)\right]\Bigg|\\
&\leq\frac{2|\sqrt[3]{u+12}|}{47\exp(u^2+1)}\left|\frac{|\Psi_1(u)|}{0.5+|\Psi_1(u)|}-\frac{|\Psi_1^*(u)|}{0.5+|\Psi_1^*(u)|}\right|\\
&+\frac{3|u|}{94\exp(u^2+1)}\left|\sin^{-1}(\Psi_2(u))-\sin^{-1}(\Psi_2^*(u))\right|\\
&+\frac{|u|}{47\exp(u^2+1)}\left|\frac{\tan^{-1}(\Psi_3(u))}{\sqrt{5}+\tan^{-1}(\Psi_3(u))}-\frac{\tan^{-1}(\Psi_3^*(u))}{\sqrt{5}+\tan^{-1}(\Psi_3^*(u))}\right|\\
&+\frac{2|u|}{47\exp(u^2+1)}\left|\frac{\sin(\Psi_4(u))}{8+\sin(\Psi_4(u))}-\frac{\sin(\Psi_4^*(u))}{8+\sin(\Psi_4^*(u))}\right|\\
&\leq\frac{1}{\exp(u^2+1)}\left(\frac{2|\sqrt[3]{u+12}|}{47}|\Psi_1(u)-\Psi_1^*(u)|+\frac{3|u|}{94}|\Psi_2(u)-\Psi_2^*(u)|\right.\\
&\left.+\frac{|u|}{47}|\Psi_3(u)-\Psi_3^*(u)|+\frac{2|u|}{47}|\Psi_4(u)-\Psi_4^*(u)|\right).
\end{aligned}
$$

So, we can choose $\xi(u)=47\exp(u^2+1)$, $\sigma_1=2|\sqrt[3]{u+12}|$, $\sigma_2=\frac{3|u|}{2}$, *and* $\sigma_3=|u|$, $\sigma_4=2|u|$. *Additionally, for* $u\in J$,

$$
\omega_i^*=\sup_{u\in[a,\delta_i]}\|\omega_i(u)\|=\begin{cases}0.264157, & i=1,\\ 0.105197, & i=2,\\ 0.082484, & i=3,\\ 0.063488, & i=4.\end{cases}\tag{43}
$$

Now, we consider four cases for ϱ:

$$
\varrho_1(u)=2^u,\quad \varrho_2(u)=u,\quad \varrho_3(u)=\ln(u+0.1),\quad \varrho_4(u)=\sqrt{u}.
$$

By using Equations (16)–(18), $\alpha=\alpha_4+\alpha_3+\alpha_2+\alpha_1\simeq 1.76527$, *and by choosing suitable values of* $\breve{Q}$ *and* $\breve{L}$ *in the four cases, we have*

$$
\begin{aligned}
&\sup_{u\in J}\left\{\frac{1}{\xi(u)}\mathfrak{I}_{a^+}^{\alpha;\varrho}[\sigma_1(u)+\sigma_2(u)+\sigma_3(u)+\sigma_4(u)]\right\}\\
&\simeq\left\{\begin{array}{ll}0.004048, & \varrho_1(u)=2^u,\\ 0.004703, & \varrho_2(u)=u,\\ 0.000975, & \varrho_3(u)=\ln(u+0.1),\\ 0.000363, & \varrho_4(u)=\sqrt{u},\end{array}\right\}\leq\breve{Q}<1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{u\in J}\left\{\frac{1}{\xi(u)}\left(\frac{(\varrho(u)-\varrho(a))^{\eta_1-1}}{\Gamma(\eta_1)}\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)}\right.\right.\\
&\left.\left.+\frac{(\varrho(u)-\varrho(a))^{\eta_2-1}}{\Gamma(\eta_2)}\left(\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)}\right)\right.\right.
\end{aligned}
$$

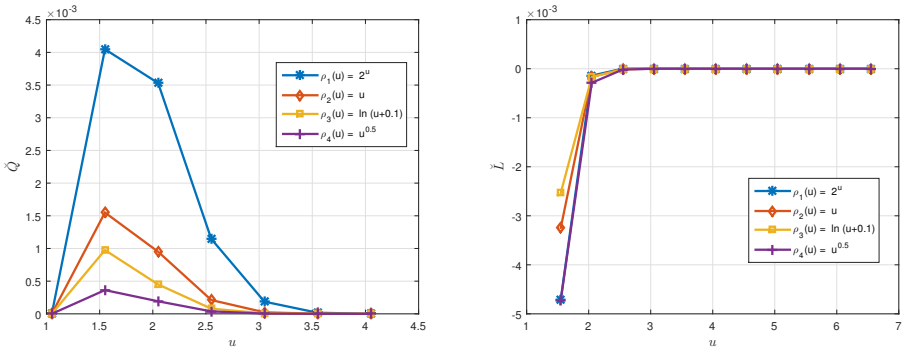
$$+\frac{(\varrho(u)-\varrho(a))^{\eta_3-1}}{\Gamma(\eta_3)}\left(\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)}\right)$$
$$+\frac{(\varrho(u)-\varrho(a))^{\eta_4-1}}{\Gamma(\eta_4)}\left(\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)}\right)+\mathfrak{I}_{a^+}^{\alpha;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\|\Bigg)\Bigg\}$$
$$\simeq\begin{cases}0.0000026, & \varrho_1(u)=2^u,\\ 0.0000007, & \varrho_2(u)=u,\\ 0.0000009, & \varrho_3(u)=\ln(u+0.1),\\ 0.0000000, & \varrho_4(u)=\sqrt{u},\end{cases}\leq\breve{L}<\infty,$$

for $u\in J$. *By considering* $\breve{Q}=0.05\in(0,1)$ *and* $\breve{L}>0.05$, *assumptions* (HP_1) *and* (HP_2) *hold.*

Table 1 shows the numerical results of $\breve{Q}$ *and* $\breve{L}$ *for* $u\in[1.05,\infty)$. *These values are also shown in Figure 1a,b for the four cases of* ϱ. *In all four cases of the function* ϱ, *we saw that all requirements of Theorem 2 were fulfilled. Therefore, this guaranteed that for all four cases of* ϱ, *the* ϱ*-Hilfer fractional snap dynamic system (38) admitted one solution on the unbounded domains* $J=[1.05,\infty)$.

Table 1. Suitable numerical values of $\breve{Q}$ and $\breve{L}$ for four cases of $\varrho_i(u)$.

	$\varrho_1(u)=2^u$		$\varrho_2(u)=u$		$\varrho_3(u)=\ln(u+0.1)$		$\varrho_4(u)=\sqrt{u}$	
u	$\leq\breve{Q}$	$\leq\breve{L}$	$\leq\breve{Q}$	$\leq\breve{L}$	$\leq\breve{Q}$	$\leq\breve{L}$	$\leq\breve{Q}$	$\leq\breve{L}$
1.05	0.000000	*NaN*	0.000000	*NaN*	0.000000	*NaN*	0.000000	*NaN*
1.55	0.004048	−0.004703	0.001550	−0.003246	0.000975	−0.002532	0.000363	−0.004722
2.05	0.003531	−0.000141	0.000950	−0.000164	0.000450	−0.000169	0.000192	−0.000287
2.55	0.001146	0.000002	0.000211	−0.000007	0.000079	−0.000010	0.000038	−0.000016
3.05	0.000186	0.000001	0.000023	0.000000	0.000007	0.000000	0.000004	−0.000001
3.55	0.000017	0.000000	0.000001	0.000000	0.000000	0.000000	0.000000	0.000000



(**a**) $\breve{Q}$ (**b**) $\breve{L}$

Figure 1. Graphical representation of $\breve{Q}$ and $\breve{L}$ for the four cases of $\varrho_i(u)$ when $u\in J$ in Example 1.

In the next example, we examine the correctness of the results obtained with Theorems 1 and 3. Of course, we shall consider the case in which $\varrho(u)=u$ (Caputo type) for three different orders α_1 and show the obtained results computationally and graphically.

Example 2. *We consider a* ϱ*-Hilfer fractional snap dynamic system (38) on the domains* $J=[1.05,\infty)$, $\varrho(u)=u$ *(Caputo type), where*

$$\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4)=\frac{|\cos(\pi u)|}{39(\sqrt{5}+|\cos(\pi u)|)}+\frac{1}{\exp(u)+\sqrt{3}}\left[\frac{4\sqrt[3]{u}}{3+39\sqrt{|u|}}\right.$$

$$\times\left(\frac{|\sin^{-1}(\Psi_1(u))|}{1.6+|\sin^{-1}(\Psi_1(u))|}\right)+\frac{u(\Psi_2(u))^2}{78(9+(\Psi_2(u))^2)}$$

$$+\frac{(\tan^{-1}(\Psi_3(u)))}{11(39+2u^2)}+\frac{\sqrt{5}u}{39}\left(\frac{\sin(\Psi_4(u))}{\sqrt{5}+\sin(\Psi_4(u))}\right)\Bigg], \tag{44}$$

for $u \in J$, $a = 1.05 \geq 0$, *and* $\alpha \in \{\frac{2}{11}, \frac{1}{2}, \frac{19}{20}\}$, *under the* ϱ*-Riemann–Liouville (R-L) fractional integral conditions* (40), *where* $\omega_1(u) = \frac{\sqrt{2}}{3}|u|$, $\omega_2(u) = \frac{1}{5}u$, $\omega_3(u) = \frac{1}{4}\sqrt[3]{u}$ *and* $\omega_4(u) = \frac{1}{\sqrt{3}}u^2$. *Clearly,*

$${}_k\alpha_1 \leq \gamma_1 = \begin{cases} 0.4436, & k=1,\ {}_1\alpha_1 = \frac{2}{11} \\ 0.6600, & k=2,\ {}_2\alpha_1 = \frac{1}{2}, \\ 0.9660, & k=3,\ {}_3\alpha_1 = \frac{19}{20}, \end{cases} \quad 1-\gamma_1 = \begin{cases} 0.5563, & {}_1\alpha_1 = \frac{2}{11}, \\ 0.3400, & {}_2\alpha_1 = \frac{1}{2}, \\ 0.0340, & {}_3\alpha_1 = \frac{19}{20}, \end{cases}$$

and for $i = 2, 3, 4$,

$$\alpha_i \leq \gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i = \begin{cases} \frac{963}{1000}, & i=2, \\ \frac{61}{100}, & i=3, \\ \frac{519}{1000}, & i=4, \end{cases} \quad 1-\gamma_i = \begin{cases} \frac{37}{1000}, & i=2, \\ \frac{39}{100}, & i=3, \\ \frac{481}{1000}, & i=4. \end{cases}$$

Now, we have

$$\begin{aligned}
&\left|\mathcal{G}(u,\Psi_1,\Psi_2,\Psi_3,\Psi_4)-\mathcal{G}(u,\Psi_1^*,\Psi_2^*,\Psi_3^*,\Psi_4^*)\right| \\
&\quad=\left|\frac{1}{\exp(u)+\sqrt{3}}\Bigg[\frac{4\sqrt[3]{u}|\sin^{-1}(\Psi_1(u))|}{(3+39\sqrt{|u|})(1.6+|\sin^{-1}(\Psi_1(u))|)}\right. \\
&\qquad+\frac{u(\Psi_2(u))^2}{78(9+(\Psi_2(u))^2)}+\frac{\tan^{-1}(\Psi_3(u))}{11(39+2u^2)} \\
&\qquad\left.+\frac{\sqrt{5}u}{39}\left(\frac{\sin(\Psi_4(u))}{\sqrt{5}+\sin(\Psi_4(u))}\right)\Bigg]\right. \\
&\qquad-\frac{1}{\exp(u)+\sqrt{3}}\left(\frac{4\sqrt[3]{u}|\sin^{-1}(\Psi_1^*(u))|}{(3+39\sqrt{|u|})(1.6+|\sin^{-1}(\Psi_1^*(u))|)}\right. \\
&\qquad+\frac{u(\Psi_2^*(u))^2}{78(9+(\Psi_2^*(u))^2)}+\frac{\tan^{-1}(\Psi_3^*(u))}{11(39+2u^2)} \\
&\qquad\left.\left.+\frac{\sqrt{5}u}{39}\left(\frac{\sin(\Psi_4^*(u))}{\sqrt{5}+(\sin(\Psi_4^*(u)))^2}\right)\right)\right| \\
&\quad\leq\frac{1}{\exp(u)+\sqrt{3}}\left(\left|\frac{4\sqrt[3]{u}}{3+39\sqrt{|u|}}\right|\left|\frac{|\sin^{-1}(\Psi_1(u))|}{1.6+|\sin^{-1}(\Psi_1(u))|}\right.\right. \\
&\qquad\left.-\frac{|\sin^{-1}(\Psi_1^*(u))|}{1.6+|\sin^{-1}(\Psi_1^*(u))|}\right| \\
&\qquad+\left|\frac{u}{78}\right|\left|\frac{(\Psi_2(u))^2}{9+(\Psi_2(u))^2}-\frac{(\Psi_2^*(u))^2}{9+(\Psi_2^*(u))^2}\right| \\
&\qquad+\left|\frac{1}{11(39+2u^2)}\right|\left|\tan^{-1}(\Psi_3(u))-\tan^{-1}(\Psi_3^*(u))\right| \\
&\qquad\left.+\left|\frac{\sqrt{5}u}{39}\right|\left|\frac{\sin(\Psi_4(u))}{\sqrt{5}+\sin(\Psi_4(u))}-\frac{\sin(\Psi_4^*(u))}{\sqrt{5}+\sin(\Psi_4^*(u))}\right|\right)
\end{aligned}$$

$$\leq \left| \frac{4\frac{1}{\exp(u)+\sqrt{3}}\left(\sqrt[3]{u}\right.}{3+39\sqrt{|u|}} \right| |\Psi_1(u)-\Psi_1^*(u)| + \left|\frac{u}{78}\right| |\Psi_2(u)-\Psi_2^*(u)| + \left| \frac{1}{11(39+2u^2)} \right| |\Psi_3(u)-\Psi_3^*(u)| + \left|\frac{\sqrt{5}u}{39}\right| |\Psi_4(u)-\Psi_4^*(u)| \Bigg).$$

Indeed, we can consider $\xi(u) = 78(\exp(u)+\sqrt{3})$ *and*

$$\sigma_1 = \frac{312|\sqrt[3]{u}|}{3+39\sqrt{|u|}}, \quad \sigma_2 = |u|, \quad \sigma_3 = \frac{78}{11(39+2u^2)}, \quad \sigma_4 = 2\sqrt{5}|u|.$$

Additionally, we can consider ω_i^*, *which was defined by* (40) *in the previous example. Now, we consider three cases:* $\alpha_1 \in \{\frac{2}{11}, \frac{1}{2}, \frac{19}{20}\}$. *By using Equations* (16)–(18), *we obtain*

$$\sup_{u\in J}\left\{ \frac{1}{\xi(u)} \mathfrak{I}_{a^+}^{\alpha;\varrho}[\sigma_1(u)+\sigma_2(u)+\sigma_3(u)+\sigma_4(u)] \right\} \simeq \left\{ \begin{array}{ll} 0.0207, & {}_1\alpha_1 = \frac{2}{11}, \\ 0.0204, & {}_2\alpha_1 = \frac{1}{2}, \\ 0.0186, & {}_3\alpha_1 = \frac{19}{20}, \end{array} \right\} \leq \breve{Q} < 1,$$

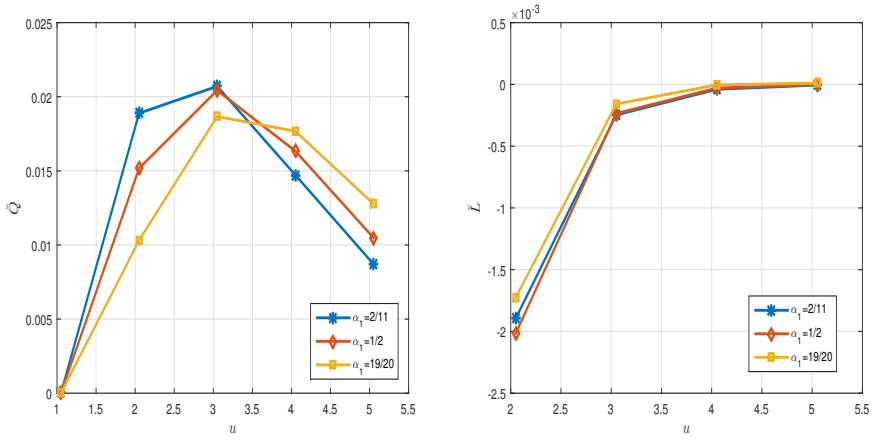
and

$$\sup_{u\in J}\Bigg\{ \frac{1}{\xi(u)} \Bigg(\frac{(\varrho(u)-\varrho(a))^{\eta_1-1}}{\Gamma(\eta_1)} \left(\frac{|\lambda_1|\omega_1^*(\varrho(\delta_1)-\varrho(a))^{\theta_1}}{\Gamma(\theta_1+1)} \right) + \frac{(\varrho(u)-\varrho(a))^{\eta_2-1}}{\Gamma(\eta_2)} \left(\frac{|\lambda_2|\omega_2^*(\varrho(\delta_2)-\varrho(a))^{\theta_2}}{\Gamma(\theta_2+1)} \right) + \frac{(\varrho(u)-\varrho(a))^{\eta_3-1}}{\Gamma(\eta_3)} \left(\frac{|\lambda_3|\omega_3^*(\varrho(\delta_3)-\varrho(a))^{\theta_3}}{\Gamma(\theta_3+1)} \right) + \frac{(\varrho(u)-\varrho(a))^{\eta_4-1}}{\Gamma(\eta_4)} \left(\frac{|\lambda_4|\omega_4^*(\varrho(\delta_4)-\varrho(a))^{\theta_4}}{\Gamma(\theta_4+1)} \right) + \mathfrak{I}_{a^+}^{\alpha;\varrho}\|\breve{\mathcal{G}}_{\Psi_1}(0)\| \Bigg) \Bigg\} \simeq \left\{ \begin{array}{ll} 0.000001, & {}_1\alpha_1 = \frac{2}{11}, \\ 0.000001, & {}_2\alpha_1 = \frac{1}{2}, \\ 0.000011, & {}_3\alpha_1 = \frac{19}{20}. \end{array} \right. \leq \breve{L} < \infty.$$

We can consider the value of $\breve{Q}$ *to belong to* $(0,1)$ *whenever* $u \in (1,\ 4.05)$, $(1,\ 4.05)$, *and* $(1,\ 3.55)$ *for* ${}_i\alpha_1 = \frac{2}{11}, \frac{1}{2}, \frac{19}{20}$, *respectively. Thus,* (HP_1) *and* (HP_2) *hold for the three cases of* α_1. *Table 2 shows these results. These results are also plotted in Figure 2. In all three cases in the order* q_i, *we see that all requirements of Theorems 1 and 4 are fulfilled. Therefore, this guarantees that for all three different cases, in terms of the order* α_1, *the solution of the* ϱ*-Hilfer fractional snap dynamic system* (38)–(40) *on the domains* $J = [1.05, \infty)$ *with* (44) *is* $\mathcal{U}.\mathcal{H}.\mathcal{R}$ *stable and, accordingly, is* $\mathcal{U}.\mathcal{H}$ *stable.*

Table 2. Suitable numerical values of $\check{Q}$ and $\check{L}$ for the three case of α_1.

	$_1\boldsymbol{\alpha_1} = \frac{2}{11}$		$_2\boldsymbol{\alpha_1} = \frac{1}{2}$		$_3\boldsymbol{\alpha_1} = \frac{19}{20}$	
u	$\leq \check{Q}$	$\leq \check{L}$	$\leq \check{Q}$	$\leq \check{L}$	$\leq \check{Q}$	$\leq \check{L}$
1.05	0.000000	NaN	0.000000	NaN	0.000000	NaN
2.05	0.018888	−0.001894	0.015172	−0.002016	0.010312	−0.001726
3.05	0.020704	−0.000248	0.020433	−0.000236	0.018668	−0.000159
4.05	0.014720	−0.000040	0.016335	−0.000030	0.017674	−0.000004
5.05	0.008707	−0.000005	0.010489	0.000000	0.012776	0.000011



(**a**) $\check{Q}$ (**b**) $\check{L}$

Figure 2. Graphical representation of $\check{Q}$ and $\check{L}$ whenever $\alpha_1 \in \{\frac{2}{11}, \frac{1}{2}, \frac{19}{20}\}$ for $u \in [1.05, 6]$ in Example 2.

6. Conclusions

In this paper, we defined a new fractional mathematical model of a dynamic system consisting of a ϱ-Hilfer fractional snap equation with the ϱ-R-L fractional integral conditions in the framework of the generalized sequential ϱ-operators. To confirm the existence criterion, we used the valid fixed point theorems. In the end, we designed examples; by assuming different cases for the function ϱ and order α_1, we obtained numerical results for these two suggested ϱ-Hilfer fractional snap dynamic systems. It is notable that, by considering the different types of $\varrho(u)$, one can discuss the rest of the qualitative properties of such an extended ϱ-Hilfer fractional snap dynamic system by regarding other generalized models in future projects.

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