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Alternate Admissibility LMI Criteria for Descriptor Fractional Order Systems with $0 < \alpha < 2$

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Abstract: The paper focuses on the admissibility problem of descriptor fractional-order systems (DFOSs). The alternate admissibility criteria are addressed for DFOSs with order in $(0, 2)$ which involve a non-strict linear matrix inequality (LMI) method and a strict LMI method, respectively. The forms of non-strict and strict LMIs are brand new and distinguished with the existing literature, which fills the gaps of studies for admissibility. These necessary and sufficient conditions of admissibility are available to the order in $(0, 2)$ without separating the order ranges into $(0, 1)$ and $[1, 2)$. Based on the special position of singular matrix, the non-strict LMI criterion has an advantage in handling the DFOSs with uncertain derivative matrices. For the strict LMI form, a method involving least real decision variables is derived which is more convenient to process the practical solution. Three numerical examples are given to illustrate the validity of the proposed results.

Keywords: descriptor fractional order systems; admissibility; unified criterion; linear matrix inequality



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1. Introduction

Fractional calculus has a long period of development which has been broadly applied in different areas of engineering applications, such as systems theory [1], signal processing [2] and image fusion [3]. Recently, fractional-order systems (FOSs) have aroused extensive attention from scholars because more and more practical problems based on engineering requirements are well described by fractional calculus [4]. Many fundamental notions and crucial research achievements on integer order systems have been expanded to FOSs successfully, and hugely fruitful research has been published in stability analysis [5] and sliding mode control [6,7]. Furthermore, the special characteristics of fractional-order are broadly applied in electrical systems [8,9], power systems [10,11], economic systems [12] and fuzzy systems [13,14].

Stability analysis is a fundamental issue for all control systems, certainly including FOSs. Since the stability region of FOSs is quite different from integer order systems, exploration on stability becomes more challenging. Experts have developed many research works and a number of theories are obtained. In [15], based on the FOS stability region, the LMI stability conditions are first proposed for fractional order $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively. However, the criterion for $0 < \alpha < 1$ contains complex numbers and is difficult to solve. In [16], the stability of FOSs with order in $(0, 1)$ is studied and a method for the robust asymptotical stability with real matrices is proposed, but the result is inapplicable to the system eigenvalues on positive real part. In [17], the authors provide a unified LMI formulation to ensure the stability of FOSs for a given order in $(0, 2)$ without separating into $(0, 1)$ and $[1, 2)$. Those criteria are necessary and sufficient conditions with least real decision variables of LMIs. In [18], the FOSs with arbitrary real order between 0 and 2 are considered and a method to quickly and robustly estimate the fractional integrals and derivatives of positions is presented. Robust stability analysis of an interval fractional-order plant with an interval time delay is investigated by a general form of fractional-order

controllers in [19]. For the discrete-time systems, the finite-dimensional feedback control of FOSs with additive state disturbance is addressed in [20] and the stability regions of FOSs with interval uncertainties are analyzed in [21]. In [22], by using a boundary layer technique without any global and unknown information, the robust adaptive fault-tolerant consensus control for nonlinear fractional-order multi-agent systems is addressed.

Descriptor systems (also called singular systems, generalized state-space systems and implicit systems) are a special class of systems with a wider range of applications than normal systems, many of which have good performance [23,24]. With the emergence of FOSs, DFOSs have aroused great attention of scholars in different research directions. A great number of attainments on admissibility have been achieved because admissibility is one of the most important properties in DFOSs. For the fractional order in $[1, 2)$, the new admissibility conditions of DFOSs expressed in a set of strict LMIs are given in [25,26], and H_∞ control problems have been solved by designing a state feedback control based on bounded real lemma in [27]. For the fractional order in $(0, 1)$, the authors in [28] provide necessary and sufficient conditions for admissibility of DFOSs and an observer-based controller is proposed to guarantee the system admissibility. However, the LMIs in the results involve complex matrices and complex numbers which cause difficulty in solving. In [29], the admissibility and robust stabilization of DFOSs with order in $(0, 1)$ are investigated, and an approach with strict LMIs with real matrices is presented. In [30], a different method for admissibility is reported for order in $(0, 1)$ and $[1, 2)$, respectively, and robust stabilization problem of DFOSs with uncertain derivative matrices is solved. Although there are many papers on admissibility, most of the existing theorems divide order α into $(0, 1)$ and $[1, 2)$ to analysis admissibility, respectively, as shown above. A unified form of admissibility for a given fractional order interval $(0, 2)$ is valuable and has been considered in few studies. In [31], a unified framework for admissibility and quadratic admissibility is provided in terms of LMIs, but it does not satisfy the condition of least real decision variables which can be improved.

Motivated by the above observations, we study the admissibility of DFOSs. The main contributions of this paper are summarized as follows:

- (i) The alternate admissibility criteria with order in $(0, 2)$ are presented with non-strict LMIs and strict LMIs, respectively. These necessary and sufficient conditions are fresh and have many good features of existing results, some of which are that these theorems do not involve any complex variables and are able to deal with the eigenvalues of system matrix with positive real part.
- (ii) The methods are applicable to the order interval $(0, 2)$ directly without separating the order ranges into $(0, 1)$ and $[1, 2)$ when discussing the admissibility of DFOSs, which overcomes the drawback brought by interval separation in [25–30].
- (iii) A strict LMI approach with only one real decision variable is provided, which owns a simpler expression and is easier to simulate compared with the methods in [25–31].
- (iv) When $E = I$ or $\alpha = 1$, the criteria in this paper are consistent with the results in [15,29] and [32], respectively. In additions, the method of non-strict LMIs is applicable to the DFOSs with uncertainties in singular matrix.

The paper is organized as follows. Section 2 presents some preliminaries which are applicable to the rest of the paper. In Section 3, for the fractional order α in $(0, 2)$, two different methods of admissibility for DFOSs are proposed with non-strict and strict LMI formulations, respectively. Three numerical examples are given in Section 4 and a brief conclusion is provided in Section 5.

Notations : $\mathbb{R}^{n \times m}$ denotes the set of all $n \times m$ real matrices. $P < 0$ ($P \leq 0$, respectively) means that P is negative definite (negative semi-definite, respectively). P^T represents the transpose of matrix P . $P > 0$ denotes that P is positive definite. $\det(A)$ is the determinant of A . $[x_{ij}]_{n \times n}$ represents a $n \times n$ matrix where x_{ij} is the element of the i -th row and the j -th column. $\text{sym}(P)$ and $\text{asym}(P)$ denote the expressions of $P + P^T$ and $P - P^T$, respectively. I_n is the identity matrix of order n . $\otimes$ stands for the Kronecker product of two matrices. $\text{spec}(E, A)$ is the spectrum of $\det(sE - A) = 0$. $\arg(z)$ denotes the argument of a complex

number z . $\Gamma(\cdot)$ indicates Euler Gamma function. $\lfloor \alpha \rfloor$ ($\lceil \alpha \rceil$) is the floor function (ceiling function, respectively) which denotes the nearest integer less (greater) than or equal to α . The symbol $*$ represents the symmetric part of a matrix. The symbol $\star$ in a matrix denotes the part which is unrelated to the discussion. For convenience, let $\Theta = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$ in the sequel.

2. Problem Formulation and Preliminaries

Consider the following DFOS

$$ED^\alpha x(t) = Ax(t), \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ is a constant matrix; $E \in \mathbb{R}^{n \times n}$ is singular with $\text{rank}(E) = r < n$; $x(t) \in \mathbb{R}^n$ is state of the system; $D^\alpha x(t)$ is the Caputo fractional order derivative with the following definition

$$D^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,$$

where $m-1 < \alpha \leq m$, m is an integer. If $E = I$, (1) is simplified to FOS

$$D^\alpha x(t) = Ax(t). \quad (2)$$

Lemma 1 ([29]). Assume that System (1) is regular, and two invertible matrices G and W are found such that

$$GEW = \begin{bmatrix} I_r & 0 \\ 0 & J_{n-r} \end{bmatrix}, GAW = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad (3)$$

where J_{n-r} is a nilpotent matrix; then, we have:

(a) System (1) is impulse-free if $J_{n-r} = 0$.

(b) System (1) is stable if $|\arg(\text{spec}(\bar{A}_1))| > \alpha \frac{\pi}{2}$.

(c) System (1) is admissible if $J_{n-r} = 0$ and $|\arg(\text{spec}(\bar{A}_1))| > \alpha \frac{\pi}{2}$.

Lemma 2 ([29]). FOS (2) with $0 < \alpha < 1$ is stable if there exist matrices $X, Y \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \quad (4)$$

$$\text{sym}(A(aX - bY)) < 0. \quad (5)$$

Lemma 3 ([15]). FOS (2) with $1 \leq \alpha < 2$ is stable if there exist positive matrix P such that

$$\text{sym}(\Theta \otimes AP) < 0. \quad (6)$$

3. Main Results

In this section, for the DFOSs with order in $(0, 2)$, new approaches of admissibility based on non-strict LMIs and strict LMIs are addressed, respectively.

3.1. Criteria of Admissibility Based on Non-Strict LMIs

Theorem 1. DFOS (1) with $0 < \alpha < 1$ is admissible if A is nonsingular and there exist matrices $X, Y, Q \in \mathbb{R}^{n \times n}$, $Q > 0$ satisfying (4) and

$$A(aX - bY)E^T + E(aX - bY)^T A^T + EQE^T \leq 0. \quad (7)$$

Proof. (Sufficiency). Since A is nonsingular, $\det(s^{\alpha}E - A) \neq 0$ for some $s \in \mathbb{C}$, so the system (1) is regular. Then, there exist two nonsingular matrices G and W such that (3) holds.

Let

$$X = W \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} W^T, \quad X_1 = X_1^T > 0, \quad X_3 = X_3^T > 0, \quad (8)$$

$$Y = W \begin{bmatrix} Y_1 & Y_2 \\ -Y_2^T & Y_3 \end{bmatrix} W^T, \quad Y_1^T = -Y_1, \quad Y_3^T = -Y_3, \quad (9)$$

$$Q = W \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} W^T, \quad Q_1 = Q_1^T > 0, \quad Q_3 = Q_3^T > 0. \quad (10)$$

From the condition (4)

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} X_1 & X_2 & Y_1 & Y_2 \\ X_2^T & X_3 & -Y_2^T & Y_3 \\ -Y_1 & -Y_2 & X_1 & X_2 \\ Y_2^T & -Y_3 & X_2^T & X_3 \end{bmatrix} \begin{bmatrix} W^T & 0 \\ 0 & W^T \end{bmatrix} > 0, \quad (11)$$

it is easy to obtain

$$\begin{bmatrix} X_1 & Y_1 \\ -Y_1 & X_1 \end{bmatrix} > 0. \quad (12)$$

Substituting (3), (8), (9) and (10) into (7), we get

$$U = \begin{bmatrix} U_1 & U_2 \\ U_2^T & U_3 \end{bmatrix} \leq 0, \quad (13)$$

where

$$\begin{aligned} U_1 &= \bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^T \bar{A}_1^T + Q_1, \\ U_2 &= \bar{A}_1(aX_2 - bY_2)J_{n-r}^T + (aX_2 + bY_2) + Q_2J_{n-r}^T, \\ U_3 &= (aX_3 - bY_3)J_{n-r}^T + J_{n-r}(aX_3 - bY_3)^T + J_{n-r}Q_3J_{n-r}^T. \end{aligned}$$

(13) implies $U_1 < 0$ or $U_1 \leq 0$. Considering $Q_1 > 0$, we have

$$\bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^T \bar{A}_1^T < 0. \quad (14)$$

Hence, X_1, Y_1 satisfy (12) and (14). By Lemma 2, we obtain that (1) is stable.

Without loss of generality, assume that the nilpotent matrix $J_{n-r} \neq 0$ which owns the Jordan form

$$J_{n-r} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix}. \quad (15)$$

Write X_3, Y_3, Q_3 as

$$X_3 = [x_{ij}]_{t \times t} > 0, \quad x_{ij} = x_{ji}, \quad (16)$$

$$Y_3 = [y_{ij}]_{t \times t}, \quad y_{ij} = -y_{ji} \ (i \neq j), \quad y_{ii} = 0, \quad (17)$$

$$Q_3 = [q_{ij}]_{t \times t} > 0, \quad q_{ij} = q_{ji}, \quad (18)$$

where $t = n - r$ and $i, j = 1, \dots, n$. Then U_3 is rewritten as

$$\begin{bmatrix} 0 & ax_{11} & \star & \star \\ ax_{11} & 2(ax_{12} - by_{12}) + q_{11} & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}.$$

Since

$$\det \begin{bmatrix} 0 & ax_{11} \\ ax_{11} & 2(ax_{12} - by_{12}) + q_{11} \end{bmatrix} < 0,$$

U_3 contains two eigenvalues with different sign, so U is neither negative definite (negative semi-definite) nor positive definite (positive semi-definite) which conflicts with $U \leq 0$. It follows that $J_{n-r} = 0$. By Lemma 1, we get that (1) is impulse-free. This, together with the regularity and stability of (1), deduces that (1) is admissible.

(Necessity). According to Lemma 1 and the admissible condition of (1), there exist nonsingular matrices G, W such that (3) holds with $J_{n-r} = 0$, and $D^\alpha x_1(t) = \bar{A}_1 x_1(t)$ is stable. By Lemma 2, there exist matrices X_1, Y_1 satisfying (12) and (14) which deduces that $\bar{A}_1$ is invertible. Then, there exists a matrix $Q_1 > 0$ such that

$$\bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^T \bar{A}_1^T + Q_1 \leq 0. \quad (19)$$

From the invertible matrix $\bar{A}_1$ and (3), we get that A is also invertible.

Let

$$X = W \begin{bmatrix} X_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} W^T, \quad Y = W \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix} W^T, \quad Q = W \begin{bmatrix} Q_1 & 0 \\ 0 & I \end{bmatrix} W^T. \quad (20)$$

From (12) and (20), we get

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} X_1 & 0 & Y_1 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ -Y_1 & 0 & X_1 & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} W^T & 0 \\ 0 & W^T \end{bmatrix} > 0.$$

Substituting (3) and (20) into (7), we obtain

$$\begin{aligned} & A(aX - bY)E^T + E(aX - bY)^T A^T + EQE^T \\ &= G^{-1} \begin{bmatrix} A_1(aX_1 - bY_1) + (aX_1 - bY_1)^T A_1^T + Q_1 & 0 \\ 0 & 0 \end{bmatrix} G^{-T} \leq 0. \end{aligned}$$

□

Corollary 1. DFOS (1) with $0 < \alpha < 1$ is admissible if A is nonsingular and there exist matrices $X, Y, \hat{Q} \in \mathbb{R}^{n \times n}$, $\hat{Q} > 0$ satisfying (4) and

$$\begin{bmatrix} \text{sym}(A(aX - bY)E^T) & E \\ * & -\hat{Q} \end{bmatrix} \leq 0. \quad (21)$$

Proof. Setting $\hat{Q} = Q^{-1}$, by Schur complement, it is easy to get that (21) is equivalent to (7). □

Remark 1. The formulations in Theorem 1 are new and different from the existing literature; they enrich the research methods of admissibility for DFOSs. Moreover, this new approach can easily deal with the uncertainties in singular matrix based on the special position of E in (21).

Corollary 2. DFOS (1) with $0 < \alpha < 1$ is admissible if A is nonsingular and there exist matrices P and $Q > 0$ such that

$$\begin{bmatrix} \text{sym}(P) & \text{asym}(P) \\ -\text{asym}(P) & \text{sym}(P) \end{bmatrix} > 0, \quad (22)$$

$$A\Phi E^T + E\Phi^T A^T + EQE^T \leq 0, \quad (23)$$

where $\Phi = a \cdot \text{sym}(P) - b \cdot \text{asym}(P)$.

Proof. According to (22), we get $\text{sym}(P) > 0$ and $(\text{asym}(P))^T = -\text{asym}(P)$.

Let $\text{sym}(P) = X$, $\text{asym}(P) = Y$, where $X > 0$ and $Y^T = -Y$. It is easy to get

$$P = \frac{1}{2}(X + Y), \quad P^T = \frac{1}{2}(X - Y). \quad (24)$$

Therefore, (22) and (23) are equivalent to (4) and (7), respectively. By Theorem 1, one concludes that (1) is admissible. $\square$

Remark 2. Corollary 2 provides a method to reduce the count of decision variables. Matrices X and Y are replaced by a single matrix P which simplifies the expressions in Theorem 1 and is easier to solve than the exiting approach.

Theorem 2. DFOS (1) with $1 \leq \alpha < 2$ is admissible if A is nonsingular and there exist two positive matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$\Theta \otimes APE^T + \Theta^T \otimes EPA^T + I_2 \otimes EQE^T \leq 0. \quad (25)$$

Proof. Using the similar proof method of Theorem 1, we deduce that Theorem 2 holds. $\square$

Corollary 3. DFOS (1) with $1 \leq \alpha < 2$ is admissible if A is nonsingular and there exist two positive matrices $P, \hat{Q} \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} \text{sym}(\Theta \otimes APE^T) & I_2 \otimes E \\ * & -I_2 \otimes \hat{Q} \end{bmatrix} \leq 0. \quad (26)$$

Proof. The proof of Corollary 3 is similar to that in Corollary 1 and is omitted. $\square$

Remark 3. For the order in $[1, 2)$, a different criterion for admissibility is proposed in Theorem 2. Considering the special form of equation (26), this method is convenient to handle the DFOSs with uncertain singular matrix E .

Theorem 3. DFOS (1) with $0 < \alpha < 2$ is admissible if A is nonsingular and there exist matrices P and $Q > 0$ such that

$$\begin{bmatrix} \text{sym}(P) & \text{asym}(P) \\ [\alpha - 1]\text{asym}(P) & \text{sym}(P) \end{bmatrix} > 0, \quad (27)$$

$$\text{sym}(\Theta_\alpha \otimes AP_\alpha E^T) + I_{[\alpha]} \otimes EQE^T \leq 0, \quad (28)$$

where $\Theta_\alpha = \Theta([\alpha])$, $\Theta(1) = \det(\Theta)$, $\Theta(2) = \Theta$, $P_\alpha = a^{-[\alpha-1]} \cdot \text{sym}(P) + b[\alpha - 1]\text{asym}(P)$.

Proof. When $0 < \alpha < 1$, it is easy to get $\Theta_\alpha = 1$, $P_\alpha = a \cdot \text{sym}(P) - b \cdot \text{asym}(P)$. Then, we obtain that (27) and (28) yield to (4) and (7). Thus, Theorem 3 is equivalent to Theorem 1 in this case.

When $1 \leq \alpha < 2$, we have $\Theta_\alpha = \Theta$, $P_\alpha = \text{sym}(P)$. From equations (27) and (28), we obtain

$$\begin{bmatrix} P_\alpha & 0 \\ 0 & P_\alpha \end{bmatrix} > 0, \quad (29)$$

$$\text{sym}(\Theta \otimes AP_\alpha E^T) + I_2 \otimes EQE^T \leq 0. \quad (30)$$

It is easy to see that (29) and (30) are equivalent to $P_\alpha > 0$ and (25) in Theorem 2. $\square$

Remark 4. Theorem 3 is an unified form of Theorems 1 and 2, which are necessary and sufficient conditions without complex number and are able to handle eigenvalues of system matrix A with positive real part. In the case of $E = I$, Theorem 3 is consistent with Lemmas 2 and 3 for α in $(0, 1)$ and $[1, 2)$, respectively. When $\alpha = 1$, Theorem 3 is the same as admissibility conditions for integer systems in [32].

Remark 5. The inequality (28) cannot be solved directly because it contains an equality constraint. In order to tackle this issue, nonsingular matrices G , W and a column full rank matrix S which satisfies $ES = 0$ are introduced in the following theorems. Nonsingular matrices G and W satisfy (3) which is deduced by the nonsingular condition of A . Then, a strict LMI without equality constraint is further formulated.

3.2. Criteria of Admissibility Based on Strict LMIs

Theorem 4. DFOS (1) with $0 < \alpha < 1$ is admissible if A is nonsingular and there exist matrices X, Y satisfying (4) and

$$A(aX - bY)E^T + E(aX - bY)^T A^T - (WG)^{-1}SS^T(WG)^{-T} < 0, \quad (31)$$

where W, G satisfy (3), and S is any matrix with full column rank which satisfies $ES = 0$.

Proof. The proof is similar to Theorem 1 and is simplified as follows:

(Sufficiency) Since A is nonsingular, there exist two invertible matrices G and W satisfying (3). Set the form of X, Y as (8) and (9), respectively. Under the condition (4), it is easy to obtain (12).

Let

$$S = W \begin{bmatrix} 0 \\ S_2 \end{bmatrix}, \quad (32)$$

which satisfies $ES = 0$. Therefore, $J_{n-r}S_2 = 0$. Substituting (3), (8), (9) and (32) into (31), we have

$$\tilde{U} = \begin{bmatrix} \tilde{U}_1 & \tilde{U}_2 \\ \tilde{U}_2^T & \tilde{U}_3 \end{bmatrix} < 0, \quad (33)$$

where

$$\tilde{U}_1 = \bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^T \bar{A}_1^T,$$

$$\tilde{U}_2 = \bar{A}_1(aX_2 - bY_2)J_{n-r}^T + (aX_2 - bY_2)^T \bar{A}_1^T,$$

$$\tilde{U}_3 = (aX_3 - bY_3)J_{n-r}^T + J_{n-r}(aX_3 - bY_3)^T - S_2S_2^T.$$

(33) implies $\tilde{U}_1 < 0$. According to $\tilde{U}_1 < 0$ and (12), by Lemma 2, one has that (1) is stable.

Without loss of generality, suppose that the nilpotent matrix $J_{n-r} \neq 0$ which owns the Jordan form (15).

Choose S_2 as

$$S_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ s_{1t} & s_{2t} & \cdots & s_{tt} \end{bmatrix},$$

where $t = n - r$, $s_{it} \neq 0$ is an arbitrary real number for $i = 1, \dots, n$, and S_2 satisfies $J_{n-r}S_2 = 0$.

Write X_3, Y_3 as (16), (17). Then, $\tilde{U}_3$ is rewritten as

$$\begin{bmatrix} 0 & ax_{11} & \star & \star \\ ax_{11} & 2(ax_{12} - by_{12}) & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}.$$

Since

$$\det\left(\begin{bmatrix} 0 & ax_{11} \\ ax_{11} & 2(ax_{12} - by_{12}) \end{bmatrix}\right) < 0,$$

$\tilde{U}_3$ contains two eigenvalues with different sign which is conflicted with $\tilde{U} < 0$. It follows that $J_{n-r} = 0$. Applying Lemma 1, one concludes that (1) is impulse-free. This together with the regularity and stability of (1) deduces that (1) is admissible.

(Necessity) According to Lemma 1 and the admissible condition of (1), there exist nonsingular matrices G, W such that (3) holds. By Lemma 2, there exist matrices X_1, Y_1 satisfying (12) and (14) which deduces that A is invertible. Setting the expressions of X, Y as (20), it is easy to see that (4) holds.

Choose S as

$$S = W \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad (34)$$

which satisfies $ES = 0$. Substituting (3), (20) and (34) into (31), one has

$$\begin{aligned} & A(aX - bY)E^T + E(aX - bY)^T A^T - (WG)^{-1} \hat{E} \hat{E}^T (WG)^{-T} \\ &= G^{-1} \begin{bmatrix} A_1(aX_1 - bY_1) + (aX_1 - bY_1)^T A_1^T & 0 \\ 0 & -I \end{bmatrix} G^{-T} < 0. \end{aligned}$$

□

Theorem 5. DFOS (1) with $0 < \alpha < 1$ is admissible if A is nonsingular and there exists a matrix P satisfying (22) and

$$A\Phi E^T + E\Phi^T A^T - (WG)^{-1} S S^T (WG)^{-T} < 0, \quad (35)$$

where $\Phi = a \cdot \text{sym}(P) - b \cdot \text{asym}(P)$.

Proof. The proof is similar to that in Corollary 2 and is therefore omitted. □

Theorem 6. DFOS (1) with $1 \leq \alpha < 2$ is admissible if A is nonsingular and there exists a matrix $P > 0$ such that

$$\Theta \otimes A P E^T + \Theta^T \otimes E P A^T - I_2 \otimes (WG)^{-1} S S^T (WG)^{-T} < 0, \quad (36)$$

where W, G and S are given in Theorem 4.

Proof. Using the method of Theorem 4, we can easily obtain Theorem 6. □

Theorem 7. DFOS (1) with $0 < \alpha < 2$ is admissible if A is nonsingular and there exist a matrix P satisfying (27) and

$$\text{sym}(\Theta_\alpha \otimes A P_\alpha E^T) - I_{[\alpha]} \otimes (WG)^{-1} S S^T (WG)^{-T} < 0, \quad (37)$$

where Θ_α and P_α are given in Theorem 3.

Proof. The proof is similar to that in Theorem 3, so the details are omitted. $\square$

Remark 6. In Theorem 7, a strict LMI criterion of admissibility is proposed. This new approach is an unified form of Theorems 5 and 6, and it is easy to cope with the eigenvalues with positive real part. In addition, the LMIs in (27) and (37) only contain one real decision variable which is easy to solve. In the case of $E = I$, Theorem 7 is also consistent with Lemmas 2 and 3 for α in $(0, 1)$ and $[1, 2)$, respectively.

4. Numerical Examples

Three numerical examples are provided to illustrate the effectiveness of theorems based on non-strict LMIs and strict LMIs.

Example 1. Consider system (1) with $\alpha = 0.4$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By solving non-strict LMIs in Theorem 3, we obtain the feasible solutions as follows:

$$P = \begin{bmatrix} 0.5621 & -0.4766 & 0 \\ 0.4766 & 0.5621 & 0 \\ 0 & 0 & 0.5000 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.8814 & 0 & 0 \\ 0 & 0.8814 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

Remark 7. The correctness of Theorem 3 is verified in Example 1 by solving Equations (27) and (28). Since (28) contains an equality constraint which cannot be solved by any simulation software directly, we need to convert the matrices A and E into the form of (3). A feasible approach to get the solution is solving the following strict LMI

$$\bar{A}_1(aX_1 - bY_1) + (aX_1 - bY_1)^T \bar{A}_1^T + Q_1 < 0.$$

we can get the values of X_1 , Y_1 and Q_1 . Construct matrices X , Y and Q with the form of (20). Then, the feasible solutions of P and Q is obtained, where $P = \frac{1}{2}(X + Y)$.

Example 2. Consider a DFOS in (1) with

$$\alpha = 0.5, \quad E = \begin{bmatrix} -3 & -1 & 2 & 5 \\ 1 & 1 & -1 & -3 \\ -3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 2 & 2 & 1 \\ -3 & -2 & -4 & -5 \\ 10 & 1 & -7 & -16 \\ 3 & 4 & -2 & -7 \end{bmatrix}.$$

Since A is nonsingular, the system (1) is regular. Due to $\deg(\det(sE - A)) = \text{rank}(E) = 3$, the system (1) is impulse-free. The roots of $\det(sE - A) = 0$ are -2 and $1 \pm 3j$. By Lemma 1, it is easy to verify that the DFOS with parameters in Example 2 is admissible. By solving strict LMIs (27) and (37) in Theorem 7, we get the following feasible solution of real matrix P :

$$P = \begin{bmatrix} 2.3934 & -1.7708 & -0.7703 & 2.3777 \\ 1.8095 & 8.1164 & -2.3734 & 2.8106 \\ 0.2597 & -0.8788 & 4.7747 & -2.3535 \\ 2.1066 & -1.3637 & -2.5610 & 3.3455 \end{bmatrix}.$$

The state responses are displayed in Figure 1.

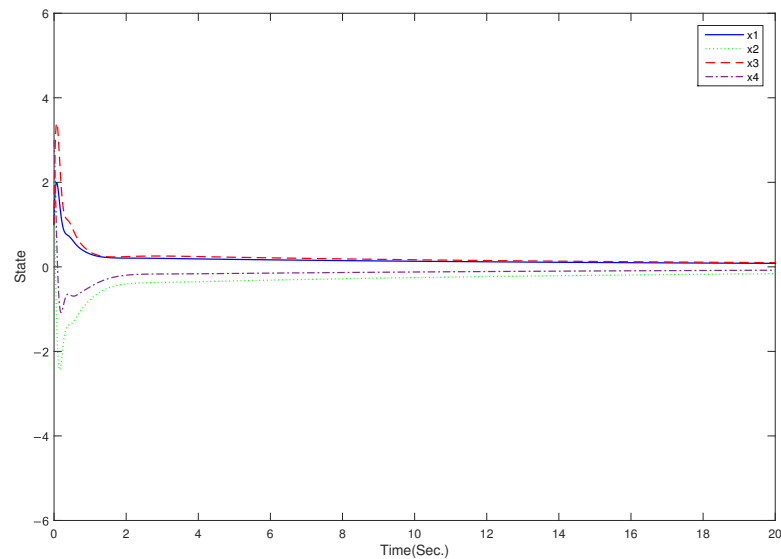


Figure 1. The state responses with $\alpha = 0.5$ in Example 2.

Example 3. Consider a DFOS in (1) with $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.5$, respectively, and

$$E = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -5 & -2 & -6 \\ 2 & 1 & 4 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & -6 & 1 & -7 \end{bmatrix}.$$

By Lemma 1, it is easy to get that the DFOS in Example 3 is also admissible. By Theorem 7, we obtain the following feasible solution:

Case $\alpha = 0.5$,

$$P = \begin{bmatrix} 0.7020 & -0.0885 & -0.3003 & 0.0798 \\ 0.1775 & 0.4749 & 0.0800 & -0.4463 \\ -0.2594 & 0.0377 & 0.2139 & -0.0457 \\ -0.1554 & -0.4179 & -0.1077 & 0.4209 \end{bmatrix};$$

Case $\alpha = 1$,

$$P = \begin{bmatrix} 7.6362 & -3.4359 & -4.4911 & 3.2509 \\ -3.4359 & 3.4431 & 2.0794 & -3.1028 \\ -4.4911 & 2.0794 & 3.0249 & -2.0289 \\ 3.2509 & -3.1028 & -2.0289 & 2.9524 \end{bmatrix};$$

Case $\alpha = 1.5$,

$$P = \begin{bmatrix} 12.1598 & -6.1003 & -7.1349 & 5.7156 \\ -6.1003 & 5.0981 & 3.4141 & -4.4487 \\ -7.1349 & 3.4141 & 5.2904 & -3.3692 \\ 5.7156 & -4.4487 & -3.3692 & 4.2455 \end{bmatrix}.$$

From the state responses in Figure 2 which are simulated based on the data above, we can see that the system quickly reaches stability after 10 s.

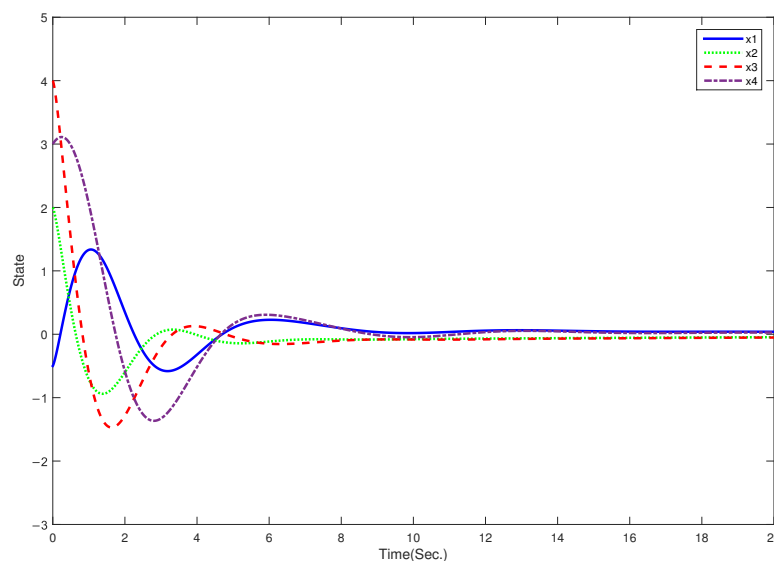


Figure 2. The state responses with $\alpha = 1.5$ in Example 3.

Remark 8. For the arbitrary $\alpha \in (0, 2)$, the effectiveness of Theorem 7 is verified by the simulation results in Example 3. Compared with the methods addressed in [15,28] with multiple complex variables, the LMIs in Theorems 3 and 7 are all composed of real matrices, which avoids the difficulty in solving brought by complex numbers. On the basis of the general approaches in [25,29,30], which divide order α into $(0, 1)$ and $[1, 2)$ to discuss admissibility, we propose a unified form without interval separation and enrich the theoretical research on admissibility.

Remark 9. The approaches reported in [33] are inapplicable to the case of matrix A with eigenvalues on positive real part, but our method has no the limitation of eigenvalues which has a wider range of applications. Compared with the necessary and sufficient criteria proposed in [25–31] with multiple decision variables, the Theorem 7 based on LMIs in this paper contains only one real decision variable in which is easy to obtain the feasible solution.

5. Conclusions

This paper analyzes the problem concerning synthesis of DFOSs without any order interval separation. Novel and alternate admissibility approaches based on non-strict LMIs and strict LMIs are proposed, which fill the vacancy of previous achievements and have theoretical research value. For the non-strict LMI criterion, the position of singular matrix E is different from existing studies, which is convenient to tackle the uncertainties in derivative matrix. For the strict LMI criterion, the research object of using the least real decision variable to solve the problem of admissibility is achieved. All the methods are necessary and sufficient conditions without complex numbers and are applicable to the system eigenvalues with positive real part. When $E = I$ or $\alpha = 1$, the Theorems established in this paper are consistent with related FOSs results or descriptor integer system results in [15,29,32], which are regarded as the extensions of Lyapunov stability. Further works will focus on the controller design for the DFOSs with uncertain derivative matrices based on the methods established above.

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