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Article

L-Index in Joint Variables: Sum and Composition of an Entire Function with a Function With a Vanished Gradient

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Abstract: The composition $H(z) = f(\Phi(z))$ is studied, where f is an entire function of a single complex variable and Φ is an entire function of n complex variables with a vanished gradient. Conditions are presented by the function Φ providing boundedness of the **L**-index in joint variables for the function H, if the function f has bounded l-index for some positive continuous function l and $\mathbf{L}(z) = l(\Phi(z))(\max\{1, |\Phi'_{z_1}(z)|\}, \ldots, \max\{1, |\Phi'_{z_n}(z)|\})$, $z \in \mathbb{C}^n$. Such a constrained function **L** allows us to consider a function Φ with a nonempty zero set. The obtained results complement earlier published results with $\Phi(z) \neq 0$. Also, we study a more general composition $H(\mathbf{w}) = G(\mathbf{\Phi}(\mathbf{w}))$, where $G : \mathbb{C}^n \to \mathbb{C}$ is an entire function of the bounded **L**-index in joint variables, $\mathbf{\Phi} : \mathbb{C}^m \to \mathbb{C}^n$ is a vector-valued entire function, and $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a continuous function. If the **L**-index of the function G equals zero, then we construct a function $\widetilde{\mathbf{L}} : \mathbb{C}^m \to \mathbb{R}^m_+$ such that the function H has bounded $\widetilde{\mathbf{L}}$-index in the joint variables $z_1, \ldots, z_n$. The other group of our results concern a sum of entire functions in several variables. As a general case, a sum of functions with bounded index is not of bounded index. The same is also valid for the multidimensional case. We found simple conditions proving that $f_1(z_1) + f_2(z_2)$ belongs to the class of functions having bounded index in joint variables z_1, z_2. We formulate some open problems based on the deduced results and on the usage of fractional differentiation operators in the theory of functions with bounded index.

Keywords: entire function; several complex variables; composition; bounded index in joint variables; sum of functions; gradient

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1. Introduction

We examine some compositions of entire functions with usage of the notion of **L**-index in joint variables. Our investigation is an extension of the paper in [1] to the case where the first-order partial derivatives of the inner function can vanish, i.e., at least one of them (or each of them) has nonempty zero set.

Let us recall some standard notations from [1,2]. Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be n-dimensional real and complex vector spaces, respectively, $n \in \mathbb{N}$. Denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$, $\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{-th place}}, 0, \ldots, 0)$. For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, let us write $\|K\| = k_1 + \cdots + k_n$, $K! = k_1! \cdot \ldots \cdot k_n!$. For $A = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions $A \pm B = (a_1 \pm b_1, \ldots, a_n \pm b_n)$, $AB = (a_1 b_1, \cdots, a_n b_n)$, $A/B = (a_1/b_1, \ldots, a_n/b_n)$, and if $A, B \in \mathbb{R}^n$, then $A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\mathbf{max}\{A; B\} = (\max\{a_1; b_1\}, \max\{a_2; b_2\}, \ldots, \max\{a_n; b_n\})$, and the notation $A < B$ means that $a_j < b_j$ for all $j \in \{1, \dots, n\}$. Similarly, the relation $A \leq B$ is defined.

We denote the K-th order partial derivative of the entire function $F(z) = F(z_1, \dots, z_n)$ by

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1+\dots+k_n} F}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}, \text{ where } K = (k_1, \dots, k_n) \in \mathbb{Z}_+^n.$$

We suppose $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$ and every component $l_j(z)$ is a positive continuous function.

An entire function F of n complex variables is called *a function of bounded* $\mathbf{L}$*-index in joint variables* [3] if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \dots, j_n) \in \mathbb{Z}_+^n$, one has

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leq \max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : K \in \mathbb{Z}_+^n, \|K\| \leq m \right\}. \quad (1)$$

The least integer m for which inequality (1) holds is called the $\mathbf{L}$-index in joint variables of the function F and is denoted by $N(F, \mathbf{L})$. If $l_j(z_j) \equiv 1$, $j \in \{1, 2, \dots, n\}$, then the entire function F is called a function of bounded index in joint variables or function of bounded index [4–7]. If $n = 1$ then we obtain a notion of the l-index for entire functions with a continuous function $l : \mathbb{C} \to \mathbb{R}_+$ It first appeared in [8] as an extension of the index of an entire function [9] to a wider class of entire functions whose growth exceed the growth of exponential-type functions. A. Kuzyk and M. Sheremeta [8] put the j-th power of the function l in the denominator of the Taylor coefficient from (1). This allowed them to consider the concept of the index boundedness for any entire function such that its zero multiplicities are uniformly bounded. The fact shows a big role of the denominator of the fraction in the theory.

Many papers are devoted to a composition of two holomorphic functions belonging to different classes and various definitions of the index. Nowadays, the most exhaustively investigated cases are case of the bounded l-index for entire functions of single variable [8,10,11] and analytic in a disc functions [12,13] and the case of the bounded L-index in a direction for multivariate entire functions [14] and analytic functions in a unit ball [15].

The case of the bounded index in joint variables is more difficult. This complexity is due to the consideration of all possible partial derivatives for the composition instead of all directional derivatives for a given direction as in the previous case. There are two papers on this case. In the first paper, the authors deal with the composition of a single-variate entire function with a finite l-index and an n-variate entire function [1]. The second paper is devoted to the composition of an entire function of bounded l-index and an analytic function in the unit ball [2]. But the authors studied composition in the case that every first-order partial derivative of the inner function does not equal zero. Recently, for the finite directional L-index, an approach was developed [14] to study the composition without the condition that the zero set of the first-order directional derivative of the inner function is empty.

Here, we will try to implement the mentioned approach for the notion of bounded $\mathbf{L}$-index in joint variables. We obtained Theorems 4 and 5. The first theorem can be applied to consider a nonlinear partial differential equation. For example, we have a composite PDE, make changes to the variables, and transform the equation to a simpler form. If the simpler equation has analytic solutions of bounded index in some sense (see [16,17]), then we can apply Theorem 4 to learn the properties of the entire solutions of the composite PDE. Our second result (Theorem 5) can become a base for the introduction of the concept of index for monogenic functions in a finite-dimensional commutative algebra [18,19], regular functions of a quaternionic variable [20,21], or slice regular functions of a quaternionic variable [22,23].

2. Materials and Methods

To prove the main theorem, we need an auxiliary proposition. For $R \in \mathbb{R}^n_+$, $j \in \{1,\dots,n\}$ and $\mathbf{L}(z) = (l_1(z),\dots,l_n(z))$, we define

$$D^n\Big[z^0, R/\mathbf{L}(z^0)\Big] = \{z \in \mathbb{C}^n : |z_j - z_j^0| \le r_j/l_j(z^0), j \in \{1,\dots,n\}\}$$
$$\lambda_{1,j}(z_0,R) = \inf\Big\{l_j(z)/l_j(z^0) \colon z \in D^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\},\ \lambda_{1,j}(R) = \inf_{z^0\in\mathbb{C}^n} \lambda_{1,j}(z_0,R),$$
$$\lambda_{2,j}(z_0,R) = \sup\Big\{l_j(z)/l_j(z^0) \colon z \in D^n\Big[z^0, R/\mathbf{L}(z^0)\Big]\Big\},\quad \lambda_{2,j}(R) = \sup_{z^0\in\mathbb{C}^n} \lambda_{2,j}(z_0,R),$$

By Q^n we denote a class of functions $\mathbf{L}(z)$ for which every $R \in \mathbb{R}^n_+$ satisfies the condition $0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < +\infty$. If $n = 1$, then $Q \equiv Q^1$.

Theorem 1 ([24]). *Let $\mathbf{L} \in Q^n$. An entire function F has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $C_1 \in \mathbb{R}_+$ such that for each $z \in \mathbb{C}^n$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \,:\, \|J\| = p+1\right\} \le C_1 \cdot \max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} \,:\, \|K\| \le p\right\}. \tag{2}$$

Theorem 1 was fisrtly deduced by W.K. Hayman [25] for single-variate entire functions having bounded index ($n = 1$, $\mathbf{L}(z) \equiv 1$). M.M. Sheremeta [13] proved it for analytic functions of one variable with finite bounded l-index. Here we will use the result for multivariate entire functions with finite $\mathbf{L}$-index in joint variables. Note that Hayman's Theorem is very convenient for investigating the properties of entire solutions of differential equations [13,26].

Let us denote

$$\nabla\Phi(z) = \Big(\frac{\partial\Phi(z)}{\partial z_1},\dots,\frac{\partial\Phi(z)}{\partial z_n}\Big),\quad |\nabla|\Phi(z) = \Big(\Big|\frac{\partial\Phi(z)}{\partial z_1}\Big|,\dots,\Big|\frac{\partial\Phi(z)}{\partial z_n}\Big|\Big). \tag{3}$$

We recall known results on the composition of entire functions having bounded $\mathbf{L}$-index in joint variables. Similar results are also known for a unit ball (see [2]).

Theorem 2 ([1]). *Let $l \in Q$, $l(w) \ge 1$, $f : \mathbb{C} \to \mathbb{C}$ and $\Phi : \mathbb{C}^n \to \mathbb{C}$ be entire functions such that all partial derivatives of the first order for the function Φ are nonvanishing. Suppose $\mathbf{L} \in Q^n$, where $\mathbf{L}(z) = l(\Phi(z))|\nabla|\Phi(z)$.*

In addition, for the function Φ and for p chosen in (2) there exists $C > 0$ such that for all $z \in \mathbb{C}^n$ and for all $J = (j_1,\dots,j_n) \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}$, $\|J\| \le p+1$, one has $|\Phi^{(J)}(z)| \le C|\nabla|\Phi(z)^J$. The entire function $F(z) = f(\Phi(z))$ has bounded $\mathbf{L}$-index in joint variables if and only if the entire function f has bounded l-index.

Theorem 3 ([1]). *Let $\mathbf{L} \in Q^2$, $G : \mathbb{C}^2 \to \mathbb{C}$ be an entire function of bounded $\mathbf{L}$-index in joint variables, $\Phi_1 : \mathbb{C}^n \to \mathbb{C}$ and $\Phi_2 : \mathbb{C}^m \to \mathbb{C}$ be entire functions such that $\frac{\partial\Phi_1(z)}{\partial z_k} \ne 0$, $\frac{\partial\Phi_2(w)}{\partial w_l} \ne 0$, $k \in \{1,\dots,n\}$, $l \in \{1,\dots,m\}$. If for all $z \in \mathbb{C}^n$, $w \in \mathbb{C}^m$, $J \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}$, $I \in \mathbb{Z}^m_+ \setminus \{\mathbf{0}\}$, $\|J\| \le N(G,\mathbf{L}) + 1$, $\|I\| \le N(G,\mathbf{L}) + 1$, one has*

$$|\Phi_1^{(J)}(z)| \le C|\nabla|\Phi_1(z)^J,\quad |\Phi_2^{(I)}(w)| \le C|\nabla|\Phi_2(w)^I,\quad C \equiv \text{const} > 0,$$

and $\widetilde{\mathbf{L}} \in Q^{n+m}$, then $H(z,w) = G(\Phi_1(z),\Phi_2(w))$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables in the space $\mathbb{C}^{n+m}$, where $\widetilde{\mathbf{L}}(z,w) = (l_1(\Phi_1(z))|\nabla|\Phi_1(z), l_2(\Phi_2(w))|\nabla|\Phi_2(w))$.

There exists a simple example of functions which do not satisfy the conditions in Theorems 2 and 3.

Example 1. *We choose $f(w) = e^w$ and $\Phi(z_1, z_2) = \sin(z_1)\cos(z_2)$. The index of the function f equals 0 because $f^{(p)}(w) = e^w$. Let us consider the composite function $F(z_1, z_2) = f(\Phi(z_1, z_2)) = e^{\sin(z_1)\cos(z_2)}$. Calculate the gradient of the function Φ :*

$$\nabla\Phi(z_1, z_2) = (\cos(z_1)\cos(z_2), -\sin(z_1)\sin(z_2)).$$

One should observe that the zero sets of the functions $\cos(z_1)\cos(z_2)$ and $-\sin(z_1)\sin(z_2)$ are not empty. Therefore, Theorem 2 is not applicable to this composition. After the proof of Theorem 4, we will return to this example.

3. Results on Composition of Entire Functions

Removing the condition $\frac{\partial\Phi(z)}{\partial z_k} \neq 0$ in Theorem 2 and slightly increasing the function $\mathbf{L}$, we deduce a new result.

Theorem 4. *Let $l \in Q$ such that $l(z) \geq 1$ for all $z \in \mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ be an entire function of bounded l-index, and $\Phi : \mathbb{C}^n \to \mathbb{C}$ be an entire function, $n \geq 2$, such that*

$$\mathbf{L} \in Q^n, \quad \mathbf{L}(w) = \mathbf{max}\{\mathbf{1}, |\nabla|\Phi(w)\}l(\Phi(w)). \tag{4}$$

If there exists $C_2 \geq 1$ such that for all $w \in \mathbb{C}^n$ and for all $J \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}$ with $\|J\| \leq N(g, l) + 1$, one has

$$|\Phi^{(J)}(w)| \leq C_2(l(\Phi(w)))^{1/(N(g,l)+1)}(|\nabla|\Phi(w))^J, \tag{5}$$

then the entire function $H(w) = g(\Phi(w))\colon \mathbb{C}^n \to \mathbb{C}$ has bounded $\mathbf{L}$-index in joint variables.

Proof. The following formula was proved in [1]:

$$H^{(K)}(w) = g^{(\|K\|)}(\Phi(w))(\nabla\Phi(w))^K + \sum_{j=1}^{\|K\|-1} g^{(j)}(\Phi(w))Q_{j,K}(w), \tag{6}$$

where

$$Q_{j,K}(w) = \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+K p_K=K \\ 0\leq p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n}\leq j-1}} c_{j,K,p_{\mathbf{1}_1},\ldots,p_K}(\Phi^{(\mathbf{1}_1)}(w))^{p_{\mathbf{1}_1}}\ldots(\Phi^{(\mathbf{1}_n)}(w))^{p_{\mathbf{1}_n}}\ldots(\Phi^{(K)}(w))^{p_K},$$

and $c_{j,K,p_{\mathbf{1}_1},\ldots,p_K} \in \mathbb{Z}_+$ are some coefficients, $K \in \mathbb{Z}^n_+$.

Also, it was deduced [1] that

$$g^{(k)}(\Phi(w)) = \frac{H^{(k\mathbf{1}_i)}(w)}{(\Phi^{(\mathbf{1}_i)}(w))^k} + \frac{1}{(\Phi^{(\mathbf{1}_i)}w))^{2k}}\sum_{j=1}^{k-1} H^{(j\mathbf{1}_i)}(w)(\Phi^{(\mathbf{1}_i)}(w))^j\widetilde{Q}_{j,k}(w), \tag{7}$$

where

$$\widetilde{Q}_{j,k}(w) = \sum_{m_1+\ldots+km_k=2(k-j)} b_{j,k,m_1,\ldots,m_k}(\Phi^{(\mathbf{1}_i)}(w))^{m_1}\ldots(\Phi^{(k\mathbf{1}_i)}(w))^{m_k}, \tag{8}$$

and $b_{j,k,m_1,\ldots,m_k} \in \mathbb{Z}$ are some coefficients, $i \in \{1, \ldots, n\}$, $k \in \mathbb{Z}_+$.

We suppose that the hypothesis of the theorem is satisfied. It means that the entire function $g : \mathbb{C} \to \mathbb{C}$ is of bounded l-index, and the entire function $\Phi : \mathbb{C}^n \to \mathbb{C}$ obeys (5). Denote $\widetilde{\mathbf{L}}(w) = |\nabla|\Phi(w)l(\Phi(w))$. Replacing K in (6) by J and dividing it by $(\widetilde{\mathbf{L}}(w))^J$, for any $J \in \mathbb{Z}^m_+ \setminus \{\mathbf{0}\}$ we have

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq \frac{|g^{(\|J\|)}(\Phi(w))|}{(\widetilde{\mathbf{L}}(w))^J} \cdot |\nabla|\Phi(w))^J + \sum_{k=1}^{\|J\|-1} \frac{|g^{(k)}(\Phi(w))|}{(\widetilde{\mathbf{L}}(w))^J} \cdot |Q_{k,J}(w)|.$$

Substituting $\widetilde{\mathbf{L}}(w) = |\nabla|\Phi(w) l(\Phi(w))$ in the last estimate, we deduce

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J}$$
$$\leq \frac{|g^{(\|J\|)}(\Phi(w))|}{(|\nabla|\Phi(w))^J (l(\Phi(w)))^{\|J\|}} \cdot |\nabla|\Phi(w))^J + \sum_{k=1}^{\|J\|-1} \frac{|g^{(k)}(\Phi(w))|}{(|\nabla|\Phi(w))^J (l(\Phi(w)))^{\|J\|}} \cdot |Q_{k,J}(w)|$$
$$\leq \frac{|g^{(\|J\|)}(\Phi(w))|}{(l(\Phi(w)))^{\|J\|}} + \sum_{k=1}^{\|J\|-1} \frac{|g^{(k)}(\Phi(w))|}{(|\nabla|\Phi(w))^J (l(\Phi(w)))^{\|J\|}} \cdot |Q_{k,J}(w)|. \tag{9}$$

By Theorem 1, inequality (2) is valid for $F = g$, $p = N(g,l):$

$$\frac{|g^{(N(g,l)+1)}(z)|}{(l(z))^{N(g,l)+1}} \leq C_1 \cdot \max\left\{ \frac{|g^{(k)}(z)|}{(l(z))^k} : 0 \leq k \leq N(g,l) \right\}.$$

Applying this inequality with $z = \Phi(w)$ to (9), we obtain for $\|J\| = N(g,l)+1:$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq \max\left\{ \frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k} :\ 0 \leq k \leq N(g,l) \right\}$$
$$\times \max_{\|J\|=N(g,l)+1} \left(C_1 + \sum_{k=1}^{\|J\|-1} \frac{|Q_{k,J}(w)|}{(|\nabla|\Phi(w))^J (l(\Phi(w)))^{\|J\|-k}} \right). \tag{10}$$

In view of (5), it is possible to deduce the upper estimate of $|Q_{k,J}(w)|$

$$|Q_{k,J}(w)| \leq \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J = J \\ 0 \leq p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n} \leq k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}| |\Phi^{(\mathbf{1}_1)}(w)|^{p_{\mathbf{1}_1}} \ldots |\Phi^{(\mathbf{1}_n)}(w)|^{p_{\mathbf{1}_n}} \ldots |\Phi^{(J)}(w)|^{p_J}$$
$$\leq \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J = J \\ 0 \leq p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n} \leq k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}| C_2^{\|J\|} (l(\Phi(w)))^{\|J\|/(N(g,l)+1)} (|\nabla|\Phi(w))^J$$
$$\leq \hat{c}_{k,J} (l(\Phi(w)))^{\|J\|/(N(g,l)+1)} (|\nabla|\Phi(w))^J, \tag{11}$$

where $\hat{c}_{k,J} = C_2^{\|J\|} \sum_{\substack{\mathbf{1}_1 p_{\mathbf{1}_1}+\ldots+\mathbf{1}_n p_{\mathbf{1}_n}+\ldots+J p_J = J \\ 0 \leq p_{\mathbf{1}_1}+\ldots+p_{\mathbf{1}_n} \leq k-1}} |c_{k,J,p_{\mathbf{1}_1},\ldots,p_J}|$. We substitute estimate (11) in (10) and use $\|J\| = N(g,l)+1$, that is, $(l(\Phi(w)))^{\|J\|/(N(g,l)+1)} = l(\Phi(w))$. Therefore, the following inequality is valid:

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq \max\left\{ \frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k} :\ 0 \leq k \leq N(g,l) \right\}$$
$$\times \max_{\|J\|=N(g,l)+1} \left(C_1 + \sum_{k=1}^{\|J\|-1} \frac{\hat{c}_{k,J} (l(\Phi(w)))^{\|J\|/(N(g,l)+1)} (|\nabla|\Phi(w))^J}{(|\nabla|\Phi(w))^J (l(\Phi(w)))^{\|J\|-k}} \right)$$
$$= \max\left\{ \frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k} :\ 0 \leq k \leq N(g,l) \right\} \max_{\|J\|=N(g,l)+1} \left(C_1 + \sum_{k=1}^{\|J\|-1} \frac{\hat{c}_{k,J}}{(l(\Phi(w)))^{\|J\|-1-k}} \right).$$

Since $l(\Phi(w)) \geq 1$, one has $(l(\Phi(w)))^{\|J\|-1-k} \geq 1$ for $k \leq \|J\|-1$. Thus, for $\|J\| = N(g,l)+1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq C_3 \max\left\{ \frac{|g^{(k)}(\Phi(w))|}{(l(\Phi(w)))^k} :\ 0 \leq k \leq N(g,l) \right\}, \tag{12}$$

where $C_3 = \max_{\|J\|=N(g,l)+1}\left(C_1 + \sum_{k=1}^{N(g,l)} \hat{c}_{k,J}\right)$. Dividing equality (7) by $l^k(\Phi(w))$ and estimating by the modulus, we deduce for each $i \in \{1,\dots,n\}$

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))}$$
$$\leq \frac{|H^{(k\mathbf{1}_i)}(w)|}{l^k(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^k} + \frac{1}{|\Phi^{(\mathbf{1}_i)}(w)|^{2k} l^k(\Phi(w))}\sum_{j=1}^{k-1} |H^{(j\mathbf{1}_i)}(w)||\Phi^{(\mathbf{1}_i)}(w)|^j |\widetilde{Q}_{j,k}(w)|$$
$$= \frac{|H^{(k\mathbf{1}_i)}(w)|}{l^k(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^k} + \sum_{j=1}^{k-1} \frac{\widetilde{Q}_{j,k}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^{2k-2j} l^{k-j}(\Phi(w))} \frac{|H^{(j\mathbf{1}_i)}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^j l^j(\Phi(w))}.$$

Introducing the maximum of the fraction $\frac{|H^{(j\mathbf{1}_i)}(w)|}{|\Phi^{(\mathbf{1}_i)}(w)|^j l^j(\Phi(w))}$ over $j \in \{1,\dots,k\}$, we can increase the previous estimate

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \leq \max_{1\leq j\leq k}\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j}\right\}\left(1 + \sum_{j=1}^{k-1} \frac{|\widetilde{Q}_{j,k}(w)|}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\right)$$

Substituting the expression from (8) instead of $\widetilde{Q}_{j,k}(w)$, we obtain

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \leq \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \leq j \leq k\right\}$$
$$\times\left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}| \frac{|\Phi^{(\mathbf{1}_i)}(w)|^{m_1}\ldots|\Phi^{(k\mathbf{1}_i)}(w)|^{m_k}}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\right). \quad (13)$$

Estimating (5) and $l(z) \geq 1$ gives us

$$|\Phi^{(s\mathbf{1}_i)}(w)| \leq C_2 l^{s/2}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^s,$$

because $s/2 \geq 1/(N(g,l)+1)$ for $s \in \{1,2,\dots,N(g,l)+1\}$. Substituting the right-hand side of this inequality in (13), we deduce

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \leq \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \leq j \leq k\right\}\left(1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\right.$$
$$\left.|b_{j,k,m_1,\ldots,m_k}|(C_2)^{m_1+m_2+\ldots+m_k} \frac{(l(\Phi(w)))^{(m_1+2m_2+\ldots+km_k)/2}|\Phi^{(\mathbf{1}_i)}(w)|^{m_1+2m_2+\ldots+km_k}}{l^{k-j}(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}}\right).$$

Since $m_1 + 2m_2 + \ldots + km_k = 2(k-j)$, we obtain $(l(\Phi(w)))^{k-j}|\Phi^{(\mathbf{1}_i)}(w)|^{2(k-j)}$ in the nominator under the sum. The expression matches with the denominator and reduces with it. Thus, it yields

$$\frac{|g^{(k)}(\Phi(w))|}{l^k(\Phi(w))} \leq C_4 \max\left\{\frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 1 \leq j \leq k\right\}, \quad (14)$$

where

$$C_4 = 1 + \sum_{j=1}^{k-1} \sum_{m_1+2m_2+\ldots+km_k=2(k-j)} |b_{j,k,m_1,\ldots,m_k}|(C_2)^{m_1+m_2+\ldots+m_k}.$$

Then, from inequality (12) and (14), we obtain for each $i \in \{1, \ldots, n\}$ and $\|J\| = N(g, l) + 1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq C_5 \max\left\{ \frac{|H^{(j\mathbf{1}_i)}(w)|}{l^j(\Phi(w))|\Phi^{(\mathbf{1}_i)}(w)|^j} : 0 \leq j \leq N(g, l) \right\}, \tag{15}$$

where $C_5 = C_3 C_4$. Estimate (15) is established by the assumption that every component of the gradient $\nabla\Phi$ does not vanish, i.e., $\Phi^{(\mathbf{1}_i)}(w) \neq 0$. Our proof is significant for equality (7), providing an estimate of the k-th order derivative of the function g by smaller-order partial derivatives of the function H in the variable w_i. Similarly to [1], by the method of mathematical induction an analog of (7) can be proved for the mixed partial derivative $J \in \mathbb{Z}_+^n$:

$$g^{(\|J\|)}(w) = \frac{H^{(J)}(w)}{(\nabla\Phi(w))^J} + \frac{1}{(\nabla\Phi(w))^{2J}} \sum_{\substack{0 < \|K\| \leq \|J\| - 1, \\ K \leq J}} H^{(K)}(w)(\nabla\Phi(w))^K Q^*(w; J, K),$$

where $Q^*(w; J, K)$ is constructed by analogy to $Q_{j,K}(w)$, $\widetilde{Q}_{j,k}(w)$. Then, repeating considerations from (9) to (15) as in [1], we deduce for $\|J\| = N(g, l) + 1$

$$\frac{|H^{(J)}(w)|}{(\widetilde{\mathbf{L}}(w))^J} \leq C \max\left\{ \frac{|H^{(K)}(w)|}{(l(\Phi(w)))^{\|K\|}|\nabla\Phi(w)|^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\}, \tag{16}$$

where $C > 1$ is a constant.

We should like to point out that $\mathbf{L}(w) = l(\Phi(w))\mathbf{max}\{\mathbf{1}, |\nabla|\Phi(w)\}$. We introduce the function $\mathbf{L}$ to inequality (16) for $\|J\| = N(g, l) + 1$ in the following form:

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \cdot \frac{(\mathbf{L}(w))^J}{(\widetilde{\mathbf{L}}(w))^J} \leq C \max\left\{ \frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} \cdot \frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\}.$$

Inverting the fraction $\frac{(\mathbf{L}(w))^J}{(\widetilde{\mathbf{L}}(w))^J}$, one has

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \leq C \frac{(\widetilde{\mathbf{L}}(w))^J}{(\mathbf{L}(w))^J} \max\left\{ \frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} \cdot \frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\}.$$

Applying $\max\limits_{a,b \in \mathcal{A} \subset \mathbb{R}_+} \{a \cdot b\} \leq \max\limits_{a \in \mathcal{A} \subset \mathbb{R}_+} \{a\} \cdot \max\limits_{b \in \mathcal{A} \subset \mathbb{R}_+} \{b\}$ with a finite set $\mathcal{A}$ to the right-hand side of the inequality, we establish

$$\begin{aligned} \frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \leq C \frac{(\widetilde{\mathbf{L}}(w))^J}{(\mathbf{L}(w))^J} &\max\left\{ \frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\} \\ &\times \max\left\{ \frac{(\mathbf{L}(w))^K}{(\widetilde{\mathbf{L}}(w))^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\}. \end{aligned}$$

Since $\max\{a : \ a \in \mathcal{A} \subset \mathbb{R}_+\} = \frac{1}{\min\{1/a: \ a \in \mathcal{A} \subset \mathbb{R}_+\}}$, the last estimate can be rewritten as

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \leq \frac{C(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^J \max\left\{ \frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} : 0 < \|K\| \leq N(g, l), K \leq J \right\}}{\min\{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^K : 0 < \|K\| \leq N(g, l), K \leq J\}}. \tag{17}$$

Let $T_0 = T(w) = \widetilde{\mathbf{L}}(w)/\mathbf{L}(w) = \frac{|\nabla|\Phi(w)}{\mathbf{max}\{\mathbf{1}, |\nabla|\Phi(w)\}} \in \mathbb{R}_+^n$ and $K_0 \leq J$, $0 < \|K_0\| \leq N(g, l)$ ($K_0 \in \mathbb{Z}_+^n$) be such that $(T_0)^{K_0} = \min\{T_0^K : \ 0 < \|K\| \leq N(g, l), K \leq J\}$. One should observe that $T_0 \in (0,1]^n$ and $\|J - K_0\| \geq N(g,l) + 1 - N(g,l) = 1$, and $J - K_0 \geq \mathbf{1}_s$ for some $s \in \{1,\ldots,n\}$. Hence,

$$\frac{T_0^J}{T_0^{K_0}} = T_0^{J-K_0} \leq T_0^{\mathbf{1}_s} \leq 1.$$

Therefore,

$$\frac{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^J}{\min\{(\widetilde{\mathbf{L}}(w)/\mathbf{L}(w))^K:\ 0<\|K\|\leq N(g,l), K\leq J\}} = T_0^{J-K_0} \leq T_0^{\mathbf{1}_s} \leq 1.$$

Thus, from inequality (17), we obtain

$$\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} \leq C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} : 0<\|K\|\leq N(g,l), K\leq J\right\}. \tag{18}$$

Let $J^* \in \mathbb{Z}._+$ be such that

$$\max_{\|J\|=N(g,l)+1} \frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} = \frac{|H^{(J^*)}(w)|}{(\mathbf{L}(w))^{J^*}}$$

and $\|J^*\| = N(g,l)+1$. Then, we deduce from (18)

$$\begin{aligned}\max\left\{\frac{|H^{(J)}(w)|}{(\mathbf{L}(w))^J} : \|J\| = N(g,l)+1\right\} &= \frac{|H^{(J^*)}(w)|}{(\mathbf{L}(w))^{J^*}} \\ \leq C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} : 0<\|K\|\leq N(g,l), K\leq J^*\right\} \\ \leq C\max\left\{\frac{|H^{(K)}(w)|}{(\mathbf{L}(w))^K} : 0<\|K\|\leq N(g,l)\right\}\end{aligned} \tag{19}$$

for all w such that $\Phi^{(\mathbf{1}_i)}(w) \neq 0$.

If $\Phi^{(\mathbf{1}_i)}(w) = 0$ for some i, then for any $J \in \mathbb{Z}^n_+$ with $\|J\| \leq N(g,l)+1$ and $J_i \neq 0$ inequality (5) implies $\Phi^{(J)}(w) = 0$. In view of (6), it means $H^{(J)}(w) = 0$. Thus, inequality (19) also holds for the points w belonging to zero for at least one component of the gradient $\nabla\Phi$.

Therefore, by Theorem 1, we conclude that the function H belongs to the class of functions with finite $\mathbf{L}$-index in joint variables. □

Example 2. *Returning to Example 1, we can apply Theorem 4 to that composition. The function* $\Phi(z_1,z_2) = \sin(z_1)\cos(z_2)$ *satisfies* (5) *for* $J = (0,1)$ *and* $J = (1,0)$. *Thus, the function* $e^{\sin(z_1)\cos(z_2)}$ *has bounded* **L**-*index in joint variables, where the function* **L** *is constructed by* (4): $\mathbf{L}(z_1,z_2) = (\max\{1,|\cos(z_1)\cos(z_2)|\}, \max\{1,|\sin(z_1)\sin(z_2)|\}$.

Now we consider a more general composition if the entire function g is multivariate.

For $\mathbf{L} = (l_1,\ldots,l_n) : \mathbb{C}^n \to \mathbb{R}^n_+$ and $\mathbf{\Phi} = (\phi_1,\ldots,\phi_n)$ with $\phi_s : \mathbb{C}^{m_s} \to \mathbb{C}$ for $s \in \{1,\ldots,n\}$, we denote

$$\mathbf{\Phi}(\mathbf{w}) = (\phi_1(w_{1,1},\ldots,w_{1,m_1})), \phi_2(w_{2,1},\ldots,w_{2,m_2}),\ldots,\phi_n(w_{n,1},\ldots,w_{n,m_n}))),$$

$$\mathbf{L}(\mathbf{\Phi}(\mathbf{w})) = (\underbrace{l_1(\mathbf{\Phi}(\mathbf{w}))),\ldots,l_1(\mathbf{\Phi}(\mathbf{w}))}_{m_1 \text{ times}},\ldots,\underbrace{l_n(\mathbf{\Phi}(\mathbf{w})),\ldots,l_n(\mathbf{\Phi}(\mathbf{w}))}_{m_n \text{ times}}),$$

$$\nabla\mathbf{\Phi}(\mathbf{w}) = (\nabla\phi_1(\mathbf{w}_1), \nabla\phi_2(\mathbf{w}_2),\ldots,\nabla\phi_n(\mathbf{w}_n)),$$

$$|\nabla|\mathbf{\Phi}(z) = (|\nabla|\phi_1(\mathbf{w}_1), |\nabla|\phi_2(\mathbf{w}_2),\ldots,|\nabla|\phi_n(\mathbf{w}_n)),$$

where $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$, $\mathbf{w}_s = (w_{s,1}, w_{s,2}, \dots, w_{s,m_s}) \in \mathbb{C}^{m_s}$, $\nabla\phi_s(\mathbf{w}_s)$, and $|\nabla|\phi_s(\mathbf{w}_s)$ are defined in (3), $s \in \{1, \dots, n\}$. Similarly, $\mathbf{J} = (\mathbf{j}_1, \dots, \mathbf{j}_n)$ and $\mathbf{j}_s = (j_{s,1}, j_{s,2}, \dots, j_{s,m_s}) \in \mathbb{Z}_+^{m_s}$.

Removing the conditions $\frac{\partial\Phi_1(z)}{\partial z_k} \neq 0$, $\frac{\partial\Phi_2(w)}{\partial w_l} \neq 0$ in Theorem 3 we can also deduce a new result with the greater function $\widetilde{\mathbf{L}}$. But the result is obtained only in the case that the $\mathbf{L}$-index of the outer function for the composition equals zero (see below Remark 1 for the explanation).

Proposition 1. *Let $\mathbf{L} \in Q^n$, $G : \mathbb{C}^n \to \mathbb{C}$ be an entire function of bounded $\mathbf{L}$-index in joint variables with $N(G, \mathbf{L}) = 0$, $\phi_s : \mathbb{C}^{m_s} \to \mathbb{C}$ be entire functions, $l_s(z) \geq 1$ for each $s \in \{1, \dots, n\}$ and for all $z \in \mathbb{C}^n$. If $\widetilde{\mathbf{L}} \in Q^m$ with $\widetilde{\mathbf{L}}(\mathbf{w}) = \mathbf{max}\{\mathbf{1}, |\nabla|\mathbf{\Phi}(\mathbf{w})\}\mathbf{L}(\mathbf{\Phi}(\mathbf{w}))$ and $m = \sum_{j=1}^n m_j$, then the function $H(\mathbf{w}) = G(\mathbf{\Phi}(\mathbf{w}))$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables.*

Proof. Let $N(G, \mathbf{L}) = 0$. Then, by Hayman's Theorem for the function G from (2) with $p = 0$, one has for every $z \in \mathbb{C}^n$ and $s \in \{1, \dots, n\}$,

$$\frac{|G^{(\mathbf{1}_s)}(z)|}{l_s(z)} \leq C|G(z)|. \tag{20}$$

Denote $\mathbf{1}_{s,j} = (0, \dots, 0, \underbrace{1}_{j+m_1+m_2+\dots+m_{s-1}\text{-th place}}, 0 \dots, 0)$ with $s \in \{1, \dots, n\}$, $j \in \{1, \dots, m_s\}$. Then, for the function $H(\mathbf{w}) = G(\mathbf{\Phi}(\mathbf{w}))$, we evaluate the first-order partial derivative: $H^{(\mathbf{1}_{s,j})}(\mathbf{w}) = G^{(\mathbf{1}_s)}(\mathbf{\Phi}(\mathbf{w}))\phi_s^{(\mathbf{1}_j)}(\mathbf{w}_s)$. In view of (20), for $z = \mathbf{\Phi}(\mathbf{w})$ we obtain

$$\frac{|H^{(\mathbf{1}_{s,j})}(\mathbf{w})|}{(\widetilde{\mathbf{L}}(\mathbf{w}))^{\mathbf{1}_{s,j}}} = \frac{|G^{(\mathbf{1}_s)}(\mathbf{\Phi}(\mathbf{w}))| \cdot |\phi_s^{(\mathbf{1}_j)}(\mathbf{w}_s)|}{\max\{1, |\phi_s^{(\mathbf{1}_j)}(\mathbf{w}_s)|\} l_s(\mathbf{\Phi}(\mathbf{w}))} \leq C|G(\mathbf{\Phi}(\mathbf{w}))| = C|H(\mathbf{w})|, \tag{21}$$

because $l_s(z) \geq 1$ and $\frac{|\phi_s^{(\mathbf{1}_j)}(\mathbf{w}_s)|}{\max\{1, |\phi_s^{(\mathbf{1}_j)}(\mathbf{w}_s)|\}} \leq 1$. Then by, Theorem 1, estimate (21) implies that the function H has the desired property. □

Actually, in Proposition (1) and Theorem 4, the vector-valued entire functions having bounded index in joint variables have implicitly appeared. There are few papers on this class of functions of a single variable [27–29], and of several variables [26,30].

4. Results on Sum of Entire Functions Having Bounded Index

The product of entire functions of bounded index has the same property [31,32]. But W. Pugh [33] demonstrated that the addition of functions having bounded index can go out of the class. This fact is based on the idea that every entire function with unbounded multiplicities of zeros has unbounded index. Moreover, the class of entire functions having bounded index is not closed under differentiation (see an example in [34]). W. Pugh, R. Roy, and S. Shah [33,35] deduced different conditions providing index boundedness for a sum of two entire functions of bounded index. Later, their assertions were transferred for a multidimensional case for functions of bounded L-index in direction, and for functions of bounded $\mathbf{L}$-index in joint variables [3]. But these conditions are very cumbersome and difficult to verify. Therefore, we found new simple conditions for a partial case.

Theorem 5. *If $f_1(z_1)$, $f_2(z_2)$ are entire transcendental functions, and their derivatives $f_1'(z_1)$, $f_2'(z_2)$ are functions having bounded index, then the function $F(z_1, z_2) = f_1(z_1) + f_2(z_2)$ is also of bounded index in joint variables and $N(F) \leq 1 + \max\{N(f_1'), N(f_2')\}$.*

Proof. Suppose that $f_1'(z_1)$, $f_2'(z_2)$ are functions of bounded index. In view of the definition of index boundedness (see inequality (1) for $n = 1$ and $\mathbf{L} \equiv 1$), the functions $f_1(z_1)$, $f_2(z_2)$

are also functions of bounded index. Then this means that for any integer $p \geq 0$ and all $z_j \in \mathbb{C}$, one has

$$\frac{|f_j^{(p)}(z_j)|}{p!} \leq \max_{0 \leq k \leq N(f_j)} \frac{|f_j^{(k)}(z_j)|}{k!} \tag{22}$$

and for any integer $p \geq 0$ and all $z_j \in \mathbb{C}$ one has

$$\frac{|(f_j'(z_j))^{(p)}|}{p!} \leq \max_{0 \leq k \leq N(f_j')} \frac{|(f_j'(z_j))^{(k)}|}{k!}$$

or equivalently

$$\frac{|(f_j^{(s)}(z_j))|}{s!} \leq \max_{1 \leq k \leq 1+N(f_j')} \frac{k|f_j^{(k)}(z_j)|}{s \cdot k!}. \tag{23}$$

Let us consider the function $F(z_1, z_2) = f_1(z_1) + f_2(z_2)$. Calculate its partial derivatives:

$$\begin{aligned} F^{(k,0)}(z_1, z_2) &= f_1^{(k)}(z_1) \quad \text{for any } k \geq 1, \\ F^{(0,s)}(z_1, z_2) &= f_2^{(s)}(z_2) \quad \text{for any } s \geq 1, \\ F^{(k,s)}(z_1, z_2) &= 0 \quad \text{for any } k \geq 1, s \geq 1. \end{aligned} \tag{24}$$

Taking into account (22) and (23), we deduce for $k \geq 1$, $s \geq 1$ that

$$\begin{aligned} \frac{|F^{(k,0)}(z_1, z_2)|}{k!} &= \frac{|f_1^{(k)}(z_1)|}{k!} \leq \max_{1 \leq m \leq N(f_1')+1} \frac{m|f_1^{(m)}(z_1)|}{k \cdot m!} \leq \max_{1 \leq m \leq 1+N(f_1')} \frac{m|F^{(m,0)}(z_1, z_2)|}{k \cdot m!}, \\ \frac{F^{(0,s)}(z_1, z_2)}{s!} &= \frac{|f_2^{(s)}(z_2)|}{s!} \leq \max_{1 \leq p \leq N(f_2')+1} \frac{p|f_2^{(p)}(z_2)|}{s \cdot p!} \leq \max_{1 \leq p \leq 1+N(f_2')} \frac{p|F^{(0,p)}(z_1, z_2)|}{s \cdot p!}, \\ \frac{|F^{(k,s)}(z_1, z_2)|}{k!s!} &= 0 \leq \max_{1 \leq p+m \leq 1+\max\{N(f_1'), N(f_2')\}} \frac{|F^{(m,p)}(z_1, z_2)|}{p!m!}. \end{aligned} \tag{25}$$

For $k \in \{1, 2, \ldots, 1 + N(f_1')\}$, the inequality $\frac{|F^{(k,0)}(z_1,z_2)|}{k!} \leq \max_{1 \leq m \leq 1+N(f_1')} \frac{|F^{(m,0)}(z_1,z_2)|}{m!}$ is obvious. At the same time, for $k > 1 + N(f_1')$ and $1 \leq m \leq 1 + N(f_1')$ one has $\frac{m}{k} < 1$. Then, from (25), it yields for any $k \in \mathbb{N}$ that

$$\frac{|F^{(k,0)}(z_1, z_2)|}{k!} \leq \max_{1 \leq m \leq 1+N(f_1')} \frac{|F^{(m,0)}(z_1, z_2)|}{m!}.$$

By analogy, we also establish

$$\frac{|F^{(0,s)}(z_1, z_2)|}{s!} \leq \max_{1 \leq p \leq 1+N(f_2')} \frac{|F^{(0,p)}(z_1, z_2)|}{p!}.$$

Combining the inequalities for $|F^{(0,s)}(z_1, z_2)|/s!$, $|F^{(k,0)}(z_1, z_2)|/k!$, and $|F^{(k,s)}(z_1, z_2)|/(k!s!)$, we have for $k \geq 0$, $s \geq 0$

$$\frac{|F^{(k,s)}(z_1, z_2)|}{k!s!} \leq \max_{0 \leq p+m \leq 1+\max\{N(f_1'), N(f_2')\}} \frac{|F^{(m,p)}(z_1, z_2)|}{p!m!}.$$

It yields that the function F has bounded index in joint variables and its index $N(F)$ does not exceed the maximum of the indexes of the functions f_1' and f_2' increased by one. □

5. Discussion

Note that $N(f) \leq 1 + N(f')$, $j \in \{1, 2\}$ (see Theorem 3 in [34]). The inequality $N(f) \leq 1 + N(f')$ is sharp. To demonstrate it, Shah S. M. [34] proposed such a function $f(z) = e^z - \sum_{s=0}^{n} \frac{z^s}{s!}$. Then, $f'(z) = e^z - \sum_{s=0}^{n-1} \frac{z^s}{s!}$, $f''(z) = e^z - \sum_{s=0}^{n-2} \frac{z^s}{s!}$, and so on. Then $N(f) = n + 1$, $N(f') = n$. In this case, we obtain the equality $N(f) = 1 + N(f')$. But if we consider $f(z) = \cos z$, then $f'(z) = -\sin z$, $f''(z) = -\cos z$, and so on. Hence, $N(f) = 1$, $N(f') = 1$. In this case, we obtain the strict inequality $N(f) = 1 < N(f') + 1 = 2$. More generally, for the function $f(z) = \sum_{k=0}^{n} \exp(z \exp(i \cdot 2\pi k/n))$, we have $f^{(n)}(z) = f(z)$ with $i^2 = -1$. Therefore, $N(f) = n - 1$, $N(f') = n - 1$ and $N(f) < 1 + N(f')$.

In Theorem (5), we state $N(F) \leq 1 + \max\{N(f_1'), N(f_2')\}$ for $F(z_1, z_2) = f_1(z_1) + f_2(z_2)$. We demonstrate it below. In view of the explanations from the previous paragraph, we choose $f_1(z_1) = e^{z_1} - 1$, $f_2(z_2) = e^{z_2} - 1$ and $F(z_1, z_2) = e^{z_1} + e^{z_2} - 2$. Then, $f_1'(z_1) = e^{z_1}$, $f_2'(z_2) = e^{z_2}$. In this case, the corresponding indexes equal $N(f_1') = 0$, $N(f_2') = 0$, and $N(F) = 1$, that is $N(F) = 1 + \max\{N(f_1'), N(f_2')\}$. Thus, this is a sharp inequality.

Now we show that for some functions the strict inequality is valid, i.e., $N(F) < 1 + \max\{N(f_1'), N(f_2')\}$. We choose $f_1(z_1) = e^{z_1}$, $f_2(z_2) = z_2 e^{z_2}$ and $F(z_1, z_2) = e^{z_1} + z_2 e^{z_2}$. Then $F^{(s,0)}(z_1, z_2) = e^{z_1}$ for any integer $s \geq 0$, and $F^{(s,m)}(z_1, z_2) = 0$ for any integer $s \geq 1$, $m \geq 1$. Therefore, $N(f_1') = 0$. For any $m \geq 1$, one has $F^{(0,m)}(z_1, z_2) = f_2^{(m)}(z_2) = (z_2 + m)e^{z_2}$. We will check inequality (1) for derivatives of the F. One should observe that $\frac{|F^{(s,m)}(z_1,z_2)|}{s!m!} \in \{0; \frac{|e^{z_1}|}{s!}\}$ for any $s \geq 1$ and $m \geq 0$. Then, $\frac{|F^{(s,m)}(z_1,z_2)|}{s!m!} \leq |e^{z_1}| = \frac{|F^{(1,0)}(z_1,z_2)|}{1!}$.

We will investigate whether

$$\frac{|F^{(0,m)}(z_1, z_2)|}{m!} \leq \max_{0 \leq p \leq m-1} \frac{|F^{(0,p)}(z_1, z_2)|}{p!} \tag{26}$$

for all $z_1, z_2 \in \mathbb{C}$. The last inequality is equivalent to

$$\frac{|z_2 + m|}{m!} \cdot |e^{z_2}| \leq \max_{0 \leq p \leq m-1} \frac{|z_2 + p|}{p!} \cdot |e^{z_2}|.$$

Hence,

$$\frac{|z_2 + m|}{m!} \leq \max_{0 \leq p \leq m-1} \frac{|z_2 + p|}{p!}.$$

Using algebraic transformations, we obtain, with $z_2 = x + iy$, $(x, y \in \mathbb{R})$,

$$\sqrt{(x+m)^2 + y^2} \leq \max_{0 \leq p \leq m-1} \frac{m!}{p!} \sqrt{(x+p)^2 + y^2} \iff$$

$$(x+m)^2 \leq \max_{0 \leq p \leq m-1} \frac{m!^2}{p!^2}(x+p)^2 + y^2\left(\frac{m!^2}{p!^2} - 1\right). \tag{27}$$

The last inequality must be satisfied for all $y \in \mathbb{R}_+$. Since $\frac{m!^2}{p!^2} - 1 > 0$ for $0 \leq p \leq m - 1$, the whole expression $y^2\left(\frac{m!^2}{p!^2} - 1\right)$ is non-negative. Therefore, inequality (27) is valid if

$$(x+m)^2 \leq \max_{0 \leq p \leq m-1} \frac{m!^2}{p!^2}(x+p)^2.$$

Hence, $|x + m| \leq \max_{0 \leq p \leq m-1} \frac{m!}{p!}|x + p|$ for all $x \in \mathbb{R}$. Using elementary methods, it is not difficult to establish that

1. For $m = 1$ and $m = 2$, the inequality does not hold at some points $x \in \mathbb{R}$ (for example, it is false for $m = 2$ and $x \in (-2/3; 0)$).
2. For $m = 3$, the inequality is fulfilled for all $x \in \mathbb{R}$.

Hence, the index of the function F equals three, $N(F) = 3$.

Next, the index of the function $f_2'(z_2) = (z_2 + 1)e^{z_2}$ can be found. We will repeat all above considerations from (26) for this inequality $\frac{|(f_2'(z_2))^{(m)}|}{m!} \le \max_{0 \le p \le m-1} \frac{|(f_2'(z_2))^{(p)}|}{p!}$. By analogy, the inequality yields $|x + m + 1| \le \max_{0 \le p \le m-1} \frac{m!}{p!} |x + p + 1|$ for all $x \in \mathbb{R}$. Checking $m = 1$ and $m = 2$, we show that the last inequality is false (for example, for $m = 2$ and $x \in (-5/3, 1)$). For $m = 3$, we found that $|x + 4| \le \max\{6|x + 1|, 6|x + 2|, 3|x + 3|\}$ is true for all $x \in \mathbb{R}$. Therefore, $N(f_2') = 3$.

Combining all results, we see $N(F) = 3 < 1 + \max\{N(f_1'), N(f_2')\} = 4$. Hence, we conclude that inequality $N(F) \le 1 + \max\{N(f_1'), N(f_2')\}$ cannot be improved in the general case.

Theorem 4 leads to the following problem:

Problem 1 (Open Problem)**.** *Is it possible to obtain an analog of Theorem 4 if we replace $\Phi : \mathbb{C}^n \to \mathbb{C}$ by $(\phi_1(w_1), \phi_2(w_2), \ldots, \phi(w_n))$?*

Remark 1. *We can not deduce analog of Theorem 4 or remove assumption $N(G, \mathbf{L}) = 0$ in Proposition 1 because under usage of methods from these assertions we obtain that $|\varphi_1'(w_1)/\varphi_2'(w_2)|$ or similar expression must be bounded in w_1 and w_2.*

Another problem in this topic appears if we introduce some operator of fractional differentiation into (1).

Problem 2. *Is it possible to replace ordinary derivatives by the fractional derivatives and deduce (or improve) all known properties of entire functions of bounded index?*

We are currently unable to provide comprehensive answers to these questions.

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