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Topological Properties of Solution Sets for τ-Fractional Non-Instantaneous Impulsive Semi-Linear Differential Inclusions with Infinite Delay [†]

Zainab Alsheekhhussain [1,*], Ahmed Gamal Ibrahim [2] and Yousef Jawarneh [1]

[1] Department of Mathematic, College of Science, Ha'il University, Hail 55476, Saudi Arabia; y.jawaneh@uoh.edu.sa
[2] Department of Mathematics, College of Sciences, Cairo University, Giza 12613, Egypt; agamal2000@yahoo.com
* Correspondence: za.hussain@uoh.edu.sa

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Abstract: The knowledge of fractional calculus can be useful in developing models that allow us to better understand and manage some phenomena. In the present paper, we study the topological structure of the mild solution set for a semi-linear differential inclusion containing the τ-Caputo fractional derivative in the presence of non-instantaneous impulses and an infinite delay. We demonstrate that this set is non-empty and an R_δ-set. We use a recent result regarding the existence of solutions for τ-Caputo fractional semi-linear differential inclusions. Unlike many results, we do not suppose that the generating semigroup is compact. An illustrative example is given as an application of our results.

Keywords: non-instantaneous impulses; τ-caputo derivative; semi-linear differential inclusions; mild solutions; R_δ-sets

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1. Introduction

The subject of fractional calculus has many applications in industry, fluid flows, dynamic systems in control theory, electrical circuits with fractance, generalized voltage dividers, viscoelasticity, multipoles with fractional-order multipoles in electromagnetism, electrochemistry, tracers in fluid flows, biological models of neurons, engineering, polymer science, organic dielectric materials, viscoelastic materials, engineering, rheology, diffusive transport, electrical, networks, electromagnetic theory and physics [1–12].

Impulsive differential inclusions (IDIs) are good tools for describing events where states change rapidly at specific times and have many applications in physics and biology. Differential equations with impulses were considered for the first time by Milman and Myshkis [13], and this was then followed by a period of active research on this subject. When the action of impulses continues on a finite interval, they are called non-instantaneous impulses. For recent papers on fractional differential inclusions (FDIs) with non-instantaneous impulses, we refer to [14–22].

It is known that the set of solutions or mild solutions for a differential inclusion (i.e., the right-hand side is a multi-valued function) is typically not a singleton. Motivated by this fact, many scientists have proven, under suitable conditions, that the set of solutions or mild solutions for different differential inclusions is an R_δ-sets, meaning that this set is a homotopy equivalent to a point from the perspective of algebraic topology. Therefore, this topic is very important and is reasonable and practical to study. Among these studies we mention the following: DeBlasi [23], Papageorgiou [24] and Zhou et al. [25] considered differential inclusions; Gabor et al. [26], Djebali et al. [27], Zhang et al. [28] and Ma et al. [29] studied IDIs; Alsheekhhussain et al. [30] and Wang et al. [31] looked at semi-linear FDIs with non-instantaneous impulses; and Zhou et al. [32,33] considered fractional stochastic differential inclusions.

For other contributions on the same subject see, for instance, refs. [34–36] and the references therein.

In refs. [37–41] the authors demonstrated that the solution set for different kinds of FDIs is compact.

Almeida [42] introduced the concept of the τ-Caputo fractional derivative that generalized the Caputo fractional derivative. Jarad et al. [43] presented some properties for this definition. Suechoei et al. [44] applied the results in [42,43] and investigated the existence and stability of mild solutions for fractional semi-linear differential inclusions containing τ-Caputo fractional derivatives.

To date, there is no work in the literature regarding the study of the topological properties of a mild solution set for semi-linear differential inclusions containing τ-Caputo fractional derivatives, with infinite delay and a linear infinitesimal generator of a semi group of operators which are not compact.

Motivated by this fact and the aforementioned works, we prove in the present work that the mild solution set, $\Sigma_{\Psi}^{\mathcal{F},\tau}(-\infty, b]$, for the following problem:

$$\begin{cases} {}^{c}D^{\gamma,\tau}_{s_i,\omega}x(\omega) \in Tx(\omega) + F(\omega, x_{\rho(\omega,x_\omega)}),\ a.e.\ \omega \in \cup_{i=0}^{i=m}(s_i,\ \theta_{i+1}], \\ x(\theta_i^+) = \Upsilon_i(\theta_i, x(\theta_i^-)),\ i = 1,\ldots\ldots m, \\ x(\omega) = \Upsilon_i(\omega, x(\theta_i^-)),\ \omega \in \cup_{i=1}^{i=m}(\theta_i, s_i], \\ x(\omega) = \Psi(\omega), \omega \in (-\infty,\ 0], \end{cases} \tag{1}$$

is non-empty and an R_δ-set, where $\gamma \in (0,1)$, $\tau : J \to \mathbb{R}$ is continuously differentiable and an increasing function with $\acute{\tau}(t) \neq 0$, $\forall t \in J$, ${}^{c}D^{\gamma,\tau}_{s_i,\omega}$ is the τ-Caputo derivative of order γ with a lower limit at s_i [42], T is the infinitesimal generator of a C_0-semigroup $\{T(\omega) :\ \omega \geq 0\}$ defined on a real Banach space E, B is a phase space, $F : J \times B \to 2^E - \{\phi\}$ is a multifunction, $\rho : J \times B \to (-\infty, b]$, $\tau \in C^1(J)$ is an increasing function with $\tau'(t) \neq 0$, $\forall t \in J$, $0 = s_0 < \theta_1 \leq s_1 < \theta_2 \leq s_2 < \ldots < s_m < \theta_{m+1} = b, \Upsilon_i : [\theta_i\ s_i] \times E \to E$ and $\Psi \in B$ is fixed with $\Psi(0) = 0$. Furthermore, for any $\omega \in (-\infty, b]$ and any $x : (-\infty, b] \to E$ with $x_{|_{(-\infty,0]}} \in B$, x_ω is an element in B defined by $(x_\omega)(\theta) = x(\omega + \theta)$; $\theta \in (-\infty, 0]$.

It is worth noting that Alsheekhhussain et al. [30] recently demonstrated that the mild solution set for a similar type for Problem (1) is non-empty and an R_δ-set in the special cases where $\tau(t) = t; t \in J$, $\rho(w, x_w) = w$ and the delay is finite.

It is important to mention that, due to the presence of non-instantaneous impulses and an infinite delay that depends on the function ρ in the considered problem, there are many difficulties in the proofs that are different from similar previous works, and we will use an appropriate technique to overcome these difficulties. Therefore, many of the strategies used in this paper are novel.

The following is a summary of this study's key contributions.

- A new class of differential inclusions (the right-hand side is a multi-valued function) is formulated containing τ-Caputo derivatives in the presence of non-instantaneous impulses and infinite delay in infinite-dimensional Banach spaces.
- We prove that the mild solution set for Problem (1), $\Sigma_{\Psi}^{F,\tau}(-\infty, b]$, is non-empty and an R_δ-set.
- Our work generalizes what was conducted by Wang et al. [31], in which Problem (1) was considered without delay ($\rho(w, x_w) = 0$) and $\tau(t) = t$, $\forall t \in J$, and by Alsheekhhussain et al. [30], in which a similar type for Problem (1) was considered in special cases where $\tau(t) = t$, $\forall t \in J$, $\rho(w, x_w) = w$ with finite delay. Moreover, this work generalizes Theorem 4.1 in [44] when the right-hand side is a multi-valued function in the presence of both non-instantaneous impulses and infinite delay.

- This work is novel and interesting because the linear part is an operator that generates a non-compact semi-group, the non-linear part is a multi-valued function, and the studied problem contains the τ-Caputo derivative with non-instantaneous impulses and infinite delay.
- Our technique helps any researcher interested in extending the results in [23–30,32,33] to cases where the right-hand side is a multi-valued function in the presence of both non-instantaneous impulses and infinite delay, while the left-hand side contains the τ-Caputo derivative.

For the directions of future work, we suggest proving that the set of solutions for the considered problems in [19,37–41] is non-empty and an R_δ-set.

In Section 2, we collate concepts and known results which will be used later. In Section 3, the non-emptiness and compactness of $\Sigma_{\Psi}^{F,\tau}(-\infty,b]$ is proven. Section 4 demonstrates that $\Sigma_{\Psi}^{F,\tau}(-\infty,b]$ is an R_δ-set. An example is presented in Section 5 to illustrate the applicability of the obtained results.

2. Preliminaries and Notation

Let $P_{ck}(E)$ denote the family of non-empty, convex and compact subsets of E; $P_{cc}(E)$ is the family of non-empty closed convex subsets of E; $AC(J,E)$ is the Banach space of absolutely continuous functions from J to E; and Γ is the Euler gamma function, whereby

$$AC^{1,\tau}(J,E) := \{x : J \to E,\ [\frac{1}{\tau\prime(t)}\frac{d}{dt}]\ x \in AC(J,E)\}.$$

Definition 1 ([42]). *The τ-Caputo fractional derivative is of order γ, where the lower limit at a, for a function $g \in AC^{1,\tau}(J,E)$, is defined by ${}^{c}D_{a+}^{\gamma,\tau}g(t) := D_{a+}^{\gamma,\tau}[g(t)-g(a)], t \in J,$*

$$AC^{1,\tau}(J,E) := \{x : J \to E, [\frac{1}{\tau\prime(t)}\frac{d}{dt}]\ x \in AC(J,E)\},$$

where

$$D_{a+}^{\gamma,\tau}g(t) := \frac{1}{\tau\prime(t)}\frac{d}{dt}\ I_{a+}^{1-\gamma,\tau}\ g\ (t), t \in J,$$

and

$$I_{a+}^{\gamma,\tau}g(t) := \int_0^t \frac{(\tau(t)-\tau(s))^{\gamma-1}\tau\prime(s)}{\Gamma(\gamma)}g(s)ds,\ t \in J.$$

Remark 1. *If $\tau(t) = t$, we obtain the Caputo fractional derivative, and if $\tau(t) = \ln t$, we obtain the Caputo–Hadamard fractional derivative. Moreover, Almeida [42] presented an application of the τ-Caputo fractional derivative in population growth.*

Definition 2 ([44]). *Let $h : J \to E$ and T be the infinitesimal generator of a C_0-semigroup $\{T(\theta) : \theta \geq 0\}$. The function $x \in C(J,E)$ is called a mild solution for the problem*

$$\begin{cases} {}^{c}D_{0,\omega}^{\gamma,\tau}x(\omega) = Tx(\omega) + h(\omega),\ \omega \in J, \\ x(0) = x_0 \in E, \end{cases}$$

if

$$x(\omega) = K_1^{\tau}(\omega,0)(x_0) + \int_0^{\omega}(\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)h(v)dv,\ \omega \in J,$$

where for $0 \leq v \leq \omega$,

$$K_1^{\tau}(\omega,v) = \int_0^{\infty}\xi_{\gamma}(\theta)T(\tau(\omega)-\tau(v))^{\gamma}\theta)d\theta,$$

$$K_2^\tau(\omega,v)=\gamma\int_0^\infty \theta\xi_\gamma(\theta)T(\tau(\omega)-\tau(v))^\gamma\theta)d\theta,$$

$$\xi_\gamma(\theta)=\frac{1}{\gamma}\theta^{-1-\frac{1}{\gamma}}w_\gamma(\theta^{-\frac{1}{\gamma}})\geq 0,$$

and

$$w_\gamma(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\gamma n-1}\frac{\Gamma(n\,\gamma+1)}{n!}\sin(n\pi\gamma),\ \theta\in(0,\infty).$$

Notice that $\int_0^\infty \xi_\gamma(\theta)d\theta=1$.

Lemma 1 ([44])**.** *The operators $K_1^\tau(\omega,v)$ and $K_2^\tau(\omega,v)$ have the following properties:*

1. *$K_1^\tau(\omega,\omega)$ is the identity operator for $\omega\geq 0$.*
2. *For any $0\leq v\leq\omega$, $K_1^\tau(\omega,v)$ and $K_2^\tau(\omega,v)$ are bounded linear operators with $||K_1^\tau(\omega,v)x||\leq M||x||$ and $||K_2^\tau(\omega,v)x||\leq\frac{M}{\Gamma(\gamma)}||x||,\forall x\in E$.*
3. *For any $0\leq v\leq\omega_1\leq\omega_2\leq b$ and any $x\in E$,*

$$\lim_{\omega_1\to\omega_2}||K_1^\tau(\omega_1,v)x-K_1^\tau(\omega_2,v)x||=0, \text{and} \lim_{\omega_1\to\omega_2}||K_2^\tau(\omega_1,v)x-K_2^\tau(\omega_1,v)x||=0.$$

4. *If any $t>0,T(t)$ is compact, then $K_1^\tau(\omega,v)$ and $K_2^\tau(\omega,v)$ are compact for $\omega,v>0$.*

Next, let $I_0=\{0,1,\dots,m\}$, $I_1=\{1,\ 2,\ ..,m\}$ and consider the vectors spaces

$$\begin{aligned} PC(J,E)\quad &: \;=\{u:J\to E: u_{|J_i}\in C(J_i,E), i\in I_0 \text{ and } u(\theta_i^+),\\ u(\theta_i)\quad &=\quad u(\theta_i^-) \text{ which are finite for each } i\in I_1\},\end{aligned}$$

and

$$B_b:=\{x:(-\infty,b]\to E \text{ such that } x_0\in B, x_{|_J}\in PC(J,E)\},$$

where $J_0=[0,\theta_1]$ and $J_i=(\theta_i\,,\theta_{i+1}]$. A semi-norm on B_b is defined by $||x||_{B_b}=||x_0||_B+\sup_{v\in J}||x(v)||$. Moreover, let

$$H:=\{x\in B_b: x_0(\theta)=0,\forall\theta\in(-\infty,0]\}.$$

It is known that $(H,||.||_H)$ and $(PC(J,E),||.||_{PC(J,E)})$ are Banach spaces where $||x||_H=sup_{\omega\in J}||x(\omega)||$, $||x||_{PC(J,E)}=\sup_{\omega\in J}||x(\omega)||$, and the Hausdorff measure of non-compactness on $PC(J,E)$ is defined by

$$\chi_{PC}(D):=\max_{i\in I_0}\chi_i(D_{|\,\overline{J_i}}),$$

where $D\subseteq PC(J,E)$ is bounded and χ_i is the Hausdorff measure of non-compactness on $C(\overline{J_i},E)$ and

$$D_{|\overline{J_i}}=\{x^*:\overline{J_i}\to E: x^*(\omega)=x(\omega),\omega\in J_i \text{ and } x^*(\theta_i)=x(\theta_i^+),x\in D\}.$$

It can be easily seen that the Hausdorff measure of non-compactness on H can be given by

$$\chi_H(D):=\max_{i\in I_0}\chi_i(D_{|\,\overline{J_i}}),$$

where $D\subseteq H$ is bounded.

Definition 3 ([45,46])**.** *A phase space is a vector space B whose elements are functions $x:(-\infty,0]\to E$ equipped with a semi-norm $||.||_B$ such that*

1. *If $x:(-\infty,b]\to E$ is such that $x|_J\in PC(J,E)$ and $x_0\in B$, then for any $\omega\in[0,b]$, the next properties hold:*

(i) $x_\omega \in B$;

(ii) $\eta > 0$ *exists with* $||x(\omega)|| \leq \eta||x_\omega||_B$;

(iii) *There is a continuous function* $L_1 : [0,\infty) \to [0,\infty)$ *and a locally bounded function* $L_2 : [0,\infty) \to [0,\infty)$ *such that*

$$||x_\omega||_B \leq L_1(\omega) \sup\{||x(v)|| : v \in [0,\omega]\} + L_2(\omega)||x_0||_B. \tag{2}$$

(iv) *The function* $\omega \to x_\omega$ *is continuous from* J *to* B.

2. *B is complete.*

Following arguments used in the proof of Lemma 3.3 in ([47]), we have the next lemma.

Lemma 2. *Let* $\rho : J \times B \to (-\infty, b]$ *be continuous,* $\Psi \in B - \{0\}$ *and set* $R(\rho\text{-}) = \rho(J \times B) \cap (-\infty, 0]$. *Assume that:*

$(H\rho)$ *The function* $\omega \to \Psi_\omega$ *is continuous from* $R(\rho\text{-})$ *to* B, *and there exists a bounded continuous function* $j^\Psi : R(\rho) \to (0,\infty)$, *such that*

$$||\Psi_\omega||_B \leq j^\Psi(\omega)||\Psi||_B, \forall \omega \in R(\rho^-).$$

Then, for any $x : (-\infty, b] \to E$, *such that* $x_0 = \Psi$ *and* $x|_J \in PC(J,E)$, *one has*

$$||x_\omega||_B \leq \xi_1 \sup\{||x(v)|| : v \in [0, \max\{0,\omega]\} + \xi_2||\Psi||_B, \omega \in R(\rho^-) \cup J, \tag{3}$$

where $\xi_1 = \sup\{L_1(v) : v \in J\}$ *and* $\xi_2 = \sup\{L_2(v) : v \in J\} + \sup\{j^\Psi(v) : v \in R(\rho^-)\}$.

Proof. Let $\omega \in J$. Due to (2), it follows that

$$\begin{aligned} ||x_\omega||_B &\leq L_1(\omega) \sup\{||x(v)|| : v \in [0,\omega]\} + L_2(\omega)||x_0||_B \\ &\leq \xi_1 \sup\{||x(v)|| : v \in [0,\omega]\} + L_2(\omega)||\Psi||_B. \end{aligned}$$

If $\omega \in R(\rho^-)$, then by $(H\rho)$, one has

$$||x_\omega||_B \leq ||\Psi_\omega||_B \leq \sup\{j^\Psi(v) : v \in R(\rho^-)||\Psi||_B.$$

By combining the last two inequalities, we arrive at (3). □

Remark 2 ([47], Remark 3.2)**.** $(H\rho)$ *is satisfied if* Ψ *is continuous and bounded.*

Definition 4. *A function* $z \in B_b$ *is said to be a mild solution of (1) if there is* $g \in L^1(J,E)$ *with* $g(\omega) \in F(\omega, z_{\rho(\omega,x_\omega)})$ *absolutely everywhere, such that*

$$z(\omega) = \begin{cases} \Psi(\omega), \omega \in (-\infty,\ 0] \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv, \omega \in [0,\ \theta_1] \\ \Upsilon_i(\omega, z(\theta_i^-)),\ \omega \in \cup_{i\in I_1}(\theta_i,\ s_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i, z(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv,\ \omega \in \cup_{i\in I_1}(s_i,\ \theta_{i+1}]. \end{cases}$$

Notice that this solution function is continuous on $(\theta_i,\ \theta_{i+1}], i = 1,\ldots,m$.

We will use the next lemmas later.

Lemma 3 (([48], p. 350) (Mazur's lemma))**.** *Let* $(X, \|\cdot\|)$ *be a normed vector space and let* $(u_n)_{n\in N}$ *be a sequence in* X *that converges weakly to* $u_0 \in X$; *then, there is a sequence* $(v_n)_{n\in N}$ *such that* v_n *is a convex combination of* $u_n, u_{n+1}, \ldots.., u_{k(n)}$ *and* v_n *converges strongly to* u_0.

Lemma 4 ([49], Corollary 3.3.1 and Proposition 3.5.1)**.** *Let* $W \in P_{cc}(E)$ *and* $R : W \to P_{ck}(E)$ *be a closed multifunction which is* χ*-condensing, where* χ *is a non-singular measure of non-*

compactness defined on E. Then, $Fix(R)$ is non-empty. Moreover, if γ is monotone and $Fix(R)$ is bounded, then it is compact.

Lemma 5 (([50], Theorem 1) (generalized Young's inequality))**.** *Suppose $r > 0, a : J \to [0, \infty)$ is locally integrable and $g : J \to [0, L]$ is non-decreasing continuous function, $L > 0$, and $u : J \to [0, \infty)$ is locally integrable with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{r-1} u(s) ds \, , \; \forall t \in J.$$

Then,

$$u(t) \leq a(t) + \int_0^t [\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(r))^n}{\Gamma(nr)} (t-s)^{nr-1} a(s)] ds, t \in J.$$

Definition 5 ([51])**.** *A subset D of a metric space Y is said to be contractible if there is a point $x_0 \in D$ and a continuous function $Z : [0,1] \times D \to D$, such that $Z(0, x) = x$ and $Z(1, x) = x_0, \forall x \in D$.*

Definition 6 ([51])**.** *A metric space Y is called an R_δ-set if $Y = \cap_{n=1}^{\infty} K_n$, where (K_n) is a decreasing sequence of non-empty compact contractible subsets.*

Remark 3 ([52], Example 1.2.12)**.** *An R_δ-set does not need to be contractible. For more on R_δ-sets, we refer the reader to [53].*

3. The Compactness of $\Sigma_\Psi^{F,\tau}(-\infty, b]$

This section shows that $\Sigma_\Psi^{F,\tau}(-\infty, b]$ is non-empty and compact in B_b.
Let $\overline{x} \in B_b$ with

$$\overline{x}(\omega) = \begin{cases} \Psi(\omega), \omega \in (-\infty, 0], \\ x(\omega), \omega \in [0, b]. \end{cases} \tag{4}$$

Then, a function $\overline{x} \in B_b$ with $\overline{x}(\omega) = \Psi(\omega); \omega \in (-\infty, 0]$ belongs to $\Sigma_\Psi^{F,\tau}(-\infty, b]$ if and only if the function x verifies the integral equation:

$$x(\omega) = \begin{cases} 0, \; \omega \in (-\infty, 0] \\ K_1^\tau(\omega, 0)\Psi(0) + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) g(v) dv, \omega \in [0, \theta_1] \\ \Upsilon_i(\omega, x(\theta_i^-)), \; \omega \in \cup_{i \in I_1} (\theta_i \; s_i], \\ K_1^\tau(\omega, s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) g(v) dv, \; \omega \in \cup_{i \in I_1} (s_i, \; \theta_{i+1}], \end{cases}$$

where $g \in L^1(J, E)$ with $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega, \overline{x}_\omega)})$ almost everywhere.

Theorem 1. *Suppose the following conditions hold:*

(HT) $T : D(T) \subseteq E \to E$ is a linear closed operator generating an equicontinuous semi-group $\{T(\omega) : \omega \geq 0\}$ of bounded linear operators, and $M \geq 1$, such that $\sup_{\omega \geq 0} ||T(\omega)|| \leq M$.

(HF) $F : J \times B \to P_{ck}(E)$, such that

(HF_1) for every $z \in B$, the multifunction $\omega \longrightarrow F(\omega, z)$ admits a strongly measurable selection, and for almost every $\omega \in J$, the multifunction $z \longrightarrow F(\omega, z)$ is upper semi-continuous.

For (HF_2), there exists a function $\varphi \in L^P(I, \mathbb{R}^+)(P > \frac{1}{\gamma})$ such that, for any $z \in B$

$$\|F(\omega, z)\| \leq \varphi(\omega) \; (1 + \|z\|_B), \; a.e. \; \omega \in J.$$

For (HF_3), there is a $\beta \in L^P([0,b],E)$, $p > \frac{1}{\gamma}$ such that, for every bounded subset $D \subset B$, we have

$$\chi_E(F(\omega,D)) \le \beta(\omega) \sup_{\theta\in(-\infty,0]} \chi_E\{z(\theta) : z \in D\}, a.e.\ for\ \omega \in J. \tag{5}$$

(H) for any $i = 1,..,m$, $\Upsilon_i : [\theta_i, s_i] \times E \to E$ is uniformly continuous on bounded sets, $\Upsilon_i(\omega,.)$; $\omega \in J$ maps the bounded sets to relatively compact subsets, and $\sigma_i > 0$ with

$$||\Upsilon_i(\omega,x)|| \le \sigma_i||x||, \forall x \in E. \tag{6}$$

Then, $\Sigma_{\Psi}^{F,\tau}(-\infty,b]$ is not void and a compact subset of B_b, provided that

$$M(\sigma + \frac{\xi_1}{\Gamma(\gamma)}||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b) < 1, \tag{7}$$

and

$$\frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)} < 1 \tag{8}$$

where $\kappa = (\max_{v\in[0,b]} \tau'(v))^{\frac{1}{p-1}}$ and $\eta_b = \frac{M}{\Gamma(\gamma)}(\frac{p-1}{\gamma p-1})^{\frac{p-1}{p}}(\tau(b)-\tau(0))^{\gamma-\frac{1}{p}}$.

Proof. Let $x \in H$. Due to (HF_1), $g \in L^1(J,E)$ with $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ almost everywhere; therefore, we can define a multifunction Φ on H as $y \in \Phi(x)$ if and only if

$$y(\omega) = \begin{cases} 0, \omega \in (-\infty,0] \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv, \omega \in [0,\theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i\ s_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i,x(\theta_i^-)) \\ +\int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv,\ \omega \in \cup_{i\in I_1}(s_i,\theta_{i+1}], \end{cases} \tag{9}$$

where $\overline{x}$ is defined by (3). Notice that if x is a fixed point for Φ, then $\overline{x}$ is a solution for Problem (1).

Step 1. Let

$$v = \frac{M||x_0|| + \frac{M(1+\xi_2||\Psi||_B)}{\Gamma(\gamma)}||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b}{1 - [M(\sigma + \frac{\xi_1}{\Gamma(\gamma)}||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b)]}, \tag{10}$$

and $D_\nu = \{z \in PC(J,E) : ||z|| \le v\}$. We show that $\Phi(D_\nu) \subseteq D_\nu$. Note that, due to (3), for any $\omega \in J$ and any $x \in H$, one has

$$||\overline{x}_{\rho(\omega,\overline{x}_\omega)}||_B \le \xi_1\nu + \xi_2||\Psi||_B. \tag{11}$$

Let $x \in D_v$ and $y \in \Phi(x)$. Then, $g \in L^1(J,E)$ with $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ almost everywhere, such that y satisfies (9). Using (HF_2), Lemma (1), (9), (11) and Holder's inequality, it follows for $\omega \in (0,\theta_1]$ that

$$
\begin{aligned}
||y(\omega)|| \leq \quad & M||x_0|| + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)\varphi(v)dv \\
\leq \quad & M||x_0|| + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)} \times \\
& (\int_0^\omega (\tau(\omega)-\tau(v))^{\frac{(\gamma-1)p}{p-1}} (\tau'(v))^{\frac{p}{p-1}} dv)^{\frac{p-1}{p}} \\
\leq \quad & M||x_0|| + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa \times \\
& (\int_0^\omega ((\omega)-\tau(v))^{\frac{(\gamma-1)p}{p-1}} \tau'(v)dv)^{\frac{p-1}{p}} \\
\leq \quad & M||x_0|| + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b.
\end{aligned} \tag{12}
$$

Next, by (6), one has

$$
\sup_{i=1,2,..,m} \sup_{\omega\in[\theta_1,s_i]} ||\Upsilon_i(\omega, x(\theta_i^-))|| \leq \sigma v. \tag{13}
$$

Moreover, as in (12), on has, for $\theta \in (s_i,\theta_{i+1}], i \in I_1$,

$$
||y(\omega)|| \leq M\,\sigma v + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b. \tag{14}
$$

Combining (10) and (12)–(14), we obtain

$$
\begin{aligned}
||y||_H \quad \leq \quad & M||x_0|| + \frac{M(1+\xi_2||\Psi||_B)}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b \\
& + vM(\sigma + \frac{\xi_1}{\Gamma(\gamma)} ||\varphi||_{L^p(J,\mathbb{R}^+)}\kappa\eta_b) \\
= \quad & v.
\end{aligned}
$$

Step 2. Φ has a closed graph on D_ν.

Assume that $x_n, y_n \in D_\nu$ with $y_n \in \Phi(x_n); n \geq 1, x_n \to x \in D_\nu$ and $y_n \to y \in D_\nu$. Then, $g_n \in L^1(J,E); n \geq 1$ with $g_n(\omega) \in F(\omega, (\overline{x}_n)_{\rho(\omega,(\overline{x}_n)_\omega)})$ almost everywhere, such that

$$
y_n(\omega) = \begin{cases}
0, \omega \in (-\infty, 0], \\
K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g_n(v)dv,\ \omega \in [0,\theta_1], \\
\Upsilon_i(\omega, x_n(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i\ s_i], \\
K_1^\tau(\omega,s_i)\Upsilon_i(s_i, x_n(\theta_i^-)) \\
+ \int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g_n(v)dv, \omega \in \cup_{i\in I_1}(s_i,\theta_{i+1}].
\end{cases} \tag{15}
$$

Due to $(HF2)$ and (11), we obtain

$$
||g_n(v)|| \leq \varphi(v)(1+\xi_1\nu+\xi_2||\Psi||_B), a.\ e.\ v \in J. \tag{16}
$$

Then, (g_n) is bounded in $L^P(J,E)$, and thus, there is a subsequence of (g_n) denoted, again, by (g_n) such that $g_n \rightharpoonup g \in L^P(J,E)$. From Lemma 3 (Mazur's Lemma), we can find a sequence $(z_n)_{n\geq 1}$ such that each z_n is a convex combination of $g_n, g_{n+1}, \ldots, g_{k(n)}$ and that z_n converges strongly to g in $L^P(J,E)$. Let

$$\widetilde{y}_n(\omega) = \begin{cases} 0, \omega \in (-\infty, 0], \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v)dv, \ \omega \in [0,\theta_1], \\ \Upsilon_i(\omega, x_n(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i\ s_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i, x_n(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v)dv, \omega \in \cup_{i\in I_1}(s_i,\theta_{i+1}]. \end{cases}$$

By (16), for every $\omega \in J, v \in (0,\omega]$ and every $n \geq 1$, one has

$$||(\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)z_n(v)|| \leq (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)\ \varphi(v) \in L^P((0,\omega],\mathbb{R}^+).$$

Since $\Upsilon_i(\omega,.)$ is uniformly continuous on bounded sets, by Lebesgue's dominated convergence theorem, it yields $\widetilde{y}_n(\omega) \to y_0(\omega); \omega \in J$, where

$$y_0(\omega) = \begin{cases} 0, \omega \in (-\infty, 0] \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv, \ \omega \in [0,\theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i\ s_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv, \omega \in \cup_{i\in I_1}(s_i,\theta_{i+1}]. \end{cases}$$

Note that $(\widetilde{y}_n)$ is a subsequence of (y_n), such that $y = y_0$.
Next, by (3), for any $\omega \in R(\rho^-) \cup J$, one has

$$\begin{aligned} \lim_{n\to\infty} ||(\overline{x}_n)_\omega - \overline{x}_\omega|| &= ||(\overline{x}_n - \overline{x})_\omega|| \\ &\leq \lim_{n\to\infty} \xi_1 ||\overline{x}_n - \overline{x}||_H + \xi_2 ||(\overline{x}_n - \overline{x})_0||_B \\ &= 0. \end{aligned} \tag{17}$$

Then, by the continuity of ρ on $J \times B$, it yields $\lim_{n\to\infty} \rho(\omega,(\overline{x}_n)_\omega) = \rho(\omega,\overline{x}_\omega)$; hence, by the second axiom of Definition (2), $\lim_{n\to\infty} ||\overline{x}_{\rho(\omega,(\overline{x}_n)_\omega)} - \overline{x}_{\rho(\omega,\overline{x}_\omega)}|| = 0$. Consequently, again by (17), we obtain

$$\begin{aligned} &\lim_{n\to\infty} ||(\overline{x}_n)_{\rho(\omega,(\overline{x}_n)_\omega)} - \overline{x}_{\rho(\omega,\overline{x}_\omega)}||_B \\ \leq\ &\lim_{n\to\infty} ||(\overline{x}_n)_{\rho(\omega,(\overline{x}_n)_\omega)} - \overline{x}_{\rho(\omega,(\overline{x}_n)_\omega)}|| + \lim_{n\to\infty} ||\overline{x}_{\rho(\omega,(\overline{x}_n)_\omega)} - \overline{x}_{\rho(\omega,\overline{x}_\omega)}|| \\ \leq\ &\xi_1 \lim_{n\to\infty} ||\overline{x}_n - \overline{x}_\omega||_H \\ =\ &0. \end{aligned} \tag{18}$$

Thus, from (18) and the upper semi-continuity of $F(\omega,.)$; *a. e.* $\omega \in J$, it follows that $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ almost everywhere, and hence, $y \in \Phi(x)$.
Step 3. $\Phi(x)$ is compact for any $x \in D_\nu$.
Let $y_n \in D_\nu$ with $y_n \in \Phi(x); n \geq 1$. Using the same arguments in step 3, there is a convergent subsequence of (y_n) converging in D_ν, proving that $\Phi(x)$ is relatively compact. Note that step 3 implies that $\Phi(x)$ is closed, and therefore it is compact.
Step 4. We demonstrate that the subsets $Z_{|\overline{J_i}}$ $(i \in I_0)$ are equicontinuous, where $Z = \Phi(D_v)$ and

$$Z_{|\overline{J_i}} = \{y^* \in C(\overline{J_i}, E) : y^*(\omega) = y(\omega), \omega \in J_i, y^*(\theta_i) = y(\theta_i^+), y \in \Phi(x), x \in D_v\}.$$

Case 1. Suppose $i = 0, y^* \in Z_{|\overline{J_0}}$. Then, there is $x \in D_v$ and $g \in L^p([0,\theta_1], E)$ with $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ almost everywhere, such that

$$y^*(\omega) = K_1^\tau(\omega,0)x_0 + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)g(v)dv,\ \omega \in [0,\theta_1].$$

Let t_1, t_2 be $0 \le t_1 < t_2 \le \theta_1$. Then, by the second statement of Lemma 1, it follows that

$$\begin{aligned}
& \|y^*(t_2) - y^*(t_1)\| \\
\le\ & ||K_1^\tau(t_2,0)x_0 - K_1^\tau(t_1,0)x_0|| \\
& + || \int_0^{t_2} (\tau(t_2) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(t_2,v)g(v)dv \\
& - \int_0^{t_1} (\tau(t_1) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(t_1,v)g(v)dv|| \\
\le\ & ||K_1^\tau(t_2,0)x_0 - K_1^\tau(t_1,0)x_0|| \\
& + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} \int_{t_1}^{t_2} (\tau(t_2) - \tau(v))^{\gamma-1}\tau'(v)\varphi(v)dv \\
& + \frac{M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)} \\
& \times \int_0^{\theta_1} |(\tau(t_2) - \tau(v))^{\gamma-1} - (\tau(t_1) - \tau(v))^{\gamma-1}|\tau'(v)\varphi(v)dv \\
& + || \int_0^{\theta_1} (\tau(t_1) - \tau(v))^{\gamma-1}\tau'(v)||K_2^\tau(t_2,v)g(v) - K_2^\tau(t_1,v)g(v)||\, dv. \\
=\ & \sum_{i=1}^{i=4} I_i.
\end{aligned} \tag{19}$$

Due to the third statement of Lemma (1), $\lim\limits_{t_2 \to t_1} I_1 = 0$. For I_2, we have

$$\begin{aligned}
\lim_{t_2 \to t_1} I_2 &= \frac{M(1+v)}{\Gamma(\gamma)} \lim_{t_2 \to t_1} \int_{t_1}^{t_2} (\tau(t_2) - \tau(v))^{\gamma-1}\tau'(v)\varphi(v)dv \\
&\le \frac{\kappa M(1+v)}{\Gamma(\gamma)} ||\varphi||_{L^p([J,\mathbb{R}^+)} \lim_{t_2 \to t_1} (\int_{t_1}^{t_2} (\tau(t_2) - \tau(v))^{\frac{P(\gamma-1)}{P-1}}\tau'(v)dv)^{\frac{P-1}{P}} = 0.
\end{aligned}$$

For I_3, using the Holder's inequality, we have

$$\begin{aligned}
\lim_{\theta_2 \to \theta_1} I_3 &= \frac{M(1+v)}{\Gamma(\gamma)} \times \\
& \lim_{t_2 \to t_1} \int_0^{t_1} |(\tau(t_2) - \tau(v))^{\gamma-1} - (\tau(t_1) - \tau(v))^{\gamma-1}|\tau'(v)\varphi(v)dv\,. \\
&\le \frac{\kappa M(1+v)}{\Gamma(\gamma)} \|\varphi\|_{L^P_{(J,\mathbb{R}^+)}} \times \\
& [\lim_{t_2 \to t_1} \int_0^{t_1} |(\tau(t_2) - \tau(v))^{\gamma-1} - (\tau(t_1) - \tau(v)^{\gamma-1})^{\frac{p}{p-1}}|\tau'(v)dv]^{\frac{p-1}{p}}.
\end{aligned}$$

Notice that $\overline{\omega} = \frac{\gamma-1}{1-\frac{1}{P}} \in (-1,0)$. Then, for $\tau(v) < \tau(t_1) < \tau(t_2)$, we have $(\tau(t_1) - \tau(v))^{\overline{\omega}} \ge (\tau(t_2) - \tau(v))^{\overline{\omega}}$. Applying Lemma 3 in [54] and keeping in mind that $\frac{P-1}{P} \in (0,1)$, we obtain

$$
\begin{aligned}
& |\left[(\tau(t_1)-\tau(v))^{\overline{\omega}}\right]^{\frac{P-1}{P}}-\left[(\tau(t_2)-\tau(v))^{\overline{\omega}}\right]^{\frac{P-1}{P}}| \\
\leq\ & \left[(\tau(t_1)-\tau(v))^{\overline{\omega}}-(\tau(t_2)-\tau(v))^{\overline{\omega}}\right]^{\frac{P-1}{P}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
& |(\tau(t_1)-\tau(v))^{\gamma-1}-(\tau(t_2)-\tau(v))^{\gamma-1}| \\
\leq\ & \left[(\tau(t_1)-\tau(v))^{\overline{\omega}}-(\tau(t_2)-\tau(v))^{\overline{\omega}}\right]^{\frac{P-1}{P}}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
& |(\tau(t_1)-\tau(v))^{\gamma-1}-(\tau(t_2)-\tau(v))^{\gamma-1}|^{\frac{P}{P-1}} \\
\leq\ & (\tau(t_1)-\tau(v))^{\overline{\omega}}-(\tau(t_2)-\tau(v))^{\overline{\omega}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \lim_{t_2\to t_1} I_3 \\
\leq\ & \frac{\kappa M(1+\xi_1\nu+\xi_2||\Psi||_B)}{\Gamma(\gamma)}\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}}\times \\
& [\lim_{t_2\to t_1}\int_0^{t_1}((\tau(t_1)-\tau(v))^{\overline{\omega}}-(\tau(t_2)-\tau(v))^{\overline{\omega}})\tau'(v)dv]^{\frac{p-1}{p}} \\
\leq\ & \frac{\kappa M(1+\xi_1\nu+\xi_2||\Psi||_B)\|\varphi\|_{L^P_{(J,\mathbb{R}^+)}}}{\Gamma(\gamma)}\times \\
& [\lim_{t_2\to t_1}\frac{1}{\omega+1}[(\tau(t_1)-\tau(0))^{\overline{\omega}+1} \\
& +(\tau(t_2)-\tau(t_1))^{\overline{\omega}+1}-(\tau(t_1)-\tau(0))^{\overline{\omega}+1}]^{\frac{P-1}{P}} \\
=\ & 0.
\end{aligned}
$$

Next, for any $v\in[0,\theta]$, one has

$$
\begin{aligned}
& (\tau(t_1)-\tau(v))^{\gamma-1}\tau'(v)||K_2^{\tau}(t_2,v)-K_2^{\tau}(t_1,v)||g(v) \\
\leq\ & \frac{2M(\nu+1)}{\Gamma(\gamma)}(\tau(t_1)-\tau(v))^{\gamma-1}\tau'(v)\varphi(v)\in L^P(J,\mathbb{R}^+).
\end{aligned}
$$

Moreover, for any $v\in[0,\theta_1]$, the equicontinuity of $\{T(\theta):\theta>0$ leads to

$$
\begin{aligned}
& \lim_{t_2\to t_1}(\tau(t_1)-\tau(v))^{\gamma-1}\tau'(v)||K_2^{\tau}(t_2,v)g(v)-K_2^{\tau}(t_1,v)g(v)|| \\
=\ & (\tau(t_1)-\tau(v))^{\gamma-1}\tau'(v)\times \\
& \lim_{\theta_2\to\theta_1}\int_0^{\infty}\theta\xi_{\gamma}(\theta)||(T(\tau(t_2)-\tau(v))^{\gamma}\theta)g(v)-T(\tau(t_1)-\tau(v))^{\gamma}\theta)g(v)||d\theta \\
=\ & 0.
\end{aligned}
$$

Therefore, by Lebesgue's dominated convergence theorem, $\lim_{t_2\to t_1} I_4=0$. Then, Relation (19) implies $\lim_{t_2\to t_1}\|y^*(t_2)-y^*(t_1)\|=0$.

Case 2. Assume $i = 1$, $y^* \in Z_{|\overline{J_1}}$. Then, there is $x \in D_v$ and $g \in L^p([\theta_1, \theta_2], E)$ with $g(\theta) \in F(\theta, \overline{x}_{\rho(\theta, \overline{x}_\theta)})$ almost everywhere, such that

$$y^*(\theta) = \begin{cases} \Upsilon_1(\theta, x(\theta_1^-)), \theta \in (\theta_1, s_1], \\ K_1^\tau(\theta, s_1)\Upsilon_1(s_1, x(\theta_1^-)) \\ + \int_{s_1}^{\theta} (\tau(\theta) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\theta, v) g(v) dv, \theta \in (s_1, \theta_2], \end{cases} \tag{20}$$

where $y^*(\theta_1) = \lim_{\theta \to \to \theta_1} y^*(\theta)$. Let $\theta_1 < t_1 \leq t_2 \leq s_1$. From (20) and the uniform continuity of Υ_1 on bounded subsets, this yields

$$\lim_{t_2 \to t_1} \|y^*(t_2) - y^*(t_1)\| = \lim_{t_2 \to t_1} \left\|\Upsilon_1(t_2, x(\theta_1^-)) - \Upsilon_1(t_2, x(\theta_1^-))\right\| = 0.$$

Let $s_1 < t_1 \leq t_2 \leq \theta_2$. Then

$$\begin{aligned} & \lim_{t_2 \to t_1} \|y^*(t_2) - y^*(t_1)\| \\ \leq & \lim_{t_2 \to t_1} \left\|K_1^\tau(t_2, s_1)\Upsilon_1(s_1, x(\theta_1^-)) - K_1^\tau(t_1, s_1)\Upsilon_1(s_1, x(\theta_1^-))\right\| \\ & + || \int_{s_1}^{t_2} (\tau(t_2) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(t_2, v) g(v) dv \\ & - \int_{s_1}^{t_1} (\tau(t_1) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(t_1, v) g(v) dv||. \end{aligned}$$

By the second statement of Lemma (1) and by using similar arguments as in the first case, one can show that $\lim\limits_{t_2 \to t_1} \|y^*(t_2) - y^*(t_1)\| = 0$.

Therefore, $Z_{|\overline{J_i}}$ is equicontinuous for any $i \in I_0$.

Step 5. Let $D_1 = \overline{conv}\Phi(D_v)$, $D_n = \overline{conv}\Phi(D_{n-1})$, $n \geq 2$, and $D = \cap_{n=1} D_n$. Notice that D is closed, bounded, convex and $\Phi(D) \subset D$. In this step, we demonstrate that D is compact. By the generalized Cantor's intersection property [55], it is enough to show that

$$\lim_{n \to \infty} \chi_{\mathcal{H}}(D_n) = 0. \tag{21}$$

Let $\varepsilon > 0$. Due to Lemma 5 in [56], there is a sequence $(y_k)_{k \geq 1}$ in $\Phi(D_{n-1})$ such that

$$\begin{aligned} \chi_H \Phi(D_{n-1}) & \leq 2\chi_H\{y_k : k \geq 1\} + \varepsilon. \\ & = 2 \max_{i \in I_0} \chi_i \{y_{k_{|\overline{J}_i}} : k \geq 1\} + \epsilon, \end{aligned} \tag{22}$$

where χ_i is the Hausdorff measure of non-compactness on $C(\overline{J_i}, E)$. As a result of step 4, the set $\Phi(D_v)_{|J_i}(i \in I_0)$ is equicontinuous, and hence, $\chi_i\{y_{k_{|\overline{J}_i}} : k \geq 1\} = \sup_{\theta \in \overline{J_i}} \chi_E\{y_k(\theta) : k \geq 1\}$. Then, (22) becomes

$$\chi_H(D_n) \leq 2 \sup_{\theta \in [0,b]} \chi_E\{y_k(\theta) : k \geq 1\} + \varepsilon. \tag{23}$$

Suppose $x_k \in D_{n-1}$ such that $y_k \in \Phi(x_k)$, $k \geq 1$. Then, for any $k \geq 1$, there exists $g_k \in L^1(J, E)$ with $g_k(\omega) \in F(\omega, \overline{x}_{\rho(\omega, (\overline{x}_k)_\omega)})$ almost everywhere, and

$$y_k(\omega) = \begin{cases} 0, \omega \in (-\infty, 0], \\ K_1^\tau(\omega, 0)x_0 + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) g_k(v) dv, \ \omega \in [0, \theta_1], \\ \Upsilon_i(\omega, x_k(\theta_i^-)), \omega \in \cup_{i \in I_1} (\theta_i \ s_i], \\ K_1^\tau(\omega, s_i)\Upsilon_i(s_i, x_k(\theta_i^-)) \\ + \int_{s_i}^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) g_k(v) dv, \omega \in \cup_{i \in I_1} (s_i, \theta_{i+1}]. \end{cases} \tag{24}$$

Using $(HF3)$ to obtain $v \in J$ for almost everywhere,

$$\begin{aligned}
\chi_E\{g_k(v) &: k \geq 1\} \leq \chi\{F(v, (\overline{x}_k)_{\rho(v,(\overline{x}_k)_v)}) : k \geq 1\} \\
&\leq \beta(v) \sup_{\theta \in (-\infty, 0]} \chi\{\overline{x}_k(\rho(v, (\overline{x}_k)_v) + \theta) : k \geq 1\} \\
&\leq \beta(v) \sup_{\delta \in (-\infty, v]} \chi\{\overline{x}_k(\delta) : k \geq 1\} \\
&\leq \beta(v) \sup_{\delta \in [0, v]} \chi\{x_k(\delta) : k \geq 1\} \\
&\leq \beta(v) \chi_H(D_{n-1}) = \gamma(v).
\end{aligned} \tag{25}$$

In addition, from $(HF_3), ||g_k(\omega)|| \leq \varphi(\omega)\ (1 + \xi_1 \nu + \xi_2 ||\tau||_B), \forall\ k \geq 1,$ and for almost $\omega \in J$, $\{g_k : k \geq 1\}$ is integrably bounded. In view of Theorem 4.2.1 in [49] or Lemma 4 in [57], there is a compact set $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$, with measures less than ϵ, and a sequence of functions $\{z_k^\epsilon\} \subset L^P(J, E)$ for all $v \in J$, $\{z_k^\epsilon(s) : k \geq 1\} \subseteq K_\epsilon$ and

$$||g_k(v) - z_k^\epsilon(v)|| < 2\gamma(v) + \epsilon, \text{ for all } k \geq 1, \text{and all } v \in J - J_\epsilon. \tag{26}$$

By using the properties of χ, (25), (26) and Minkowski's inequality, it follows that

$$\begin{aligned}
&\chi\{\int_{J_0 - J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v)(g_k(v) - z_k^\epsilon(v)) dv : k \geq 1\} \\
\leq\ & \frac{2M}{\Gamma(\gamma)} \int_{J_0 - J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v)(2\gamma(v) + \epsilon) dv \\
\leq\ & \frac{2M}{\Gamma(\gamma)} \int_{J_0 - J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v)(2\beta(v)\chi_{PC(J,E)}(D_{n-1}) + \epsilon) dv \\
\leq\ & \frac{2M}{\Gamma(\gamma)} [2\kappa\eta_b ||\beta||_{L^P(J,\, \mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}) \\
& + \epsilon \int_{J_0 - J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) dv] \\
\leq\ & \frac{2M}{\Gamma(\gamma)} [2\kappa\eta_b ||\beta||_{L^P(J,\, \mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}) + \frac{\epsilon(\tau(b) - \tau(0))^\gamma}{\gamma}],
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\chi\{\int_{J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) g_k(v) dv : k \geq 1\} \\
\leq\ & \frac{2M}{\Gamma(\gamma)} \int_{J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) \chi\{g_k(v) : k \geq 1\} dv \\
\leq\ & \frac{2M}{\Gamma(\gamma)} \int_{J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) \gamma(v) dv \\
\leq\ & \frac{2M}{\Gamma(\gamma)} \chi_{PC(J,E)}(D_{n-1}) \int_{J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) \beta(v) dv \\
\leq\ & \frac{2\kappa M}{\Gamma(\gamma)} \chi_{PC(J,E)}(D_{n-1})\ ||\beta||_{L^p(J,\mathbb{R}^+)} (\int_{J_\epsilon} (\tau(\omega) - \tau(v))^{\frac{(\gamma-1)p}{p-1}} dv)^{\frac{p-1}{p}}.
\end{aligned} \tag{28}$$

Note that

$$\chi\{\int_{J_0 - J_\epsilon} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^\tau(\omega, v) z_k^\epsilon(v)) dv : k \geq 1\} = 0, \tag{29}$$

and

$$\begin{aligned}
&\int_{J_0} \tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)g_k(v)dv \\
\leq\ &\int_{J_\epsilon} \tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)g_k(v)dv \\
&+\int_{J-J_\epsilon} \tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)(g_k(v) - z_k^{\epsilon}(v))dv \\
&+\int_{J-J_\epsilon} \tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)z_k^{\epsilon}(v)dv.
\end{aligned} \tag{30}$$

Then, the inequalities in (27)–(30) lead to

$$\begin{aligned}
&\chi\{\int_{J_0} (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)g_k(v)dv : k \geq 1\} \\
\leq\ &\frac{2\kappa M}{\Gamma(\gamma)}\chi_{PC(J,E)}(D_{n-1})\ ||\beta||_{L^p(J,\mathbb{R}^+)}(\int_{J_\epsilon}(\tau(\omega) - \tau(v))^{\frac{(\gamma-1)p}{p-1}}dv)^{\frac{p-1}{p}} \\
&+\frac{2M}{\Gamma(\gamma)}[2\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}) + \frac{\epsilon(\tau(b) - \tau(0))^{\gamma}}{\gamma}].
\end{aligned} \tag{31}$$

Taking into account that ε is arbitrary, it follows from (31) that

$$\begin{aligned}
&\chi\{\int_{J_0} (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^{\tau}(\omega,v)g_k(v)dv : k \geq 1\} \\
\leq\ &\frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}).
\end{aligned} \tag{32}$$

Next, since $\Upsilon_i(\omega,.)$ maps bounded sets to relatively compact sets, and since K_1^{τ} is linear and bounded,

$$\chi\{K_1^{\tau}(\omega,s_i)\Upsilon_i(s_i,x_k(\theta_i^-)) : k \geq 1\} = 0, \forall i = 1,..,m. \tag{33}$$

Through (24), (32) and (33), it yields that,

$$\begin{aligned}
&\chi\{y_k(\theta) : k \geq 1\} \\
\leq\ &\frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}), \forall\theta \in J.
\end{aligned}$$

This relation with (23) implies

$$\chi_H(D_n) \leq \frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)}\ \chi_{PC(J,E)}(D_{n-1}),\ \forall n \geq 1.$$

One can obtain the following after a few steps.

$$\chi_H(D_n) \leq (\frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^p(J,\,\mathbb{R}^+)})^{n-1}\chi_{PC(J,E)}(D_1),\ \forall n \geq 1. \tag{34}$$

Using (15) and (34), we obtain (23). Applying Lemma (4) to conclude this, the fixed points for the multifunction $\Phi : D \to P_{ck}(D)$ are non-empty. Moreover, as in step 1, one can prove that, $Fix(\Phi)$ is bounded. By Lemma (4), again, $Fix(\Phi)$ is compact in H, and hence, $\Sigma_{\tau}^{F}(-\infty, b]$ is no-empty and a compact subset of B_b. □

4. $\Sigma_{\Psi}^{F,\tau}(-\infty, b]$ Is an $\mathbb{R}_{\delta}$-Set

Consider the multi-valued function $\widetilde{F} : J \times B \to P_{ck}(E)$ which is defined by

$$\widetilde{F}(t,u) = \begin{cases} F(t,u), ||u|| < v, \\ F(t, \frac{vu}{||u||}), ||u|| \geq v, \end{cases} \tag{35}$$

where v is defined by (13). Since $\widetilde{F} = F$ on D_v, the solution set of mild solutions for Problem (1) is equal to the solution set of mild solutions for the problem:

$$\begin{cases} {}^{c}D_{s_i,\theta}^{\gamma,\tau} x(\theta) \in Tx(\theta) + \widetilde{F}(\theta, x_{\rho(\theta,x_\theta)}), \ a.e.\ \theta \in \cup_{i=0}^{i=m}(s_i,\ \theta_{i+1}], \\ x(\theta_i^{+}) = \Upsilon_i(\theta_i, x(\theta_i^{-})), i \in I_1 \\ x(\theta) = \Upsilon_i(\theta, x(\theta_i^{-})), \cup_{i\in I_1}(\theta_i, s_i], \\ x(t) = \Psi, t \in (-\infty,\ 0]. \end{cases} \tag{36}$$

Obviously, $\widetilde{F}$ satisfies (HF_1) and almost for $t \in J$,

$$||\widetilde{F}(t,u)|| \leq \begin{cases} \varphi(t)(1+||u||) \leq \varphi(t)(1+v) = \xi(t), ||u|| < v, \\ \varphi(t)(1+||\frac{vu}{||u||}||) = \varphi(t)(1+v) = \xi(t), ||u|| \geq v. \end{cases}$$

Thus, we can assume, without loss of generality, that F satisfies the following condition: $(HF_2)^*$ is a function $\xi \in L^P(I, \mathbb{R}^+)(P > \frac{1}{\gamma})$ such that, for every $z \in \Theta$,

$$\|F(t,z)\| \leq \xi(t),\ a.e.\ t \in J. \tag{37}$$

We recall the following lemma (see [18,22]).

Lemma 6. *Assume that the multi-valued function F satisfies (HF_1) and $(HF_2)^*$. Then, there is $\{F_n\}_{n=1}^{\infty}$, $F_n : J \times B \to P_{ck}(E)$ satisfying*

(i) Every $F_n\ (t,.)$ is almost continuous for $t \in J$.

(ii) $F_{n+1}(t,u) \subseteq F_n(t,u)$, $\overline{co}F(t, \{y \in B : ||y-u||_B \leq 3^{1-n}\}), \forall\ n \geq 1$ and $\forall\ (t,u) \in J\ \times B$.

(ii) $F(t,u) = \cap_{n\geq 1} F_n(t,u)$.

(iv) For every $n \geq 1$, there exists a selection $z_n : J \times B \to E$ of F_i such that $\Upsilon_n(.,u)$ is measurable for any $u \in B$ and $z_n(t,.)$; $t \in J$ is locally Lipschitz.

Remark 4. *From (iv) in Lemma 6, for $t \in J$, $z_n(t,.)$, $n \geq 1$, is almost continuous.*

By the symbol $\Sigma_{\Psi}^{F_n,\tau}(-\infty,\ b]$, we denote the set of mild solutions to the following fractional neutral impulsive semi-linear differential inclusions with finite delay:

$$\begin{cases} {}^{c}D_{s_i,\omega}^{\gamma,\tau} x(\omega) \in Tx(\omega) + F_n(\omega, x_{\rho(\omega,x_\omega)}),\ a.e.\ \omega \in \cup_{i=0}^{i=m}(s_i,\ \theta_{i+1}], \\ x(\theta_i^{+}) = \Upsilon_i(\theta_i,\ x(\theta_i^{-})),\ i \in I_1, \\ x(\omega) = \Upsilon_i(\omega,\ x(\theta_i^{-})), \theta \in \cup_{i\in I_1}(\theta_i, s_i], \\ x(t) = \Psi, t \in (-\infty,\ 0]. \end{cases} \tag{38}$$

From Theorem (1) and Lemma (6), we obtain the following theorem.

Theorem 2. *Under the assumptions of Theorem 1 after replacing $(HF2)$ with $(HF2)^*$, there is a natural number N_0 such that $\Sigma_{\Psi}^{F_n,\tau}(-\infty, b]$; $n \geq N_0$ is non-empty and compact in B_b.*

Proof. We can use similar arguments to the ones used in the proof of Theorem (1) to demonstrate this theorem. Therefore, we focus on the differences. Let $\epsilon > 0$ and N_0 be a natural number with $\frac{2\eta_b M}{\Gamma(\gamma)} ||\beta||_{L^P(J,\, \mathbb{R}^+)} 3^{1-n} < \epsilon, \forall n \geq N_0$. Fix $n_0 \geq N_0$ and define a multi-operator $\Phi_{n_0} : H \to 2^H$ as follows: $y \in \Phi_{n_0}$ if and only if

$$y(\omega) = \begin{cases} 0, \omega \in (-\infty,\, 0], \\ K_1^\tau(\omega, 0)x_0 + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega, v)g^{n_0}(v)dv, \\ \ \omega \in [0,\, \theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i,\, s_i], \\ K_1^\tau(\omega, s_i)\Upsilon_i(s_i,\, x(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega, v)g^{n_0}(v)dv, \omega \in \cup_{i\in I_1}(s_i,\, \theta_{i+1}] \end{cases}$$

where $g^{n_0} \in L^1(J, E)$ such that $g^{n_0}(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ almost everywhere. Using similar arguments as the ones in the proof of Theorem (1), the values of Φ_{n_0} are convex compact and $\Phi_{n_0}(D_v) \subseteq D_v$. Moreover, Φ_{n_0} is closed and $\Phi_{n_0}(D_v)$ is equicontinuous. Let $D^{n_0} = \bigcap\limits_{r=N_0}^{\infty} D_{r,n_0}$, where $D_{1,n_0} = \overline{conv}\Phi_{n_0}(D_v)$ and $D_{r+1,n_0} = \overline{conv}\Phi_{n_0}(D_{r,n_0})$, $r \geq 2$. To show the compactness of D^{n_0}, it suffices to show that $\lim\limits_{r\to\infty} \chi_H(D_{r,n_0}) = 0$. As in step 5 in Theorem (1), we have

$$\chi_H(D_{r,n_0}) \leq 2 \sup_{\omega\in[0,b]} \chi_E\{y_k(\omega) : k \geq 1\} + \varepsilon, \tag{39}$$

where, for any $k \geq 1$, we have

$$y_k(\omega) = \begin{cases} 0, \omega \in (-\infty,\, 0], \\ K_1^\tau(\omega, 0)x_0 + \int_0^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega, v)g_k^{n_0}(v)dv,\ \omega \in [0,\, \theta_1], \\ \Upsilon_i(\omega, x_k(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i\,, s_i], \\ K_1^\tau(\omega, s_i)\Upsilon_i(s_i, x_k(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega, v)g_k^{n_0}(v)dv, \omega \in \cup_{i\in I_1}(s_i,\, \theta_{i+1}], \end{cases}$$

and $x_k \in D_{r-1,n_0}$, $y_k \in \Phi(x_k)$, $k \geq 1$ and $g_k^{n_0} \in L^1(J, E)$, such that $g_k^{n_0}(\omega) \in F_{n_0}(\omega, (\overline{x}_k\,)_{\rho(\omega,(\overline{x}_k)_\omega)})$ almost everywhere, Note that, due to Remark 4.2 in [31], it follows for almost everywhere that $v \in J$,

$$\begin{aligned} \chi_E\{g_k^{n_0}(v) &: k \geq 1\} \leq \chi\{F_{n_0}(v, (\overline{x}_k)_{\rho(v,(\overline{x}_k)_v)}) : k \geq 1\} \\ &\leq \beta(v) \sup_{\theta\in(-\infty,0]} \chi\{\overline{x}_k(\rho(v, (\overline{x}_k)_v) + \theta) : k \geq 1\} + 3^{1-n_0} \\ &\leq \beta(v) \sup_{\delta\in(-\infty,v]} \chi\{\overline{x}_k(\delta) : k \geq 1\} + 3^{1-n_0} \\ &\leq \beta(v) \sup_{\delta\in[0,v]} \chi\{x_k(\delta) : k \geq 1\} + 3^{1-n_0} \\ &\leq \beta(v)[\chi_H(D_{r-1,n_0}) + 3^{1-n_0}]. \end{aligned} \tag{40}$$

By using the arguments in (27)–(33), from (40), it yields

$$\begin{aligned} \chi_H(D_{r,n_0}) &\leq \frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^P(J,\, \mathbb{R}^+)}\chi_H(D_{r-1,n_0}) \\ &\quad + \frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^P(J,\, \mathbb{R}^+)}3^{1-n_0} \\ &\leq \frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^P(J,\, \mathbb{R}^+)}\chi_H(D_{r-1,n_0}) + 2\epsilon, \forall n \geq 1. \end{aligned} \tag{41}$$

Since ϵ is arbitrary, Relation (41) becomes

$$\chi_{\chi_H}(D_{r,n_0}) \leq \frac{4M}{\Gamma(\gamma)}\kappa\eta_b||\beta||_{L^P(J,\, \mathbb{R}^+)}\chi_H(D_{r-1,n_0}).$$

Similar to the proof of Theorem (1), we can show that $\lim_{r\to\infty} \chi_{\chi_H}(D_{r,n_0}) = 0$, and hence by the generalized Cantor's intersection property, the set D^{n_0} is non-empty and compact in H. Similar to Theorem (1), the set $\Sigma_{\Psi}^{F_n,\tau}(-\infty, b]$ is non-empty and a compact subset in B_b. $\square$

Theorem 3. *Suppose that the assumptions of Theorem (2) hold. Then,* $\Sigma_{\Psi}^{\mathcal{F},\tau}(-\infty, b] = \cap_{n=N_0}^{\infty} \Sigma_{\Psi}^{\mathcal{F}_n,\tau}(-\infty, b]$.

Proof. Due to (ii) in Lemma (6), one can conclude that $\Sigma_{x_0}^{F,\tau}[0,b] \subseteq \cap_{r=N_0}^{\infty} \Sigma_{x_0}^{F_r,\tau}[0,b]$. Let $x \in \cap_{n=N_0}^{\infty} \Sigma_{x_0}^{F_n,\tau}[0,b]$. Then, for any $n \geq N_0$, there exists $g_n \in L^1(J,E)$ such that $g_n(\omega) \in F_n(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$, for $\omega \in J$ and

$$x(\omega) = \begin{cases} 0, \omega \in (-\infty,\ 0], \\ K_1^{\tau}(\omega,0)x_0 + \int_0^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) g_n(v) dv,\ \omega \in [0,\ \theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1} (\theta_i,\ s_i], \\ K_1^{\tau}(\omega,s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) g_n(v) dv, \omega \in \cup_{i\in I_1} (s_i,\ \theta_{i+1}]. \end{cases}$$

almost everywhere.

Using similar arguments to the ones in step 2, in the proof of Theorem (1), we can show that there is a subsequence $(z_n)_{n\geq 1}$ of $(g_n)_{n\geq 1}$ that converges to g for almost everywhere. We also have

$$x(\omega) = \begin{cases} 0, \omega \in (-\infty,\ 0], \\ K_1^{\tau}(\omega,0)x_0 + \int_0^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) z_n(v) dv,\ \omega \in [0,\ \theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i=1}^{i=m} (\theta_i\ , s_i], \\ K_1^{\tau}(\omega,s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) z_n(v) dv, \omega \in \cup_{i=1}^{i=m} (s_i,\ \theta_{i+1}]. \end{cases} \tag{42}$$

Next, using Lemma 5, (ii), for $\omega \in J$, we obtain

$$z_n(\omega) \in \overline{co}F(\omega, \{y \in B : ||y - \overline{x}_{\rho(\omega,\overline{x}_\omega)}|| \leq 3^{1-n}\}), \forall n \geq N_0.$$

which implies the upper semi-continuity of $F(\omega, .)$ almost everywhere, to $g(\omega) \in F(\omega, \overline{x}_{\rho(\omega,\overline{x}_\omega)})$ for $\omega \in J$ almost everywhere. By taking the limit in (42) and applying the dominated convergence theorem, it yields

$$x(\omega) = \begin{cases} 0, \omega \in (-\infty,\ 0], \\ K_1^{\tau}(\omega,0)x_0 + \int_0^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) g(v) dv,\ \omega \in [0, \theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1} (\theta_i,\ s_i], \\ K_1^{\tau}(\omega,s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^{\omega} (\tau(\omega) - \tau(v))^{\gamma-1} \tau'(v) K_2^{\tau}(\omega,v) g(v) dv, \omega \in \cup_{i\in I_1} (s_i,\ \theta_{i+1}]. \end{cases}$$

This means that $x \in \sum_{\Psi}^{F,\tau}(-\infty,\ b]$. $\square$

Theorem 4. *In addition to the assumptions of Theorem (2), if the following condition holds:*

$(H\rho)^*$ for any $i \in I_1$, $v, y \in H$ and any $v \in [s_i, \theta_{i+1}]$, we have

$$||y_{\rho(v,y_v)} - v_{\rho(v,v_v)}||_B \leq \sup_{\varsigma\in[s_i,\ v]} ||y(\varsigma) - v(\varsigma)||_E. \tag{43}$$

Then, $\sum_{\Psi}^{F,\tau}(-\infty, b]$ is an R_δ-set.

Proof. Thanks to Theorems (1)–(3), it is enough to prove that $\sum_{\Psi}^{F_n,\tau}(-\infty,b], n \geq N_0$ is contractible. Let $n \geq N_0$ be a fixed natural number. Consider the the following impulsive semi-linear differential equation:

$$\begin{cases} {}^cD_{s_i,\omega}^{\gamma,\tau}x(\omega) \in Tx(\omega) + z_n(\omega, x_{\rho(s,x_s)}),\ a.e.\ \omega \in \cup_{i=0}^{i=m}(s_i,\ \theta_{i+1}], \\ x(\theta_i^+) = \Upsilon_i(\theta_i, x(\theta_i^-)), i \in I_1, \\ x(\omega) = \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i,\ s_i], \\ x(t) = \Psi, t \in (-\infty,\ 0]. \end{cases} \tag{44}$$

Due to Lemma (6) and Remark (5), $z_n(.,u); u \in E$ is measurable, and $\omega \in J$, $z_n(t,.)$ is almost continuous. As the multi-valued F satisfies $(F_2)^*$ and (F_3), using similar arguments to the ones in the proof of Theorem (2), the fractional differential in Equation (44) has a mild solution $y \in \sum_{\Psi}^{F_n,\tau}(-\infty,b]$. Then,

$$y(\omega) = \begin{cases} \Psi(\omega), \omega \in (-\infty,\ 0] \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v, y_{\rho(v,y_v)})dv,\ \omega \in [0,\ \theta_1], \\ \Upsilon_i(\omega, y(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i, s_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i, y(\omega_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v, y_{\rho(v,y_v)})dv, \omega \in \cup_{i=1}^{i=m}(s_i,\ \theta_{i+1}]. \end{cases} \tag{45}$$

Let $v \in \sum_{\Psi}^{F_n,\tau}(-\infty,\ b]$ be another mild solution for (44). Then,

$$v(\omega) = \begin{cases} \Psi(\omega), \omega \in (-\infty,\ 0] \\ K_1^\tau(\omega,0)\Psi(0) + \int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v, v_{\rho(v,x_v)})dv, \\ \ \omega \in [0,\theta_1], \\ \Upsilon_i(\omega, v(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i,\ v_i], \\ K_1^\tau(\omega,s_i)\Upsilon_i(s_i, v(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)K_2^\tau(\omega,v)z_n(v, v_{\rho(v,x_v)})dv, \\ \omega \in \cup_{i\in I_1}(s_i,\ \theta_{i+1}]. \end{cases} \tag{46}$$

By the second axiom of Definition 2, the function $s \to y_{\rho(s,\ y_s)}; s \in [0,\ \theta_1]$ is continuous, and hence the subset $\{y_{\rho(s,y_s)} : s \in [0,\ \theta_1]\}$ is compact. Similarly, $\{v_{\rho(s,\ v_s)}, s \in [0,\ \theta_1]\}$ is compact, and consequently, $Q(\theta_1) = \{y_{\rho(s,y_s)} : s \in [0,\ \theta_1]\} \cup \{v_{\rho(s,v_s)} : s \in [0,\ \theta_1]\}$ is compact in B. Hence, $[0,\ \theta_1] \times Q(\theta_1)$ is compact in $[0,\ \theta_1] \times B$. Therefore, by Lemma (6), there exists ς_{θ_1} such that, for any $s \in [0,\ \theta_1]$ and any $v_1,\ v_2 \in Q(\theta_1)$,

$$||z_n(s,\ v_1) - z_n(s,\ v_2)|| \leq \varsigma_{\theta_1}||v_1 - v_2||_B. \tag{47}$$

Therefore, by (2) and (43)–(47), we can obtain for $\omega \in [0,\ \theta_1]$,

$$\begin{aligned} & ||y(\omega) - v(\omega)|| \\ \leq\ & \frac{M}{\Gamma(\gamma)}\int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)||z_n(v, y_{\rho(v,\ y_v)}) - z_n(v, v_{\rho(v,\ v_v)})||dv \\ \leq\ & \frac{M\varsigma_{\theta_1}}{\Gamma(\gamma)}\int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)||y_{\rho(v,\ y_v)} - v_{\rho(v,\ v_v)}||_B dv \\ \leq\ & \frac{M\varsigma_{\theta_1}}{\Gamma(\gamma)}\int_0^\omega (\tau(\omega)-\tau(v))^{\gamma-1}\tau'(v)\xi_1 \sup_{\varsigma\in[0,\ v]} ||y(\varsigma) - v(\varsigma)||_E dv. \end{aligned} \tag{48}$$

Since both y and v are continuous on $[0,\ \theta_1]$, there exists $\delta \in [0,\ \theta_1]$ with $||y(\delta) - v(\delta)|| = \sup_{\omega\in[0,\ \theta_1]} ||y(\omega) - v(\omega)||$. Therefore, by (48), it yields that

$$\begin{aligned} & \sup_{\theta\in[0,\ \theta_1]} ||y(\omega) - v(\omega)|| \\ = & \ ||y(\delta) - v(\delta)|| \\ \leq & \ \frac{M\varsigma_\delta}{\Gamma(\gamma)} \int_0^\delta (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)\xi_1 \sup_{\varsigma\in[0,\ v]} ||y(\varsigma) - v(\varsigma)||dv \\ \leq & \ \frac{M\varsigma_\delta}{\Gamma(\gamma)} \int_0^\delta (\tau(\omega) - \tau(v))^{\gamma-1}\tau'(v)\xi_1 \sup_{\varsigma\in[0,\ \theta_1]} ||y(\varsigma) - v(\varsigma)||dv. \end{aligned} \tag{49}$$

Applying Lemma (5) it follows, from (49), that $y = v$ on $[0, \theta_1]$, and thus, $y = v$ on $[\theta_1, s_1]$. Using similar arguments, we can confirm that $y = v$ on $[s_1, \theta_2]$. We continue the same processes to show that $y = v$ on $[0, b]$.

Next, we prove that $\sum_\Psi^{F_n,\tau}(-\infty, b]$ is a homotopy equivalent to y. We have to define a continuous function $H_n : [0,1] \times \sum_\Psi^{F_n,\tau}(-\infty, b] \to \sum_\Psi^{F_n,\tau}(-\infty, b]$ with $H_n\ (0, x) = x$ and $H_n(1, x) = y,\ \forall x \in \sum_\Psi^{F_n,\tau}(-\infty, b]$. Consider the partition $D = \{0, \frac{1}{m+1}, \frac{2}{m+1}, \ldots., \frac{m+1}{m+1} = 1\}$ and let $\overline{x} \in \sum_\Psi^{F_n,\tau}(-\infty,\ b]$ be a fixed element. Then, there exists $v \in L^1(J, E)$ with $v(\omega) \in F_n(\omega, \overline{x}_{\rho(\omega, \overline{x}_\omega)})$ almost everywhere, such that

$$\overline{x}(\omega) = \begin{cases} \Psi(\omega), \omega \in (-\infty,\ 0], \\ K_1^\tau(\omega, 0)x_0 + \int_0^\omega (\tau(\omega) - \tau(s))^{\gamma-1}\tau'(s)K_2^\tau(\omega, s)v(s)ds, \\ \ \omega \in [0,\ \theta_1], \\ \Upsilon_i(\omega, x(\theta_i^-)), \omega \in \cup_{i\in I_1}(\theta_i,\ s_i], \\ K_1^\tau(\omega, s_i)\Upsilon_i(s_i, x(\theta_i^-)) \\ + \int_{s_i}^\omega (\tau(\omega) - \tau(s))^{\gamma-1}\tau'(s)K_2^\tau(\omega, s)v(s)ds, \\ \omega \in \cup_{i\in I_1}(s_i,\ \theta_{i+1}]. \end{cases} \tag{50}$$

(i) Let $\lambda \in [0, \frac{1}{m+1}]$. Put $a_\lambda^1 = b - \lambda(m+1)(b - s_m)$. As a result of the above discussion, there is a unique mild solution $\overline{x}_\lambda^1 \in \sum_\Psi^{F_n,\tau}(-\infty, b]$ for the following problem:

$$\begin{cases} {}^cD_{a_\lambda^1,\omega}^{\gamma,\tau} u(\omega) = Tu(\omega) + z_n(\omega, u_{\rho(\omega, u(\omega))}), \omega \in (a_\lambda^1, b], \\ u(\omega) = \overline{x}(\omega), \omega \in (-\infty, a_\lambda^1]. \end{cases} \tag{51}$$

Note that $a_0^1 = b$ and

$$\overline{x}_\lambda^1(\omega) = \begin{cases} \overline{x}(\omega),\ \omega \in (-\infty,\ a_\lambda^1], \\ K_1^\tau(\omega, a_\lambda^1)\overline{x}(a_\lambda^1) \\ + \int_{a_\lambda^1}^\omega (\tau(\omega) - \tau(s))^{\gamma-1}\tau'(s)K_2^\tau(\omega, s)z_n(s, (\overline{x}_\lambda^1)_{\rho(s,\ (\overline{x}_\lambda^1)_s)})ds, \\ \omega \in (a_\lambda^1,\ b]. \end{cases} \tag{52}$$

(ii) Let $\lambda \in [\frac{1}{m+1}, \frac{2}{m+1}]$. Put $a_\lambda^2 = \theta_m - (\lambda - \frac{1}{m+1})(m+1)(\theta_m - s_{m-1})$. By arguing as in case (i), there is a unique mild solution $\overline{x}_\lambda^2 \in \sum_\Psi^{F_n,\tau}(-\infty, b]$ for the impulsive semi-linear differential equation:

$$\begin{cases} x(\omega) = \overline{x}(\omega), \omega \in (-\infty,\ a_\lambda^2] \\ {}^cD_{a_\lambda^2,\omega}^{\gamma,\tau} x(\omega) = Tx(\omega) + z_n(\omega, x_{\rho(\omega, x(\omega))}), \omega \in (a_\lambda^2,\ \theta_m], \\ x(\omega) = \Upsilon_m(\omega, x(\theta_m^-)), \theta \in (\theta_m,\ s_m], \\ {}^cD_{s_m,\theta}^{\gamma,\tau} x(\omega) = Tx(\omega) + z_n(\omega, x_{\rho(\omega, x(\omega))}), \theta \in (s_m,\ \theta_{m+1}]. \end{cases}$$

Note that

$$\overline{x}^2_\lambda(\omega) = \begin{cases} \overline{x}(\omega), \omega \in (-\infty,\ a^2_\lambda], \\ K^\tau_1(\omega, a^2_\lambda)\overline{x}(a^2_\lambda) \\ +\int_{a^2_\lambda}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{x}^2_\lambda)_{\rho(s,(\overline{x}^2_\lambda)_s)})ds, \\ \omega \in (a^2_\lambda,\ \theta_m], \\ \Upsilon_m(\omega, \overline{x}^2_\lambda(\theta^-_m)), \omega \in (\theta_m,\ s_m], \\ K^\tau_1(\omega, s_m)\Upsilon_m(s_m, \overline{x}^2_\lambda(\theta^-_m)) \\ +\int_{s_m}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{x}^2_\lambda)_{\rho(s,(\overline{x}^2_\lambda)_s)})ds, \\ \omega \in (s_m,\ \theta_{m+1}]. \end{cases}$$

(iii) Continuing to the $(m+1)^{th}$ step, we obtain $\lambda \in [\frac{m}{m+1}, 1]$. Put $a^{m+1}_\lambda = \theta_1 - (\lambda - \frac{m}{m+1})(m+1)\theta_1$, and let $\overline{x}^{m+1}_\lambda \in \sum_{x_0}^{F_n,\tau}[0,b]$ be the unique mild solution for the impulsive semi-linear differential equation:

$$\begin{cases} x(\omega) = \overline{x}(\omega), t \in (-\infty,\ a^{m+1}_\lambda], \\ {}^cD^{\gamma,\tau}_{a^{m+1}_\lambda,\omega}x(\omega) = Tx(\omega) + z_n(\omega, x_{\rho(\omega,x(\omega))}), \omega \in (a^{m+1}_\lambda, \theta_1], \\ x(\omega) = \Upsilon_i(\omega, x(\theta^-_i)), \omega \in \cup_{i=1}^{i=m}(\theta_i,\ s_i], \\ {}^cD^{\gamma,\tau}_{s_i,\omega}x(\omega) = Tx(\omega) + z_n(\omega, x_{\rho(\omega,x(\omega))}), \omega \in \cup_{i=1}^{i=m}(s_i,\ \theta_{i+1}]. \end{cases}$$

Notice that $a^{m+1}_1 = 0$ and

$$\overline{x}^{m+1}_\lambda(\omega) = \begin{cases} \overline{x}(t), t \in (-\infty, a^{m+1}_\lambda], \\ K^\tau_1(\omega, a^{m+1}_\lambda)\overline{x}(a^{m+1}_\lambda) \\ +\int_{a^{m+1}_\lambda}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{x}^{m+1}_\lambda)_{\rho(s,\ (\overline{x}^2_\lambda)_s)})ds, \\ \omega \in (a^{m+1}_\lambda,\ \theta_1] \\ \Upsilon_i(\omega, \overline{x}^{m+1}_\lambda(\theta^-_i)),\ \omega \in \cup_{i\in I_1}(t_i,\ s_i] \\ K^\tau_1(\omega, s_i)l\Upsilon_i(\omega, \overline{x}^{m+1}_\lambda(\theta^-_i)) \\ +\int_{s_i}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{x}^{m+1}_\lambda)_{\rho(s,\ (\overline{x}^2_\lambda)_s)})ds, \\ \omega \in \cup_{i=1}^{i=m}(s_i,\ \omega_{i+1}]. \end{cases} \tag{53}$$

Now, we define H_n at $(\lambda,\ \overline{x})$ by

$$H_n(\lambda, \overline{x}) = \begin{cases} \overline{x}^1_\lambda,\ \lambda \in [0,\ \frac{1}{m+1}] \\ \overline{x}^2_\lambda, \lambda \in (\frac{1}{m+1},\ \frac{2}{m+1}] \\ . \\ . \\ . \\ . \\ . \\ \overline{x}^{m+1}_\lambda, \lambda \in (\frac{m}{m+1},\ \frac{m+1}{m+1}]. \end{cases} \tag{54}$$

From (51) and (53), we obtain $H_n(0,\ \overline{x}) = \overline{x}^1_0 = \overline{x}$, and from (53) and (54), we obtain $H_n(1, \overline{x}) = \overline{x}^{m+1}_1 = y$.

Now, we only need to prove the continuity of H_n. Let $(\sigma,\ w\ ), (\delta,\ v) \in [0,1] \times \sum_\Psi^{F_n,\tau}(-\infty, b]$. If $\sigma = \delta = 0$, then, by (54), $\lim_{w\to v} H_n(\sigma,\ w) = \lim_{w\to v} w = v = H_n(\delta,\ v)$. If $\sigma, \delta \in (0,\ \frac{1}{m+1}]$, then $H_n(\sigma,\ w) = \overline{w}^1_\sigma$ and $H_n(\delta,\ v) = \overline{v}^1_\delta$, where

$$\overline{w}^1_\sigma(\omega) = \begin{cases} w(\omega),\ \omega \in (-\infty,\ a^1_\sigma], \\ K^\tau_1(\omega, a^1_\sigma)w(a^1_\sigma) \\ +\int_{a^1_\sigma}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{w}^1_\sigma)_{\rho(s,(\overline{w}^1_\sigma)_s)})ds, \\ \omega \in (a^1_\sigma,\ b], \end{cases} \tag{55}$$

and

$$\overline{v}^1_\delta(\omega)=\begin{cases} v(\omega),\ \omega\in(-\infty,\ a^1_\delta],\\ K^\tau_1(\omega,\ a^1_\delta)v(a^1_\delta)\\ +\int_{a^1_\delta}^{\omega}(\tau(\omega)-\tau(s))^{\gamma-1}\tau'(s)K^\tau_2(\omega,s)z_n(s,(\overline{v}^1_\sigma)_{\rho(s,(\overline{v}^1_\sigma)_s)})ds,\\ \omega\in(a^1_\delta,\ b],\end{cases} \tag{56}$$

where $a^1_\sigma=b-\sigma(m+1)(b-s_m)$ and $a^1_\delta=b-\delta(m+1)(b-s_m)$. Obviously, $\lim_{\sigma\to\delta}a^1_\sigma=a^1_\delta$, and hence

$$\lim_{\substack{\sigma\to\delta\\ w\to v}} H_n(\sigma,\ w)(\omega)=H_n(\delta,v)(\omega),\forall\omega\in(-\infty,\ a^1_\delta].$$

By (55) and (56), and by arguing as above, we obtain

$$\lim_{\substack{\sigma\to\delta\\ w\to v}} H_n(\sigma,\ w)(\omega)=H_n(\delta,v)(\omega),\forall\omega\in(-\infty,\ b].$$

This leads to the continuity of $H_n(.,.)$, when $\sigma,\delta\in[0,\frac{1}{m+1}]$. Using similar arguments, one can prove the continuity of $H_n(.,.)$ on $[0,1]\times\sum_\Psi^{F_n,\tau}(-\infty,\ b]$, and consequently the set $\sum_\Psi^{F_n,\tau}(-\infty,\ b]$ is contractible. This completes the proof. $\square$

5. Example

Example 1. *Let $E=L^2([0,\ \pi],\mathbb{R}),J=[0,\ 1],\ m=1,s_0=0,\omega_1=\frac{1}{2},\ s_1=\frac{3}{4},\ \omega_2=1$. For every $x:J\to E=L^2([0,\ \pi],\ \mathbb{R})$, the $x(\omega,\ \omega);\omega\in J,\omega\in[0,\pi]$ denotes the value of $x(\omega)$ at ω, and $x(\omega+\theta,\ \omega)$ denotes the value of $(x_\omega)(\theta)$ at ω. Let $\varrho:(-\infty,0]\to(-\infty,0]$ be continuous with $L=\int_{-\infty}^0\varrho(s)ds<\infty$, and*

$$\begin{aligned} B_\varrho\quad &: \quad =\{u:(-\infty,\ 0]\to E: u \textit{ is bounded}\\ \textit{and measurable on } [-r,0];\ \forall r\quad &>\quad 0,\textit{ and }\int_{-\infty}^0\varrho(s)\sup_{\omega\in[s.0]}||x(\omega)||ds<\infty\}.\end{aligned}$$

It is known that B_ϱ is a Banach space where $||x||_{B_\varrho}=\int_{-\infty}^0\varrho(s)\sup_{\omega\in[s.0]}||x(\omega)||ds$ [58].

We show that B_ϱ satisfies the assumptions of Definition (2). In fact, let $t\in[0,1]$ and $x:(-\infty,b]\ \to E$ with $x|_J\in PC(J,\ E)$ and $x_0\in B_\varrho$. We have

$$\begin{aligned}\int_{-\infty}^0\varrho(s)\sup_{\omega\in[s,\ 0]}||x_t(\omega)||ds &= \int_{-\infty}^0\varrho(s)\sup_{\omega\in[s,\ 0]}||x(t+\omega)||ds\\ &= \int_{-\infty}^{-t}\varrho(s)\sup_{\omega\in[s,\ 0]}||x(t+\omega)||ds+\int_{-t}^0\varrho(s)\sup_{\omega\in[s.0]}||x(t+\omega)||ds\\ &\le \int_{-\infty}^{-t}\varrho(s)[\sup_{\delta\in[t+s,\ 0]}||x(\delta)||ds+\sup_{\delta\in[0,\ t]}||x(\delta)|||]ds\\ &\quad +\int_{-t}^0\varrho(s)\sup_{\delta\in[0,\ t]}||x(\delta)||ds\\ &\le \int_{-\infty}^0\varrho(s)\sup_{\delta\in[s,\ 0]}||x(\delta)||ds+\int_{-\infty}^0\varrho(s)\sup_{\delta\in[0,\ t]}||x(\delta)||ds\\ &\le ||x_0||_{B_\varrho}+L\sup_{\delta\in[0,\ t]}||x(\delta)||,\end{aligned}$$

which means $x_t \in B_\varrho$ and $||x_t||_{B_\gamma} \leq ||x_0||_{B_\varrho} + L\sup_{\delta\in[0,t]}||x(\delta)||$. Moreover,

$$
\begin{aligned}
||x_t||_{B_\gamma} &= \int_{-\infty}^{0} \varrho(s) \sup_{\omega\in[s.0]} ||x_t(\omega)||ds \\
&= \int_{-\infty}^{0} \varrho(s) \sup_{\omega\in[s.0]} ||x(t+\omega)||ds \\
&\geq ||x(t)|| \int_{-\infty}^{0} \varrho(s)ds = ||x(t)||L.
\end{aligned}
$$

Therefore, $||x(t)|| \leq \frac{1}{L}||x(t)||$. Finally, if $\nu_1, \nu_2 \in [0,\ 1]$, then

$$
\begin{aligned}
\lim_{\nu_1\to\nu_2} ||x_{\nu_1} - x_{\nu_2}||_{B_\varrho} &= \lim_{\nu_1\to\nu_2} \int_{-\infty}^{0} \varrho(s) \sup_{\omega\in[s,\ 0]} ||x(\nu_1+\omega) - x(\nu_2+\omega)||ds \\
&= 0.
\end{aligned}
$$

Therefore, B_ϱ is a phase space satisfying all assumptions of Definition (2). For more information about this phase space, see [58].

Next, we define an operator $T : D(T) \subseteq L^2[0,\ \pi] \to L^2[0,\ \pi]$ given by

$$
Tx(\omega,\ \omega) = -\frac{\partial^2}{\partial\omega^2}x(\omega,\ \omega),
$$

with the absolutely continuous domain $D(T) = \{u \in L^2[0,\pi] : u, u'$, and $u'' \in L^2[0,1]$, $u(\omega,\ 0) = u(\omega,\pi) = 0\}$.

It is known that [25], T generates an equicontinuous semi-group $\{T(\omega) : \omega \geq 0\}$. In addition,

$$
Tu = \sum_{n=1}^{\infty} n^2 < u,\ u_n > u_n, u \in D(T), \tag{57}
$$

where $u_n(y) = \sqrt{2}\ \sin ny, n \in \mathbb{N}$ is the orthonormal set of eigenvalues of T. Moreover, for any $u \in L^2[0,1]$, we have

$$
T(\omega)(u) = \sum_{n=1}^{\infty} e^{-n^2\omega} < u,\ u_n > u_n.
$$

Moreover, for any $u \in L^2([0,\ \pi], \mathbb{R})$,

$$
T^{\frac{-1}{2}}u = \sum_{n=1}^{\infty} \frac{1}{n} < u,\ u_n > u_n,
$$

and

$$
T^{\frac{1}{2}}u = \sum_{n=1}^{\infty} n < u,\ u_n > u_n,
$$

where the domain of $T^{\frac{1}{2}}$ is given by

$$
D(T^{\frac{1}{2}}) = \{u \in L^2([0,\ \pi], \mathbb{R}) : \sum_{n=1}^{\infty} n < u,\ u_n > u_n \in L^2([0,\ \pi], \mathbb{R})\}.
$$

Next, we define $\rho : J \times B_w \to \mathbb{R}$ by

$$
\rho(\omega,\ \varphi) = \omega - \sigma(\varphi(0)), \tag{58}
$$

where $\sigma : E \to [0,\ \infty)$ is continuous.

Next, we define $F : J \times B_{\varrho} \to P_{ck}(E)$ as

$$F(t,x) = \{z \in E : z(s) = \frac{e^{-rt}\sqrt{s_1^2+s_2^2}\sup_{\omega\in(-\infty,0]}||x(\omega)||}{v}Z, s = (s_1,\ s_2) \in \Omega\}, \quad (59)$$

where $r \in (1,\infty)$. Then, for every $\tau \in B_{\varrho}$, $t \to F(t,\ \tau)$ is strongly measurable, and for any $t \in J$, $F(t,)$ is upper semi-continuous. Moreover,

$$\begin{aligned} ||F(t,\ x)|| &= \sup_{z\in F(t,x)} ||z||_E = \sup_{z\in F(t,\ \psi)} [\int_{\Omega} ||z(s)||^2 ds]^{\frac{1}{2}} \\ &= \frac{e^{-rt}||x||}{(1+||x||)}[\int_{\Omega}(s_1^2+s_2^2)|ds]^{\frac{1}{2}} \\ &\leq e^{-rt}||x|| < e^{-rt}(||x||+1). \end{aligned}$$

In addition, let $\omega \in J$, $\psi_1,\ \psi_2 \in B_{\varrho}$ and $z_1 \in F(t,\ \psi_1)$. Then,

$$z_1 = \frac{e^{-rt}\sqrt{s_1^2+s_2^2}\ \sup_{\omega\in(-\infty,0]}||\psi_1(\omega)||\ \omega}{v}, \omega \in Z.$$

Set $z_2 = \frac{e^{-rt}\sqrt{s_1^2+s_2^2}\ \sup_{\omega\in(-\infty,0]}||\psi_2(\omega)||\ \omega}{v}$. Obviously, $z_2 \in F(t,\ \psi_2)$ and

$$\begin{aligned} ||z_1 - z_2|| &\leq e^{-rt}[\sup_{\omega\in(-\infty,\ 0]}||\psi_1(\omega)|| - \sup_{\omega\in(-\infty,\ 0]}||\psi_2(\omega)||][\int_{\Omega}|s|ds]^{\frac{1}{2}} \\ &= e^{-rt}\sup_{\omega\in(-\infty,\ 0]}(||\psi_1(\omega)|| - ||\psi_2(\omega)||) \\ &\leq e^{-rt}\sup_{\omega\in(-\infty,\ 0]}||\psi_1(\omega) - \psi_2(\omega)||, \end{aligned}$$

which yields

$$h(F(t,\psi_1), F(t,\psi_2)) \leq e^{-rt}\sup_{\omega\in(-\infty,0]}||\psi_1(\omega)-\psi_2(\omega)||,\ \forall t \in J,\ \psi_1,\ \psi_2 \in B_{\varrho}.$$

By (52), it follows that, for any bounded subset, D, of B_{ϱ}, one has

$$\chi(F(t,D)) \leq e^{-rt}\sup_{\omega\in(-\infty,\ 0]}\chi\{\psi(\omega) : \psi \in D\}.$$

Then, $(HF1)$, $(HF2)$ and $(HF3)$ are satisfied where $\varphi(\omega) = \beta(\omega) = e^{-r\omega}$; $\omega \in J$. Now, we define $\Upsilon_1 : [\omega_1,\ s_1] \times E \to E$ by:

$$\Upsilon_1(\omega,x) = \kappa\omega_1\Pi(x), \quad (60)$$

where $\Pi : E \to E$ is a linear bounded compact operator. By applying Theorems (1) and (4), the mild solution set for the following problem:

$$\begin{cases} {}^{c}D^{\gamma,\tau}_{s_i,\omega}x(\omega) \in Tx(\omega) + F(\omega, x_{\rho(\omega,x_\omega)}),\ a.e.\ \omega \in [0,\ \frac{1}{2}] \cup (\frac{3}{4},\ 1], \\ x(\omega_1^+) = \Upsilon_1(\omega_1,\ x(\omega_1^-)), \\ x(\omega) = \Upsilon_1(\omega,\ x(\omega_1^-)), \omega \in (\frac{1}{2},\ \frac{3}{4}], \\ x(\omega) = \Psi(\omega), \omega \in (-\infty,\ 0]. \end{cases}$$

is a non-empty and R_δ-set, where $A,\ F, \rho$ and Υ_i are given b (57)–(60).

6. Discussion and Conclusions

It is known that the set of mild solutions with the same initial point for a differential inclusion is typically not a singleton. Therefore, it is useful and interesting to investigate the topological structure of this set. Many researchers have performed this for different types of differential inclusions, proving that it is an R_δ-set and homotopically equivalent to a point (see, for instance, [25,26,28,30–33,36–41]). None of these works addressed the topological properties of the mild solution set for non-instantaneous impulsive semi-linear differential inclusions involving a τ-Caputo fractional derivative with infinite delay in infinite-dimensional Banach spaces.

In this paper, we have proven that the mild solution set for a non-instantaneous impulsive semi-linear differential inclusion involving a τ-Caputo fractional derivative with infinite delay in infinite-dimensional Banach spaces is non-empty and an R_δ-set. This work is novel and interesting because the linear part is an operator that generates a non-compact semi-group, while the non-linear part is a multi-valued function, and the studied problem contains the τ-Caputo derivative with non-instantaneous impulses and infinite delay. Moreover, our methodology is based on the properties of both multi-valued functions, measures of non-compactness and the infinitesimal generators of a C_0-semigroup. This study generalizes the work of Wang et al. [31], in which Problem (1) was considered without delay and $\tau(t) = t$, $\forall t \in J$. Furthermore, it generalizes Theorem (4.1) in [44] when the right-hand side is a multi-valued function in the presence of both non-instantaneous impulses and infinite delay. In addition, our technique could be used to extend the results reported in [14–22] when the Caputo derivative is replaced with a τ-Caputo fractional derivative and in [25,28–30,32,33,36–41], when the considered problems involve a τ-Caputo fractional derivative with impulsive effects and infinite delay. This could be a proposal for future work.

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