fractal and fractional

Article

Certain Novel Fractional Integral Inequalities via Fuzzy Interval Valued Functions

Miguel Vivas-Cortez [1], Rana Safdar Ali [2], Humira Saif [2], Mdi Begum Jeelani [3], Gauhar Rahman [4,*] and Yasser Elmasry [5]

[1] Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076, Apartado, Quito 17-01-2184, Ecuador; mjvivas@puce.edu.ec
[2] Department of Mathematics, University of Lahore, Lahore 54000, Pakistan; safdar.ali@math.uol.edu.pk (R.S.A.); pmat07213019@student.uol.edu.pk (H.S.)
[3] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 13314, Saudi Arabia; mbshaikh@imamu.edu.sa
[4] Department of Mathematics and Statistics, Hazara University, Mansehra 21300, Pakistan
[5] Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha 61466, Saudi Arabia; sadek@kku.edu.sa
* Correspondence: drgauhar.rahman@hu.edu.pk

Abstract: Fuzzy-interval valued functions (FIVFs) are the generalization of interval valued and real valued functions, which have a great contribution to resolve the problems arising in the theory of interval analysis. In this article, we elaborate the convexities and pre-invexities in aspects of FIVFs and investigate the existence of fuzzy fractional integral operators (FFIOs) having a generalized Bessel–Maitland function as their kernel. Using the class of convexities and pre-invexities FIVFs, we prove some Hermite–Hadamard (H-H) and trapezoid-type inequalities by the implementation of FFIOs. Additionally, we obtain other well known inequalities having significant behavior in the field of fuzzy interval analysis.

Keywords: convex (FIV) function; fuzzy fractional integral operator; pre-invex FIV function; Hermite–Hadamard (H-H)-type inequality; trapezoid-type inequality; extended generalized Bessel–Maitland function

MSC: 26D10; 26D15; 26D10; 26D53; 05A30

check for updates

Citation: Vivas-Cortez, M.; Ali, R.S.; Saif, H.; Jeelani, M.B.; Rahman, G.; Elmasry, Y. Certain Novel Fractional Integral Inequalities via Fuzzy Interval Valued Functions. *Fractal Fract.* **2023**, *7*, 580. https://doi.org/10.3390/fractalfract7080580

Academic Editor: Palle Jorgensen

Received: 9 June 2023
Revised: 4 July 2023
Accepted: 6 July 2023
Published: 28 July 2023

1. Introduction

The theory of convexity is a dynamic, addictive, and significant area of study that has made major contributions to other fields of research, such as mathematical analysis, optimization problems, control theories, economics, finance, and game theory. By the theory of convexity, researchers have created unified numerical structures that can be used to resolve the wide range of problems, which have arisen in pure and applied mathematics. Convexity has been through significant advancements, generalizations, and extensions in a few decades. The study of fractional analysis has increased the demand of fractional operators in different areas of mathematics. To fulfil this requirement, many researchers have worked to develop the fractional operators by utilizing the non-singular special functions as their kernel and obtained modified versions of inequalities. Generalized fractional operators are one of the techniques used to improve the fractional inequalities for different convexities and pre-invexities, which have significant applications in the field of analysis.

Fractional calculus is the generalization of classical calculus, which plays an important role in pure, applied, and computation fields of mathematics. The research work in the field of fractional analysis has made a great contribution in various directions, such as signal-image processing, biology, physics, control operator theory, computer structure optimizations, and fluid dynamics [1–3]. During the last few decades, most of the scientists have worked to obtain the generalized versions of well-known inequalities and discussed a huge number of applications in the fields of analysis and discrete optimization. Many authors have worked extensively [4–12] and discussed the refinements and extensions in different areas of mathematics. The advanced analysis of inequalities is possible for the development of fractional operators by means of their kernel in multi-dimension functions, which play an ideal role to create new horizons to study the behavior of inequalities in multi-discipline branches of mathematics.

There are many significant applications of fuzzy set theory, which deals with the problems incorporating ambiguous, vague, and imprecise information, and makes decisions for individual or group collaboration. This initiative developed the idea of interval analysis [13] by Moore Ramon in 1966. Many scientists are attracted towards this field because of to its decision-making evaluation. The investigation on interval analysis proved to be beneficial in global optimization and constraint solution algorithms for decision makers and in the last few decades, it has become very popular among experts. It has provided effective and valid results, minimized the errors, and improved the accuracy. Due to this motivation, several researchers started their research in inequalities to get the desired results.

Zhao et al. [14] were the first who introduced the interval-valued-function (IVF). Many mathematicians introduced a strong relation between inequalities and IVFs by the implementation of different integral operators [15–19]. Many scholars have elaborated the applications of fuzzy differential equations and fuzzy interval analysis, which deal with many mathematical or computer modules [20–24]. In different research articles, many people illustrated several inequalities, such as Hermite–Hadamard inequality, Jensen inequality, Mercer inequality, Schur inequality, and trapezoid-type inequalities with the help of a fuzzy interval valued function [25,26].

Motivation

Convexity and generalized convexity are important concepts in optimization under the fuzzy domain, which provide fuzzy variational inequalities as a result of characterizing the optimality condition of convexity. Variational inequality and fuzzy set theory have generated strong techniques, which resolved many mathematical problems related to minimization theory and interval analysis. Fuzzy mappings are also termed as fuzzy-IVFs. There are many fractional integrals that contain fuzzy-IVFs for lower and upper cases. The behavior of well known inequalities can be checked by the successful implementation of such kinds of fractional integrals. These investigations demonstrate that this strategy is important from both a theoretical and a practical standpoint to transform the actual integral inequalities to the fuzzy integral inequalities.

2. Preliminaries

In this section, we will discuss the basic definitions and results, which help to understand the concepts of our new results.

Definition 1. *[27] Let $k \in \mathbb{R}$ be convex set, then the convex function for $F : K \to \mathbb{R}$ is defined as follows:*

$$F\left[\sigma u + (1-\sigma)v\right] \leq \sigma F(u) + (1-\sigma)F(v). \tag{1}$$

for $\sigma \in [0,1]$, $\forall$ ð$, v \in K$.

The concave function for F is to have the reversed inequality, defined in Equation (1).

Definition 2. *The Hermite–Hadamard type inequality [28–31] is as follows:*

$$F\left(\frac{\jmath + \aleph}{2}\right) \leq \frac{1}{\aleph - \jmath}\int_{\jmath}^{\aleph} F(z)dz \leq \frac{F(\jmath) + F(\aleph)}{2}. \tag{2}$$

where $F : K \to R$ is the convex function, and $[\jmath, \aleph] \in K \subseteq \mathbb{R}, \jmath < \aleph, \jmath, \aleph \in R$.

The modified form of fractional H-H inequality for the convex function is as follows:

$$F\left(\frac{\jmath + \aleph}{2}\right) \leq \frac{1}{2\varsigma(v,\eth)^{\nu'+1}}\left[\mathcal{I}^{\eth+}_{v,v'}(\Omega, F) + \mathcal{I}^{v-}_{\eth,v'}(\Omega, F)\right] \leq \frac{F(\jmath) + F(\aleph)}{2} \tag{3}$$

Definition 3. *[25] Let $F : [\eth, v] \subset \mathbb{R} \to \Omega_c^+$ (a space of all positive closed and bounded intervals of $\mathbb{R}$) is a convex IVF given by $F(\alpha) = [F(\alpha)_*, F(\alpha)^*]$ for all $\alpha \in [\eth, v]$ where $F(\alpha)_*$ is convex function and $F(\alpha)^*$ is concave function. If F is Riemann integrable function, then:*

$$F\left(\frac{\alpha + \Lambda}{2}\right) \supseteq (IR)\frac{1}{\Lambda - \alpha}\int_{\alpha}^{\Lambda} F(z)dz \supseteq \frac{F(\alpha) + F(\Lambda)}{2} \tag{4}$$

The Hermite–Hadamard inequality for inclusion relation is as follows:

$$F\left(\frac{\alpha + \Lambda}{2}\right) \supseteq \frac{1}{2\varsigma(v,\eth)^{\nu'+1}}\left[\mathcal{I}^{\eth+}_{v,v'}(\Omega, F) + \mathcal{I}^{v-}_{\eth,v'}(\Omega, F)\right] \supseteq \frac{F(\alpha) + F(\Lambda)}{2} \tag{5}$$

Proposition 1. *[16] If $F, \daleth \in \mathbb{F}_0$, then for $\mathbb{F}_0$, relation " $\preccurlyeq$ " is defined as:*

$$F \preccurlyeq \daleth \text{ if and only if } [F]^\imath \leq_I [\daleth]^\imath \text{ for all } \imath \in [0,1], \tag{6}$$

the relation given above is also called partial order relation.

For $F, \daleth \in \mathbb{F}_0$, their scalar and vector properties are defined for $\imath \in [0,1]$, as follows:

$$\begin{aligned} (F \tilde{+} \daleth)^\imath &= (F)^\imath + (\daleth)^\imath \\ (F \tilde{+} \daleth)^\imath &= F + (\daleth)^\imath \\ (\wp . F)^\imath &= \wp . (F)^\imath \\ (F \tilde{\times} \daleth)^\imath &= (F)^\imath \times (\daleth)^\imath \end{aligned}$$

The Hukuhara difference of F and $\daleth$, for $\sqsupset \in \mathbb{F}_0$ and $F = \daleth \tilde{+} \sqsupset$, then $\sqsupset$ is the Hukuhara difference of F and $\daleth$, which is defined as follows:

$$\begin{aligned} (\sqsupset)^*(\imath) = (F \tilde{-} \daleth)^*(\imath) = F^*(\imath) - \daleth^*(\imath), \\ (\sqsupset)_*(\imath) = (F \tilde{-} \daleth)_*(\imath) = F_*(\imath) - \daleth_*(\imath) \end{aligned}$$

Definition 4. *[11] Let P partition be the partition on the closed interval $[\eth, v]$ as the form:*

$$P = \eth = \jmath_1 < \jmath_2 < \jmath_3 < \jmath_4 < \jmath_5 < ....\jmath_k = v.$$

The subintervals containing point P have a maximum length, which is called the mesh of a partition and defined as follows:

$$mesh(P) = max(\jmath_j - \jmath_{j-1} : j = 1, 2, 3...k) \tag{7}$$

Let $P(\sigma, [\eth, v])$ be the set of all $p \in P(\sigma, [\eth, v])$ and $mesh(p) < \sigma$. By taking arbitrary point $\daleth_j$ on each subinterval $[\jmath_{j-1}, \jmath_j]$, where $1 \leq j \leq k$, such that:

$$S(F, p, \sigma) = \sum_{j=1}^{k} F(\daleth_j)(\jmath_j - \jmath_{j-1}),$$

where $F : [\eth, v] \to \mathbb{R}_I$ is the real valued function and $S(F, p, \sigma)$ is called the Riemann sum of F corresponding to $p \in P(\sigma, [\eth, v])$.

Definition 5. *[32] A function $F : [\eth, v] \to \mathbb{R}_I$ is called the Reimann integrable (IR-integrable) on the closed interval $[\eth, v]$ if there exists $B \in \mathbb{R}_I$ and for each ϵ and $\sigma > 0$, such that:*

$$d(S(F, P, \sigma), B) < \epsilon.$$

for every Riemann sum of F corresponding to the partition $P \in p(\sigma, [\eth, v])$, and arbitrary choice of $\daleth_j \in [\jmath_{j-1}, \jmath_j]$ for $1 \leq j \leq k$; then, we have B be the IR-integral of F on $[\eth, v]$, which is denoted by $B = (IR)\int_{\eth}^{v} F(t)dt$.

Theorem 1. *[33] The real interval valued function $F : [\eth, v] \subseteq \mathbb{R} \to \mathbb{R}_I$ and $F(\jmath) = [F_*, F^*]$; then, F is the integrable function on $[\eth, v]$ if and only if F_* and F^* are both integrable functions over $[\eth, v]$, if:*

$$(IR)\int_{\eth}^{v} F(\jmath)dx = \left[(R)\int_{\eth}^{v} F_*(\jmath), (R)\int_{\eth}^{v} F^*(\jmath)dx\right]. \tag{8}$$

The representation of integrable function real valued functions and generalized integrable interval valued functions are $R_{[c,d]}, IR_{[\eth,v]}$, respectively.

Definition 6. *[34] If for each $\imath \in [0, 1]$ and let $F : k \subseteq R \to \mathbb{F}_0$ be fuzzy IVF, the $\imath$-levels define on IVF $F : k \subseteq R \to \Omega_c$ are given by $F_\imath(\jmath) = [F_*(\jmath, \imath), F^*(\jmath, \imath)]$, $\forall \jmath \in \Omega$. Both the real-valued functions $F_*(\jmath, \imath), F^*(\jmath, \imath) : \Omega \to \mathbb{R}$, called the upper and lower functions of F.*

Remark 1. *Let $F : k \subseteq R \to \mathbb{F}_0$ be the FIV function, and for each $\imath \in [0, 1]$, then $F(\jmath)$ is called the continuous function at $\jmath \in \Omega$, and both the left and right real valued functions $F_*(\jmath, \imath), F^*(\jmath, \imath)$ are continuous at $\jmath \in K$.*

Definition 7. *[11] Let all closed and bounded intervals Ω_C of $\mathbb{R}$ and $\tau \in \Omega_C$, described as follows:*

$$\tau = [\tau_*, \tau^*] = \{\alpha \in \mathbb{R} | \tau_* \preccurlyeq \alpha \preccurlyeq \tau^*\}, (\tau_*, \tau^* \in \mathbb{R}).$$

- If $\tau_* = \tau^*$, then we say that τ is degenerate.
- If $\tau_* \geq 0$, then $[\tau_*, \tau^*]$ is said to be a positive interval. All positive intervals are denoted by Ω_C^+ and defined as:

$$\Omega_C^+ = \{[\tau_*, \tau^*] : [\tau_*, \tau^*] \in \Omega_C \quad and \quad \tau_* \geq 0\}.$$

Definition 8. *[11] Let $\sigma \in \mathbb{R}$ and $\sigma\tau$ be defined as:*

$$\sigma\tau = [\sigma\tau_*, \sigma\tau^*] \ if \ \sigma \geq 0, \ [\sigma\tau^*, \sigma\tau_*] \ if \ \sigma \preccurlyeq 0. \tag{9}$$

Algebraic Minkowski properties are defined for $\tau, \varsigma \in \Omega_C$ as follows:

$$[\varsigma_*, \varsigma^*] - [\tau_*, \tau^*] = [\varsigma_* - \tau_*, \varsigma^* - \tau^*]$$
$$, [\varsigma_*, \varsigma^*] + [\tau_*, \tau^*] = [\varsigma_* + \tau_*, \varsigma^* + \tau^*]$$

The inclusion " $\subseteq$ " property is defined as:

$$\varsigma \subseteq \tau, \ then \ [\varsigma_*, \varsigma^*] \subseteq [\tau_*, \tau^*] \ implies \ that \ \varsigma_* \preccurlyeq \tau_*, \varsigma^* \preccurlyeq \tau^*.$$

Remark 2. *[35] The relation " $\preccurlyeq_1$ " is defined on Ω_C as follows:*

$$[\sigma_*, \sigma^*] \preccurlyeq_1 [\tau_*, \tau^*] \text{ if and only if } \sigma_* \preccurlyeq \tau_*, \sigma^* \preccurlyeq \tau^*$$

for all $[\sigma_*, \sigma^*], [\tau_*, \tau^*] \in \Omega_C$, is an order relation. We have $[\sigma_*, \sigma^*], [\tau_*, \tau^*] \in \Omega_C$, then $[\sigma_*, \sigma^*] \preccurlyeq_1 [\tau_*, \tau^*]$ if and only if $\sigma_* \preccurlyeq \tau_*$ or $\sigma^* \preccurlyeq \tau^*, \sigma < \tau$.

Definition 9. *[11] For $\eth, v \in K$, $\lambda \in [0,1]$, if $F : K \times K \to \mathbb{R}$ is the bi-function, then the invex set $K \subseteq \mathbb{R}$ is defined as follows:*

$$v + \lambda F(\eth, v) \in K.$$

Definition 10. *[11] Let K be an invex set with respect to ς and $F : K \to \mathbb{R}$ is the pre-invex function, defined for $\jmath, \aleph \in K$ as follows:*

$$F(\aleph + \lambda\varsigma(\jmath, \aleph)) \leq \lambda F(\jmath) + (1-\lambda)F(\aleph), \tag{10}$$

where $\lambda \in [0,1]$.

Definition 11. *[22] If $F : K \to \mathbb{R}$, then the convex fuzzy interval valued function is defined as follows:*

$$F\Big[\sigma\eth + (1-\sigma)v\Big] \preccurlyeq \sigma F(\eth)\tilde{+}(1-\sigma)F(v). \tag{11}$$

for $\sigma \in [0,1]$, $\forall \eth, v \in K$, and then we call F concave if inequality (11) is reversed.

Remark 3. *If $F_*(\jmath, \imath) = F^*(\jmath, \imath)$ and $\imath = 1$, then we obtain the inequality (1).*

Definition 12. *[23] The pre-invex FIV function $F : K \to \mathbb{R}$ is defined for $\jmath, \aleph \in K$ and $\lambda \in [0,1]$ as follows:*

$$F(\aleph + \lambda\varsigma(\jmath, \aleph)) \preccurlyeq \lambda F(\jmath)\tilde{+}(1-\lambda)F(\aleph), \tag{12}$$

where K is an open invex set with respect to ς if we reversed the inequality (12), then named as pre-incave FIV functions.

Definition 13. *[36] The gamma function is defined in an integral form as follows:*

$$\Gamma(\sigma) = \int_0^\infty z^{\sigma-1}e^{-z}dz,$$

for $\Re(t) > 0$.

Definition 14. *[36] The Pochammer's symbol is defined as follows:*

$$(\wp)_\sigma = \left\{ \begin{array}{ll} 1, & \text{for } \sigma = 0, \wp \neq 0 \\ \wp(\wp+1)\cdots(\wp+\sigma-1), & \text{for } \sigma \geq 1, \end{array} \right\}$$

For $\sigma \in \mathbb{N}$ and $\wp \in \mathbb{C}$:

$$\begin{aligned} (\wp)_n &= \frac{\Gamma(\wp+n)}{\Gamma(\wp)} \\ (\wp)_{kn} &= \frac{\Gamma(\wp+kn)}{\Gamma(\wp)} \end{aligned}$$

where Γ is the gamma function.

Definition 15. *[37] The beta function for $\Re(m) > 0$ and $\Re(n) > 0$ is defined as follows:*

$$\Lambda(g,h) = \int_0^1 \sigma^{g-1}(1-\sigma)^{h-1} d\sigma = \frac{\Gamma(g)\Gamma(h)}{\Gamma(g+h)}.$$

Definition 16. *[38] The extended version of beta functions is defined for $\Re(g) > 0$, $\Re(h) > 0$, $\Re(p) > 0$ as follows:*

$$\Lambda_p(g,h) = \int_0^1 z^{g-1}(1-z)^{h-1} exp(\frac{-p}{z(1-z)}) dz.$$

If we replace $p = 1$, then the extended beta function is going to replace the classical beta function.

Definition 17. *[39,40] The generalized Bessel–Maitland (eight-parameter) function is defined as follows:*

$$J^{\beth,\Theta,\sigma,\vartheta}_{\xi,\hbar,m,\sigma}(\aleph) = \sum_{p=0}^{\infty} \frac{(\Theta)_{\hbar p}(\vartheta)_{\sigma p}(-\aleph)^p}{\wp(\xi p + \beth + 1)(\sigma)_{mp}}, \quad (13)$$

where $F, \beth, \sigma, F, \vartheta \in \mathbb{C}$, $\Re(F) > 0$, $\Re(F) > 0$, $\Re(\nu) \geq -1$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$; $m, \hbar, \sigma \geq 0$ and $\hbar > \Re(F) + \sigma, m$.

Definition 18. *[40] The extended version of the Bessel–Maitland function is defined for $\Xi, \daleth, \nu, \wp, \rho, c \in \mathbb{C}$, $\Re(\Xi) > 0$, $\Re(\nu) \geq -1$, $\Re(\daleth) > 0$, $\Re(\rho) > 0$, $\Re(\wp) > 0$; $\hbar, \sigma, m \geq 0$ and $\hbar, m > \Re(\Xi) + \sigma$ as follows:*

$$J^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega; p) = \sum_{n=0}^{\infty} \frac{\Lambda_p(\daleth + \hbar n, c - \daleth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\daleth, c - \daleth)\wp(\Xi n + v + 1)(\rho)_{mn}} (-\Omega)^n. \quad (14)$$

Definition 19. *Let $F : K \subseteq \mathbb{R} \to \mathbb{R}$ be the positive real valued function, then the Godunova–Levin FIV function is defined as follows:*

$$F(\sigma\eth + (1-\sigma)v) \preccurlyeq \frac{F(\eth)}{\sigma} \tilde{+} \frac{F(v)}{1-\sigma},$$

where $\eth, v \in K$, $\sigma \in (0,1)$

Definition 20. *Let $h : (0,1) \to \mathbb{R}$ and $F : K \to \mathbb{R}$ be the non-negative real valued function, the h-Godunova–Levin FIV function is defined for $\sigma \in (0,1)$, $\eth, v \in K$ as follows:*

$$F(\sigma\eth + (1-\sigma)v) \preccurlyeq \frac{F(\eth)}{h(\sigma)} \tilde{+} \frac{F(v)}{h(1-\sigma)}.$$

Definition 21. *A real valued function $F : K \to \mathbb{R}$ is said to be h-Godunova–Levin pre-invex FIV function with respect to ς if for all $\eth, v \in K$, $\Phi \in (0,1)$ as follows:*

$$F(\eth + \Phi\varsigma(v,\eth)) \preccurlyeq \frac{F(\eth)}{h(1-\Phi)} \tilde{+} \frac{F(v)}{h(\Phi)}$$

holds.

Definition 22. *[40] The generalized fractional integral operators having an extended version of the Bessel–Maitland function as their kernel are defined for $\Xi, \nu, \daleth, \rho, \wp, c \in \mathbb{C}$, $\Re(\Xi) > 0$, $\Re(\nu) \geq -1$, $\Re(\rho) > 0$, $\Re(\daleth) > 0$, $\Re(\wp) > 0$; $\hbar, \sigma, m \geq 0$ and $\hbar, m > \Re(\Xi) + \sigma$ as follows:*

$$\left(\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp;p^+} f\right)(\jmath, r) = \int_p^\jmath (\jmath - t)^v \mathrm{J}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(\jmath - t)^\Xi; r) f(\eth) d\eth, (\jmath > p)$$

and

$$\left(\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp;q^-} f\right)(\jmath, r) = \int_\jmath^q (t - \jmath)^v \mathrm{J}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(t - \jmath)^\Xi; r) f(\eth) d\eth, (\jmath < q).$$

Remark 4. *The generalized fractional operators are represented in short notations [41] as follows:*

$$(\mathcal{I}^{\eth+}_{v,v'})(\Omega, F) = (\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp;\eth^+} F)(v, p)$$
$$(\mathcal{I}^{v^-}_{\eth,v'})(\Omega, F) = (\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp;v^-} F)(\eth, p).$$

and

$$\int_v^u (v - x)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega(\frac{v - x}{\varsigma(v, \eth)})^\Xi; p) dx = (\mathfrak{I}^{v^-}_{\eth,v'})(\Omega', 1)$$

3. Hermite–Hadamard (H-H) Integral Inequalities via Convex FIVF

Here, we define the left- and right-sided generalized fuzzy fractional integral operators, discuss the existence of H-H type inequality for the h-Godnova–Levin convex FIVF, and deduce some corollaries from our main results.

Definition 23. *The left- and right-sided generalized fuzzy fractional integral operators based on the left and right end point function for $\Xi, \nu, \daleth, \rho, \wp, c \in \mathbb{C}$, $\Re(\Xi) > 0$, $\Re(\nu) \geq -1$, $\Re(\daleth) > 0$, $\Re(\rho) > 0$, $\Re(\wp) > 0$; $\hbar, m, \sigma \geq 0$ and $m, \hbar > \Re(\Xi) + \sigma$ are defined as follows:*

$$\left[\left(\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp;p^+} f\right)(\jmath, r)\right]^\imath = \int_p^\jmath (\jmath - t)^v \mathrm{J}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(\jmath - t)^\Xi; r) f_\imath(\eth) d\eth$$

$$= \int_p^\jmath (\jmath - t)^v \mathrm{J}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(\jmath - t)^\Xi; r)[f_*(\eth, \imath), f^*(\eth, \imath)] d\eth, (\jmath > p)$$

where

$$\left(\mathfrak{T}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp;p^+} f_*\right)((\jmath, r), \imath) = \int_p^\jmath (\jmath - t)^v \mathrm{J}^{\Xi,\hbar,m,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(\jmath - t)^\Xi; r) f_*(\eth, \imath) d\eth, (\jmath > p)$$

$$\left(\mathfrak{T}^{\Xi,m,\hbar,\sigma,c}_{v,\daleth,\rho,\wp;p^+} f^*\right)((\jmath, r), \imath) = \int_p^\jmath (\jmath - t)^v \mathrm{J}^{\Xi,m,\hbar,\sigma,c}_{v,\daleth,\rho,\wp}(\Omega(\jmath - t)^\Xi; r) f^*(\eth, \imath) d\eth, (\jmath > p)$$

On the same pattern, we can easily define the right-sided generalized fractional integral operator based on left and right end point functions.

Remark 5. *The generalized fuzzy fractional integral operators are represented in short notations as follows:*

$$\int_u^v (x - u)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega(\frac{x - u}{\varsigma(v, \eth)})^\Xi; p) F_*(x, \imath) dx = (\mathfrak{I}^{\eth^+}_{v,v'})(\Omega'; F_*)$$
$$\int_v^u (v - x)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega(\frac{v - x}{\varsigma(v, \eth)})^\Xi; p) F_*(x, \imath) dx = (\mathfrak{I}^{v^-}_{\eth,v'})(\Omega'; F_*) \quad (15)$$

Theorem 2. *Let $h : (0,1) \to \mathbb{R}$ be a positive function, $h(\sigma) \neq 0$ and h-Godunova Levin convex fuzzy interval valued function $\mathcal{F} : [\eth, v] \to \mathbb{R}$ with $\mathcal{F} \in L_1[\eth, v]$ and $0 < \eth < v$, then with the generalized fractional integral described in (22), we have;*

$$\frac{h(\frac{1}{2})}{2}\mathcal{F}\left(\frac{\eth + v}{2}\right)\left(\Im^{v-}_{\eth,v'}\right)(\Omega', 1) \preccurlyeq \frac{1}{2}\Big[\left(\Im^{v-}_{\eth,v'}\right)(\Omega', \mathcal{F})\widetilde{+}\left(\Im^{\eth+}_{v,v'}\right)(\Omega', \mathcal{F})\Big]$$
$$\preccurlyeq \frac{\mathcal{F}(\eth)\widetilde{+}\mathcal{F}(v)}{2}\int_0^1 \Big[\frac{1}{h(\sigma)} + \frac{1}{h(1-\sigma)}\Big]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi}; p)d\sigma, \tag{16}$$

where $\Omega' = \frac{\Omega}{\varsigma(v,\eth)^{\Xi}}$

Proof. Consider $\mathcal{F}$ to be the h-Godunova–Levin convex function for $\jmath, \aleph \in [\eth, v]$, we have:

$$\mathcal{F}(\hbar\jmath + (1-\hbar)\aleph) \leq \frac{\mathcal{F}(\jmath)}{h(\hbar)} + \frac{\mathcal{F}(\aleph)}{h(1-\hbar)}$$

Replacing the values $\jmath = \sigma\eth + (1-\sigma)v$, $\aleph = (1-\sigma)\eth + \sigma v$ and $\hbar = \frac{1}{2}$, we have:

$$\mathcal{F}(\frac{\eth + v}{2}) \leq \frac{1}{h(\frac{1}{2})}\big[\mathcal{F}(\sigma\eth + (1-\sigma)v) + \mathcal{F}((1-\sigma)\eth + \sigma v)\big]$$
$$h(\frac{1}{2})\mathcal{F}(\frac{\eth + v}{2}) \leq \mathcal{F}(\sigma\eth + (1-\sigma)v) + \mathcal{F}((1-\sigma)\eth + \sigma v)$$

If we take for every $\imath \in [0,1]$, then we have:

$$\mathcal{F}(\frac{\eth + v}{2}, \imath) \leq \frac{1}{h(\frac{1}{2})}\big[\mathcal{F}_*(\sigma\eth + (1-\sigma)v, \imath) + \mathcal{F}_*((1-\sigma)\eth + \sigma v), \imath\big] \tag{17}$$

Multiplying by $\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi}; p)$ in Equation (17) and then integrating with respect to σ over the interval $[0,1]$, we have:

$$h(\frac{1}{2})\mathcal{F}_*(\frac{\eth + v}{2}, \imath)\int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi}; p)d\sigma \leq \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi}; p)\mathcal{F}_*(\sigma\eth + (1-\sigma)v, \imath)d\sigma$$
$$+ \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi}; p)\mathcal{F}_*((1-\sigma)\eth + \sigma v, \imath)d\sigma$$

$$h(\frac{1}{2})\mathcal{F}_*(\frac{\eth + v}{2}, \imath)\sum_{n=0}^{\infty}\frac{\Lambda_p(\daleth + \hbar n, c - \daleth)(c)_{\hbar n}(\imath)_{\sigma n}}{\Lambda(\daleth, c - \daleth)\imath(\Xi n + v' + 1)(\rho)_{mn}}(-\Omega)^n\int_0^1 \sigma^{v'+\Xi n}d\sigma$$
$$\leq \sum_{n=0}^{\infty}\frac{\Lambda_p(\daleth + \hbar n, c - \daleth)(c)_{\hbar n}(\imath)_{\sigma n}}{\Lambda(\daleth, c - \daleth)\imath(\Xi n + v' + 1)(\rho)_{mn}}(-\Omega)^n\Big[\int_0^1 \sigma^{v'+\Xi n}\mathcal{F}_*(\sigma\eth + (1-\sigma)v, \imath)d\sigma$$
$$+ \int_0^1 \sigma^{v'+\Xi n}\mathcal{F}_*((1-\sigma)\eth + \sigma v, \imath)d\sigma\Big]. \tag{18}$$

After simplification of the integral inequality (18) by using (15), we obtain:

$$\frac{h(\frac{1}{2})}{2}\mathcal{F}_*(\frac{\eth + v}{2}, \imath)\left(\Im^{v-}_{\eth,v'}\right)(\Omega', 1) \leq \frac{1}{2}\Big[\left(\Im^{\eth+}_{v,v'}\right)(\Omega'; \mathcal{F}_*) + \left(\Im^{v-}_{\eth,v'}\right)(\Omega'; \mathcal{F}_*)\Big]. \tag{19}$$

Now, again consider the h-Godunova–Levin convex on $\mathcal{F}$:

$$\mathcal{F}(\sigma\eth + (1-\sigma)v) \leq \frac{\mathcal{F}(\eth)}{h(\sigma)} + \frac{\mathcal{F}(v)}{h(1-\sigma)} \tag{20}$$

Re-writing Equation (20), we have:

$$F((1-\sigma)\eth+\sigma v) \leq \frac{F(\eth)}{h(1-\sigma)}+\frac{F(v)}{h(\sigma)}. \tag{21}$$

Adding Equations (20) and (21) gives the following inequality:

$$F(\sigma\eth+(1-\sigma)v)+F((1-\sigma)\eth+\sigma v) \leq (F(\eth)+F(v))[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}].$$

If we take for every $\imath \in [0,1]$, then we are given the following inequality:

$$F_*(\sigma\eth+(1-\sigma)v,\imath)+F_*((1-\sigma)\eth+\sigma v,\imath) \leq (F_*(\eth,\imath)+F_*(v,\imath))[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}] \tag{22}$$

Multiplying by $\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\omega\sigma^{\Xi};p)$ in Equation (22), then integrating with respect to σ over the interval $[0,1]$, we get:

$$\begin{aligned}\int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)F_*(\sigma\eth+(1-\sigma)v,\imath)d\sigma+\int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)F_*((1-\sigma)\eth+\sigma v,\imath)d\sigma \\ \leq (F_*(\eth,\imath)+F_*(v,\imath))\int_0^1[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma.\end{aligned} \tag{23}$$

After simplification of the inequality (23), we have:

$$\frac{1}{2}\Big[(\Im^{\eth+}_{v,v'})(\Omega';F_*)+(\Im^{v-}_{\eth,v'})(\Omega';F_*)\Big] \leq \frac{F_*(\eth,\imath)+F_*(v,\imath)}{2}\int_0^1[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma. \tag{24}$$

Combining (19) and (24), we get inequality.

$$\begin{aligned}\frac{h(\frac{1}{2})}{2}F_*(\frac{\eth+v}{2},\imath)(\Im^{v-}_{\eth,v'})(\Omega',1) &\leq \frac{1}{2}\Big[(\Im^{\eth+}_{v,v'})(\Omega';F_*)+(\Im^{v-}_{\eth,v'})(\Omega';F_*)\Big] \\ &\leq \frac{F_*(\eth,\imath)+F_*(v,\imath)}{2}\int_0^1[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma.\end{aligned} \tag{25}$$

Similarly for F^*:

$$\begin{aligned}\frac{h(\frac{1}{2})}{2}F^*(\frac{\eth+v}{2},\imath)(\Im^{v-}_{\eth,v'})(\Omega',1) &\leq \frac{1}{2}\Big[(\Im^{\eth+}_{v,v'})(\Omega';F^*)+(\Im^{v-}_{\eth,v'})(\Omega';F^*)\Big] \\ &\leq \frac{F^*(\eth,\imath)+F^*(v,\imath)}{2}\int_0^1[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma.\end{aligned} \tag{26}$$

From Equations (25) and (26):

$$\begin{aligned}&\frac{h(\frac{1}{2})}{2}[F^*(\frac{\eth+v}{2},\imath),F_*(\frac{\eth+v}{2},\imath)](\Im^{v-}_{\eth,v'})(\Omega',1) \\ &\leq_I \Big[(\Im^{\eth+}_{v,v'})(\Omega';F_*)+(\Im^{v-}_{\eth,v'})(\Omega';F_*)\Big],\Big[(\Im^{\eth+}_{v,v'})(\Omega';F^*)+(\Im^{v-}_{\eth,v'})(\Omega';F^*)\Big] \\ &\leq_I \Big[\{F^*(\eth,\imath)+F^*(v,\imath)\},\{F_*(\eth,\imath)+F_*(v,\imath)\}\Big]\int_0^1[\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma.\end{aligned}$$

After simplification:

$$\frac{h(\frac{1}{2})}{2}F\left(\frac{\eth+v}{2}\right)(\mathfrak{I}^{v-}_{\eth,v'})(\Omega',1) \preccurlyeq \frac{1}{2}\Big[\left(\mathfrak{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathfrak{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big] \preccurlyeq \frac{F(\eth)\tilde{+}F(v)}{2}$$
$$\int_0^1 [\frac{1}{h(\sigma)}+\frac{1}{h(1-\sigma)}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\imath}(\Omega\sigma^{\Xi};p)d\sigma,$$

□

Corollary 1. *We obtain H-H-type inequality for the p-type FIV function if we substitute $h(\sigma)=1$ in Theorem 2:*

$$\frac{1}{2}F\left(\frac{\eth+v}{2}\right)(\mathcal{I}^{v-}_{\eth,v'})(\Omega',1) \preccurlyeq \frac{1}{2}\Big[\left(\mathcal{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big]$$
$$\preccurlyeq (F(\eth)\tilde{+}F(v))(\mathcal{I}^{\eth+}_{v,v'})(\Omega',1)$$

Corollary 2. *We obtained H-H-type inequality for the s-Godunova–Levin FIV function after replacing $h(\sigma)=\sigma^s$ in Theorem 2.*

$$\frac{(\frac{1}{2})^s}{2}F\left(\frac{\eth+v}{2}\right)(\mathcal{I}^{v-}_{\eth,v'})(\Omega','1) \preccurlyeq \frac{1}{2}\Big[\left(\mathcal{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big] \preccurlyeq \frac{F(\eth)\tilde{+}F(v)}{2}$$
$$\int_0^1 [\frac{1}{\sigma^s}+\frac{1}{(1-\sigma)^s}]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)d\sigma.$$

Corollary 3. *Putting the value $h(\sigma)=\frac{1}{\sigma}$ in Theorem 2, we obtain Hermite–Hadamard-type inequality for the FIV convex function.*

$$F\left(\frac{\eth+v}{2}\right)(\mathcal{I}^{v-}_{\eth,v'})(\Omega',1) \preccurlyeq \frac{1}{2}\Big[\left(\mathcal{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big]$$
$$\preccurlyeq \frac{F(\eth)\tilde{+}F(v)}{2}(\mathcal{I}^{\eth+}_{v,v'})(\Omega',1)$$

Corollary 4. *If we take $h(\sigma)=\sigma$ in Theorem 2, we obtain Hermite–Hadamard-type inequality for the Godunova–Levin FIV function.*

$$\frac{1}{4}F\left(\frac{\eth+v}{2}\right)(\mathcal{I}^{v-}_{\eth,v'})(\Omega',1) \preccurlyeq \frac{1}{2}\Big[\left(\mathcal{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big] \preccurlyeq \frac{F(\eth)\tilde{+}F(v)}{2}$$
$$\int_0^1 [\frac{\sigma^{v'-1}}{1-\sigma}]\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)d\sigma.$$

Corollary 5. *If we choose $h(\sigma)=\frac{1}{\sigma^s}$ in Theorem 2, we obtain Hermite–Hadamard-type inequality for the s-convex FIV function.*

$$2^{s-1}F\left(\frac{\eth+v}{2}\right)(\mathcal{I}^{v-}_{\eth,v'})(\Omega',1) \preccurlyeq \frac{1}{2}\Big[\left(\mathcal{I}^{v-}_{\eth,v'}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{\eth+}_{v,v'}\right)(\Omega',F)\Big]$$
$$\preccurlyeq \frac{F(\eth)\tilde{+}F(v)}{2}\int_0^1 [\sigma^s+(1-\sigma)^s]\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)d\sigma.$$

4. Applications of Trapezoid Type Inequalities via Pre-Invex Fuzzy Interval Valued Function (FIV)

In this section, we discuss the important result in the form of lemma, which are used to develop our main results related to the trapezoid-type inequalities by the implementation of generalized fractional operators for the h-Godunova–Levin pre-invex fuzzy interval valued function.

Lemma 1. *Let J be an open invex set with respect to $\varsigma : J \times J \to \mathbb{R}$, $\varsigma(v,\eth) > 0$ for $\eth, v \in J$, $F \in L_1[\eth, \eth + \varsigma(v,\eth)]$ be a differentiable function and $F : J = [\eth, \eth + \varsigma(v,\eth)] \to \mathbb{R}$ with $\eth, v \in \mathbb{R}$, then the following result holds:*

$$\frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}}\Big[\big(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\big)(\Omega';F)\widetilde{+} \big(\mathcal{I}^{(\eth+\beth_1(v,\eth))^-}_{\eth,v'-1}\big)(\Omega';F)\Big] = \frac{\varsigma(v,\eth)}{2}J \tag{27}$$

where $\Omega' = \frac{\Omega}{\varsigma(v,\eth)^{\Xi}}$ *and* $J = \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(\sigma)^{\Xi};p)F'(\eth+\sigma\varsigma(v,\eth))d\sigma + \int_0^1 -(1-\sigma)^{v'}$ $\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)F'(\eth+\sigma\varsigma(v,\eth))d\sigma$,

Proof. Consider the integral:

$$J = \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(\sigma)^{\Xi};p)F'(\eth+\sigma\varsigma(v,\eth))d\sigma + \int_0^1 -(1-\sigma)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)F'(\eth+\sigma\varsigma(v,\eth))d\sigma.$$

Therefore, by taking the value of $\imath \in [0,1]$, we have:

$$J = \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(\sigma)^{\Xi};p)F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma + \int_0^1 -(1-\sigma)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma. \tag{28}$$

Now, we take the integrals:

$$J_1 = \int_0^1 \sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(\sigma)^{\Xi};p)F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma$$
$$J_1 = \sum_{n=0}^{\infty}\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\int_0^1 \sigma^{v'+\Xi n}F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma.$$
$$J_2 = \int_0^1 -(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma.$$

$$J_2 = \sum_{n=0}^{\infty}\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\int_0^1 -(1-\sigma)^{v'+\Xi n}F'_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma.$$

For solving the integral J_1:

$$J_1 = \sum_{n=0}^{\infty}\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\Big[\sigma^{v'+\Xi n}\frac{F_*(\eth+\sigma\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)}\Big|_0^1 - \frac{v'+\Xi n}{\varsigma(v,\eth)}\int_0^1 \sigma^{v'+\Xi n-1}F_*(\eth+\sigma\varsigma(v,\eth),\imath)d\sigma\Big]$$
$$J_1 = \sum_{n=0}^{\infty}\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\Big[\frac{F_*(\eth+\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)} - \frac{v'+\Xi n}{\varsigma(v,\eth)}\int_0^1 \sigma^{v'+\Xi n-1}F(\eth+\sigma\varsigma(v,\eth))d\sigma\Big]$$
$$J_1 = \frac{F_*(\eth+\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}}\big(\mathcal{I}^{(\eth+\varsigma(v,\eth),\imath)^-}_{\eth,v'-1}\big)(\Omega';F_*)$$

By applying the same procedure for lower FIVF of the integral J_2, we have:

$$J_2 = \frac{F_*(\eth,\imath)}{\varsigma(v,\eth)} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}} \left(\mathcal{I}^{\eth^+}_{(\eth+\varsigma(v,\eth),\imath),v'-1}\right)(\Omega',F_*) \tag{29}$$

Substituting the values of J_1 and J_2 in (28), we have:

$$J = \frac{F_*(\eth,\imath)+F_*(\eth+\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}}$$
$$\left[\left(\mathcal{I}^{(\eth+\varsigma(v,\eth),\imath)^-}_{\eth,v'-1}\right)(\Omega',F_*) + \left(\mathcal{I}^{(\eth,\imath)^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F_*)\right]. \tag{30}$$

Similarly, solving the expression of J for upper FIVF F^*, we obtain:

$$J = \frac{F^*(\eth,\imath)+F^*(\eth+\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}} \times$$
$$\left[\left(\mathcal{I}^{(\eth+\varsigma(v,\eth),\imath)^-}_{\eth,v'-1}\right)(\Omega',F^*) + \left(\mathcal{I}^{(\eth,\imath)^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F^*)\right]. \tag{31}$$

By combining (30) and (31), we get:

$$J = \frac{F^*(\eth,\imath),F_*(\eth,\imath)+F^*(\eth+\varsigma(v,\eth),\imath),F_*(\eth+\varsigma(v,\eth),\imath)}{\varsigma(v,\eth)} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}} \times$$
$$\left[\left(\mathcal{I}^{(\eth+\varsigma(v,\eth),\imath)^-}_{\eth,v'-1}\right)(\Omega',F^*) + \left(\mathcal{I}^{(\eth,\imath)^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F^*), \left(\mathcal{I}^{(\eth+\varsigma(v,\eth),\imath)^-}_{\eth,v'-1}\right)(\omega',F^*) + \left(\mathcal{I}^{(\eth,\imath)^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F^*)\right].$$
$$J = \frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{\varsigma(v,\eth)} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{(\varsigma(v,\eth))^{v'+1}} \left[\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right.$$
$$\left.\widetilde{+}\left(\mathcal{I}^{\eth^+}_{(\eth+\varsigma(v,\eth)),v'-1}\right)(\Omega',F)\right]. \tag{32}$$

Multiplying by $\frac{\varsigma(v,\eth)}{2}$, we get the required result. □

Theorem 3. *Let $J \in \mathbb{R}$ be a differentiable function on J with a function $F : J = [\eth, \eth+\varsigma(v,\eth)] \to (0,\infty)$ for the generalized the fractional integral defined in (22) with the restricted extended generalized Bessel–Maitland function to a real valued function and suppose that $|F'|$ is a h-Godunova–Levin pre-invex FIV function on J, then we have:*

$$\left|\frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{2} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}} \left[\left(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F)\widetilde{+}\right.\right.$$
$$\left.\left.\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right]\right|$$
$$\preccurlyeq \frac{\varsigma(v,\eth)}{2}\left(|F'(\eth)|\widetilde{+}|F'(v)|\right) \int_0^1 \sum_{n=0}^{\infty} \left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\right| d\sigma.$$

Proof. By considering previous lemma and taking mod on both sides:

$$\left|\frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{2} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}} \left[\left(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F)\widetilde{+}\right.\right.$$
$$\left.\left.\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right]\right| = \left|\frac{\varsigma(v,\eth)}{2}J\right|$$

by taking the value $\imath \in [0,1]$, we have:

$$= \left|\frac{\varsigma(v,\eth)}{2}J\right|$$

$$\leq \frac{\varsigma(v,\eth)}{2}\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|\int_0^1\left|\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}\right|$$
$$|F'_*(\eth+\sigma\varsigma(v,\eth),\imath)|d\sigma$$
$$\leq \frac{\varsigma(v,\eth)}{2}\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|\int_0^1\left|\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}\right|$$
$$\left|\frac{F'_*](\eth)}{h(\sigma)}+\frac{F'_*(v)}{h(1-\sigma)}\right|d\sigma.$$

$$\leq \frac{\varsigma(v,\eth)}{2}\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left[|F'_*(\eth)|\int_0^1\left|\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}\right|\left|\frac{1}{h(\sigma)}\right|d\sigma+|F'_*(v)|\int_0^1\left|\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}\right|\left|\frac{1}{h(1-\sigma)}\right|d\sigma\right]$$
$$=\frac{\varsigma(v,\eth)}{2}\left(|F'_*(\eth,\imath)|+|F'_*(v,\imath)|\right)\int_0^1\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\right|d\sigma. \tag{33}$$

Similarly, if we solve for upper FIV function F^*, we have:

$$\leq \frac{\varsigma(v,\eth)}{2}\left(|F'^*(\eth,\imath)|+|F'^*(v,\imath)|\right)\int_0^1\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|\times$$
$$\left|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\right|d\sigma. \tag{34}$$

By combining Equations (33) and (34), we have the required result:

$$\leq \frac{\varsigma(v,\eth)}{2}\left(|F'_*(\eth,\imath)|+|F'_*(v,\imath)|, |F'^*(\eth,\imath)|+|F'^*(v,\imath)|\right)$$
$$\int_0^1\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|\left|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\right|d\sigma.$$
$$\preccurlyeq \frac{\varsigma(v,\eth)}{2}\left(|F'(\eth)|\widetilde{+}|F'(v)|\right)\int_0^1\sum_{n=0}^{\infty}\left|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\right|$$
$$\left|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\right|d\sigma.$$

□

Corollary 6. *Taking $\varsigma(v,\eth) = v - \eth$ in Theorem 3, we obtain the following inequality:*

$$\Big|\frac{F(\eth)\widetilde{+}F(v)}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2(v-\eth)^{v'}}\Big[(\mathcal{I}^{\eth+}_{v,v'-1})(\Omega',F)\widetilde{+}(\mathcal{I}^{v-}_{\eth,v'-1})(\Omega',F)\Big]\Big|$$
$$\preccurlyeq \frac{v-\eth}{2}(|F'(\eth)|\widetilde{+}|F'(v)|)\int_0^1\sum_{n=0}^{\infty}\Big|\frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth,c-\beth)\wp(\Xi n+v'+1)(\rho)_{mn}}(-\Omega)^n\Big|$$
$$\Big|\frac{\sigma^{v'+\Xi n}-(1-\sigma)^{v'+\Xi n}}{h(\sigma)}\Big|d\sigma.$$

Theorem 4. *Let $F : J = [\eth, \eth+\varsigma(v,\eth)] \to (0,\infty)$ be a differentiable function on $J \in \mathbb{R}$, and suppose that $|F'|^q$ is a h-Godunova–Levin pre-invex FIV function on J with $p > 1$ and $q = (p)(p-1)^{-1}$, then for the generalized fractional integral defined in (22) with the restricted extended generalized Bessel–Maitland function to a real valued function, we have:*

$$\Big|\frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}}\Big[(\mathcal{I}^{\eth+}_{\eth+\varsigma(v,\eth),v'-1})(\Omega',F)\widetilde{+}(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1})(\Omega',F)\Big]\Big|$$
$$\preccurlyeq \frac{\varsigma(v,\eth)}{2}(|F'(\eth)|^q\widetilde{+}|F'(v)|^q)^{\frac{1}{q}}$$
$$\Big(\int_0^1|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)|^p d\sigma\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{1}{h(\sigma)}d\sigma\Big)^{\frac{1}{q}}.$$

Proof. By using Lemma 1, we have:

$$\Big|\frac{F(\eth)\widetilde{+}F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}}\Big[(\mathcal{I}^{\eth+}_{\eth+\varsigma(v,\eth),v'-1})(\Omega',F)\widetilde{+}(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1})(\Omega',F)\Big]\Big|$$
$$= \Big|\frac{\varsigma(v,\eth)}{2}J\Big|$$

by taking the value $\imath \in [0,1]$, we have:

$$= \Big|\frac{\varsigma(v,\eth)}{2}J\Big|$$

$$\leq \frac{\varsigma(v,\eth)}{2}\int_0^1\Big|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\Big|\Big|F'_*(\eth+\sigma\varsigma(v,\eth),\imath)\Big|d\sigma$$

Using Hölder's integral inequality, we have:

$$\leq \frac{\varsigma(v,\eth)}{2}\Big(\int_0^1|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)|^p d\sigma\Big)^{\frac{1}{p}}\times$$
$$\Big(\int_0^1|F'_*(\eth+\sigma\varsigma(v,\eth),\imath)|^q d\sigma\Big)^{\frac{1}{q}} \tag{35}$$

where $p^{-1}+q^{-1}=1$.

Considering the h-Godunova–Levin pre-invex FIV function $|F'_*|^q$, we have:

$$\int_0^1|F'_*(\eth+\sigma\varsigma(v,\eth))|^q d\sigma \leq \int_0^1\Big(\frac{|F'(\eth)|^q}{h(\sigma)}+\frac{|F'(v)|^q}{h(1-\sigma)}\Big)d\sigma$$
$$\leq (|F'_*(\eth,\imath)|^q+|F'_*(v,\imath)|^q)\int_0^1\frac{1}{h(\sigma)}d\sigma. \tag{36}$$

By using Equation (36) in Equation (35), we obtain:

$$\leq \frac{\varsigma(v,\eth)}{2}\left(|F'_*(\eth,\imath)|^q + |F'_*(v,\imath)|^q\right)^{\frac{1}{q}}$$

$$\left(\int_0^1 \left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|^p d\sigma\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{h(\sigma)}d\sigma\right)^{\frac{1}{q}}. \quad (37)$$

Similarly, if we solve for upper FIV function for $|F'^*|^q$, we are given the following inequality:

$$\leq \frac{\varsigma(v,\eth)}{2}\left(|F'^*(\eth,\imath)|^q + |F'^*(v,\imath)|^q\right)^{\frac{1}{q}}$$

$$\left(\int_0^1 \left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|^p d\sigma\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{h(\sigma)}d\sigma\right)^{\frac{1}{q}}. \quad (38)$$

By combining Equations (37) and (38), we have:

$$\leq_I \frac{\varsigma(v,\eth)}{2}\left(|F'_*(\eth,\imath)|^q + |F'_*(v,\imath)|^q, |F'^*(\eth,\imath)|^q + |F'^*(v,\imath)|^q\right)^{\frac{1}{q}}$$

$$\left(\int_0^1 \left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|^p d\sigma\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{h(\sigma)}d\sigma\right)^{\frac{1}{q}}. \quad (39)$$

After replacing the value $|F'_*(\eth,\imath)|^q + |F'_*(v,\imath)|^q, |F'^*(\eth,\imath)|^q + |F'^*(v,\imath)|^q \preccurlyeq |F'(\eth)|^q \tilde{+} |F'(v)|^q$, we have the required result. □

Theorem 5. *With the assumptions of Theorem 4, we get the inequality related to Hermite–Hadamard inequality as follows:*

$$\left|\frac{F(\eth)\tilde{+}F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}}\left[\left(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right]\right|$$

$$\preccurlyeq \frac{\varsigma(v,\eth)}{2^{\frac{1}{q}}}\left(|F'(\eth)|^q\tilde{+}|F'(v)|^q\right)^{\frac{1}{q}}\left[\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v'+1,\daleth,\rho,\wp}(\Omega;p) - \left(\frac{1}{2}\right)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v'+1,\daleth,\rho,\wp}\left(\Omega\left(\frac{1}{2}\right)^{\Xi};p\right)\right]^{1-\frac{1}{q}}$$

$$\left[\int_0^1 \frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}d\sigma\right]^{\frac{1}{q}},$$

where $v', \Xi \in \mathbb{R}^+$.

Proof. Considering Lemma 1, we have:

$$\left|\frac{F(\eth)\tilde{+}F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega;p) - \frac{1}{2\varsigma(v,\eth)^{v'}}\left[\left(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F)\tilde{+}\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right]\right| = \left|\frac{\varsigma(v,\eth)}{2}J\right|$$

by taking the value $\imath \in [0,1]$, we have:

$$= \left|\frac{\varsigma(v,\eth)}{2}J\right|$$

$$\leq \frac{\varsigma(v,\eth)}{2}\int_0^1 \left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|\left|F'_*(\eth+\sigma\varsigma(v,\eth),\imath)\right|d\sigma.$$

By applying the power mean inequality, we get:

$$\left|\frac{F(\eth)+F(\eth+\varsigma(v,\eth))}{2}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega;p)-\frac{1}{2\varsigma(v,\eth)^{v'}}\left[\left(\mathcal{I}^{\eth^+}_{\eth+\varsigma(v,\eth),v'-1}\right)(\Omega',F)+\left(\mathcal{I}^{(\eth+\varsigma(v,\eth))^-}_{\eth,v'-1}\right)(\Omega',F)\right]\right|$$
$$\leq\frac{\varsigma(v,\eth)}{2}\left(\int_0^1\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|d\sigma\right)^{1-\frac{1}{q}}\times$$
$$\left(\int_0^1\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|\left|F'_*(\eth+\sigma\varsigma(v,\eth),\iota)\right|^q d\sigma\right)^{\frac{1}{q}}. \tag{40}$$

Let

$$I=\int_0^1\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|\left|F'_*(\eth+\sigma\varsigma(v,\eth),\iota)\right|^q d\sigma$$

By applying the definition of the h-Godunova–Levin pre-invex function FIV function on $\left|F'_*(\eth+\sigma\varsigma(v,\eth),\iota)\right|^q$, we have:

$$\leq\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}\left|F'_*(\eth,\iota)\right|^q$$
$$+\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(1-\sigma)}\left|F'_*(v,\iota)\right|^q\Bigg)d\sigma$$
$$=\left(|F'_*(\eth,\iota)|^q+|F'_*(v,\iota)|^q\right)$$
$$\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}. \tag{41}$$

Similarly, we solve for the upper FIV function $|F'^*|^q$:

$$I\leq\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}\left|F'^*(\eth,\iota)\right|^q$$
$$+\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(1-\sigma)}\left|F'^*(v,\iota)\right|^q\Bigg)d\sigma$$
$$=\left(|F'^*(\eth,\iota)|^q+|F'^*(v,\iota)|^q\right)\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}. \tag{42}$$

Combining the upper and lower value of FIV function for I, we have:

$$I\leq_I\left(|F'_*(\eth,\iota)|^q+|F'_*(v,\iota)|^q,|F'^*(\eth,\iota)|^q+|F'^*(v,\iota)|^q\right)$$
$$\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}.$$
$$\preccurlyeq\left(|F'(\eth)|^q\widetilde{+}|F'(v)|^q\right)\int_0^1\frac{\left|\sigma^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega\sigma^{\Xi};p)-(1-\sigma)^{v'}\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\daleth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p)\right|}{h(\sigma)}. \tag{43}$$

Now, consider the integral:

$$\int_0^1 \left| \sigma^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega\sigma^{\Xi};p) - (1-\sigma)^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v',\beth,\rho,\wp}(\Omega(1-\sigma)^{\Xi};p) \right|$$
$$= \sum_{n=0}^{\infty} \left| \frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n + v' + 1)(\rho)_{mn}} (-\Omega)^n \right| \int_0^1 \left| \sigma^{v'+\Xi n} - (1-\sigma)^{v'+\Xi n} \right| d\sigma$$
$$= \sum_{n=0}^{\infty} \left| \frac{\Lambda_p(\beth+\hbar n, c-\beth)(c)_{\hbar n}(\wp)_{\sigma n}}{\Lambda(\beth, c-\beth)\wp(\Xi n + v' + 1)(\rho)_{mn}} (-\Omega)^n \right| \left[\int_0^{\frac{1}{2}} \left((1-\sigma)^{v'+\Xi n} - \sigma^{v'+\Xi n}\right) d\sigma + \right.$$
$$\left. \int_{\frac{1}{2}}^1 \left(\sigma^{v'+\Xi n} - (1-\sigma)^{v'+\Xi n}\right) \right]$$
$$= 2\left[\mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v'+1,\beth,\rho,\wp}(\Omega;p) - (\frac{1}{2})^{v'} \mathcal{J}^{\Xi,\hbar,m,\sigma,c}_{v'+1,\beth,\rho,\wp}(\Omega(\frac{1}{2})^{\Xi};p) \right]. \quad (44)$$

Putting the values (43) and (44) in (40), we have the required result. □

5. Conclusions

In research work, we investigated the existence of inequalities such as Hermite–Hadamard-type inequalities and trapezoid-type inequalities for *h*-Godunova–Levin convex and pre-invex fuzzy interval valued functions by the implementation of FFIPs. The obtained inequalities are the generalizations and extensions by means of fuzzy interval valued functions. A lot of work could be conducted in the field of analysis by improving the convex and non-convex functions in FIVFs. In future work, we will attempt to investigate this idea for generalized convex fuzzy-IVFs and various applications in fuzzy-interval nonlinear programming. The new area of research in convex analysis and optimization theory can be found by applying this idea.

Author Contributions: Conceptualization, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; methodology, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; software, M.V.-C., R.S.A. and H.S.; validation, M.B.J., G.R. and Y.E.; formal analysis, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; investigation, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; resources, M.V.-C. and M.B.J.; data curation, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; writing—original draft preparation, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; writing—review and editing, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; visualization, M.V.-C., R.S.A., H.S., M.B.J., G.R. and Y.E.; supervision, M.V.-C., G.R. and Y.E.; project administration, Y.E.; funding acquisition, M.V.-C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not Applicable.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the Large Groups under grant number (RGP.2/120/44).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Abdeljawad, T.; Baleanu, D. Monotonicity results for fractional difference operators with discrete exponential kernels. *Adv. Differ. Equations* **2017**, *2017*, 1–9. [CrossRef]
2. Agarwal, R.; Purohit, D.S. A mathematical fractional model with nonsingular kernel for thrombin receptor activation in calcium signalling. *Math. Methods Appl. Sci.* **2019**, *42*, 7160–7171. [CrossRef]
3. Agarwal, R.; Yadav, M.P.; Baleanu, D.; Purohit, S. Existence and Uniqueness of Miscible Flow Equation through Porous Media with a Non Singular Fractional Derivative. *AIMS Math.* **2020**, *5*, 1062–1073. [CrossRef]
4. Khan, M.A.; Begum, S.; Khurshid, Y.; Chu, M.Y. Ostrowski type inequalities involving conformable fractional integrals. *J. Inequalities Appl.* **2018**, *2018*, 1–14.
5. Khan, M.A.; Chu, M.Y.; Kashuri, A.; Liko, R.; Ali, G. Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations. *J. Funct. Spaces* **2018**, *2018*, 6928130.

6. Mohammed, O.P.; Abdeljawad, T. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. *Adv. Differ. Equ.* **2020**, *2020*, 1–19. [CrossRef]
7. Mohammed, O.P.; Aydi, H.; Kashuri, A.; Hamed, S.Y.; Abualnaja, M.K. Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels. *Symmetry* **2021**, *13*, 550. [CrossRef]
8. Fejér, L. Uber die fourierreihen, II. *Math. Naturwiss. Anz. Ungar. Akad. Wiss* **1906**, *24*, 369–390.
9. Mehmood, S.; Zafar, F.; Asmin, N. New Hermite-Hadamard-Fejér type inequalities for (η_1, η_2)-convex functions via fractional calculus. *Sci. Asia* **2020**, *46*, 102–108. [CrossRef]
10. Aslani, S.M.; Delavar, M.R.; Vaezpour, S.M. Inequalities of Fejér Type Related to Generalized Convex Functions. *Int. J. Anal. Appl.* **2018**, *6*, 38–49.
11. Delavar, M.R.; Aslani, S.M.; Sen, M.D.L. Hermite-Hadamard-Fejér inequality related to generalized convex functions via fractional integrals. *J. Math.* **2018**, *2018*, 5864091.
12. Gordji, M.E.; Delavar, M.R.; Sen, M.D.L. On Φ-convex functions. *J. Math. Inequal.* **2016**, *10*, 173–183. [CrossRef]
13. Moore, E.R.; Kearfott, R.B.; Michael, J.C. *Introduction to Interval Analysis*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 2009.
14. Zhao, D.; An, T.; Ye, G.; Liu, W. New Jensen and Hermite-Hadamard type inequalities for h-convex interval-valued functions. *J. Inequalities Appl.* **2018**, *2018*, 302. [CrossRef]
15. Costa, T. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
16. Costa, T.M.; Romn-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
17. Romn-Flores, H.; Chalco-Cano, Y.; Lodwick, W. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
18. Romn-Flores, H.; Chalco-Cano, Y.; Silva, G.N. A note on Gronwall type inequality for interval-valued functions. In Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 24–28 June 2013; pp. 1455–1458.
19. Chalco-Cano, Y.; Flores-Franulic, A.; Romn-Flores, H. Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. *Comput. Appl. Math.* **2012**, *31*, 457–472.
20. Agarwal, P.A.; Baleanu, D.; Nieto, J.J.; Torres, F.D.; Zhou, Y. A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *J. Comput. Appl. Math.* **2018**, *339*, 3–29. [CrossRef]
21. Diamond, P.; Kloeden, P. Metric spaces of fuzzy sets. *Fuzzy Sets Syst.* **1990**, *35*, 241–249. [CrossRef]
22. Nanda, S.; Kar, K. Convex fuzzy mappings. *Fuzzy Sets Syst.* **1992**, *48*, 129–132. [CrossRef]
23. Noor, M.A. Fuzzy preinvex functions. *Fuzzy Sets Syst.* **1994**, *64*, 95–104. [CrossRef]
24. Bede, B.; Gal, S.G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. [CrossRef]
25. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Hamed, Y.S. New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities. *Symmetry* **2021**, *13*, 673. [CrossRef]
26. Khan, B.M.; Noor, M.A.; Shah, N.A.; Abualnaja, K.M.; Botmart, T. Some New Versions of Hermite-Hadamard Integral Inequalities in Fuzzy Fractional Calculus for Generalized Pre-Invex Functions via Fuzzy-Interval-Valued Settings. *Fractal Fract.* **2022**, *6*, 83. [CrossRef]
27. Toader, G.H. Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization*; Universitatea Cluj-Napoca: Cluj Napoca, Romania, 1984; pp. 329–338.
28. Peajcariaac, J.E.; Tong, Y.L. *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press: Cambridge, MA, USA, 1992.
29. Qiang, X.; Farid, G.; Yussouf, M.; Khan, K.A.; Rahman, A.U. New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions. *J. Inequalities Appl.* **2020**, *2020*, 1–13. [CrossRef]
30. Iscan, I.; Wu, S. Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **2014**, *238*, 237–244.
31. Ion, D.A. Some estimates on the Hermite-Hadamard inequality through quasi-convex functions. *Ann. Univ. Craiova-Math. Comput. Sci. Ser.* **2007**, *34*, 82–87.
32. Stefanini, L.; Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 1311–1328. [CrossRef]
33. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
34. Kaleva, O. Fuzzy Differential Equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
35. Khan, M.B.; Noor, M.A.; Al-Shomrani, M.M.; Abdullah, L. Some novel inequalities for LR-h-convex interval-valued functions by means of pseudo-order relation. *Math. Methods Appl. Sci.* **2022**, *45*, 1310–1340. [CrossRef]
36. Rainville, E.D. *Special Functions*; Chelsea Publ. Co.: Bronx, NY, USA, 1971.
37. Petojevic, A. A note about the Pochhammer symbol. *Math. Moravica* **2008**, **12**, 37–42. [CrossRef]
38. Mubeen, S.; Ali, S.R.S.; Nayab, I.; Rahman, G.; Abdeljawad, T.; Nisar, K.S. Integral transforms of an extended generalized multi-index Bessel function. *AIMS Math.* **2020**, *5*, 7531–7547. [CrossRef]
39. Ali, R.S.; Mubeen, S.; Nayab,I.; Araci,S.; Rahman,G.; Nisar, K.S. Some Fractional Operators with the Generalized Bessel-Maitland Fuction *Discret. Dyn. Nat. Soc.* **2020**, *2020*, 1378457. [CrossRef]

40. Ali, S.; Mubeen, S.; Ali, R.S.; Rahman, G.; Morsy, A.; Nisar, K.S.; Purohit, S.D.; Zakarya, M. Dynamical significance of generalized fractional integral inequalities via convexity. *AIMS Math.* **2021**, *6*, 9705–9730. [CrossRef]
41. Ali, R.S.; Mubeen, S.; Ali, S.; Rahman, G.; Younis, J.; Ali, A. Generalized Hermite–Hadamard-Type Integral Inequalities for-Godunova–Levin Functions. *J. Funct. Spaces* **2022**, *2022*, 9113745. [CrossRef]