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Ground State Solutions of Fractional Choquard Problems with Critical Growth

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Abstract: In this article, we investigate a class of fractional Choquard equation with critical Sobolev exponent. By exploiting a monotonicity technique and global compactness lemma, the existence of ground state solutions for this equation is obtained. In addition, we demonstrate the existence of ground state solutions for the corresponding limit problem.

Keywords: fractional Choquard equation; ground state solution; Pohožaev identity; critical growth

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1. Introduction

In this paper, we study the following fractional Choquard equation

$$(-\Delta)^\alpha u + K(x)u = (I_\beta * G(u))g(u) + |u|^{2^*_\alpha - 2}u, \quad x \in \mathbb{R}^3, \tag{1}$$

where $\alpha \in (\frac{3}{4}, 1)$, $\beta \in (2\alpha, 3)$, $2^*_\alpha = \frac{6}{3-2\alpha}$. $(-\Delta)^\alpha$ is defined as

$$(-\Delta)^\alpha u(x) := C_\alpha P.V. \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{3+2\alpha}} dy, \quad x \in \mathbb{R}^3,$$

where $C_\alpha = \left(\int_{\mathbb{R}^3} \frac{1 - cos\zeta_1}{|\zeta|^{3+2\alpha}} d\zeta \right)^{-1}$ and $P.V.$ is an abbreviation for Cauchy principal value. The Riesz potential $I_\beta : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ is defined by

$$I_\beta(x) := \frac{\Gamma(\frac{3-\beta}{2})}{\Gamma(\frac{\beta}{2})\pi^{\frac{3}{2}} 2^\beta |x|^{3-\beta}}, \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Let us state some hypothesises on K and g:

(K_1) $K \in \mathcal{C}^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\alpha K(x) + (\nabla K(x), x) \geqslant 0$ for any $x \in \mathbb{R}^3$;

(K_2) $K(x) \leqslant \liminf\limits_{|y| \to +\infty} K(y) = K_\infty \in \mathbb{R}^+$ for all $x \in \mathbb{R}^3$ and the inequality is strict in a subset of positive Lebesgue measure;

(K_3) there is a positive constant a_0 such that

$$a_0 = \inf_{u \in H^\alpha_K(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{\alpha}{2}} u|^2 + K(x)|u|^2 \right) dx}{\int_{\mathbb{R}^3} |u|^2 dx} > 0;$$

(H_1) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $G(\tau) = \int_0^\tau g(u)du \geq 0$ for all $\tau \in \mathbb{R}$ and there exits $C_0 > 0$ and $1 + \frac{\beta}{3} < q < 2^*_{\beta,\alpha}$ such that for every $\tau \in \mathbb{R}$,

$$|g(\tau)| \leqslant C_0(|\tau|^{\frac{\beta}{3}} + |\tau|^{q-1}),$$

where $2^*_{\beta,\alpha} = \frac{\beta+3}{3-2\alpha}$;

(H_2) $\lim_{|\tau|\to 0} \frac{g(\tau)}{\tau^{\frac{\beta}{3}}} = \lim_{|\tau|\to+\infty} \frac{g(\tau)}{\tau^{2^*_{\beta,\alpha}-1}} = 0$;

(H_3) $[4\alpha g(\tau)\tau - (3+\beta)G(\tau)]/\tau|\tau|^{(\beta+6-4\alpha)/4\alpha}$ is non-decreasing on both $(0,+\infty)$ and $(-\infty,0)$;

(H_4) there exist $\nu > 0$ and $p \in (2, 2^*_{\beta,\alpha})$ such that $g(\tau) \geqslant \nu\tau^{p-1}$ for any $\tau \geqslant 0$.

When $\alpha = 1$, $G(u) = |u|^p$, $K(x) \equiv 1$, Equation (1) is known as following Choquard equation

$$-\Delta u + u = (I_\beta * |u|^p)|u|^{p-2}u + f(u), \quad u \in H^1(\mathbb{R}^3). \tag{2}$$

For the case $\beta = 2$, $p = 2$ and $f(u) = 0$, Equation (2) is called the Choquard-Pekar equation. It was first proposed by Pekar [1]. Later, Choquard rediscovered it as an approximation of Hartree-Fock's theory of single-component plasma [2]. For more physical interpretation, one can refer to Penrose [3], Diosi [4], Lewin-Nam-Rougerie [5,6] and Jones [7,8]. Moroz-Van Schaftingen [9] obtained a solution to (2) when $1 + \frac{\beta}{3} < p < 3 + \beta$, where $3 + \beta$ and $1 + \frac{\beta}{3}$ are the upper and lower critical exponents. Gao-Yang [10] investigated the existence of solution for the upper critical exponent case. Yao-Chen-Radulescu-Sun [11] studied the existence of results for the double critical case. Du-Gao-Yang [12] and Yang-Radulescu-Zhou [13] considered the existence and properties of solutions with Stein-Weiss type nonlinearities. Moroz-Van Schaftingen [14] investigated the semi-classical states for $p \geqslant 2$, and set the rest case $p < 2$ for open problem. Cingolani-Tanaka [15] given a pisitive answer to the open problem. Furthermore, Su-Liu [16,17] extended the semi-classical results to the upper critical case. Afterwards, these results were extended to the more general function G or the more general potential K, see [18–28].

Recently, researchers are increasingly concerned the existence results of fractional Choquard equation

$$(-\Delta)^\alpha u + K(x)u = (I_\beta * |u|^p)|u|^{p-2}u, \quad \text{in } \mathbb{R}^3,$$

where $\frac{3+\beta}{3} < p < \frac{3+\beta}{3-2\alpha}$. For instance, Shen-Gao-Yang [29] established the existence of nonnegative ground state solution by supposing that the nonlinearities satisfied the general Berestycki-Lions type conditions. Su et al. [30] studied the existence of results for the double critical case. Li, Zhang, Wang and Teng [31] considered

$$(-\Delta)^\alpha u + K(x)u = [|x|^{-\beta} * G(u)]g(u) + \lambda[|x|^{-\beta} * |u|^p]p|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{3}$$

where $0 < \alpha < 1, N > 2\alpha, 0 < \beta < 2\alpha$, and $p \geq 2^*_{\beta,\alpha}$. They obtained that there existed ground state solutions to Equation (3) with critical or supercritical growth by the variational methods.

We notice that the work in the above literature focuses on the existence of ground state solutions to fractional Choquard equations with a Hardy-Littlewood-Sobolev critical exponent. We determine to investigate a class of fractional order Choquard equations with a critical local term. Inspired by the above work, we first deal with the case of a constant potential. Specifically, by taking a minimizing sequence from a Nehari-Pohožaev manifold, we obtain a minimum value and prove that the minimum value is a critical point. Therefore, we prove the existence of the Nehari-Pohožaev type ground state solution for the fractional Choquard equation with a constant potential. On this basis, we use a monotonicity technique and a global compactness lemma to obtain the existence of a Nehari-Pohožaev type ground state solution to Equation (1). As we've seen there is nearly not any result for the existence of nonnegative least energy solutions for the fractional Choquard Equation (1) with critical growth.

Therefore, Let's first study the limit problem

$$(-\Delta)^\alpha u + K_\infty u = (I_\beta * G(u))g(u) + |u|^{2^*_\alpha - 2}u, \quad x \in \mathbb{R}^3. \tag{4}$$

We obtain the following result.

Theorem 1. *Assume that $\alpha \in (\frac{3}{4}, 1)$, $\beta \in (2\alpha, 3)$, (H_1)–(H_4) hold.*

(i) If $p \in (2, 2_\alpha^* - 1)$, there exists $\nu_1 > 0$ such that for $\nu > \nu_1$, Equation (4) has a ground state solution.

(ii) If $p \in (2_\alpha^* - 1, 2_{\beta,\alpha}^*)$, for any $\nu > 0$, Equation (4) has a ground state solution.

Then, we can obtain the next main result.

Theorem 2. *Assume that $\alpha \in (\frac{3}{4}, 1)$, $\beta \in (2\alpha, 3)$, (K_1)–(K_3) and (H_1)–(H_4) hold.*

(i) If $p \in (2, 2_\alpha^* - 1)$, there exists $\nu_1 > 0$ such that for $\nu > \nu_1$, Equation (1) has a ground state solution.

(ii) If $p \in (2_\alpha^* - 1, 2_{\beta,\alpha}^*)$, for any $\nu > 0$, Equation (1) has a ground state solution.

Remark 1. *We will use a global compactness lemma, Jeanjean's monotonicity trick, and a general minimax principle to prove the main results. There are some troubles in proving the main theorems. The first trouble is that f doesn't satisfy Ambrosetti-Rabinowtiz condition, we can't use the Nehari manifold to obtain the ground state solution of Equation (1). Moreover, It is difficult to acquire the boundedness of (PS) sequence. To conquer it, we shall use Jeanjean's monotonicity technique [32] and establish the Pohožaev identity. The second problem is the lack of the compactness induced by the critical term. We will use some new estimates to obtain a global compactness lemma to overcome this difficulty. Because of the fractional Laplace operator and convolutional nonlinearity, these estimates are complex. Moreover, the potential $K(x)$ is not a constant, we consider the limit problem of Equation (1).*

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries results. In Section 3, we investigate the limit problem. Section 4 is devoted to the existence of ground state solutions to Equation (1). Section 5 is a brief conclusion of this paper.

2. Preliminaries

The fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$\mathcal{D}^{\alpha,2}(\mathbb{R}^3) := \left\{ u \in L^{2_\alpha^*}(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{3+2\alpha}{2}}} \in L^2\Big(\mathbb{R}^3 \times \mathbb{R}^3\Big) \right\}$$

with the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{1}{2}}.$$

It follows from Propositions 3.4 and 3.6 in [33] that

$$\int_{\mathbb{R}^3} \left|(-\Delta)^{\frac{\alpha}{2}} u\right|^2 dx = \frac{1}{C(\alpha)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2\alpha}} dxdy.$$

It is well known that the embedding $\mathcal{D}^{\alpha,2}(\mathbb{R}^3) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^3)$ is continuous.The constant S_α is defined as

$$S_\alpha = \inf_{u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^3} |u(x)|^{2_\alpha^*} dx\right)^{\frac{2}{2_\alpha^*}}}. \tag{5}$$

The fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{3+2\alpha}{2}}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\}$$

endowed with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + u^2\right)dx\right]^{\frac{1}{2}}.$$

Define the work space of Equation (1) by

$$H^\alpha_K(\mathbb{R}^3) := \left\{u \in H^\alpha(\mathbb{R}^3) : \int_{\mathbb{R}^3} K(x)|u|^2dx < +\infty\right\},$$

with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}vdx + \int_{\mathbb{R}^3} K(x)uvdx,$$

and the norm

$$\|u\|_{H^\alpha_K(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^3} K(x)u^2dx\right)^{\frac{1}{2}}.$$

From (K_2)–(K_3), the norms $\|\cdot\|_{H^\alpha(\mathbb{R}^3)}$ and $\|u\|_{H^\alpha_K(\mathbb{R}^3)}$ are equivalent.

From (H_1), Hardy-Littlewood-Sobolev inequality [34] and Sobolev embedding theorem, we can conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx\right| &\le C\|G(u)\|^2_{L^{\frac{6}{3+\beta}}(\mathbb{R}^3)} \\ &\le C\left[\int\left(|u|^{\frac{3+\beta}{3}} + |u|^q\right)^{\frac{6}{3+\beta}}\right]^{\frac{3+\beta}{3}} \\ &\le C\left(\|u\|^{\frac{2(3+\beta)}{3}}_{L^2(\mathbb{R}^3)} + \|u\|^{2q}_{L^{\frac{6q}{3+\beta}}(\mathbb{R}^3)}\right),\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}(I_\beta * G(u))g(u)vdx\right| \\ &\le C(\beta)\left(\int_{\mathbb{R}^3}|G(u)|^{\frac{6}{3+\beta}}dx\right)^{\frac{3+\beta}{6}}\left(\int_{\mathbb{R}^3}|g(u)v|^{\frac{6}{3+\beta}}dx\right)^{\frac{3+\beta}{6}} \\ &\le C\left(\|u\|^{\frac{3+\beta}{3}}_{L^2(\mathbb{R}^3)} + \|u\|^q_{L^{\frac{6q}{3+\beta}}(\mathbb{R}^3)}\right)\left(\|u\|^{\frac{\beta}{3}}_{L^2(\mathbb{R}^3)}\|v\|_{L^2(\mathbb{R}^3)} + \|u\|^{q-1}_{L^{\frac{6q}{3+\beta}}(\mathbb{R}^3)}\|v\|_{L^{\frac{6q}{3+\beta}}(\mathbb{R}^3)}\right).\end{aligned} \tag{7}$$

Hence, the energy functional $E : H^\alpha_K(\mathbb{R}^3) \to \mathbb{R}$ associated with Equation (1)

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + K(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx$$

is well defined on $H^\alpha_K(\mathbb{R}^3)$ and $E \in C^1(H^\alpha_K(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $v \in H^\alpha_K(\mathbb{R}^3)$,

$$\langle E'(u), v\rangle = \int_{\mathbb{R}^3}\left[(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v + K(x)uv - (I_\beta * G(u))g(u)v - |u|^{2^*_\alpha - 1}uv\right]dx.$$

Similar to Proposition 2.10 in [35], we get the Pohožaev type identity.

Lemma 1. *Assume that* (K_1)–(K_2), (H_1) *hold, and* $u \in H_K^\alpha(\mathbb{R}^3)$ *is a solution to Equation* (1). *Then the Pohožaev identity holds true*

$$\begin{aligned}&\frac{3-2\alpha}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}+\frac{3}{2}\int_{\mathbb{R}^3}K(x)u^2dx+\frac{1}{2}\int_{\mathbb{R}^3}(\nabla K(x),x)u^2dx\\ =&\frac{3+\beta}{2}\int_{\mathbb{R}^3}(I_\beta*G(u))G(u)dx+\frac{3}{2_\alpha^*}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx.\end{aligned}$$

Define the Nehari-Pohožaev manifold by

$$\Pi:=\{u\in H_K^\alpha(\mathbb{R}^3)\setminus\{0\}:J(u):=2\alpha\langle E'(u),u\rangle-\mathcal{L}(u)=0\},$$

where

$$\begin{aligned}\mathcal{L}(u):=&\frac{3-2\alpha}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}+\frac{3}{2}\int_{\mathbb{R}^3}K(x)u^2dx+\frac{1}{2}\int_{\mathbb{R}^3}(\nabla K(x),x)u^2dx\\ &-\frac{3+\beta}{2}\int_{\mathbb{R}^3}(I_\beta*G(u))G(u)dx-\frac{3}{2_\alpha^*}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx\end{aligned}$$

and

$$\begin{aligned}J(u):=&\frac{6\alpha-3}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)K(x)-(\nabla K(x),x)]u^2dx\\ &+\frac{1}{2}\left\{\int_{\mathbb{R}^3}(I_\beta*G(u))[(3+\beta)G(u)-4\alpha g(u)u]dx-(6\alpha-3)\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx\right\}.\end{aligned}$$

By combining Lemma 2.2 in [10] and [30], we are able to get the next Brezis-Lieb type lemma.

Lemma 2. *If* $u_n \rightharpoonup u$ *in* $H_K^\alpha(\mathbb{R}^3)$ *with* $\alpha\in(\frac{3}{4},1)$ *and* $u_n\to u$ *a.e. in* $\mathbb{R}^3$, *then*

$$E(u_n)=E(u)+E(u_n-u)+o(1),\quad J(u_n)=J(u)+J(u_n-u)+o(1),$$

and

$$E'(u_n)=E'(u)+E'(u_n-u)+o(1),$$

and

$$\langle E'(u_n),u_n\rangle=\langle E'(u),u\rangle+\langle E'(u_n-u),u_n-u\rangle+o(1).$$

At the end of this section, we set up some crucial inequalities.

Lemma 3. *Assume that* (H_1) *and* (H_3) *hold. Then for all* $\vartheta>0$ *and* $\tau\in\mathbb{R}$,

$$f(\vartheta,\tau):=\frac{3}{\vartheta^{\frac{3+\beta}{2}}}G(\vartheta^{2\alpha}\tau)+\left(1-\vartheta^{\frac{3}{2}}\right)[4\alpha g(\tau)\tau-(3+\beta)G(\tau)]-3G(\tau)\geq 0.$$

Proof. Clearly, $f(\vartheta,0)\geq 0$ for $\vartheta>0$. From (H_3), for $\tau\neq 0$, we obtain

$$\begin{aligned}&\frac{d}{d\vartheta}f(\vartheta,\tau)\\ =&\frac{3\vartheta^{\frac{1}{2}}|\tau|^{\frac{\beta+6}{4\alpha}}}{2}\left[\frac{4\alpha g(\vartheta^{2\alpha}\tau)\vartheta^{2\alpha}\tau-(3+\beta)G(\vartheta^{2\alpha}\tau)}{|\vartheta^{2\alpha}\tau|^{\frac{\beta+6}{4\alpha}}}-\frac{4\alpha g(\tau)\tau-(3+\beta)G(\tau)}{|\tau|^{\frac{\beta+6}{4\alpha}}}\right]\\ &\begin{cases}\geq 0, & \vartheta\geq 1\\ <0, & 0<\vartheta<1,\end{cases}\end{aligned}$$

which implies that $f(\vartheta,\tau)\geq f(1,\tau)=0$ for all $\vartheta\in(0,+\infty)$ and $\tau\in(-\infty,0)\cup(0,+\infty)$. □

Lemma 4. *Assume that* $\alpha \in (\frac{3}{4}, 1)$, (H_1)–(H_3) *hold. Then*

$$\frac{G(\vartheta)}{\vartheta|\vartheta|^{(8\alpha+\beta-6)/4\alpha}} \text{ is nondecreasing on both } (-\infty, 0) \text{ and } (0, +\infty).$$

Proof. Together with (H_2), Lemma 3 and $\alpha > \frac{3}{4}$, we obtain

$$\lim_{\vartheta \to 0} f(\vartheta, \tau) = 4\alpha g(\tau)\tau - (6+\beta)G(\tau) \geq 0. \tag{8}$$

It follows from (8) that

$$\frac{d}{d\vartheta}\left(\frac{G(\vartheta)}{\vartheta|\vartheta|^{(8\alpha+\beta-6)/4\alpha}}\right) = \frac{1}{4\alpha|\vartheta|^{(16\alpha+\beta-6)/4\alpha}}[4\alpha g(\vartheta)\vartheta - (12\alpha+\beta-6)G(\vartheta)] \geq 0.$$

□

Lemma 5. *Assume that* (H_1)–(H_3) *hold. Then*

$$\begin{aligned}
k(\vartheta, v) := &\int_{\mathbb{R}^3}\bigg\{\frac{6\alpha-3}{\vartheta^{3+\beta}}(I_\beta * G(\vartheta^{2\alpha}v))G(\vartheta^{2\alpha}v) \\
&+ (1-\vartheta^{6\alpha-3})(I_\beta * G(v))[4\alpha g(v)v - (3+\beta)G(v)] - (6\alpha-3)(I_\beta * G(v))G(v)\bigg\}dx \\
&\geq 0, \quad \forall \vartheta > 0,\ v \in H^\alpha_K(\mathbb{R}^3).
\end{aligned}$$

Proof. It follows from (H_1) and Lemma 4 that

$$I_\beta * \left(\frac{G(\vartheta^{2\alpha}v)}{|\vartheta|^{(12\alpha+\beta-6)/2}}\right) - I_\beta * G(v)\begin{cases} \geq 0, & \vartheta \geq 1, \\ \leq 0, & 0 < \vartheta < 1. \end{cases} \tag{9}$$

From (H_1), (H_3) and (9), we obtain

$$\begin{aligned}
&\frac{d}{d\vartheta}k(\vartheta, v) \\
&= \int_{\mathbb{R}^3}\bigg\{\frac{(6\alpha-3)4\alpha}{\vartheta^{3+\beta}}(I_\beta * G(\vartheta^{2\alpha}v))g(\vartheta^{2\alpha}v)\vartheta^{2\alpha-1}v \\
&\quad - \frac{(6\alpha-3)(3+\beta)}{\vartheta^{\beta+4}}(I_\beta * G(\vartheta^{2\alpha}u))G(\vartheta^{2\alpha}v) \\
&\quad - (6\alpha-3)\vartheta^{6\alpha-4}(I_\beta * G(v))[4\alpha g(v)v - (3+\beta)G(v)]\bigg\}dx \\
&= (6\alpha-3)\vartheta^{6\alpha-4}\int_{\mathbb{R}^3}|v|^{(\beta+6)/4\alpha}\bigg\{I_\beta * \left(\frac{G(\vartheta^{2\alpha}v)}{|\vartheta|^{(12\alpha+\beta-6)/2}}\right)\frac{4\alpha g(\vartheta^{2\alpha}v)\vartheta^{2\alpha}v - (3+\beta)G(\vartheta^{2\alpha}v)}{|\vartheta^{2\alpha}v|^{(\beta+6)/4\alpha}} \\
&\quad - (I_\beta * G(v))\frac{4\alpha g(v)v - (3+\beta)G(v)}{|v|^{(\beta+6)/4\alpha}}\bigg\}dx \\
&\begin{cases} \geq 0, & \vartheta \geq 1, \\ \leq 0, & 0 < \vartheta < 1, \end{cases}
\end{aligned}$$

which yields $k(\vartheta, v) \geq k(1, v) = 0$ for all $\vartheta > 0$ and $v \in H^\alpha_K(\mathbb{R}^3)$. □

3. The Limit Problem

Noting that $\lim_{|x|\to\infty} K(x) = K_\infty$, we consider the limit problem (4), whose energy functional is defined by

$$\begin{aligned} E_\infty(u) =&\frac{1}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{1}{2}K_\infty\|u\|^2_{L^2(\mathbb{R}^3)} \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx. \end{aligned} \tag{10}$$

By Lemma 1, if u is the critical point of E_∞ in $H^\alpha_K(\mathbb{R}^3)$, then it satisfies the Pohožaev identity

$$\begin{aligned} \mathcal{L}_\infty(u) :=& \frac{3-2\alpha}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{3}{2}K_\infty\|u\|^2_{L^2(\mathbb{R}^3)} - \frac{3+\beta}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx - \frac{3}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx \\ =& 0. \end{aligned}$$

Set $u_\vartheta = \vartheta^{2\alpha}u(\vartheta x)$. By direct calculation, we deduce that

$$\begin{aligned} h(\vartheta) = E_\infty(u_\vartheta) =& \frac{\vartheta^{6\alpha-3}}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{\vartheta^{4\alpha-3}}{2}K_\infty\|u\|^2_{L^2(\mathbb{R}^3)} \\ &- \frac{1}{2\vartheta^{3+\beta}}\int_{\mathbb{R}^3}(I_\beta * G(\vartheta^{2\alpha}u))G(\vartheta^{2\alpha}u)dx - \frac{\vartheta^{\frac{3(6\alpha-3)}{3-2\alpha}}}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx, \end{aligned} \tag{11}$$

which implies that $E_\infty(u_\vartheta) \to -\infty$ as $\vartheta \to +\infty$. As a consequence, we obtain the next lemma.

Lemma 6. *E_∞ is not bounded from below.*

Define $\Pi_\infty = \{u \in H^\alpha_K(\mathbb{R}^3)\backslash\{0\} : J_\infty(u) = 0\}$, where

$$\begin{aligned} J_\infty(u) &= 2\alpha\langle E'_\infty(u), u\rangle - \mathcal{L}_\infty(u) \\ &= \frac{6\alpha-3}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{4\alpha-3}{2}K_\infty\|u\|^2_{L^2(\mathbb{R}^3)} - \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx \\ &\quad + \frac{3+\beta}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx - 2\alpha\int_{\mathbb{R}^3}(I_\beta * G(u))g(u)udx \\ &= \frac{dE_\infty(u_\vartheta)}{d\vartheta}|_{\vartheta=1}. \end{aligned} \tag{12}$$

Set

$$\xi(\vartheta) := \frac{1-\vartheta^{4\alpha-3}}{2} - \frac{(4\alpha-3)(1-\vartheta^{6\alpha-3})}{2(6\alpha-3)}, \quad \zeta(\vartheta) := \frac{1-\vartheta^{6\alpha-3}}{2} - \frac{1-\vartheta^{\frac{3(6\alpha-3)}{3-2\alpha}}}{2^*_\alpha}, \quad \forall\vartheta > 0.$$

Clearly,

$$\xi(\vartheta) > 0, \quad \zeta(\vartheta) > 0, \quad \vartheta \in (0,1)\cup(1,+\infty). \tag{13}$$

Lemma 7. *For any $u \in H^\alpha_K(\mathbb{R}^3)$ and $\vartheta > 0$,*

$$E_\infty(u) \geq E_\infty(u_\vartheta) + \frac{1-\vartheta^{6\alpha-3}}{6\alpha-3}J_\infty(u) + \xi(\vartheta)\int_{\mathbb{R}^3}K_\infty u^2dx + \zeta(\vartheta)\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx.$$

Proof. From (10)–(13) and Lemma 5, we get that

$$\begin{aligned}
&E_\infty(u) - E_\infty(u_\vartheta) - \frac{1-\vartheta^{6\alpha-3}}{6\alpha-3} J_\infty(u) \\
&= \xi(\vartheta)\int_{\mathbb{R}^3} K_\infty u^2 dx + \frac{1}{2(6\alpha-3)}\int_{\mathbb{R}^3}\left\{\frac{6\alpha-3}{\vartheta^{3+\beta}}(I_\beta * G(\vartheta^{2\alpha}u))G(\vartheta^{2\alpha}u)\right. \\
&\quad \left. + (1-\vartheta^{6\alpha-3})(I_\beta * G(u))[4\alpha g(u)u - (3+\beta)G(u)] - (6\alpha-3)(I_\beta * G(u))G(u)\right\}dx \\
&\quad + \zeta(\vartheta)\int_{\mathbb{R}^3} |u|^{2_\alpha^*} dx \\
&\geq 0,
\end{aligned}$$

which comes to the conclusions. □

Lemma 8. *For all $u \in H_K^\alpha(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $\vartheta_0 > 0$ such that $u_{\vartheta_0} \in \Pi_\infty$. Moreover,*

$$E_\infty(u_{\vartheta_0}) = \max_{\vartheta \geq 0} E_\infty(u_\vartheta).$$

Proof. Set $u \in H_K^\alpha(\mathbb{R}^3)\backslash\{0\}$ be fixed, one has $h'(\vartheta) = 0$ is equivalent to $u_\vartheta \in \Pi_\infty$, for $\vartheta > 0$. By (K_2), (H_2), (6) and $q > 1 + \frac{\beta}{3} > 1 + \frac{\beta}{4\alpha}$, we obtain $\lim_{\vartheta \to 0^+} h(\vartheta) = 0$, for $\vartheta > 0$ small, $h(\vartheta) > 0$ and for ϑ large, $h(\vartheta) < 0$. Hence, $\max_{\vartheta > 0} h(\vartheta)$ is attained at $\vartheta = \vartheta_0(u) > 0$ such that $h'(\vartheta_0) = 0$ and $u_{\vartheta_0} \in \Pi_\infty$.

Next, we show ϑ_0 is unique for any $u \in H_K^\alpha(\mathbb{R}^3)\backslash\{0\}$. Suppose on the contrary that there exist $\vartheta_1, \vartheta_2 > 0$ such that $h'(\vartheta_1) = h'(\vartheta_1) = 0$. It follows from $J_\infty(u_{\vartheta_1}) = J_\infty(u_{\vartheta_2}) = 0$ and Lemma 7 that

$$\begin{aligned}
&E_\infty(\vartheta_1^{2\alpha}u(\vartheta_1 x)) \\
&\geq E_\infty(\vartheta_2^{2\alpha}u(\vartheta_2 x)) + \frac{\vartheta_1^{6s-3} - \vartheta_2^{6\alpha-3}}{(6\alpha-3)\vartheta_1^{6\alpha-3}} J_\infty(\vartheta_1^{2\alpha}u(\vartheta_1 x)) \\
&\quad + \xi(\vartheta_2/\vartheta_1)\vartheta_1^{4\alpha-3}\int_{\mathbb{R}^3} K_\infty u^2 dx + \zeta(\vartheta_2/\vartheta_1)\vartheta_1^{\frac{(6\alpha-3)3}{3-2\alpha}}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx \\
&\geq E_\infty(\vartheta_2^{2\alpha}u(\vartheta_2 x)) + \xi(\vartheta_2/\vartheta_1)\vartheta_1^{4s-3}\int_{\mathbb{R}^3} K_\infty u^2 dx + \zeta(\vartheta_2/\vartheta_1)\vartheta_1^{\frac{(6\alpha-3)3}{3-2\alpha}}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&E_\infty(\vartheta_2^{2\alpha}u(\vartheta_2 x)) \\
&\geq E_\infty(\vartheta_1^{2\alpha}u(\vartheta_1 x)) + \frac{\vartheta_2^{6\alpha-3} - \vartheta_1^{6\alpha-3}}{(6\alpha-3)\vartheta_2^{6\alpha-3}} J_\infty(\vartheta_2^{2\alpha}u(\vartheta_2 x)) \\
&\quad + \xi(\vartheta_1/\vartheta_2)\vartheta_2^{4\alpha-3}\int_{\mathbb{R}^3} K_\infty u^2 dx + \zeta(\vartheta_1/\vartheta_2)\vartheta_2^{\frac{(6\alpha-3)3}{3-2\alpha}}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx \\
&\geq E_\infty(\vartheta_1^{2\alpha}u(\vartheta_1 x)) + \xi(\vartheta_1/\vartheta_2)\vartheta_2^{4\alpha-3}\int_{\mathbb{R}^3} K_\infty u^2 dx + \zeta(\vartheta_1/\vartheta_2)\vartheta_2^{\frac{(6\alpha-3)3}{3-2\alpha}}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx.
\end{aligned} \tag{15}$$

Therefore, from (14) and (15), we obtain $\vartheta_1 = \vartheta_2$. That is, ϑ_0 is unique for any $u \in H_K^\alpha(\mathbb{R}^3)\backslash\{0\}$. □

Lemma 9. *The manifold Π_∞ satisfies the following properties:*

(1) *there exists $\sigma > 0$ such that $\|u\|_{H_K^\alpha(\mathbb{R}^3)} \geq \sigma$, $\forall u \in \Pi_\infty$.*

(2) *$c_\infty = \inf_{u \in \Pi_\infty} E_\infty(u) > 0$.*

Proof. (1) By (6)–(7), we have

$$\frac{4\alpha-3}{2}\|u\|^2_{H^\alpha_K(\mathbb{R}^3)} \le \frac{6\alpha-3}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{4\alpha-3}{2}\int_{\mathbb{R}^3} K_\infty u^2 dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))[4\alpha g(u)u - (3+\beta)G(u)]dx + \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx$$
$$\le C(\|u\|^{2+\frac{2\beta}{3}}_{H^\alpha_K(\mathbb{R}^3)} + \|u\|^{2q}_{H^\alpha_K(\mathbb{R}^3)} + \|u\|^{2^*_\alpha}_{H^\alpha_K(\mathbb{R}^3)}),$$

which implies

$$\|u\|_{H^\alpha_K(\mathbb{R}^3)} \ge \sigma, \quad \forall u \in \Pi_\infty. \tag{16}$$

(2) Let $\{u_n\} \subset \Pi_\infty$ be such that $E_\infty(u_n) \to c_\infty$. There exist two possible scenarios:

(i) $\inf_{n\in\mathbb{N}} \|u_n\|_{L^2(\mathbb{R}^3)} > 0$, or

(ii) $\inf_{n\in\mathbb{N}} \|u_n\|_{L^2(\mathbb{R}^3)} = 0$.

Case (i) $\inf_{n\in\mathbb{N}} \|u_n\|_{L^2(\mathbb{R}^3)} = \sigma_1 > 0$. It follows from (8), (10) and (12) that

$$\begin{aligned} c_\infty &= E_\infty(u_n) = E_\infty(u_n) - \frac{1}{6s-3}J_\infty(u_n) \\ &= \frac{\alpha}{6\alpha-3}\int_{\mathbb{R}^3} K_\infty u_n^2 dx + \left(\frac{1}{2}-\frac{1}{2^*_\alpha}\right)\int_{\mathbb{R}^3}|u_n|^{2^*_\alpha}dx \\ &\quad + \frac{1}{2(6\alpha-3)}\int_{\mathbb{R}^3}(I_\beta * G(u_n))[4\alpha g(u_n)u_n - (6\alpha+\mu)G(u_n)]dx \\ &\ge \frac{\alpha K_\infty}{6\alpha-3}\sigma_1^2. \end{aligned} \tag{17}$$

Case (ii) $\inf_{n\in\mathbb{N}} \|u_n\|_{L^2(\mathbb{R}^3)} = 0$. According to (16), passing to a sub-sequence, we obtain

$$\|u_n\|_{L^2(\mathbb{R}^3)} \to 0, \quad \|u_n\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \ge \frac{\sigma}{2}. \tag{18}$$

It follows from (H_1)–(H_2) that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|G(\tau)| \le C_\epsilon|\tau|^{1+\frac{\beta}{3}} + \epsilon|\tau|^{2^*_{\beta,\alpha}}, \quad \forall \tau \in \mathbb{R}. \tag{19}$$

From (5), (6) and (19), we deduce that

$$\int_{\mathbb{R}^3}(I_\beta * G(u))G(u) \le CC_\epsilon\|u\|^{2+\frac{2\beta}{3}}_{L^2(\mathbb{R}^3)} + C\epsilon S_\alpha^{-\frac{2^*_\alpha}{2}}\|u\|^{\frac{2^*_\alpha(3+\beta)}{3}}_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}. \tag{20}$$

Let $\vartheta_n = \left[(2^*_\alpha)^{\frac{3-2\alpha}{2\alpha}} S_\alpha^{\frac{3}{2\alpha}} \|u_n\|^{-2}_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}\right]^{\frac{1}{6\alpha-3}}$. Then, due to (18), $\{\vartheta_n\}$ is bounded. Applying Lemma 7, (11), (18) and (20), we deduce that

$$\begin{aligned} c_\infty + o_n(1) &= E_\infty(u_n) \ge E_\infty((u_n)_{\vartheta_n}) \\ &= \frac{\vartheta_n^{6\alpha-3}}{2}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{\vartheta_n^{4\alpha-3}}{2}\int_{\mathbb{R}^3} K_\infty u_n^2 dx \\ &\quad - \frac{1}{2\vartheta_n^{3+\beta}}\int_{\mathbb{R}^3}(I_\beta * G(\vartheta_n^{2\alpha}u))G(\vartheta_n^{2\alpha}u)dx - \frac{\vartheta_n^{\frac{3(6\alpha-3)}{3-2\alpha}}}{2^*_\alpha}\int_{\mathbb{R}^3}|u_n|^{2^*_\alpha}dx \\ &\ge \frac{\vartheta_n^{6\alpha-3}}{2}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} - CC_\epsilon\left(\vartheta_n^{4\alpha-3}\|u_n\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{3+\beta}{3}} \\ &\quad - C\epsilon S_\alpha^{-\frac{2^*_\alpha}{2}}\left(\vartheta_n^{6\alpha-3}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}\right)^{\frac{3+\beta}{3-2\alpha}} - \frac{S_\alpha^{-\frac{2^*_\alpha}{2}}}{2^*_\alpha}\left(\vartheta_n^{6\alpha-3}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}\right)^{\frac{3}{3-2\alpha}} \\ &= \frac{\vartheta_n^{6s-3}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}}{4}\left[2 - \frac{S_\alpha^{-\frac{2^*_\alpha}{2}}}{2^*_\alpha}\left(\vartheta_n^{6\alpha-3}\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)}\right)^{\frac{2\alpha}{3-2\alpha}}\right] + o_n(1) \\ &= \frac{1}{4}(2^*_\alpha)^{\frac{3-2\alpha}{2\alpha}} S_\alpha^{\frac{3}{2\alpha}} + o_n(1). \end{aligned}$$

It follows from Case (i) and Case (ii) that $c_\infty = \inf_{u\in\Pi_\infty} E_\infty(u) > 0$. $\square$

By using Lemmas 8 and 9, we can find the next result.

Lemma 10. *The following equality holds*

$$c_\infty = \bar{c}_\infty := \inf_{u\neq 0}\max_{\vartheta>0} E_\infty(u_\vartheta).$$

We shall use Proposition 2.8 (the general mini-max principle) in [36] to obtain a Cerami sequence for the functional E_∞ with $J_\infty(u_n) \to 0$, where $E_\infty,\ J_\infty(u_n)$ are given in (10) and (12), respectively.

Lemma 11. *There exists a sequence* $\{u_n\} \subset H^\alpha_K(\mathbb{R}^3)$ *such that*

$$E_\infty(u_n) \to \check{c}_\infty > 0, \quad \|E'_\infty\|_{H^{-\alpha}_K(\mathbb{R}^3)}(1 + \|u_n\|_{H^\alpha_K(\mathbb{R}^3)}) \to 0 \text{ and } J_\infty(u_n) \to 0, \tag{21}$$

where

$$\check{c}_\infty := \inf_{\mu\in\Gamma}\max_{\tau\in[0,1]} E_\infty(\mu(\tau)), \quad \Gamma := \Big\{\mu \in \mathcal{C}([0,1], H^\alpha_K(\mathbb{R}^3)) : \mu(0) = 0, E_\infty(\mu(1)) < 0\Big\}.$$

Proof. For $u \in H^\alpha_K(\mathbb{R}^3)\backslash\{0\}$, By (H_2), one has $E_\infty(\tau u) \to -\infty$ as $\tau \to +\infty$. By the standard arguments, we have $\Gamma \neq \emptyset$ and $\check{c}_\infty < \infty$. Furthermore, it is easy to check that there exist $\sigma_0, \delta_0 > 0$ such that $E_\infty(u) \geq 0$ for all u with $\|u\|_{H^\alpha_K(\mathbb{R}^3)} \leq \sigma_0$ and $E_\infty(u) \geq \delta_0$ for all u with $\|u\|_{H^\alpha_K(\mathbb{R}^3)} = \sigma_0$. Together with the definition of Γ, we obtain $\|\mu(1)\|_{H^\alpha_K(\mathbb{R}^3)} > \sigma_0$. From the continuity of $\mu(\tau)$ and the intermediate value theorem, there exists $\tau_\mu \in (0,1)$ such that $\|\mu(\tau_\mu)\|_{H^\alpha_K(\mathbb{R}^3)} = \sigma_0$. Hence, we obtain

$$\max_{\tau\in[0,1]} E_\infty(\mu(\tau)) \geq E_\infty(\mu(\tau_\mu)) \geq \delta_0 > 0,$$

which implies

$$\infty > \check{c}_\infty = \inf_{\mu\in\Gamma}\max_{\tau\in[0,1]} E_\infty(\mu(\tau)) \geq \delta_0 > 0.$$

Define the continuous map

$$\eta : \mathbb{R}\times H^\alpha_K(\mathbb{R}^3) \to H^\alpha_K(\mathbb{R}^3), \quad \eta(\tau,u)(x) = e^{2\alpha\tau}u(e^\tau x), \quad \text{for } \tau\in\mathbb{R},\ u \in H^\alpha_K(\mathbb{R}^3), x\in\mathbb{R}^3,$$

where $\mathbb{R}\times H^\alpha_K(\mathbb{R}^3)$ is the Banach space with the product norm $\|(\tau,u)\|_{\mathbb{R}\times H^\alpha_K(\mathbb{R}^3)} := (|\tau|^2 + \|u\|^2_{H^\alpha_K(\mathbb{R}^3)})^{\frac{1}{2}}$. We define the following auxiliary functional:

$$\begin{aligned}
\widetilde{E}_\infty(\tau,u) &= E_\infty(\eta(\tau,u)) \\
&= \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}\eta(\tau,u)|^2 + K_\infty|\eta(\tau,u)|^2)dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(\eta(\tau,u)))G(\eta(\tau,u))dx - \frac{1}{2^*_\alpha}\int_{\mathbb{R}^3}|\eta(\tau,u)|^{2^*_\alpha}dx \\
&= \frac{e^{(6\alpha-3)\tau}}{2}\|u\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \frac{e^{(4\alpha-3)\tau}}{2}K_\infty\|u\|^2_{L^2(\mathbb{R}^3)} \\
&\quad - \frac{e^{-(3+\beta)\tau}}{2}\int_{\mathbb{R}^3}(I_\beta * G(e^{2\alpha\tau}u))G(e^{2\alpha\tau}u)dx - \frac{e^{\frac{2^*_\alpha(6\alpha-3)}{2}\tau}}{2^*_\alpha}\|u\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)}.
\end{aligned}$$

Moreover, by direct calculations, we obtain $\widetilde{E}_\infty \in \mathcal{C}^1(\mathbb{R}\times H^\alpha_K(\mathbb{R}^3), \mathbb{R})$, and

$$\partial_\tau\widetilde{E}_\infty(\tau,u) = J_\infty(\eta(\tau,u)), \quad \partial_u\widetilde{E}_\infty(\tau,u)w = E'_\infty(\eta(\tau,u))\eta(\tau,w)$$

for all $\tau \in \mathbb{R}$ and $u, w \in H_K^\alpha(\mathbb{R}^3)$. We define the mini-max value $\tilde{c}_\infty$ for $\widetilde{E}$ by

$$\tilde{c}_\infty = \inf_{\tilde{\mu} \in \tilde{\Gamma}} \max_{\tau \in [0,1]} \widetilde{E}_\infty(\tilde{\mu}(\tau)),$$

where $\widetilde{\Gamma} = \{\tilde{\mu} \in \mathcal{C}([0,1], \mathbb{R} \times H_K^\alpha(\mathbb{R}^3)) : \tilde{\mu}(0) = (0,0), \widetilde{E}_\infty(\tilde{\mu}(1)) < 0\}$. Since $\Gamma = \{\eta \circ \tilde{\mu} : \tilde{\mu} \in \widetilde{\Gamma}\}$, we deduce that $\check{c}_\infty = \tilde{c}_\infty$. By the definition of $\check{c}_\infty$, there exists $\mu_n \in \Gamma$ such that for any $n \in \mathbb{N}$,

$$\max_{\tau \in [0,1]} \widetilde{E}_\infty(0, \mu_n(\tau)) = \max_{\tau \in [0,1]} E_\infty(\mu_n(\tau)) \leq \check{c}_\infty + \frac{1}{n^2}.$$

Applying Proposition 2.8 (the general mini-max principle) in [36] to $\widetilde{E}_\infty$, setting $D = [0,1]$, $D_0 = \{0,1\}$, $\mathbb{B} = \mathbb{R} \times H_K^\alpha(\mathbb{R}^3)$, $\sigma = \frac{1}{n^2}$, $\delta = \frac{1}{n}$ and $\tilde{\mu}_n(\tau) = (0, \mu_n(\tau))$, we conclude that there exist $(\tau_n, u_n) \in \mathbb{R} \times H_K^\alpha(\mathbb{R}^3)$ such that

$$\widetilde{E}_\infty(\tau_n, u_n) \to \check{c}_\infty,$$

and

$$\|\widetilde{E}'_\infty(\tau_n, u_n)\|_{\mathbb{B}'}(1 + \|(\tau_n, u_n)\|_{\mathbb{B}}) \to 0, \tag{22}$$

and

$$\operatorname{dist}((\tau_n, u_n), \{0\} \times \mu_n([0,1])) \to 0, \tag{23}$$

as $n \to \infty$. Thus, (23) implies $\tau_n \to 0$. It is easy to see that for all $(t, w) \in \mathbb{R} \times H_K^\alpha(\mathbb{R}^3)$,

$$\langle \widetilde{E}'_\infty(\tau_n, u_n), (t, w) \rangle = \langle E'_\infty(\eta(\tau_n, u_n)), \eta(\tau_n, w) \rangle + J_\infty(\eta(\tau_n, u_n))t. \tag{24}$$

Set $v_n = \eta(\tau_n, u_n)$. If we take $t = 1$ and $w = 0$ in (24), we obtain $J_\infty(v_n) \to 0$ as $n \to \infty$. For each $u \in H_K^\alpha(\mathbb{R}^3)$, let $t = 0$ and $w_n = e^{-2\alpha\tau_n} u(e^{-\tau_n} x)$ in (24). We deduce from (22)–(23) that

$$\begin{aligned}
&\left|\langle E'_\infty(v_n), u \rangle\right|(1 + \|v_n\|_{H_K^\alpha(\mathbb{R}^3)}) \\
=&|\langle E_\infty(\eta(\tau_n, u_n)), \eta(\tau_n, w_n) \rangle|(1 + \|v_n\|_{H_K^\alpha(\mathbb{R}^3)}) \\
=&o(1)\|w_n\|_{H_K^\alpha(\mathbb{R}^3)},
\end{aligned}$$

as $n \to \infty$. Therefore, (21) holds. □

Lemma 12. *$\check{c}_\infty < \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}}$, where S_α is given in (5).*

Proof. Let $\phi(x) \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function such that $0 \leq \phi(x) \leq 1$ in $\mathbb{R}^3$, $\phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ in $\mathbb{R}^3 \setminus B_2(0)$. As is known to all, S_α is achieved by

$$\tilde{u} := \kappa(\sigma^2 + |x - x_0|^2)^{-\frac{3-2\alpha}{2}}$$

for any $\kappa \in \mathbb{R}$, $\sigma > 0$ and $x_0 \in \mathbb{R}^3$. Hence, setting $x_0 = 0$, we define

$$u_\epsilon(x) := \phi(x) U_\epsilon(x), \quad x \in \mathbb{R}^3,$$

where

$$U_\epsilon(x) = \epsilon^{-\frac{(3-2\alpha)}{2}} u^*(x/\epsilon), \quad u^*(x) = \frac{\tilde{u}\left(x/S_\alpha^{1/(2\alpha)}\right)}{\|\tilde{u}\|_{L^{2_\alpha^*}(\mathbb{R}^3)}}.$$

As in [23], we obtain

$$\mathcal{A}_\epsilon := \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_\epsilon|^2 dx \leq S_\alpha^{\frac{3}{2\alpha}} + O(\epsilon^{3-2\alpha}), \tag{25}$$

and

$$\mathcal{B}_\epsilon := \int_{\mathbb{R}^3} |u_\epsilon|^{2^*_\alpha} dx = S_\alpha^{\frac{3}{2\alpha}} + O(\epsilon^3). \tag{26}$$

By a simple calculation, we observe

$$\mathcal{C}_\epsilon := \int_{\mathbb{R}^3} |u_\epsilon|^2 dx = O(\epsilon^{3-2\alpha}), \tag{27}$$

and

$$\mathcal{D}_\epsilon := \int_{\mathbb{R}^3} |u_\epsilon|^s dx = \begin{cases} O(\epsilon^{\frac{3(2-s)+2\alpha s}{2}}), & s > \frac{3}{3-2\alpha}, \\ O(\epsilon^{\frac{3(2-s)+2\alpha s}{2}} |\log \epsilon|), & s = \frac{3}{3-2\alpha}, \\ O(\epsilon^{\frac{(3-2\alpha)s}{2}}), & s < \frac{3}{3-2\alpha}. \end{cases} \tag{28}$$

By virtue of (6) and (28), for any $p \in (2, 2^*_\alpha)$ we obtain

$$\begin{aligned} \mathcal{H}_\epsilon &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\epsilon(x)|^p |u_\epsilon(y)|^p}{|x-y|^{3-\beta}} dxdy \\ &\geq C \int_{B_1(\frac{x_0}{\epsilon})} \int_{B_1(\frac{x_0}{\epsilon})} |U_\epsilon(x)|^p |U_\epsilon(y)|^p dxdy \\ &= C \left(\epsilon^{3-\frac{(3-2\alpha)p}{2}} \int_0^{\frac{1}{\epsilon S_\alpha^{1/(2\alpha)}}} \frac{r^2}{(\sigma^2+r^2)^{\frac{(3-2\alpha)p}{2}}} dr \right)^2 \\ &= O(\epsilon^{6-(3-2\alpha)p}). \end{aligned} \tag{29}$$

Since $\sup_{\tau \geq 0} E_\infty(\tau u_\epsilon) = E_\infty(\tau_\epsilon u_\epsilon) \geq \delta_0 > 0$, there exists $T_1 > 0$ such that $\tau_\epsilon > T_1$. Moreover, we infer from $E_\infty(\tau u_\epsilon) \to -\infty$ as $\tau \to \infty$ that there exists $T_2 > 0$ such that $\tau_\epsilon < T_2$. Then $T_1 < \tau_\epsilon < T_2$. Note that

$$\begin{aligned} E_\infty(\tau u_\epsilon) &\leq \frac{\tau^2}{2} \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{\alpha}{2}} u_\epsilon|^2 + K_\infty |u_\epsilon|^2) dx \\ &\quad - \frac{\tau^{2p}\nu^2}{2p^2} \int_{\mathbb{R}^3} (I_\beta * |u_\epsilon|^p) |u_\epsilon|^p dx - \frac{\tau^{2^*_\alpha}}{2^*_\alpha} \int_{\mathbb{R}^3} |u_\epsilon|^{2^*_\alpha} dx \\ &= \frac{\tau^2}{2} \mathcal{A}_\epsilon + \frac{\tau^2}{2} K_\infty \mathcal{C}_\epsilon - \frac{\tau^{2p}}{2p^2} \nu^2 \mathcal{H}_\epsilon - \frac{\tau^{2^*_\alpha}}{2^*_\alpha} \mathcal{B}_\epsilon. \end{aligned} \tag{30}$$

Define

$$Q_\epsilon(t) := \frac{\tau^2}{2} \mathcal{A}_\epsilon - \frac{\tau^{2^*_\alpha}}{2^*_\alpha} \mathcal{B}_\epsilon.$$

According to (25)–(26), it is easy to verify that

$$\sup_{\tau \geq 0} Q_\epsilon(\tau) \leq \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} + O(\epsilon^{3-2\alpha}). \tag{31}$$

It follows from (27), (29)–(31) that

$$\check{c}_\infty \leq E_\infty(\tau u_\epsilon) \leq \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} + O(\epsilon^{3-2\alpha}) - O(\nu^2 \epsilon^{6-(3-2\alpha)p}).$$

If $2^*_{\beta,\alpha} > p > 2^*_\alpha - 1$, then $0 < 6 - (3-2\alpha)p < 3 - 2\alpha$, which implies that for any fixed $\nu^2 > 0$, $\check{c}_\infty < \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}}$ for $\epsilon > 0$ small. If $2 < p \leq 2^*_\alpha - 1$ and $\nu \geq \epsilon^{\frac{(3-2\alpha)p-4-2\alpha}{2}}$, we also obtain $\check{c}_\infty < \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}}$. $\square$

Lemma 13. *The following equality holds*

$$\inf_{\mu\in\Gamma}\sup_{\tau\in[0,1]} E_\infty(\mu(\tau)) = \check{c}_\infty = \bar{c}_\infty = \inf_{u\neq 0}\max_{\vartheta>0} E_\infty(u_\vartheta).$$

Proof. From Lemma 6, we can see that $E_\infty(u_\vartheta) < 0$ for $u \in H^\alpha_K(\mathbb{R}^3)\setminus\{0\}$ and ϑ large enough. This implies $\check{c}_\infty \le \bar{c}_\infty$. Then, we show $\check{c}_\infty \ge \bar{c}_\infty$. We claim that for any $\mu \in \Gamma$, $\mu([0,1])\cap\Pi_\infty \neq \emptyset$. Indeed, from (17), we obtain for any $\mu\in\Gamma$,

$$J_\infty(\mu(1)) \le (6\alpha-3)E_\infty(\mu(1)) - \alpha K_\infty \sigma_1^2 < 0.$$

For any $u \in H^\alpha_K(\mathbb{R}^3)\setminus\{0\}$, from (6), (7) and (12), we deduce that

$$\begin{aligned} J_\infty(u) =& \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{4\alpha-3}{2}\int_{\mathbb{R}^3}K_\infty u^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))[4\alpha g(u)u-(3+\beta)G(u)]dx - \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx \\ \ge& \frac{4\alpha-3}{2}\|u\|^2_{H^\alpha_K(\mathbb{R}^3)} - C\|u\|^{2+\frac{2\beta}{3}}_{H^\alpha_K(\mathbb{R}^3)} - C\|u\|^{2q}_{H^\alpha_K(\mathbb{R}^3)} - C\|u\|^{2^*_\alpha}_{H^\alpha_K(\mathbb{R}^3)}, \end{aligned}$$

which yields there exists $\sigma_2 \in (0, \|\mu(1)\|_{H^\alpha_K(\mathbb{R}^3)})$ and $\rho_2 > 0$ such that $J_\infty(u) \ge \rho_2$ for $\|u\|_{H^\alpha_K(\mathbb{R}^3)} = \sigma_2$. This implies there exists $\tau_0 \in (0,1)$ such that $J_\infty(\mu(\tau_0)) \ge \rho_2$. Therefore, the curve $\mu\in\Gamma$ must cross Π_∞, which indicates $\check{c}_\infty \ge c_\infty$. Together with Lemma 10, we obtain $\check{c}_\infty \ge \bar{c}_\infty$. □

Proof of Theorem 1. By Lemma 11, there exists a sequence $\{u_n\}\subset H^\alpha_K(\mathbb{R}^3)$ satisfying (21). It follows from (8) and (17) that

$$\begin{aligned} \check{c}_\infty = E_\infty(u_n) =& E_\infty(u_n) - \frac{1}{6\alpha-3}J_\infty(u_n) \\ \ge& \frac{\alpha}{6\alpha-3}\int_{\mathbb{R}^3}K_\infty u_n^2dx + \left(\frac{1}{2}-\frac{1}{2^*_\alpha}\right)\int_{\mathbb{R}^3}|u_n|^{2^*_\alpha}dx. \end{aligned}$$

Combining with the Hölder inequality, we deduce that $\{u_n\}$ is bounded in $L^r(\mathbb{R}^3)$ for $r\in[2,2^*_\alpha]$. Then, by $J_\infty(u_n)\to 0$, (6) and (7), we can see that

$$\begin{aligned} &\frac{6\alpha-3}{2}\|u_n\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \\ &\le \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u_n))[4\alpha g(u_n)u_n-(3+\beta)G(u_n)]dx + \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|u_n|^{2^*_\alpha}dx \\ &\le C(\|u_n\|^{2+\frac{2\beta}{3}}_{L^2(\mathbb{R}^3)} + \|u_n\|^{2q}_{L^{\frac{6q}{3+\beta}}(\mathbb{R}^3)} + \|u_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)}) \le C. \end{aligned}$$

This implies $\{u_n\}$ is bounded in $H^\alpha_K(\mathbb{R}^3)$.
Now, we claim that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|u_n|^2dx > 0.$$

If it does not occur, then it follows fromLemma 2.4 (a fractional version of Lions vanishing lemma) in [37] that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $r\in(2,2^*_\alpha)$. Hence, we obtain

$$o_n(1) = \langle E'_\infty(u_n), u_n\rangle = \|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + K_\infty\|u_n\|^2_{L^2(\mathbb{R}^3)} - \|u_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)} + o_n(1),$$

as $n \to \infty$. Since $\{u_n\}$ is bounded in $H^\alpha_K(\mathbb{R}^3)$ and $\check{c}_\infty > 0$, we may assume that up to a sub-sequence, as $n \to \infty$, for some $l > 0$,

$$\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + K_\infty \|u_n\|^2_{L^2(\mathbb{R}^3)} \to l, \quad \|u_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)} \to l. \tag{32}$$

In view of (5) and (32), we obtain

$$\begin{aligned} l &= \lim_{n\to\infty} \left(\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + K_\infty \|u_n\|^2_{L^2(\mathbb{R}^3)} \right) \\ &\geq \lim_{n\to\infty} \|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \\ &\geq S_\alpha \|u_n\|^2_{L^{2^*_\alpha}(\mathbb{R}^3)} = S_\alpha l^{\frac{2}{2^*_\alpha}}, \end{aligned}$$

which implies

$$l \geq S_\alpha^{\frac{3}{2\alpha}}. \tag{33}$$

Combining the fact

$$\begin{aligned} \check{c}_\infty + o_n(1) =& E_\infty(u_n) \\ =& \frac{1}{2} \left(\|u_n\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + K_\infty \|u_n\|^2_{L^2(\mathbb{R}^3)} \right) - \frac{1}{2^*_\alpha} \|u_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)} + o_n(1) \\ =& \frac{\alpha}{3} l + o_n(1) \end{aligned}$$

and (33), we observe that $\check{c}_\infty \geq \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}}$, which contradicts Lemma 12. Hence, there exists $\sigma > 0$ and a sequence $\{z_n\} \subset \mathbb{R}^3$ such that $\int_{B_1(z_n)} |v_n|^2 dx > \sigma$. Let $\check{v}_n(x) = v_n(x + z_n)$, then as $n \to \infty$,

$$E_\infty(\check{v}_n) \to \check{c}_\infty, \quad E'_\infty(\check{v}_n) \to 0, \quad J_\infty(\check{v}_n) \to 0$$

and $\int_{B_1(0)} |\check{v}_n|^2 dx > \delta$. Hence, passing to a sub-sequence, there exists $\check{v} \in H^\alpha_K(\mathbb{R}^3)$ such that

$$\begin{cases} \check{v}_n \rightharpoonup \check{v} & \text{in } H^\alpha_K(\mathbb{R}^3), \\ \check{v}_n \to \check{v} & \text{in } L^r_{loc}(\mathbb{R}^3) \text{ for } r \in [1, 2^*_\alpha), \\ \check{v}_n \to \check{v} & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

By using standard arguments, we obtain that $E'_\infty(\check{v}) = 0$ and $E_\infty(\check{v}) \geq \bar{c}_\infty$. Therefore, $\check{v}$ is a nontrivial solution to (4). In view of Lemma 13, (8) and Fatou's Lemma, we obtain

$$\begin{aligned} \bar{c}_\infty = \check{c}_\infty &= \lim_{n\to\infty} E_\infty(\check{v}_n) = \lim_{n\to\infty} \left[E_\infty(\check{v}_n) - \frac{1}{6\alpha - 3} J_\infty(\check{v}_n) \right] \\ &= \lim_{n\to\infty} \left\{ \frac{\alpha}{6\alpha - 3} \int_{\mathbb{R}^3} K_\infty \check{v}_n^2 dx + \frac{1}{2(6\alpha - 3)} \int_{\mathbb{R}^3} (I_\beta * G(u))[4\alpha g(\check{v}_n)\check{v}_n - (6\alpha + \beta) G(\check{v}_n)] dx \right. \\ &\quad \left. + \left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \int_{\mathbb{R}^3} |\check{v}_n|^{2^*_\alpha} dx \right\} \\ &\geq \frac{\alpha}{6\alpha - 3} \int_{\mathbb{R}^3} K_\infty \check{v}^2 dx + \frac{1}{2(6\alpha - 3)} \int_{\mathbb{R}^3} (I_\beta * G(u))[4\alpha g(\check{v})\check{v} - (6\alpha + \beta) G(\check{v})] dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^*_\alpha} \right) \int_{\mathbb{R}^3} |\check{v}|^{2^*_\alpha} dx \\ &= E_\infty(\check{v}) - \frac{1}{6\alpha - 3} J_\infty(\check{v}) \geq \bar{c}_\infty, \end{aligned}$$

which implies that $E_\infty(\check{v}) = \bar{c}_\infty = c_\infty$, recalling Lemma 10. □

4. Existence of Ground State Solution to (1)

In this section, our aim is to find ground state solution to (1), whose potential is not a constant. In order to use a delicate method exploited by Jeanjean [32] (Theorem 1.1), for $\lambda \in [\frac{1}{2}, 1]$, we study a family of functional $E_\lambda : H^\alpha_K(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$E_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + K(x)u^2)dx - \frac{\lambda}{2}\left[\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx + \frac{2}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx\right].$$

We get the next lemma, which is analogue to Lemma 1.

Lemma 14. *Assume that (K_1)–(K_2) and (H_1) hold. Let u be a critical point of E_λ in $H^\alpha_K(\mathbb{R}^3)$, then the next Pohožaev type identity holds*

$$\begin{aligned}\mathcal{L}_\lambda(u) := \frac{3-2\alpha}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}K(x)u^2dx + \frac{1}{2}\int_{\mathbb{R}^3}(\nabla K(x), x)u^2dx \\ - \frac{(3+\beta)\lambda}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx - \frac{3\lambda}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx = 0.\end{aligned}$$

Due to Lemma 14, set $J_\lambda(u) = 2\alpha\langle E'_\lambda(u), u\rangle - \mathcal{L}_\lambda(u)$ for all $\lambda \in [\frac{1}{2}, 1]$. Then

$$\begin{aligned}J_\lambda(u) = \frac{6\alpha-3}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}[(4\alpha-3)K(x) - (\nabla K(x), x)]u^2dx \\ + \frac{\lambda}{2}\left\{\int_{\mathbb{R}^3}(I_\beta * G(u))[(3+\beta)G(u) - 4\alpha g(u)u]dx - (6\alpha-3)\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx\right\}.\end{aligned} \tag{34}$$

Let us set $E_\lambda(u) = A(u) - \lambda B(u)$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + K(x)u^2)dx \to +\infty,$$

as $\|u\|_{H^\alpha_K(\mathbb{R}^3)} \to +\infty$ and

$$B(u) = \frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx + \frac{1}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx \geq 0.$$

Lemma 15. *(i) There is $e > 0$ such that $E_\lambda(e) < 0$ for any $\lambda \in [\frac{1}{2}, 1]$.*
(ii) $c_\lambda = \inf_{\mu\in\Gamma}\max_{\tau\in[0,1]} E_\lambda(\mu(\tau)) > \max\{E_\lambda(0), E_\lambda(e)\}$ for all $\lambda \in [\frac{1}{2}, 1]$, where

$$\Gamma = \{\mu \in \mathcal{C}([0,1], H^\alpha_K(\mathbb{R}^3)) : \mu(0) = 0, \mu(1) = e\}.$$

Proof. (i) For $u \in H^\alpha_K(\mathbb{R}^3)\backslash\{0\}$ fixed and $\lambda \in [\frac{1}{2}, 1]$, define

$$\begin{aligned}E^\lambda_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + K_\infty u^2)dx \\ - \lambda\left[\frac{1}{2}\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx + \frac{1}{2^*_\alpha}\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx\right].\end{aligned} \tag{35}$$

Thus, from (K_2), we obtain $E_\lambda(u) \leq E^\infty_\lambda(u)$. Set $u_\vartheta = \vartheta^{2\alpha}u(\vartheta x)$, $\forall\vartheta > 0$. It follows from Lemma 6 that

$$E^\infty_\lambda(u_\vartheta) \to -\infty, \quad \text{as } \vartheta \to +\infty.$$

Take $e = u_\vartheta$ for ϑ large enough, (i) follows immediately.

(ii) From (6), we observe

$$E_\lambda(u) \geq \frac{1}{2}\|u\|^2_{H^\alpha_K(\mathbb{R}^3)} - C(\|u\|^{2+\frac{2\beta}{3}}_{H^\alpha_K(\mathbb{R}^3)} + \|u\|^{2q}_{H^\alpha_K(\mathbb{R}^3)} + \|u\|^{2^*_\alpha}_{H^\alpha_K(\mathbb{R}^3)}),$$

which implies that there exist $\delta > 0$ and $\sigma > 0$ such that

$$E_\lambda(u) \geq \delta > 0, \quad \forall \, \|u\|_{H^\alpha_K(\mathbb{R}^3)} = \sigma, \text{ for any } \lambda \in [\frac{1}{2}, 1].$$

Thus, for any $\lambda \in [\frac{1}{2}, 1]$, there exists $\tau_0 \in (0,1)$ such that $\|\mu(\tau_0)\|_{H^\alpha_K(\mathbb{R}^3)} = \sigma$ and

$$\max_{\tau \in [0,1]} E_\lambda(\mu(\tau)) \geq E_\lambda(\mu(\tau_0)) \geq \delta > \max\{E_\lambda(0), E_\lambda(e)\},$$

which yields $c_\lambda > 0$.
□

Taking into account of Theorem 1.1 in [32] and Lemma 15, for any $\lambda \in [\frac{1}{2}, 1]$, we get a sequence $\{v_n\} \subset H^\alpha_K(\mathbb{R}^3)$ which is bounded and satisfies

$$E_\lambda(v_n) \to c_\lambda, \quad E'_\lambda(v_n) \to 0.$$

To prove the above sequence $\{v_n\}$ satisfies the (PS) condition, we consider the following limit problem

$$(-\Delta)^\alpha v + K_\infty v = \lambda(I_\beta * G(v))g(v) + \lambda|v|^{2^*_\alpha - 2}v, \quad \text{in } \mathbb{R}^3. \tag{36}$$

By Theorem 1, Equation (36) admits a ground state solution $v^\lambda_\infty \in H^\alpha_K(\mathbb{R}^3)$, i.e., for any $\lambda \in [\frac{1}{2}, 1]$, there exists $v^\lambda_\infty \in \Pi^\lambda_\infty$ such that

$$(E^\lambda_\infty)'(v^\lambda_\infty) = 0, \quad E^\lambda_\infty(v^\lambda_\infty) = c^\lambda_\infty = \inf_{v \in \in \Pi^\lambda_\infty} E^\lambda_\infty(u),$$

where $E^\lambda_\infty(u)$ defined in (35),

$$\Pi^\lambda_\infty = \{u \in H^\alpha_K(\mathbb{R}^3) \setminus \{0\} : J^\lambda_\infty(u) = 0\},$$

$$\begin{aligned} J^\lambda_\infty(u) = &\frac{6\alpha - 3}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \frac{4\alpha - 3}{2} \int_{\mathbb{R}^3} K_\infty u^2 dx \\ &+ \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\beta * G(u))[(3+\beta)G(u)dx - 4\alpha g(u)u]dx - \frac{(6\alpha - 3)\lambda}{2} \int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx. \end{aligned}$$

Lemma 16. *For any $\lambda \in [\frac{1}{2}, 1]$ fixed,*

$$c_\lambda < m^\lambda_\infty,$$

where c_λ is defined in Lemma 15 and $m^\lambda_\infty = \inf_{u \in \Pi^\lambda_\infty} E^\lambda_\infty(u)$.

Proof. Take u^λ_∞ as the minimizer of m^λ_∞. By Lemmas 10 and 15 and (K_2), we observe that for all $\lambda \in [\frac{1}{2}, 1]$ and ϑ large enough,

$$c_\lambda \leq \max_{\vartheta > 0} E_\lambda(\vartheta^{2\alpha} u^\lambda_\infty(\vartheta x)) < \max_{\vartheta > 0} E^\lambda_\infty(\vartheta^{2\alpha} u^\lambda_\infty(\vartheta x)) = E^\lambda_\infty(u^\lambda_\infty) = m^\lambda_\infty.$$

□

Lemma 17. *Let $\{v_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence of E_λ. Then there exist a sub-sequence of $\{v_n\}$, and integer $l \in \mathbb{N} \cup \{0\}$, a sequence $\{z^j_n\} \subset \mathbb{R}^3$, $\omega^j \in H^\alpha_K(\mathbb{R}^3)$ for $1 \leq j \leq l$ such that*

(i) $v_n \rightharpoonup v_\lambda$ *with* $E'_\lambda(v_\lambda) = 0$;
(ii) $z^j_n \to +\infty$ *and* $|z^i_n - z^j_n| \to +\infty$ *for* $i \neq j$;
(iii) $\omega^i \neq 0$ *and* $(E^\lambda_\infty)'(\omega^i) = 0$ *for* $1 \leq i \leq l$;
(iv) $\|v_n - v_\lambda - \sum_{j=1}^{l} \omega^j(\cdot - z^j_n)\|_{H^\alpha_K(\mathbb{R}^3)} \to 0$;

(v) $E_\lambda(v_n) \to E_\lambda(v_\lambda) + \sum_{j=1}^{l} E_\infty^\lambda(\omega^j)$.

In addition, we agree that in the case $l = 0$ the above hold without w^j and z_n^j.

Proof. Since $\{v_n\} \subset H_K^\alpha(\mathbb{R}^3)$ is a bounded sequence satisfying

$$E_\lambda(v_n) \to c_\lambda > 0, \quad E'_\lambda(v_n) \to 0.$$

Then, there exists $v_\lambda \in H_K^\alpha(\mathbb{R}^3) \setminus \{0\}$ satisfying

$$\begin{cases} v_n \rightharpoonup v_\lambda & \text{in } H_K^\alpha(\mathbb{R}^3), \\ v_n \to v_\lambda & \text{in } L_{loc}^r(\mathbb{R}^3) \text{ for } r \in [1, 2_s^*), \\ v_n \to v_\lambda & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Moreover, we can show that $E'_\lambda(v_\lambda) = 0$, and so $J_\lambda(v_\lambda) = 0$. We deduce from (8), (34) and (K_1) that

$$\begin{aligned} E_\lambda(v_\lambda) &= E_\lambda(v_\lambda) - \frac{1}{6\alpha - 3} J_\lambda(v_\lambda) \\ &= \frac{1}{6\alpha - 3} \int_{\mathbb{R}^3} [\alpha K(x) + (\nabla K(x), x)] v_\lambda^2 dx + \frac{\alpha\lambda}{3} \int_{\mathbb{R}^3} |v_\lambda|^{2_\alpha^*} dx \\ &\quad + \frac{\lambda}{2(6\alpha - 3)} \int_{\mathbb{R}^3} (I_\beta * G(v_\lambda))[4\alpha g(u_\lambda) v_\lambda - (6\alpha + \beta) G(v_\lambda)] dx \\ &\geq 0. \end{aligned} \tag{37}$$

Set $u_n^1 = v_n - v_\lambda$, then we have $u_n^1 \rightharpoonup 0$. In the sequel, one of two conclusions of u_n^1 holds:
Case 1: $u_n^1 \to 0$ in $H_K^\alpha(\mathbb{R}^3)$, or
Case 2: there exists a sequence $\{y_n^1\} \in \mathbb{R}^3$, $R_0 > 0, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_{R_0}(y_n^1)} |u_n^1|^2 dx \geq \delta > 0. \tag{38}$$

In fact, suppose that **Case 2** does not occur. Hence, for any $R > 0$, we get

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y_n^1)} |u_n^1|^2 dx = 0.$$

Thus, Lemma 2.4 in [37] implies that $u_n^1 \to 0$ in $L^s(\mathbb{R}^3)$, $s \in (2, 2_\alpha^*)$. In view of (6), we see that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} (I_\beta * G(u_n^1)) G(u_n^1) dx = 0. \tag{39}$$

Moreover, we infer from Lemma 2 and (37) that

$$\lim_{n\to\infty} E_\lambda(u_n^1) = \lim_{n\to\infty} E_\lambda(v_n) - E_\lambda(v_\lambda) \leq c_\lambda \tag{40}$$

and

$$\lim_{n\to\infty} (E_\lambda)'(u_n^1) = \lim_{n\to\infty} (E_\lambda)'(v_n) - (E_\lambda)'(v_\lambda) = 0. \tag{41}$$

By virtue of (39) and (41), we see

$$0 = \lim_{n\to\infty} \langle (E_\lambda)'(u_n^1), u_n^1 \rangle = \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n^1|^2 dx + \int_{\mathbb{R}^3} K(x)(u_n^1)^2 dx - \lambda \int_{\mathbb{R}^3} |u_n^1|^{2_\alpha^*} dx \right).$$

Since $\{u_n^1\}$ is bounded in $H_K^\alpha(\mathbb{R}^3)$, then we can suppose that up to a subsequence, as $n \to \infty$,

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n^1|^2 dx + \int_{\mathbb{R}^3} K(x)(u_n^1)^2 dx \to \chi, \quad \lambda \int_{\mathbb{R}^3} |u_n^1|^{2_\alpha^*} dx \to \chi \tag{42}$$

for some $\chi \geq 0$. If $\chi > 0$, in view of (5), we obtain

$$\|u_n^1\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} + \int_{\mathbb{R}^3} K(x)(u_n^1)^2 dx \geq \|u_n^1\|^2_{\mathcal{D}^{\alpha,2}(\mathbb{R}^3)} \geq S_\alpha \|u_n^1\|^2_{L^{2_\alpha^*}(\mathbb{R}^3)}.$$

This together with (42) gives that

$$\chi \geq S_\alpha^{\frac{3}{2\alpha}} \lambda^{-\frac{3-2\alpha}{2\alpha}}, \quad \text{for any } \lambda \in [\frac{1}{2}, 1].$$

However, (40) implies that

$$\begin{aligned} c_\lambda &\geq \lim_{n\to\infty} E_\lambda(u_n^1) \\ &= \lim_{n\to\infty} \left[\frac{1}{2} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_n^1|^2 dx + \int_{\mathbb{R}^3} K(x)(u_n^1)^2 dx \right) - \frac{\lambda}{2_\alpha^*} \int_{\mathbb{R}^3} |u_n^1|^{2_\alpha^*} dx \right] \\ &\geq \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} \lambda^{-\frac{3-2\alpha}{2\alpha}}. \end{aligned} \tag{43}$$

By using similar argument as in Lemmas 10, 12 and 13, we show that

$$m_\infty^\lambda < \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} \lambda^{-\frac{3-2\alpha}{2\alpha}}.$$

Combining with (43) and Lemma 16, we obtain

$$\frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} \lambda^{-\frac{3-2\alpha}{2\alpha}} \leq c_\lambda < m_\infty^\lambda < \frac{\alpha}{3} S_\alpha^{\frac{3}{2\alpha}} \lambda^{-\frac{3-2\alpha}{2\alpha}}, \quad \text{for all } \lambda \in [\frac{1}{2}, 1].$$

which is a contradiction. Thus, $\chi = 0$. From (42), we conclude that $\|u_n^1\|_{H_K^\alpha(\mathbb{R}^3)} \to 0$, that is, $v_n \to v$ in $H_K^\alpha(\mathbb{R}^3)$ and Lemma 17 holds with $\chi = 0$ if **Case 2** does not occur.

In the following, we suppose that **Case 2** is true, that is (38) holds. Then, up to a sub-sequence, we obtain

$$|z_n^1| \to +\infty, \quad u_n^1(\cdot + z_n^1) \rightharpoonup \omega^1 \neq 0, \quad (E_\lambda^\infty)'\omega^1 = 0.$$

Indeed, consider $\widehat{u_n^1} := u_n^1(\cdot + z_n^1)$. Note that $\{u_n^1\}$ is bounded. Then together with (38), we deduce that $\widehat{u_n^1} \rightharpoonup \omega^1 \neq 0$. Therefore, it follows from $u_n^1 \rightharpoonup 0$ in $H_K^\alpha(\mathbb{R}^3)$ that $\{z_n^1\}$ is unbounded, up to a subsequence, $|z_n^1| \to +\infty$. Now we shall prove $(E_\lambda^\infty)'(\omega^1) = 0$. It suffices to prove that $\langle (E_\lambda^\infty)'(\widehat{u_n^1}), \psi \rangle \to 0$ for any $\psi \in C_0^\infty(\mathbb{R}^3)$.

According to (41), we obtain

$$|\langle E_\lambda'(u_n), \psi\rangle - \langle E_\lambda'(u_\lambda), \psi\rangle - \langle E_\lambda'(u_n^1), \psi\rangle| \leq o_n(1)\|\psi\|_{H_K^\alpha(\mathbb{R}^3)},$$

which implies $|\langle E_\lambda'(u_n^1), \psi\rangle| = o_n(1)\|\psi\|_{H_K^\alpha(\mathbb{R}^3)}$. Note that

$$\begin{aligned}
&\langle E_\lambda'(u_n^1), \psi(\cdot - z_n^1)\rangle \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u_n^1(x) - u_n^1(y))(\psi(x - z_n^1) - \psi(y - z_n^1))}{|x-y|^{3+2\alpha}} dxdy \\
&\quad + \int_{\mathbb{R}^3} K(x)u_n^1(x)\psi(x - z_n^1)dx - \lambda \int_{\mathbb{R}^3} (I_\beta * G(u_n^1))g(u_n^1)\psi(x - z_n^1)dx \\
&\quad - \lambda \int_{\mathbb{R}^3} |u_n^1|^{2_\alpha^*-2}u_n^1(x)\psi(x - z_n^1)dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\widehat{u_n^1}(x) - \widehat{u_n^1}(y))(\psi(x) - \psi(y))}{|x-y|^{3+2\alpha}} dxdy + \int_{\mathbb{R}^3} K(x + z_n^1)\widehat{u_n^1}(x)\psi(x)dx \\
&\quad - \lambda \int_{\mathbb{R}^3} (I_\beta * G(\widehat{u_n^1}))g(\widehat{u_n^1})\psi(x)dx - \lambda \int_{\mathbb{R}^3} |\widehat{u_n^1}|^{2_\alpha^*-2}\widehat{u_n^1}(x)\psi(x)dx \\
&= o_n(1)\|\psi\|_{H_K^\alpha(\mathbb{R}^3)}.
\end{aligned} \tag{44}$$

Since $|z_n^1| \to +\infty$ and $\psi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} [K(x + z_n^1) - K_\infty]\widehat{u_n^1}(x)\psi(x)dx \to 0. \tag{45}$$

Combining (44) and (45), we obtain that for any $\psi \in C_0^\infty(\mathbb{R}^3)$, $\langle (E_\infty^\lambda)'(\widehat{u_n^1}), \psi\rangle \to 0$. By (K_2) and $v_n \to v_\lambda$ in $L^2_{loc}(\mathbb{R}^3)$, we can see

$$\int_{\mathbb{R}^3} (K(x) - K_\infty)(u_n^1)^2 dx \to 0. \tag{46}$$

It follows immediately from (40) and (46) that

$$E_\lambda(u_n^1) \to c_\lambda - E_\lambda(v_\lambda), \quad E_\lambda(v_n) - E_\lambda(v_\lambda) - E_\infty^\lambda(u_n^1) \to 0. \tag{47}$$

Set $u_n^2(\cdot) := u_n^1(\cdot) - \omega^1(\cdot - z_n^1)$, then $u_n^2 \rightharpoonup 0$ in $H_K^\alpha(\mathbb{R}^3)$. Noting that $\widehat{u_n^1} \rightharpoonup \omega^1 \neq 0$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)|u_n^2|^2 dx &= \int_{\mathbb{R}^3} K(x)|u_n^1|^2 dx + \int_{\mathbb{R}^3} K(x + z_n^1)|\omega^1(x)|^2 dx \\
&\quad - 2\int_{\mathbb{R}^3} K(x + z_n^1)u_n^1(x + z_n^1)\omega^1(x)dx \\
&= \int_{\mathbb{R}^3} K(x)|v_n|^2 dx - \int_{\mathbb{R}^3} K(x)|v_\lambda|^2 dx - \int_{\mathbb{R}^3} K_\infty|\omega^1|^2 dx + o_n(1).
\end{aligned} \tag{48}$$

From (48), Brezis-Lieb Lemma, Lemma 2.6 in [35] and Lemma 2.9 in [38], we deduce that

$$\begin{cases}
E_\lambda(u_n^2) = E_\lambda(u_n) - \Phi_\lambda(u_\lambda) - E_\infty^\lambda(\omega^1) + o_n(1), \\
E_\infty^\lambda(u_n^2) = \Phi_\lambda(u_n^1) - E_\infty^\lambda(\omega^1) + o_n(1), \\
\langle E_\lambda'(u_n^2), \psi\rangle = \langle E_\lambda'(v_n), \psi\rangle - \langle E_\lambda'(v_\lambda), \psi\rangle - \langle (E_\infty^\lambda)'(\omega^1), \psi\rangle + o_n(1) = o_n(1).
\end{cases}$$

Therefore, together with (47), we obtain

$$E_\lambda(v_n) = E_\lambda(v_\lambda) + E_\infty^\lambda(u_n^1) + o_n(1) = E_\lambda(v_\lambda) + E_\infty^\lambda(\omega^1) + E_\infty^\lambda(u_n^2) + o_n(1).$$

It follows from (37) and Lemma 16 that

$$E_\infty^\lambda(u_n^2) = c_\lambda - E_\lambda(u_\lambda) - E_\infty^\lambda(\omega^1) \le c_\lambda.$$

Please notice that one of **Case 1** and **Case 2** is true for v_n^2. If **Case 1** holds, then Lemma 17 holds with $l = 1$. If **Case 2** occurs, we repeat the above arguments. By iterating this process we have sequences of $\{z_n^j\} \subset \mathbb{R}^3$ such that $|z_n^j| \to +\infty$, $|z_n^j - z_n^i| \to +\infty$ for $i \neq j$ and $u_n^j = u_n^{j-1} - \omega^{j-1}(\cdot - z_n^{j-1})$ with $j \geq 2$ satisfying

$$u_n^j \rightharpoonup 0 \text{ in } H_K^\alpha(\mathbb{R}^3), \quad (E_\infty^\lambda)'(\omega^j) = 0$$

and

$$\begin{cases} \quad \|v_n\|^2_{H_K^\alpha(\mathbb{R}^3)} - \|v_\lambda\|^2_{H_K^\alpha(\mathbb{R}^3)} - \sum_{j=1}^l \|\omega^j(\cdot - z_n^j)\|^2_{H_K^\alpha(\mathbb{R}^3)} \\ = \|v_n - v_\lambda - \sum_{j=1}^l \omega^j(\cdot - z_n^j)\|_{H_K^\alpha(\mathbb{R}^3)} + o_n(1), \\ E_\lambda(v_n) - E_\lambda(v_\lambda) - \sum_{j=1}^{l-1} E_\infty^\lambda(\omega^j) - E_\infty^\lambda(u_n^l) = o_n(1). \end{cases} \tag{49}$$

In view of $\{v_n\}$ is bounded in $H_K^\alpha(\mathbb{R}^3)$, (49) yields that the iteration stops at some l. That is, $u_n^{l+1} \to 0$ in $H_K^\alpha(\mathbb{R}^3)$. From (49), it is easy to check that (iv) and (v) are true. The proof is complete. □

Lemma 18. *For almost every $\lambda \in [\frac{1}{2}, 1]$, let $\{v_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence of E_λ. Then there exists a subsequence $\{v_n\}$ converges to a nontrivial $v_\lambda \in H_K^\alpha(\mathbb{R}^3) \setminus \{0\}$ satisfying*

$$E_\lambda(v_\lambda) = c_\lambda, \quad (E_\lambda)'(v_\lambda) = 0.$$

Proof. From Lemma 17, up to a sub-sequence, there exists $v_\lambda \in H_K^\alpha(\mathbb{R}^3)$, nontrivial critical points ω^j, $j = 1, \ldots, l$ of E_∞^λ, $l \in \mathbb{N} \cup \{0\}$ and $\{z_n^j\} \subset \mathbb{R}^3$ with $|z_n^j| \to +\infty$, $1 \leq j \leq l$ such that

$$E_\lambda^{'}(v_\lambda) = 0, \quad v_n \rightharpoonup v_\lambda, \quad E_\lambda(v_n) \to E_\lambda(v_\lambda) + \sum_{j=1}^{l} E_\infty^\lambda(\omega^j).$$

Together with (37), we infer that if $l \neq 0$,

$$c_\lambda = \lim_{n\to\infty} E_\lambda(v_n) = E_\lambda(v_\lambda) + \sum_{j=1}^{l} E_\infty^\lambda(\omega^j) \geq m_\infty^\lambda,$$

which contradicts with Lemma 16. Therefore, this lemma follows. □

Proof of Theorem 2. Taking a sequence $\{\lambda_n\} \subset [\frac{1}{2}, 1]$ satisfying $\lambda_n \to 1$, from Lemma 15, there is a sequence of nontrivial critical points v_{λ_n} (we may still denote by $\{v_n\}$) for E_{λ_n} and $E_{\lambda_n}(v_n) = c_{\lambda_n}$. Now, we prove that $\{v_n\}$ is bounded. It follows from (8) and $\beta > 2\alpha$ that for every $\tau \in \mathbb{R}$,

$$g(\tau)\tau - 2G(\tau) > g(\tau)\tau - \frac{6\alpha + \beta}{4\alpha} G(\tau) \geq 0.$$

Combining $\langle E'_{\lambda_n}(v_n), v_n \rangle = 0$ and $3 < 4\alpha$ we infer that

$$\begin{aligned} c_{\frac{1}{2}} \geq & c_{\lambda_n} \\ = & E_{\lambda_n}(v_n) - \frac{1}{4}\langle E'_{\lambda_n}(v_n), v_n \rangle \\ = & \frac{1}{4}\|v_n\|^2_{H_K^\alpha(\mathbb{R}^3)} + \frac{\lambda_n}{4}\int_{\mathbb{R}^3} (I_\beta * G(v_n))[g(v_n)v_n - 2G(v_n)]dx + \left(\frac{\alpha}{3} - \frac{1}{4}\right)\lambda_n \int_{\mathbb{R}^3} |v_n|^{2_\alpha^*} dx \\ \geq & \frac{1}{4}\|v_n\|^2_{H_K^\alpha(\mathbb{R}^3)}, \end{aligned}$$

which means that $\{v_n\}$ is bounded in $H_K^\alpha(\mathbb{R}^3)$. Hence, by Theorem 1.1 in [32], we obtain that

$$\begin{aligned}&\lim_{n\to\infty} E(v_n)\\&=\lim_{n\to\infty}\left\{E_{\lambda_n}(v_n)+\frac{\lambda_n-1}{2}\left[\int_{\mathbb{R}^3}(I_\beta*G(v_n))G(v_n))dx+\frac{2}{2^*_\alpha}\int_{\mathbb{R}^3}|v_n|^{2^*_\alpha}dx\right]\right\}\\&=\lim_{n\to\infty} c_{\lambda_n}=c_1,\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\langle E'(v_n),\psi\rangle\\&=\lim_{n\to\infty}\left\{\langle E'_{\lambda_n}(v_n),\psi\rangle+(\lambda_n-1)\left[\int_{\mathbb{R}^3}(I_\beta*G(v_n))g(v_n)\psi dx+\int_{\mathbb{R}^3}|v_n|^{2^*_\alpha-2}v_n\psi dx\right]\right\}\\&=0,\end{aligned}$$

which implies that $\{v_n\}$ is a bounded $(PS)_{c_1}$ sequence of E. Hence, in view of Lemma 18, there is a nontrivial critical point $v_0\in$ for E with $E(v_0)=c_1$.

At last, we prove there is a ground state solution to Equation (1). Let

$$m=\inf\{E(u):u\neq 0,\ E'(u)=0\}.$$

It is easy to see that $0\le m\le E(v_0)=c_1<+\infty$. For any v satisfying $E'(v)=0$ and $\varrho>0$, we see that $\|u\|_{H_K^\alpha(\mathbb{R}^3)}\ge\varrho$. While, it follows from (K_1), $J(v)=0$ and (8) that

$$\begin{aligned}E(v)&=E(v)-\frac{1}{6\alpha-3}J(v)\\&=\frac{1}{6\alpha-3}\int_{\mathbb{R}^3}[\alpha K(x)+(\nabla K(x),x)]v^2dx\\&\quad+\frac{1}{2(6\alpha-3)}\int_{\mathbb{R}^3}(I_\beta*G(v))[4\alpha g(v)v-(6\alpha+\beta)G(v)]dx+\frac{\alpha}{3}\int_{\mathbb{R}^3}|v|^{2^*_\alpha}dx,\end{aligned}$$

which implies $m\ge 0$. Suppose $m=0$, then one has a critical point sequence $\{v_n\}$ of E with $E(v_n)\to 0$. Consequently,

$$\lim_{n\to\infty}\|v_n\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^3)}=0. \tag{50}$$

Similar as (20), we infer that

$$\int_{\mathbb{R}^3}(I_\beta*G(v_n))g(v_n)v_n\le\epsilon\|v_n\|^{2+\frac{2\beta}{3}}_{L^2(\mathbb{R}^3)}+C_\epsilon\|v_n\|^{2q}_{L^{2^*_\alpha}(\mathbb{R}^3)},$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(I_\beta*G(v_n))g(v_n)v_n=0,$$

as $\epsilon\to 0$. Combining with (50) and $\langle E'(v_n),v_n\rangle=0$, we obtain $\lim_{n\to\infty}\|v_n\|_{H_K^\alpha(\mathbb{R}^3)}=0$, which contradicts with $\|v_n\|_{H_K^\alpha(\mathbb{R}^3)}\ge\varrho$. Therefore, $0<m<+\infty$. Then let $\{v_n\}$ be a sequence such that $E'(v_n)=0, E(v_n)\to m$. Similarly, we observe that $\{v_n\}$ is bounded. Using a similar proof of Lemma 18, we infer that there is $v\in H_K^\alpha(\mathbb{R}^3)$ satisfying $E'(v)=0$, $E(v)=m$. $\square$

5. Conclusions

The main purpose of this paper is to study the existence of ground state solution for the fractional Choquard equation with critical Sobolev exponent. To prove Theorem 1, we first establish a key inequality

$$E_\infty(u)\ge E_\infty(u_\vartheta)+\frac{1-\vartheta^{6\alpha-3}}{6\alpha-3}J_\infty(u)+\xi(\vartheta)\int_{\mathbb{R}^3}K_\infty u^2dx+\zeta(\vartheta)\int_{\mathbb{R}^3}|u|^{2^*_\alpha}dx. \tag{51}$$

Using (51), we can prove Lemmas 8–9, which investigate some properties of Π_∞. Then we show $m_\infty = \bar{c}_\infty := \inf_{u\neq 0}\max_{\vartheta>0} E_\infty(u_\vartheta)$. We use the general mini-max principle Proposition 2.8 in [36] to obtain a Cerami sequence for the functional E_∞ with $J_\infty(v_n) \to 0$, where E_∞, $J_\infty(v_n)$ are given in (10) and (12), respectively. Finally, we conclude that c_∞ is achieved by using an important estimate $c_\infty < \frac{\alpha}{3}S_\alpha^{\frac{3}{2\alpha}}$.

Next, we aim to find ground state solution to Equation (1). Due to the potential is not a constant, we use Jeanjean's monotonicity trick. Define a family of functional

$$E_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{\alpha}{2}}u|^2 + K(x)u^2)dx - \frac{\lambda}{2}\left[\int_{\mathbb{R}^3}(I_\beta * G(u))G(u)dx + \frac{2}{2_\alpha^*}\int_{\mathbb{R}^3}|u|^{2_\alpha^*}dx\right].$$

We show that

$$c_\lambda = \inf_{\mu\in\Gamma}\max_{\tau\in[0,1]} E_\lambda(\mu(\tau)) > \max\{E_\lambda(0), E_\lambda(e)\}$$

for all $\lambda \in [\frac{1}{2}, 1]$, where

$$\Gamma = \{\mu \in \mathcal{C}([0,1], H_K^\alpha(\mathbb{R}^3)) : \mu(0) = 0, \mu(1) = e\}.$$

Together with Jeanjean's monotonicity trick, we obtain a bounded sequence $\{u_n\} \subset H_K^\alpha(\mathbb{R}^3)$ such that

$$E_\lambda(u_n) \to c_\lambda, \quad E'_\lambda(u_n) \to 0.$$

To prove that the above sequence $\{v_n\}$ satisfies the (PS) condition, we consider the following limit problem

$$(-\Delta)^\alpha u + K_\infty u = \lambda(I_\beta * G(u))g(u) + \lambda|u|^{2_\alpha^*-2}u, \quad \text{in } \mathbb{R}^3$$

and conclude that $c_\lambda < m_\infty^\lambda$. Then we can obtain a global compactness result, i.e., Lemma 17, which implies that there exists a nontrivial critical point v for E.

In the proof, the restriction on α is very crucial, we do not know whether the solution can still exist for $\alpha \in (0, \frac{3}{4})$. This is a question that we need to further consider.

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