fractal and fractional

Article

The Analytical Fractional Solutions for Coupled Fokas System in Fiber Optics Using Different Methods

Wael W. Mohammed [1,2,*], Clemente Cesarano [3], Elsayed M. Elsayed [4] and Farah M. Al-Askar [5]

1 Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
4 Mathematics Department, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
5 Department of Mathematical Science, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: The Fokas system with M-truncated derivative (FS-MTD) was considered in this study. To get analytical solutions of FS-MTD in the forms of elliptic, rational, hyperbolic, and trigonometric functions, we employed the extend $\mathcal{F}$-expansion approach and the Jacobi elliptic function method. Since nonlinear pulse transmission in monomode optical fibers is explained by the Fokas system, the derived solutions may be utilized to analyze a broad range of important physical processes. In order to comprehend the impacts of MTD on the solutions, the dynamic behavior of the various generated solutions are shown using 2D and 3D figures.

Keywords: Fokas system; optical solutions; analytical solutions

MSC: 35Q51; 83C15

Citation: Mohammed, W.W.; Cesarano, C.; Elsayed, E.M.; Al-Askar, F.M. The Analytical Fractional Solutions for Coupled Fokas System in Fiber Optics Using Different Methods. *Fractal Fract.* **2023**, 7, 556. https://doi.org/10.3390/fractalfract7070556

Academic Editor: Carlo Cattani

Received: 14 June 2023
Revised: 10 July 2023
Accepted: 17 July 2023
Published: 18 July 2023

1. Introduction

Most phenomena and models in the fields of computer science, medicine, botany, zoology, ecology, human biology, oceanography, engineering, quantum mechanics, applied physics, plasma physics, meteorology, applied mathematics, electricity, and fluid dynamics, are well described by partial differential equations (PDEs). Over the last several decades, there has been an increase in study on the robustness, stability, uniqueness and existence and application of solutions to PDEs in many disciplines. Consequently, the exact solution of PDEs is one of the main topics of nonlinear science. Many specific and simple methods for getting the exact solutions to PDEs have been introduced and improved in recent years, for example the (G'/G)-expansion approach [1,2], mapping method [3,4], Jacobi elliptic function expansion [5], exp-function method [6], sine-cosine procedure [7], auxiliary equation scheme [8], first-integral method [9], generalized Kudryashov approach [10], $exp(-\phi(\varsigma))$-expansion method [11], etc.

In this study, we look at the FS-MTD in the following form

$$\begin{aligned} &i\mathcal{M}_{k,t}^{\delta,\beta}\mathcal{U} + \theta_1\mathcal{U}_{xx} + \theta_2\mathcal{U}\mathcal{V} = 0, \\ &\theta_3\mathcal{V}_y = \theta_4(|\mathcal{U}|^2)_x\,, \end{aligned} \tag{1}$$

where $\mathcal{V} = \mathcal{V}(x,y,t)$ and $\mathcal{U} = \mathcal{U}(x,y,t)$ are complex functions, and $\theta_1,\theta_2,\theta_3$ and θ_4 are arbitrary constants. $\mathcal{M}_{k,t}^{\delta,\beta}$ is the M-truncated derivative operator for $\delta \in (0,1]$, and $\beta > 0$, and will be defined in the next section. If we put $\delta = 1, \beta \to 0$, then we have the Fokas system (FS):

$$
\begin{aligned}
&i\mathcal{U}_t + \theta_1 \mathcal{U}_{xx} + \theta_2 \mathcal{U}\mathcal{V} = 0, \\
&\theta_3 \mathcal{V}_y = \theta_4 (|\mathcal{U}|^2)_x \,.
\end{aligned} \tag{2}
$$

Fokas [12] and Shulman [13] introduced the 2 (FS) for examining NLSE in (2 + 1) dimensions. Due to the importance of FS (2), many authors have acquired the exact solutions by employing numerous techniques, such as Hirota's bilinear method [14], extended rational sine-cosine and sinh-cosh methods [15], Jacobi elliptic function expansion [16], exp-function method [17], He's frequency formulation method, variational method, tanh-function, and the simplified extended [18], generalized Kudryashov method [19].

Recently, there have been some studies about the obtained solutions of PDEs with MTD, for instance the Boiti–Leon–Manna–Pempinelli equation [20], Fokas equation [21], KdV equation [22], Kraenkel–Manna–Merle system [23], complex Ginzburg-Landau equation [24], Phi-4 equation [25], etc.

The goal of this research is to obtain the exact solutions of the FS-MTD (1). We utilize the extended $\mathcal{F}$-expansion method (EFE method) and Jacobi elliptic function method (JEF method) to obtain the solutions of FS-MTD (1). Since the Fokas system is utilized to explain nonlinear pulse transmission in monomode optical fibers, the attained solutions can be utilized for the analysis of a wide variety of crucial physical process. The dynamic behavior of the various obtained solutions are simulated in 3D and 2D in order to interpret the effects of MTD on the solutions.

The paper has the following structure: Next, we identify MTD and describe its features. In Section 3, we find the wave equation for the FS-MTD (1). After then, we employ two various methods to attain the exact solutions of FS-MTD (1) in Section 4, while the impact of MTD on the solution of FS-MTD (1) is investigated in Section 5. Finally, the conclusion of the paper is provided.

2. M-Truncated Derivative

Many authors have presented different types of fractional derivatives. Those offered by Riemann–Liouville, Riesz, Erdelyi, and Hadamard and Caputo [26–29], are the most popular. Classical derivative rules, including the quotient rule, chain rule, and product rule, cannot be applied to the huge variety of fractional derivative forms. In recent years, Sousa et al. [30] have suggested a novel derivative known as MTD.

Definition 1. *If $\varphi : [0,\infty) \to \mathbb{R}$ then the MTD of order $\delta \in (0,1]$ is known as*

$$
\mathcal{M}_{k,t}^{\delta,\beta}\varphi(t) = \lim_{h\to 0} \frac{\varphi(t\mathcal{E}_{k,\beta}(ht^{-\delta})) - \varphi(t)}{h}, \text{ for } t > 0,
$$

where $\mathcal{E}_{k,\beta}$ is known as

$$
\mathcal{E}_{k,\beta}(z) = \sum_{j=0}^{k} \frac{z^j}{\Gamma(j\beta+1)},
$$

for $\beta > 0$, $z \in \mathbb{C}$ and Γ is a gamma function.

The next theorem describes the characteristics of MTD:

Theorem 1. *If ψ and φ are δ-differentiable functions for $\delta \in (0,1]$, $\beta > 0$, then*
(1) $\mathcal{M}_{k,t}^{\delta,\beta}(a\varphi + b\psi) = a\mathcal{M}_{k,t}^{\delta,\beta}(\varphi) + b\mathcal{M}_{k,t}^{\delta,\beta}(\psi)$;
(2) $\mathcal{M}_{k,t}^{\delta,\beta}(t^\nu) = \frac{\nu}{\Gamma(\beta+1)} t^{\nu-\delta}$;
(3) $\mathcal{M}_{k,t}^{\delta,\beta}(\varphi\psi) = \varphi\mathcal{M}_{k,t}^{\delta,\beta}\psi + \psi\mathcal{M}_{k,t}^{\delta,\beta}\varphi$;
(4) $\mathcal{M}_{k,t}^{\delta,\beta}(\varphi)(t) = \frac{t^{1-\delta}}{\Gamma(\beta+1)} \frac{d\varphi}{dt}$;
(5) $\mathcal{M}_{k,t}^{\delta,\beta}(\psi \circ \varphi)(t) = \psi'(\varphi(t))\mathcal{M}_{k,t}^{\delta,\beta}\varphi(t)$,
where a, b, ν are real constants.

3. Traveling Wave Equation for FS-MTD

To get the wave equation for FS-MTD (1), we utilize

$$\mathcal{U}(x,y,t) = \Phi(\mu)e^{i\phi},\ \mathcal{V}(x,y,t) = \Psi(\mu),$$
with
$$\phi = \phi_1 x + \phi_2 y + \phi_3 \tfrac{\Gamma(\beta+1)}{\delta} t^{\delta} \ \text{and}\ \mu = \mu_1 x + \mu_2 y + \mu_3 \tfrac{\Gamma(\beta+1)}{\delta} t^{\delta}, \tag{3}$$

where Φ and Ψ are real functions, and $\phi_k,\ \mu_k$ are non-zero constants for $k = 1,\ 2,\ 3$. We note that

$$\begin{aligned}
\mathcal{M}^{\delta,\beta}_{k,t}\mathcal{U} &= [\mu_3\Phi' + i\phi_3\Phi]e^{i\phi}, \\
\mathcal{U}_x &= (\mu_1\Phi' + i\phi_1\Phi)e^{i\phi},\ (|\mathcal{U}|^2)_x = \mu_1(\Phi^2)', \\
\mathcal{U}_{xx} &= (\mu_1^2\Phi'' + 2i\phi_1\mu_1\Phi' - \phi_1^2\Phi)e^{i\phi},\ \mathcal{V}_y = \mu_2\Psi'.
\end{aligned} \tag{4}$$

Plugging Equation (4) into Equation (1), we have, for the real part

$$(\theta_1\mu_1^2)\Phi'' + (-\phi_3 - \theta_1\phi_1^2)\Phi + \theta_2\Phi\Psi = 0, \tag{5}$$

$$\theta_3\mu_2\Psi' = \theta_4\mu_1(\Phi^2)', \tag{6}$$

and for the imaginary part

$$(2\theta_1\phi_1\mu_1 + \mu_3)\Phi' = 0. \tag{7}$$

Setting that:

$$\mu_3 = -2\theta_1\phi_1\mu_1.$$

Then, Equation (7) vanishes. Integrating (6) once and ignoring the integral constant, we get

$$\Psi = \frac{\theta_4\mu_1}{\theta_3\mu_2}\Phi^2. \tag{8}$$

Substituting Equation (8) into Equation (5), we have

$$\Phi'' + A\Phi + B\Phi^3 = 0, \tag{9}$$

where

$$A = \frac{-\phi_3 - \theta_1\phi_1^2}{\theta_1\mu_1^2},\ \text{and}\ B = \frac{\theta_2\theta_4}{\theta_3\theta_1\mu_1\mu_2}.$$

4. Exact Solutions of FS-MTD

To solve Equation (9), two different approaches are used: the EFE method and the JEF method. Then, the solutions to the FS-MTD (1) are derived.

4.1. EFE Method

Assuming the solution Φ of Equation (9) is:

$$\Phi(\mu) = a_0 + \sum_{k=1}^{N}(a_k\mathcal{F}^k(\mu) + \frac{b_k}{\mathcal{F}^k(\mu)}), \tag{10}$$

where $\mathcal{F}$ solves

$$\mathcal{F}' = \mathcal{F}^2 + \varpi, \tag{11}$$

where ϖ is a real constant. Hence, Equation (11) has the solutions:

$$\mathcal{F}(\mu) = \sqrt{\varpi}\tan(\sqrt{\varpi}\mu)\ \text{ or } \mathcal{F}(\mu) = -\sqrt{\varpi}\cot(\sqrt{\varpi}\mu), \tag{12}$$

if $\varpi > 0$, or

$$\mathcal{F}(\mu) = -\sqrt{-\varpi}\tanh(\sqrt{-\varpi}\mu) \text{ or } \mathcal{F}(\mu) = -\sqrt{-\varpi}\coth(\sqrt{-\varpi}\mu), \tag{13}$$

if $\varpi < 0$, or

$$\mathcal{F}(\mu) = \frac{-1}{\mu}, \tag{14}$$

if $\varpi = 0$.

To determine N, we balance Φ'' with Φ^3 in Equation (9) as follows:

$$N + 2 = 3N \Rightarrow N = 1.$$

Equation (10) becomes

$$\Phi(\mu) = a_0 + a_1\mathcal{F} + \frac{b_1}{\mathcal{F}}. \tag{15}$$

Setting Equation (15) into Equation (9), we attain

$$\begin{aligned}
&(2a_1 + Ba_1^3)\mathcal{F}^3 + (3a_0a_1^2)\mathcal{F}^2 + (2\varpi a_1 + 3Ba_0^2a_1 \\
&+3Ba_1^2b_1 + Aa_1)\mathcal{F} + (Ba_0^3 + 6Ba_0a_1b_1 + Aa_0) \\
&+(2\varpi b_1 + 3Ba_0^2b_1 + 3Ba_1b_1^2 + Ab_1)\mathcal{F}^{-1} + \\
&+(3a_0a_1^2)\mathcal{F}^{-2} + (2b_1\varpi^2 + Bb_1^3)\mathcal{F}^{-3} = 0.
\end{aligned}$$

Comparing to zero the coefficients of every power of $\mathcal{F}$:

$$2a_1 + Ba_1^3 = 0,$$

$$3a_0a_1^2 = 0,$$

$$2\varpi a_1 + 3Ba_0^2a_1 + 3Ba_1^2b_1 + Aa_1 = 0,$$

$$Ba_0^3 + 6Ba_0a_1b_1 + Aa_0 = 0,$$

$$2\varpi b_1 + 3Ba_0^2b_1 + 3Ba_1b_1^2 + Ab_1 = 0,$$

$$3a_0b_1^2 = 0,$$

and

$$2b_1\varpi^2 + Bb_1^3 = 0.$$

The following are the three families of solutions obtained by solving these equations:

First family:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{-2}{B}},\ b_1 = 0,\ \mu_1 = \pm\sqrt{\frac{\phi_3 + \theta_1\phi_1^2}{2\theta_1\varpi}}. \tag{16}$$

Second family:

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{-2}{B}},\ b_1 = \pm\varpi\sqrt{\frac{-2}{B}},\ \mu_1 = \pm\sqrt{\frac{\phi_3 + \theta_1\phi_1^2}{8\theta_1\varpi}}. \tag{17}$$

Third family:

$$a_0 = 0,\ a_1 = \mp\sqrt{\frac{-2}{B}},\ b_1 = \pm\varpi\sqrt{\frac{-2}{B}},\ \mu_1 = \pm\sqrt{\frac{\phi_3 + \theta_1\phi_1^2}{4\theta_1\varpi}}. \tag{18}$$

First family: Equations (8) and (9) have the following solutions:

$$\Phi(\mu) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\mathcal{F}(\mu),$$

and

$$\Psi(\mu) = \frac{-2\theta_1\mu_1^2}{\theta_2}\mathcal{F}^2(\mu).$$

Consequently, by using Equation (3), the solution of FS-MTD (1) is

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\mathcal{F}(\mu)e^{i\phi} \text{ and } \mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\mathcal{F}^2(\mu). \tag{19}$$

where $\phi = \phi_1 x + \phi_2 y + \phi_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$ and $\mu = \mu_1 x + \mu_2 y + \mu_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$. For $\mathcal{F}(\mu)$, there are three different cases:

Case 1: If $\varpi > 0$, then Equation (19), using (12), has the form

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\tan(\sqrt{\varpi}\mu)e^{i\phi}, \tag{20}$$

$$\mathcal{V}(x,y,t) = \frac{-2\varpi\theta_1\mu_1^2}{\theta_2}\tan^2(\sqrt{\varpi}\mu), \tag{21}$$

and

$$\mathcal{U}(x,y,t) = \mp\sqrt{\frac{-2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\cot(\sqrt{\varpi}\mu)e^{i\phi}, \tag{22}$$

$$\mathcal{V}(x,y,t) = \frac{-2\varpi\theta_1\mu_1^2}{\theta_2}\cot^2(\sqrt{\varpi}\mu). \tag{23}$$

Case 2: If $\varpi < 0$, then Equation (19), using (13), becomes

$$\mathcal{U}(x,y,t) = \mp\sqrt{\frac{2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\tanh(\sqrt{-\varpi}\mu)e^{i\phi}, \tag{24}$$

$$\mathcal{V}(x,y,t) = \frac{2\varpi\theta_1\mu_1^2}{\theta_2}\tanh^2(\sqrt{-\varpi}\mu), \tag{25}$$

and

$$\mathcal{U}(x,y,t) = \mp\sqrt{\frac{2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\coth(\sqrt{-\varpi}\mu)e^{i\phi}, \tag{26}$$

$$\mathcal{V}(x,y,t) = \frac{2\varpi\theta_1\mu_1^2}{\theta_2}\coth^2(\sqrt{-\varpi}\mu). \tag{27}$$

Case 3: If $\varpi = 0$, then Equation (19), using (14), takes the type

$$\mathcal{U}(x,y,t) = \mp\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\frac{1}{\mu}e^{i\phi} \text{ and } \mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\frac{1}{\mu^2}, \tag{28}$$

where $\phi = \phi_1 x + \phi_2 y + \phi_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$ and $\mu = \mu_1 x + \mu_2 y + \mu_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$.

Second family: Equations (8) and (9) have the following solutions:

$$\Phi(\mu) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\mathcal{F}(\mu) + \frac{\varpi}{\mathcal{F}(\mu)}\right),$$

and

$$\Psi(\mu) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\mathcal{F}(\mu) + \frac{\varpi}{\mathcal{F}(\mu)}\right)^2.$$

Consequently, by using Equation (3), the solution of FS-MTD (1) is

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\mathcal{F}(\mu) + \frac{\varpi}{\mathcal{F}(\mu)}\right)e^{i\phi}, \tag{29}$$

and

$$\mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\mathcal{F}(\mu) + \frac{\varpi}{\mathcal{F}(\mu)}\right)^2, \tag{30}$$

where $\phi = \phi_1 x + \phi_2 y + \phi_3 \frac{\Gamma(\beta+1)}{\delta} t^\delta$ and $\mu = \mu_1 x + \mu_2 y + \mu_3 \frac{\Gamma(\beta+1)}{\delta} t^\delta$. For $\mathcal{F}(\mu)$, there are three cases:

Case 1: If $\varpi > 0$, then Equations (29) and (30), using (12), become

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\tan(\sqrt{\varpi}\mu) + \cot(\sqrt{\varpi}\mu)\right)e^{i\phi}, \tag{31}$$

and

$$\mathcal{V}(x,y,t) = \frac{-2\varpi\theta_1\mu_1^2}{\theta_2}\left(\tan(\sqrt{\varpi}\mu) + \cot(\sqrt{\varpi}\mu)\right)^2. \tag{32}$$

Case 2: If $\varpi < 0$, then Equations (29) and (30), using (13), have the forms

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\tanh(\sqrt{-\varpi}\mu) + \coth(\sqrt{-\varpi}\mu)\right)e^{i\phi}, \tag{33}$$

and

$$\mathcal{V}(x,y,t) = \frac{2\varpi\theta_1\mu_1^2}{\theta_2}\left(\tanh(\sqrt{-\varpi}\mu) + \coth(\sqrt{-\varpi}\mu)\right)^2. \tag{34}$$

Case 3: If $\varpi = 0$, then Equations (29) and (30), using (14), become

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\frac{1}{\mu} + \mu\right)e^{i\phi} \text{ and } \mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\frac{1}{\mu} + \mu\right)^2. \tag{35}$$

Third family: Equations (8) and (9) have the following solutions:

$$\Phi(\mu) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\mathcal{F}(\mu) - \frac{\varpi}{\mathcal{F}(\mu)}\right),$$

and

$$\Psi(\mu) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\mathcal{F}(\mu) - \frac{\varpi}{\mathcal{F}(\mu)}\right)^2.$$

Consequently, by using Equation (3), the solution of FS-MTD (1) is

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\mathcal{F}(\mu) - \frac{\varpi}{\mathcal{F}(\mu)}\right)e^{i\phi}, \tag{36}$$

and

$$\mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\mathcal{F}(\mu) - \frac{\varpi}{\mathcal{F}(\mu)}\right)^2, \tag{37}$$

where $\phi = \phi_1 x + \phi_2 y + \phi_3 \frac{\Gamma(\beta+1)}{\delta} t^\delta$ and $\mu = \mu_1 x + \mu_2 y + \mu_3 \frac{\Gamma(\beta+1)}{\delta} t^\delta$. For $\mathcal{F}(\mu)$, there are three cases:

Case 1: If $\varpi > 0$, then Equations (36) and (37), using (12), become

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\Big(\tan(\sqrt{\varpi}\mu) - \cot(\sqrt{\varpi}\mu)\Big)e^{i\phi}, \tag{38}$$

and

$$\mathcal{V}(x,y,t) = \frac{-2\varpi\theta_1\mu_1^2}{\theta_2}\Big(\tan(\sqrt{\varpi}\mu) - \cot(\sqrt{\varpi}\mu)\Big)^2. \tag{39}$$

Case 2: If $\varpi < 0$, then Equations (36) and (37), using (13), take the types

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2\varpi\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\Big(\tanh(\sqrt{-\varpi}\mu) - \coth(\sqrt{-\varpi}\mu)\Big)e^{i\phi}, \tag{40}$$

and

$$\mathcal{V}(x,y,t) = \frac{2\varpi\theta_1\mu_1^2}{\theta_2}\Big(\tanh(\sqrt{-\varpi}\mu) - \coth(\sqrt{-\varpi}\mu)\Big)^2. \tag{41}$$

Case 3: If $\varpi = 0$, then Equations (36) and (37), using (14), has the form

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{-2\theta_3\theta_1\mu_1\mu_2}{\theta_2\theta_4}}\left(\frac{1}{\mu} - \mu\right)e^{i\phi} \text{ and } \mathcal{V}(x,y,t) = \frac{-2\theta_1\mu_1^2}{\theta_2}\left(\frac{1}{\mu} - \mu\right)^2. \tag{42}$$

4.2. JEF Method

Assuming that the solutions of Equation (9), (with $N = 1$), is

$$\Phi(\mu) = a + bcn(\lambda\mu, m), \tag{43}$$

where $a,\ b,\ \lambda$ are unknown constants and $cn(\lambda\mu, m)$ is Jacobi elliptic cosine function for $0 < m < 1$, then differentiating Equation (43) twice, we attain

$$\Phi''(\mu) = -(2m^2 - 1)b\lambda^2 cn(\lambda\mu, m) - 2m^2 b\lambda^2 cn^3(\lambda\mu, m), \tag{44}$$

where $[cn(\lambda\mu, m)]' = -\lambda sn(\lambda\mu, m)dn(\lambda\mu, m)$, $[dn(\lambda\mu, m)]' = -\lambda m^2 sn(\lambda\mu, m)cn(\lambda\mu, m)$ and $[sn(\lambda\mu, m)]' = \lambda cn(\lambda\mu, m)dn(\lambda\mu, m)$. Since sn is Jacobi elliptic sine function and dn is the delta amplitude function, then plugging Equations (43) and (44) into Equation (9), we have

$$\begin{aligned}(Bb^3 - 2m^2b\lambda^2)cn^3(\lambda\mu, m) + 3Bab^2cn^2(\lambda\mu, m)\\ +[3Ba^2b - (2m^2-1)b\lambda^2 + Ab]cn(\lambda\mu, m) + (Ba^3 + aA) = 0.\end{aligned}$$

Balancing the coefficient of $[cn(\lambda\mu, m)]^n$ to zero for $n = 0, 1, 2, 3$, we obtain

$$Ba^3 + aA = 0,$$

$$-(2m^2 - 1)b\lambda^2 + 3Ba^2b + Ab = 0,$$

$$3Bab^2 = 0,$$

and

$$Bb^3 - 2m^2b\lambda^2 = 0.$$

The solution to the above equations is

$$a = 0,\ b = \pm\sqrt{\frac{2m^2A}{(2m^2-1)B}},\ \lambda^2 = \frac{A}{(2m^2-1)}.$$

Thus, Equation (9), by using (43), has the solution

$$\Phi(\mu) = \pm\sqrt{\frac{2m^2 A}{(2m^2-1)B}}cn(\lambda\mu, m).$$

Hence, the solution of FS-MTD (1) is

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2m^2 A}{(2m^2-1)B}}cn\Big(\sqrt{\frac{A}{(2m^2-1)}}\mu, m\Big)e^{i\phi}, \quad (45)$$

$$\mathcal{V}(x,y,t) = \frac{2\theta_4\mu_1 m^2 A}{(2m^2-1)\theta_3\mu_2 B}cn^2\Big(\sqrt{\frac{A}{(2m^2-1)}}\mu, m\Big), \quad (46)$$

where $\frac{B}{(2m^2-1)} > 0$, $\phi = \phi_1 x + \phi_2 y + \phi_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$ and $\mu = \mu_1 x + \mu_2 y + \mu_3 \frac{\Gamma(\beta+1)}{\delta}t^\delta$. If $m \to 1$, the solution of FS-MTD (1) is

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2A}{B}}\operatorname{sech}\Big(\sqrt{A}\mu\Big)e^{i\phi}. \quad (47)$$

$$\mathcal{V}(x,y,t) = \frac{2\theta_4\mu_1 A}{\theta_3\mu_2 B}\operatorname{sech}^2\Big(\sqrt{A}\mu\Big), \quad (48)$$

In similar way, we can replace cn in (43) by sn and dn to get the next solutions of Equation (9):

$$\Phi(\mu) = \pm\sqrt{\frac{2m^2 A}{(2-m^2)B}}dn(\frac{A}{(2-m^2)}\mu, m),$$

and

$$\Phi(\mu) = \pm\sqrt{\frac{2m^2 B}{(m^2+1)A}}sn(\frac{B}{(m^2+1)}\mu, m),$$

respectively. Thus, the solutions of FS-MTD (1) are

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2m^2 B}{(2m^2-1)A}}dn\Big(\sqrt{\frac{B}{(2m^2-1)}}\mu, m\Big)e^{i\phi}, \quad (49)$$

$$\mathcal{V}(x,y,t) = \frac{2\theta_4\mu_1 m^2 B}{(2m^2-1)\theta_3\mu_2 A}dn^2\Big(\sqrt{\frac{B}{(2m^2-1)}}\mu, m\Big), \quad (50)$$

for $\frac{B}{(2m^2-1)} > 0$, and

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2m^2 B}{(m^2+1)A}}sn\Big(\sqrt{\frac{B}{(m^2+1)}}\mu, m\Big)e^{i\phi}, \quad (51)$$

$$\mathcal{V}(x,y,t) = \frac{2\theta_4\mu_1 m^2 B}{(m^2+1)\theta_3\mu_2 A}sn^2\Big(\sqrt{\frac{B}{(m^2+1)}}\mu, m\Big), \quad (52)$$

for $B > 0$, respectively. If $m \to 1$, then the solutions (49) and (51) are

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{2B}{A}}\operatorname{csch}\Big(\sqrt{B}\mu\Big)e^{i\phi}, \quad (53)$$

$$\mathcal{V}(x,y,t) = \frac{2\theta_4\mu_1 B}{\theta_3\mu_2 A}\operatorname{csch}^2\Big(\sqrt{B}\mu\Big), \quad (54)$$

and for $B > 0$

$$\mathcal{U}(x,y,t) = \pm\sqrt{\frac{B}{A}}\tanh\left(\sqrt{\frac{B}{2}}\mu\right)e^{i\phi}, \tag{55}$$

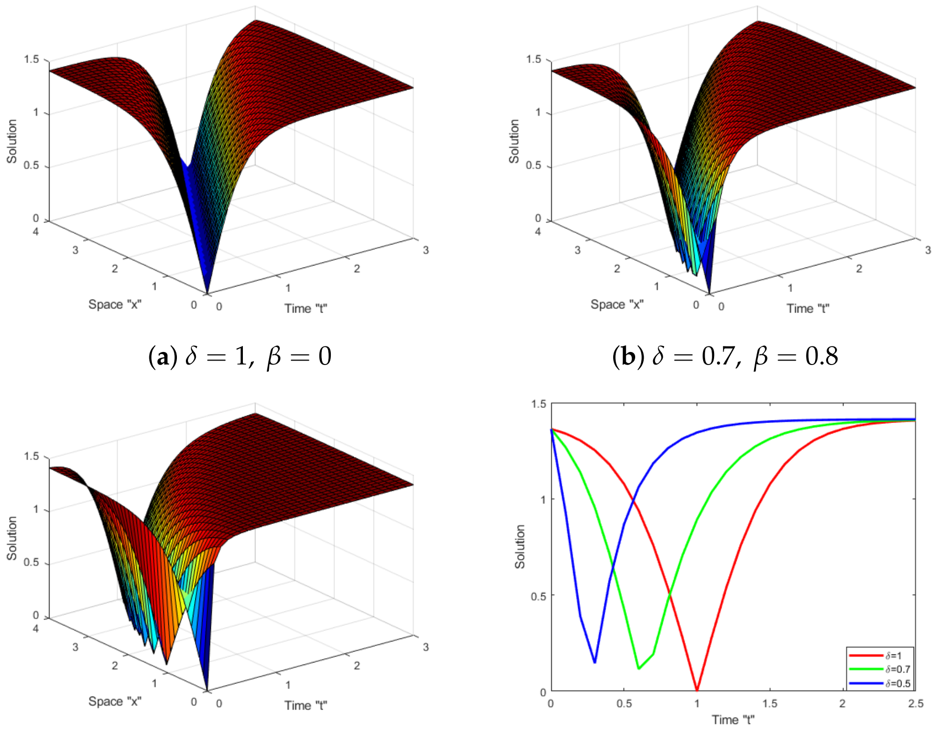
$$\mathcal{V}(x,y,t) = \frac{\theta_4\mu_1 B}{\theta_3\mu_2 A}\tanh^2\left(\sqrt{\frac{B}{2}}\mu\right), \tag{56}$$

where $\phi = \phi_1 x + \phi_2 y + \phi_3\frac{\Gamma(\beta+1)}{\delta}t^{\delta}$ and $\mu = \mu_1 x + \mu_2 y + \mu_3\frac{\Gamma(\beta+1)}{\delta}t^{\delta}$.

5. Effects of MTD on the Solutions

Here, we discuss the effects of MTD on the exact solution of the FS (1). To demonstrate the behavior of certain found solutions, several diagrams are presented, such as (24), (25), and (45)–(48). Let us fix the parameters $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$, $\mu_2 = -1$, $\mu_3 = -2$, $y = 0$, $x \in [0,4]$, and $t \in [0,3]$ to simulate these figures.

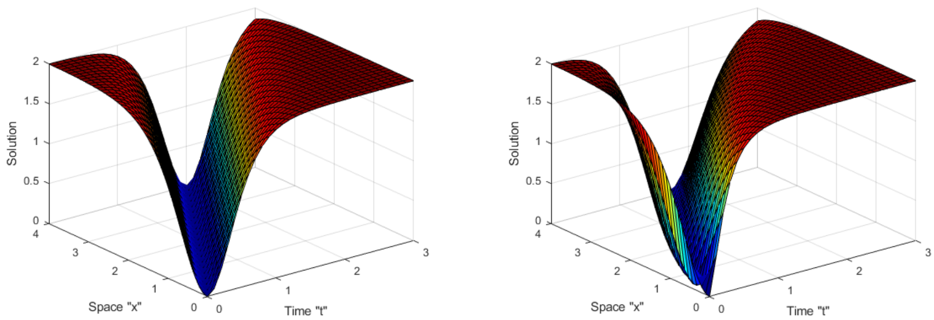
Now we conclude from Figures 1–6 that the surface shifts to the right when the order of M-truncated derivatives increases.



(**a**) $\delta = 1,\ \beta = 0$ (**b**) $\delta = 0.7,\ \beta = 0.8$

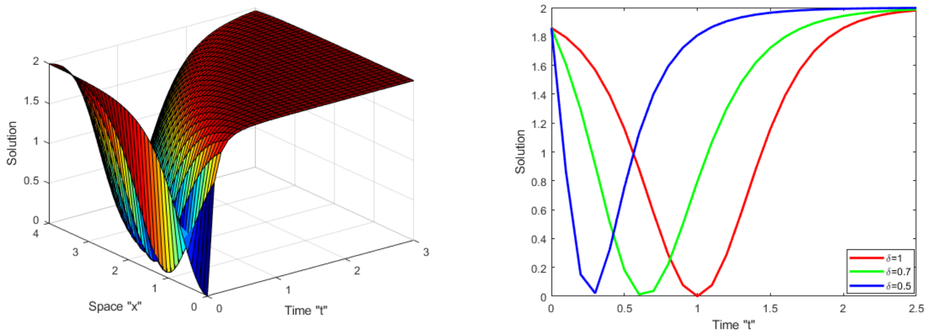
(**c**) $\delta = 0.5,\ \beta = 0.8$ (**d**) $\delta =\ 0.5,\ 0.7,\ 1$

Figure 1. (**a**–**c**) display 3D profile of solution $|\mathcal{U}(x,y,t)|$ in Equation (24) with $\mu_1 = 1$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D-style of Equation (24) with different δ.



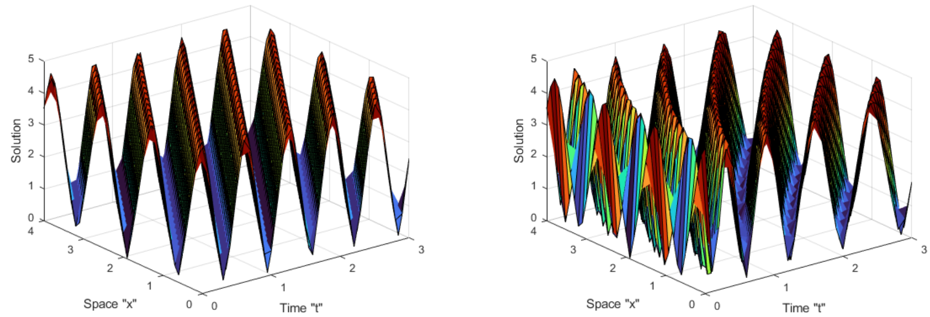
(**a**) $\delta = 1,\ \beta = 0$ (**b**) $\delta = 0.7,\ \beta = 0.8$

Figure 2. *Cont.*

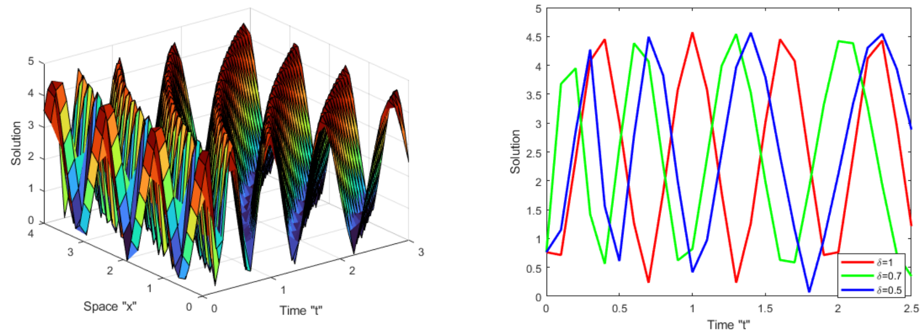


(**c**) $\delta = 0.5,\ \beta = 0.8$ (**d**) $\delta =\ 0.5,\ 0.7,\ 1$

Figure 2. (**a**–**c**) display 3D profile of solution $|\mathcal{V}(x, y, t)|$ in Equation (25) with $\mu_1 = 1$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D profile of Equation (25) with different δ.

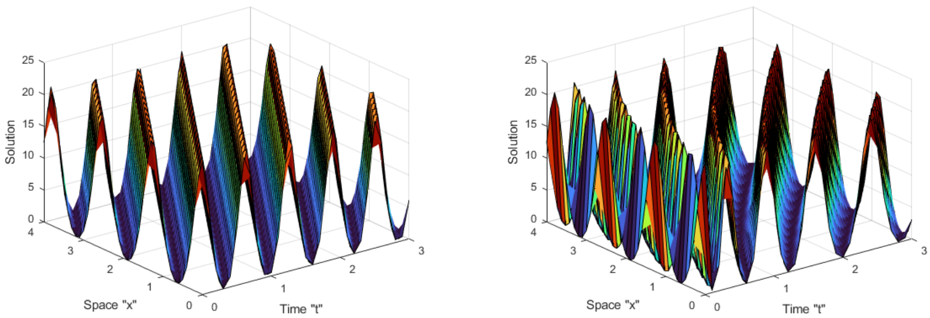


(**a**) $\delta = 1,\ \beta = 0$ (**b**) $\delta = 0.7,\ \beta = 0.8$



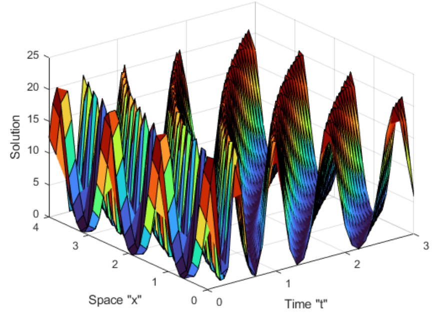
(**c**) $\delta = 0.5,\ \beta = 0.8$ (**d**) $\delta =\ 0.5,\ 0.7,\ 1$

Figure 3. (**a**–**c**) display 3D profile of solution $|\mathcal{U}(x, y, t)|$ in Equation (45) with $\mu_1 = 1$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D profile of Equation (45) with different δ.

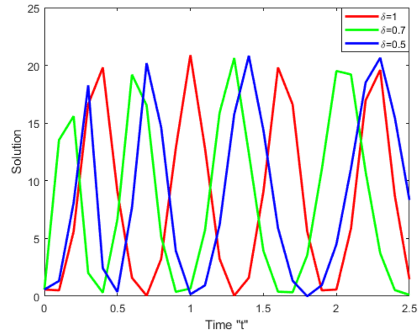


(**a**) $\delta = 1,\ \beta = 0$ (**b**) $\delta = 0.7,\ \beta = 0.8$

Figure 4. *Cont.*

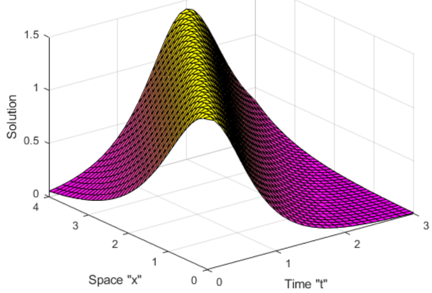


(**c**) $\delta = 0.5,\ \beta = 0.8$

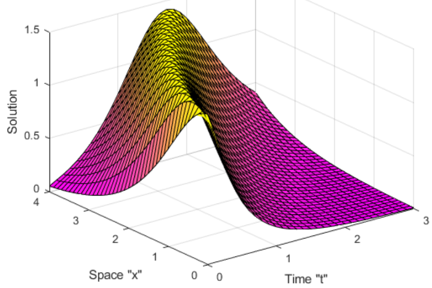


(**d**) $\delta =\ 0.5,\ 0.7,\ 1$

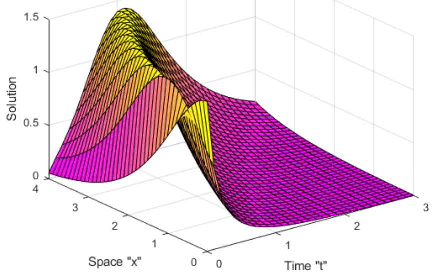
Figure 4. (**a**–**c**) display 3D profile of solution $|\mathcal{V}(x,y,t)|$ in Equation (46) with $\mu_1 = 1$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D profile of Equation (46) with different δ.



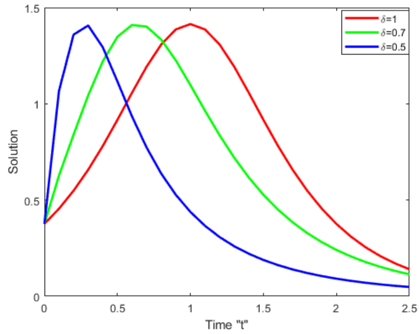
(**a**) $\delta = 1,\ \beta = 0$



(**b**) $\delta = 0.7,\ \beta = 0.8$

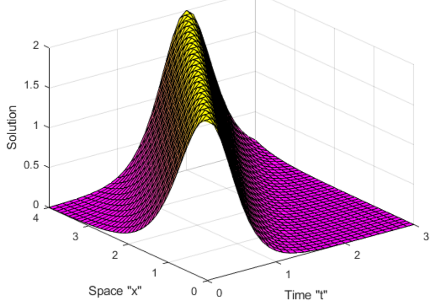


(**c**) $\delta = 0.5,\ \beta = 0.8$

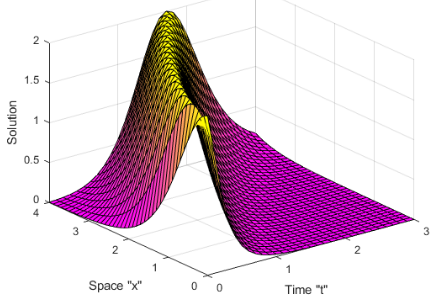


(**d**) $\delta =\ 0.5,\ 0.7,\ 1$

Figure 5. (**a**–**c**) display 3D profile of solution $|\mathcal{U}(x,y,t)|$ in Equation (47) with $\mu_1 = 2$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D profile of Equation (47) with different δ.

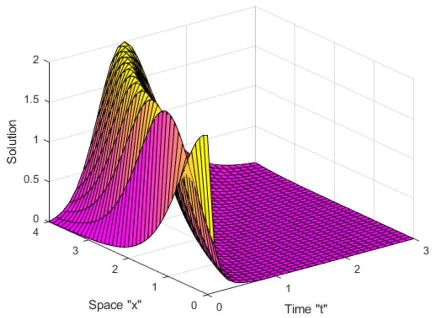


(**a**) $\delta = 1,\ \beta = 0$

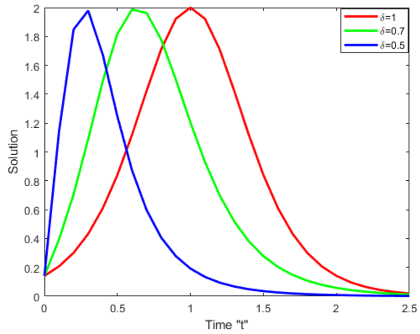


(**b**) $\delta = 0.7,\ \beta = 0.8$

Figure 6. *Cont.*



(c) $\delta = 0.5,\ \beta = 0.8$



(d) $\delta =\ 0.5,\ 0.7,\ 1$

Figure 6. (**a**–**c**) display 3D profile of solution $|\mathcal{V}(x,y,t)|$ in Equation (48) with $\mu_1 = 2$ and $\delta =\ 0.5,\ 0.7,\ 1$. (**d**) shows 2D profile of Equation (48) with different δ.

6. Conclusions

In this study, the Fokas system with M-truncated derivative (FS-MTD) is considered. We used extend $\mathcal{F}$-expansion method and Jacobi elliptic function method to get the exact solutions of FS-MTD (1) in the form of rational, elliptic, hyperbolic, and trigonometric functions. Furthermore, we can use various other methods, such as the $\exp(-\varphi)$-expansion method, improved $tan(\frac{\phi(\rho)}{2})$-expansion method, Hirota bilinear method, complex hyperbolic-function method, Painleve approach, extended trial equation, Weierstrass elliptic function expansion method, etc. to acquire some different solutions.

Since the Fokas system is utilized to explain nonlinear pulse transmission in monomode optical fibers, the acquired solutions can be applied to the analysis of a wide variety of crucial physical phenomena. The dynamic performances of the various obtained solutions are depicted using 3D and 2D curves in order to interpret the effects of MTD on the solutions. We deduced that the surface shifts to the left when the order of M-truncated derivatives decreases.

Author Contributions: Data curation, W.W.M., F.M.A.-A. and E.M.E.; formal analysis, W.W.M., E.M.E., F.M.A.-A. and C.C.; funding acquisition, F.M.A.-A.; methodology, C.C.; project administration, W.W.M.; software, W.W.M.; supervision, C.C.; visualization, E.M.E. and F.M.A.-A.; writing—original draft, W.W.M., E.M.E. and F.M.A.-A.; writing—review and editing, W.W.M. and C.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Princess Nourah bint Abdulrahman University Researcher Supporting Project number (PNURSP2023R 273), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Wang, M.L.; Li, X.Z.; Zhang, J.L. The (G'/G) -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **2008**, *372*, 417–423. [CrossRef]
2. Zhang, H. New application of the (G'/G)-expansion method. *Commun. Nonlinear Sci. Numer. Simul.* **2009**, *14*, 3220–3225. [CrossRef]
3. Mohammed, W.W.; Al-Askar, F.M.; Cesarano, C. The Analytical Solutions of the Stochastic mKdV Equation via the Mapping Method. *Mathematics* **2022**, *10*, 4212. [CrossRef]
4. Al-Askar, F.M.; Cesarano, C.; Mohammed, W.W. Multiplicative Brownian Motion Stabilizes the Exact Stochastic Solutions of the Davey-Stewartson Equations. *Symmetry* **2022**, *14*, 2176. [CrossRef]
5. Yan, Z.L. Abunbant families of Jacobi elliptic function solutions of the dimensional integrable Davey-Stewartson-type equation via a new method. *Chaos Solitons Fractals* **2003**, *18*, 299–309. [CrossRef]
6. He, J.H.; Wu, X.H. Exp-function method for nonlinear wave equations. *Chaos Solitons Fractals* **2006**, *30*, 700–708. [CrossRef]

7. Wazwaz, A.M. The sine-cosine method for obtaining solutions with compact and noncompact structures. *Appl. Math. A Comput.* **2004**, *159*, 559–576. [CrossRef]
8. Jiong, S. Auxiliary equation method for solving nonlinear partial differential equations. *Phys. Lett. A* **2003**, *309*, 387-396.
9. Lu, B. The first integral method for some time fractional differential equations. *J. Math. Anal. Appl.* **2012**, *395*, 684–693. [CrossRef]
10. Arnous, A.H.; Mirzazadeh, M. Application of the generalized Kudryashov method to Eckhaus equation. *Nonlinear Anal. Model. Control* **2016**, *21*, 577–586. [CrossRef]
11. Khan, K.; Akbar, M.A. The $exp(-\phi(\varsigma))$-expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *Int. J. Dyn. Syst. Differ. Equ.* **2014**, *5*, 72–83.
12. Fokas, A.S. On the simplest integrable equation in 2+ 1. *Inverse Probl.* **1994**, *10*, L19. [CrossRef]
13. Shulman, E.I. On the integrability of equations of Davey Stewartson type. *Teor. Mat. Fiz.* **1983**, *56*, 131–136. [CrossRef]
14. Rao, J.; Mihalache, D.; Cheng, Y.; He J. Lump-soliton solutions to the Fokas system. *Phys. Lett. A* **2019**, *383*, 1138–1142. [CrossRef]
15. Wang, K.J.; Liu, J.H.; Wu, J. Soliton solutions to the Fokas system arising in monomode optical fibers. *Optik* **2022**, *251*, 168319. [CrossRef]
16. Tarla, S.; Ali, K.K.; Sun, T.C.; Yilmazer, R.; Osman, M.S. Nonlinear pulse propagation for novel optical solitons modeled by Fokas system in monomode optical fibers. *Results Phys.* **2022**, *36*, 1053. [CrossRef]
17. Wang, K.J. Abundant exact soliton solutions to the Fokas system. *Optik* **2022**, *249*, 168265. [CrossRef]
18. Zhang, P.L.; Wang, K.J. Abundant optical soliton structures to the Fokas system arising in monomode optical fibers. *Open Phys.* **2022**, *20*, 493–506. [CrossRef]
19. Kaplan, M.; Akbulut, A.; Alqahtani, R.T. New Solitary Wave Patterns of the Fokas System in Fiber Optics. *Mathematics* **2023** , *11*, 1810. [CrossRef]
20. Al-Askar, F.M.; Cesarano, C.; Mohammed, W.W. Abundant Solitary Wave Solutions for the Boiti–Leon–Manna–Pempinelli Equation with M-Truncated Derivative. *Axioms* **2023**, *12*, 466. [CrossRef]
21. Mohammed, W.W.; Al-Askar, F.M.; Cesarano, C. Solutions to the (4+1)-Dimensional Time-Fractional Fokas Equation with M-Truncated Derivative. *Mathematics* **2022**, *11*, 194. [CrossRef]
22. Mohammed, W.W.; Cesarano, C.; Al-Askar, F.M.; El-Morshedy, M. Solitary Wave Solutions for the Stochastic Fractional-Space KdV in the Sense of the M-Truncated Derivative. *Mathematics* **2022**, *10*, 4792. [CrossRef]
23. Alshammari, M.; Hamza, A.E.; Cesarano, C.; Aly, E.S.; Mohammed, W.W. The Analytical Solutions to the Fractional Kraenkel–Manna–Merle System in Ferromagnetic Materials. *Fractal Fract.* **2023**, *7*, 523. [CrossRef]
24. Yusuf, A.; Inc, M.; Baleanu, D. Optical Solitons with M-Truncated and Beta Derivatives in Nonlinear Optics. *Front. Phys.* **2019**, *7*, 126. [CrossRef]
25. Akram, G.; Sadaf, M.; Zainab, I. Observations of fractional effects of β-derivative and M-truncated derivative for space time fractional Phi-4 equation via two analytical techniques. *Chaos Solitons Fractals* **2022**, *154*, 111645. [CrossRef]
26. Katugampola, U.N. New approach to a generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
27. Katugampola, U.N. New approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
28. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2016.
29. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives, Theory and Applications*; Gordon and Breach: Yverdon, Switzerland, 1993.
30. Sousa, J.V.; de Oliveira, E.C. A new truncated Mfractional derivative type unifying some fractional derivative types with classical properties. *Int. J. Anal. Appl.* **2018**, *16*, 83–96.