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Upper and Lower Solution Method for a Singular Tempered Fractional Equation with a p-Laplacian Operator

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Abstract: In this paper, we consider the existence of positive solutions for a singular tempered fractional equation with a p-Laplacian operator. By constructing a pair of suitable upper and lower solutions of the problem, some new results on the existence of positive solutions for the equation including singular and nonsingular cases are established. The asymptotic behavior of the solution is also derived, which falls in between two known curves. The interesting points of this paper are that the nonlinearity of the equation may be singular in time and space variables and the corresponding operator can have a singular kernel.

Keywords: positive solutions; asymptotic analysis of solution; singular boundary value problem; integral boundary condition

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1. Introduction

In this paper, we focus on the existence of positive solutions for the following singular tempered fractional equation with a p-Laplacian operator

$$\begin{cases} {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}x(t)\right)\right) = f(t,x(t)), \\ x(0)=0,\ {}_0^R D_t^{\beta,\lambda}x(0)=0,\ x(1)=\int_0^1 e^{-\lambda(1-t)}x(t)dt, \end{cases} \tag{1}$$

where $0<\alpha\le 1, 1<\beta\le 2$, λ is a positive constant, $\varphi_p(s)=|s|^{p-2}s$ with $\frac{1}{p}+\frac{1}{q}=1, p>1$ is a p-Laplacian operator, $f(\cdot,\cdot)$ is decreasing with respect with the second variable, ${}_0^R D_t^{\alpha,\lambda}$ and ${}_0^R D_t^{\alpha,\lambda}$ denotes tempered fractional derivatives related to the Riemann–Liouville fractional derivative by

$${}_0^R D_t^{\alpha,\lambda}x(t) = e^{-\lambda t}{}_0^R\mathscr{D}_t^{\alpha}(e^{\lambda t}x(t)),$$

where ${}_0^R\mathscr{D}_t^{\alpha}x(t) = \frac{d^n}{dt^n}\left({}_0I_t^{n-\alpha}x(t)\right)$ denotes the standard Riemann–Liouville fractional derivative, and ${}_0I_t^{n-\alpha}$ is the Riemann–Liouville fractional integral operator defined by

$${}_0I_t^{n-\alpha}x(t) = \int_0^t (t-s)^{\alpha-1}x(s)ds,$$

and for more details, we refer the reader to [1].

It is well known that the fractional-order derivative possesses nonlocal characteristics, which provides a possibility to inherit long-term memory in many large range dynamic processes. Thus, the fractional-order model overcomes the limitations or restrictions of locality of many integer-order models; it is more accurate than the integer order for long-term and large-range physical phenomena [2–4]. Because of this advantage of the fractional derivatives, in the past decades, various type fractional derivatives and integrals such as Riemann–Liouville, Caputo, tempered, Hadamard, Erdelyi–Kober, Caputo–Fabrizio, Hilfer, Riesz derivatives and so on have been introduced to describe different physical phenomena. In fact, each definition has own conditions and properties, and many of them are not equivalent to each other. In practical application, the physical system under consideration determines the selection of a suitable fractional operator. Therefore, it is logical that we study and develop the specific type of equation and operator for modeling different physical system. In comparison, the tempered derivative is a nonlocal fractional derivative with an exponential tempering factor, which possesses stronger nonlinearity. So, the study for tempered-type fractional differential equations is relatively difficult; for more background of tempered fractional operators, we refer the reader to [5–7].

Mathematical research has shown that many physical phenomena exhibit a characteristic of semi-long-range dependence. For example, in a stochastic propagation process, the tempered fractional Brownian motion involves a tempered fractional Gaussian noise, which follows a power-law operation at moderate time scales but eventually reduces to a short-term dependent at a long time scale [7]. This implies that the Brownian motion of the particle jump density in the tempered diffusion adopts an exponential tempering factor. Recently, Cartea and Negrete [8] showed that the probability density of tempered Lévy flights is governed by the tempered fractional diffusion equation, which provides a complete set of tools for statistical physics and numerical analysis. In [9], Chakrabarty and Meerschaert showed that random walks with exponentially tempered power-law jumps converge to a tempered stable motion. During the tempered stabilization process, the price fluctuations of the semi-heavy tail conform to a pure power law on moderate time scales but converge to a Gaussian distribution on long time scales [10].

On the other hand, a p-Laplacian equation can model turbulent flow in a porous medium [11–15]; in particular, when the equation contains tempered fractional derivatives, it can model turbulent velocity fluctuations of porous medium with features of power-law behavior at infinity and infinite divisibility [16]. Therefore, in the process of analyzing the statistical data and and modeling the basic physical phenomena in turbulent flow, Brownian motion, tempered Lévy flight, tempered stable laws are an useful tool. Because the Equation (1) not only contains tempered fractional derivatives but also includes a p-Laplacian operator, it is a mathematical model to describe turbulent velocity fluctuations of the porous medium. Thus, in this paper, we focus on the existence of positive solutions for the model (1) in a singular case. In fact, singularity may occur in the transmission process of a turbulent flow in highly heterogeneous porous media, as some unpredictable factors force the transmission process from a phase into another different phase or state. In past decades, many works have been completed for various singular nonlinear equations; for more details, we refer the reader to [17–25].

Due to the widespread application of differential equations in practice, in recent decades, many theories and methods of nonlinear analysis, such as the spaces theories [26–31], smoothness theories [32–35], operator theories [36–38], fixed-point theorems [18,21,24,25,39–41], sub-super solution methods [17,42–45], monotone iterative techniques [12,46–53] and the variational method [54–58], have been developed to study various differential equations. For example, by adopting the fixed point theorem of the mixed monotone operator, Zhou et. al [13] established the existence and uniqueness of positive solutions for the following tempered fractional differential equation

$$\begin{cases} {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} x(t)\right)\right) = f(t, x(t), x(t)) + g(t.u(t)), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ \varphi_p\left({}_0^R D_t^{\beta,\lambda} x(0)\right) = 0, \\ {}_0^R D_t^{\beta,\lambda} x(1) = \delta \displaystyle\int_0^1 e^{-\lambda(1-t)} x(t)dt, \\ {}_0^R D_t^{\gamma_1,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} x\right)\right)(1) = \displaystyle\int_0^\eta a(t) {}_0^R D_t^{\gamma_2,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} x(t)\right)\right) dA(t), \end{cases} \tag{2}$$

where $1 < \alpha \leq 2, n-1 < \beta \leq n, n \geq 4, 0 < \gamma_2 < \gamma_1 < \alpha - 1, \delta < \beta, \lambda > 0$ is a constant, $\varphi_p, p > 1$ is a p-Laplacian operator, ${}_0^R D_t{}^{\alpha,\lambda}$ is a tempered fractional derivative, $\int_0^1 \cdot dA(t)$ denotes a Riemann–Stieltjes integral and A is a function of bounded variation. In our recent work [14], we studied the existence of extremal solutions for a tempered fractional turbulent flow Equation (2) in the case where $\gamma_2 = \gamma_1, \delta = 1$, and the nonlinearity takes the special form $h(t)f(x(t))$. In virtue of iterative techniques, some new results of the existence of maximum and minimum solutions were established; moreover, iterative properties of the extremal solutions such as the iterative sequences and the asymptotic estimates of solutions were also obtained.

However, the nonlinearity of the equations in [13,14] does not allow singularity in space variables, and also, the order of the tempered fractional derivative must be between 1 and 2, namely $1 < \alpha \leq 2$. this implies the results of [13,14] are not applicable for Equation (1). On the other hand, if the order of the fractional derivative is less than 1, the corresponding integral operator will possess a singular kernel, which is very difficult to deal with. So, for the convenience of handling, most of the existing works on fractional equations require the order of the fractional derivative to be greater than 1. Thus, the contribution of this paper is to solve the singular problem including the nonlinearity of the equation with singularity in space variables and with a singular kernel of the corresponding operator.

2. Preliminaries and Lemmas

In this section, we give some preliminaries and lemmas to be used in the rest of the paper.

Lemma 1 ([13]).

(1) Let $g(t) \in L^1[0,1] \cap C[0,1], \gamma > 0$, then

$$ {}_0I_t^{\gamma}\,{}_0^R\mathscr{D}_t{}^{\gamma}(g(t)) = g(t) + b_1 t^{\gamma-1} + b_2 t^{\gamma-2} + \cdots + b_m t^{\gamma-m}, $$

where $b_i \in R, i = 1,2,3,...,m, (m = [\gamma]+1)$.

(2) If $u \in L^1(0,1), \alpha > \beta > 0$, then

$$ {}_0I_t^{\alpha}\,{}_0I_t^{\beta}u(t) = {}_0I_t^{\alpha+\beta}u(t), \;\; {}_0^R\mathscr{D}_t{}^{\beta}\,{}_0I_t^{\alpha}u(t) = {}_0I_t^{\alpha-\beta}u(t), \;\; {}_0^R\mathscr{D}_t{}^{\beta}\,{}_0I_t^{\beta}u(t) = u(t). $$

(3) If $\rho > 0, \mu > 0$, then

$$ {}_0^R\mathscr{D}_t{}^{\rho}t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)}t^{\mu-\rho-1}. $$

The following Lemma has been proved in Lemma 2.3 of [13].

Lemma 2. *Suppose $g(t)$ is a positive continuous function in $[0,1]$; then, the linear tempered fractional equation*

$$ \begin{cases} {}_0^R D_t{}^{\beta,\lambda}x(t) - g(t) = 0, \\ x(0) = 0, \;\; x(1) = \displaystyle\int_0^1 e^{-\lambda(1-t)}x(t)dt, \end{cases} \tag{3} $$

has the unique positive solution

$$ x(t) = \int_0^1 H(t,s)g(s)ds, \tag{4} $$

where

$$H(t,s)=\begin{cases}\dfrac{[\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}t^{\beta-1}-\beta(\beta-1)e^{\lambda s}(t-s)^{\beta-1}]e^{-\lambda t}}{(\beta-1)\Gamma(\beta+1)},\ 0\le s\le t\le 1;\\ \dfrac{[\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}t^{\beta-1}e^{-\lambda t},\ 0\le t\le s\le 1.\end{cases}\tag{5}$$

is the Green function of (3).

Remark 1. *In the proof of Lemma 2.3 in [13], there is a mistake in using Lemma 1; it should be:*

$${}_0I_t^{\gamma}{}_0^R\mathscr{D}_t^{\beta}(e^{\lambda t}x(t))=e^{\lambda t}x(t)+b_1t^{\beta-1}+b_2t^{\beta-2}+\cdots+b_nt^{\beta-n}=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds,$$

that is

$$e^{\lambda t}x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds-b_1t^{\beta-1}-b_2t^{\beta-2}-\cdots-b_nt^{\beta-n},$$

which leads to the equation lacking a minus sign.

From Lemmas 1 and 2, we have the following lemma.

Lemma 3. *Let $g(t)$ be a positive continuous function in $[0,1]$; then, the associated linear tempered fractional equation*

$$\begin{cases}{}_0^RD_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^RD_t^{\beta,\lambda}x(t)\right)\right)=g(t),\\ x(0)=0,\ {}_0^RD_t^{\beta,\lambda}x(0)=0,\ x(1)=\displaystyle\int_0^1e^{-\lambda(1-t)}x(t)dt,\end{cases}\tag{6}$$

has the unique positive solution

$$x(t)=\int_0^1H(t,s)\left(\int_0^s\frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}g(\tau)e^{\lambda\tau}d\tau\right)^{q-1}ds.\tag{7}$$

Proof. Firstly, let $v={}_0^RD_t^{\beta,\lambda}x(t),\ \ u=\varphi_p(v)$; then, it follows from Lemma 1 and the definition of tempered fractional derivative that the solution of the initial value problem

$$\begin{cases}{}_0^RD_t^{\alpha,\lambda}u(t)=g(t),\ t\in(0,1),\\ u(0)=0,\end{cases}\tag{8}$$

is

$$e^{\lambda t}u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds-c_1t^{\alpha-1},t\in[0,1].$$

Since $u(0)=0,0<\alpha\le 1$, one has $c_1=0$,

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds,t\in[0,1].\tag{9}$$

On the other hand, by ${}_0^RD_t^{\beta,\lambda}x(t)=v,\varphi_p(v)=u$, we have

$$u=\varphi_p\left({}_0^RD_t^{\beta,\lambda}x(t)\right)=\int_0^t\frac{(t-s)^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}g(s)e^{\lambda s}ds,$$

which implies that the solution of (6) satisfies

$$\begin{cases} {}_0^R D_t^{\beta,\lambda} x(t)) = \varphi_p^{-1}\left(\int_0^t \frac{(t-s)^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} g(s) e^{\lambda s} ds \right), \\ x(0) = 0, \ \ x(1) = \int_0^1 e^{-\lambda(1-t)} x(t) dt, \end{cases} \tag{10}$$

It follows from Lemma 2 that

$$x(t) = \int_0^1 H(t,s) \varphi_p^{-1}\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} g(\tau) e^{\lambda \tau} d\tau \right) ds, \ \ t \in [0,1].$$

Since $g(s) \geq 0, s \in [0,1]$, we have

$$\varphi_p^{-1}\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} g(\tau) e^{\lambda \tau} d\tau \right) = \left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} g(\tau) e^{\lambda \tau} d\tau \right)^{q-1}, \ \ s \in [0,1],$$

which implies that Equation (6) has a unique solution that can be expressed by

$$x(t) = \int_0^1 H(t,s) \left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)} (s-\tau)^{\alpha-1} g(\tau) e^{\lambda \tau} d\tau \right)^{q-1} ds, \ t \in [0,1]. \tag{11}$$

□

Lemma 4. *The Green function $H(t,s)$ in (5) has the following properties:*

(1) $H(t,s)$ is non-negative and continuous for $(t,s) \in [0,1] \times [0,1]$.

(2) For any $t, s \in [0,1]$, $H(t,s)$ satisfies

$$m_1(s) e^{-\lambda t} t^{\beta-1} \leq H(t,s) \leq M_1(s) e^{-\lambda t} t^{\beta-1}, \tag{12}$$

where

$$M_1(s) = \frac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}, m_1(s) = \frac{\beta s(1-s)^{\beta-1} e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}.$$

From (10) and (11), we have the following analogous maximum value principle.

Lemma 5. *If $x \in C([0,1], \mathbb{R})$ satisfies*

$$\begin{cases} {}_0^R D_t^{\beta,\lambda} x(t) \geq 0, \ \ t \in [0,1], \\ x(0) = 0, \ \ x(1) = \int_0^1 e^{-\lambda(1-t)} x(t) dt, \end{cases}$$

then $x(t) \geq 0, \ t \in [0,1]$.

3. Main Results

In this section, we firstly list the hypotheses used in this paper.

(A1) $f \in C((0,1) \times (0,\infty), [0,+\infty))$, and $f(t,z)$ is decreasing in $z > 0$;
(A2) For any $\rho > 0$, $f(t,\rho) \not\equiv 0$, and there exists a constant $0 < \sigma < \alpha$ such that

$$0 < \int_0^1 e^{\frac{\lambda}{\sigma} t} f^{\frac{1}{\sigma}}(t, \rho e^{-\lambda t} t^{\beta-1}) dt < +\infty. \tag{13}$$

Denote the Banach space $E = C[0,1]$ with the maximum norm

$$||x|| = \max\{x(t) : t \in [0,1]\}.$$

Now, define a cone $K = \{x \in E : x(t) \geq 0\}$ and a subset K^* of K

$$K^* = \Big\{x \in E : \text{there exists a number } \ 0 < h_x < 1 \ \text{ such that} \\ h_x e^{-\lambda t} t^{\beta-1} \leq x(t) \leq h_x^{-1} e^{-\lambda t} t^{\beta-1}, \ t \in [0,1]\Big\},$$

and then define an operator S in E,

$$(Sx)(t) = \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,x(\tau))d\tau\right)^{q-1} ds, \tag{14}$$

then, the fixed point of operator S in E is the solution of the singular tempered fractional Equation (1).

Lemma 6. *Assume that* (**A1**)–(**A2**) *hold. Then,* $S : K^* \to K^*$ *is a completely continuous operator.*

Proof. Firstly, it follows from the definition of K^* that for any $x \in K^*$, there exists a number $0 < h_x < 1$ such that

$$h_x e^{-\lambda t} t^{\beta-1} \leq x(t) \leq h_x^{-1} e^{-\lambda t} t^{\beta-1}, \ t \in [0,1]. \tag{15}$$

Since S is increasing with respect to x, by (14), (15), Hölder inequality and Lemma 4, we have

$$\begin{aligned}
(Sx)(t) &= \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,x(\tau))d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,x(\tau))d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)} \\
&\quad\times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} ds \\
&= \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)} e^{-\lambda t}t^{\beta-1}\int_0^1 \frac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1} s^{(\alpha-\sigma)(q-1)} \\
&\quad\times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} ds \\
&\leq \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)} e^{-\lambda t}t^{\beta-1}\int_0^1 \frac{\beta(1-s)^{\beta-1}(\beta-1+s)e^{\lambda s}}{(\beta-1)\Gamma(\beta+1)}\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1} s^{(\alpha-\sigma)(q-1)} ds \\
&\quad\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&\leq \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)} e^{-\lambda t}t^{\beta-1}\int_0^1 \frac{\beta^2 e^{\lambda}}{(\beta-1)\Gamma(\beta+1)}\left(\frac{1}{\Gamma(\alpha)}\right)^{q-1} ds \\
&\quad\times\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&\leq M_1^* e^{-\lambda t}t^{\beta-1}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&< +\infty,
\end{aligned} \tag{16}$$

where

$$M_1^* = \frac{\beta^2 e^{\lambda}}{\Gamma^{q-1}(\alpha)(\beta-1)\Gamma(\beta+1)}.$$

On the other hand, for any $\rho > 0$, since $f(t,\rho) \not\equiv 0$, we have $f(t, h_x^{-1}) \not\equiv 0$. Thus, by the local inheriting order property of continuous functions, there exists $[a,b] \subset (0,1)$ such that

$$\int_a^b m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1})d\tau\right)^{q-1}ds > 0. \tag{17}$$

As $0 < \alpha < 1$, by (16) and (17), we have

$$\begin{aligned}
0 &< \int_a^b m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1})d\tau\right)^{q-1}ds \\
&\leq \int_0^1 m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds \\
&\leq \int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds \\
&< +\infty.
\end{aligned} \tag{18}$$

Thus, it follows from Lemma 4 and (18) that

$$\begin{aligned}
(Sx)(t) &\geq e^{-\lambda t}t^{\beta-1}\int_0^1 m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,x(\tau))d\tau\right)^{q-1}ds \\
&\geq e^{-\lambda t}t^{\beta-1}\int_0^1 m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds \\
&\geq e^{-\lambda t}t^{\beta-1}\int_a^b m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds.
\end{aligned} \tag{19}$$

Take

$$\begin{aligned}
h_x^* = \min\Bigg\{&\frac{1}{3}, \left(M_1^*\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,h_x e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)}\right)^{-1}, \\
&\int_a^b m_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,h_x^{-1}e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1}ds\Bigg\}.
\end{aligned}$$

Then, we have

$$h_x^* e^{-\lambda t}t^{\beta-1} \leq S(x)(t) \leq \frac{1}{h_x^*}e^{-\lambda t}t^{\beta-1},$$

which implies that $S(K^*) \subset K^*$ is well defined and uniformly bounded.

On the other hand, it is easy to see that S is continuous in E and also equicontinuous on any bounded set of K^*. Thus, according to the Arezela–Ascoli theorem, $S : K^* \to K^*$ is completely continuous. □

In what follows, we introduce the definition of the upper and lower solutions of the tempered fractional Equation (1).

Definition 1. *Suppose the function $\xi \in E$ satisfies*

$$\begin{cases}
{}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\xi(t)\right)\right) \leq f(t,\xi(t)), \\
\xi(0) \geq 0, \ {}_0^R D_t^{\beta,\lambda}\xi(0) \geq 0, \ \xi(1) \geq \int_0^1 e^{-\lambda(1-t)}\xi(t)dt,
\end{cases} \tag{20}$$

then, $\xi(t)$ is called a lower solution of the tempered fractional Equation (1).

Definition 2. *Suppose the function $\eta \in E$ satisfies*

$$\begin{cases} {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right)\right) \geq f(t,\eta(t)), \\ \eta(0) \leq 0,\ {}_0^R D_t^{\beta,\lambda}\eta(0) \leq 0,\ \eta(1) \leq \int_0^1 e^{-\lambda(1-t)}\eta(t)dt, \end{cases} \tag{21}$$

then, $\eta(t)$ is called an upper solution of the tempered fractional Equation (1).

Theorem 1. *Assume that the conditions* (**A1**)–(**A2**) *are satisfied. Then, the singular tempered fractional Equation (1) has at least one positive solution $w(t)$, and there exist two constants $k_1, k_2 > 0$ such that*

$$k_1 e^{-\lambda t} t^{\beta-1} \leq w(t) \leq k_2 e^{-\lambda t} t^{\beta-1}.$$

Proof. Firstly, by Lemma 6, $S : K^* \to K^*$ is completely continuous. Thus, it follows from Lemma 3 and (14) that

$$\begin{cases} {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(Sx)(t)\right)\right) = f(t,x(t)), \\ (Sx)(0) = 0,\ {}_0^R D_t^{\beta,\lambda}(Sx)(0) = 0,\ (Sx)(1) = \int_0^1 e^{-\lambda(1-t)}(Sx)(t)dt. \end{cases} \tag{22}$$

Now, we shall construct a pair of lower and upper solutions for the tempered fractional Equation (1). For this, take

$$\begin{aligned} \kappa(t) &= \min\left\{e^{-\lambda t}t^{\beta-1}, S(e^{-\lambda t}t^{\beta-1})\right\}, \\ \theta(t) &= \max\left\{e^{-\lambda t}t^{\beta-1}, S(e^{-\lambda t}t^{\beta-1})\right\}. \end{aligned} \tag{23}$$

If $e^{-\lambda t}t^{\beta-1} = S(e^{-\lambda t}t^{\beta-1})$, then $e^{-\lambda t}t^{\beta-1}$ is a positive solution of the tempered fractional Equation (1) and thus, the proof of Theorem 1 is completed. Otherwise, one has $\theta(t),\ \kappa(t) \in K^*$ and

$$\kappa(t) \leq e^{-\lambda t}t^{\beta-1} \leq \theta(t). \tag{24}$$

Letting

$$\xi(t) = S\theta(t),\ \ \eta(t) = S\kappa(t). \tag{25}$$

we assert that $\xi(t)$ and $\eta(t)$ are a pair of lower and upper solutions for the tempered fractional Equation (1).

In fact, since S is a decreasing operator with respect to x due to the monotonicity of f, it follows from (23)–(25) that $\xi(t), \eta(t) \in K^*$ and

$$\begin{aligned} \xi(t) &= S\theta(t) \leq S\kappa(t) = \eta(t), \\ \xi(t) &= S\theta(t) \leq S(e^{-\lambda t}t^{\beta-1}) \leq \theta(t), \\ \eta(t) &= S\kappa(t) \geq S(e^{-\lambda t}t^{\beta-1}) \geq \kappa(t). \end{aligned} \tag{26}$$

Consequently, by (22)–(26), one obtains

$$\begin{aligned} &{}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\xi(t)\right)\right) - f(t,\xi(t)) \\ &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\theta)(t)\right)\right) - f(t,(S\theta)(t)) \\ &\leq {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\theta)(t)\right)\right) - f(t,\theta(t)) = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned}
&{}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right)\right) - f(t,\eta(t)) \\
&= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\kappa)(t)\right)\right) - f(t,(S\kappa)(t)) \\
&\geq {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\kappa)(t)\right)\right) - f(t,\kappa(t)) = 0.
\end{aligned} \tag{28}$$

Obviously, (22) and (25) imply that η and ξ satisfy

$$\begin{aligned}
&\xi(0) = 0,\ {}_0^R D_t^{\beta,\lambda}\xi(0) = 0,\ \xi(1) = \int_0^1 e^{-\lambda(1-t)}\xi(t)dt \\
&\eta(0) = 0,\ {}_0^R D_t^{\beta,\lambda}\eta(0) = 0,\ \eta(1) = \int_0^1 e^{-\lambda(1-t)}\eta(t)dt.
\end{aligned} \tag{29}$$

Thus, (26)–(29) guarantee that the function $\eta(t)$ and $\xi(t)$ are a pair of upper and lower solutions of Equation (1) satisfying $\xi(t), \eta(t) \in K^*$.

Now, define an auxiliary function

$$F(t,x) = \begin{cases} f(t,\xi(t)), & x < \xi(t), \\ f(t,x(t)), & \xi(t) \leq x \leq \eta(t), \\ f(t,\eta(t)), & x > \eta(t). \end{cases} \tag{30}$$

Clearly, $F[0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. Then, we consider the following modified tempered fractional equation

$$\begin{cases} {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}x(t)\right)\right) = F(t,x(t)),\ 0 < t < 1 \\ x(0) = 0,\ {}_0^R D_t^{\beta,\lambda}x(0) = 0,\ x(1) = \displaystyle\int_0^1 e^{-\lambda(1-t)}x(t)dt. \end{cases} \tag{31}$$

For this, define an operator B in E

$$(Bx)(t) = \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,x(\tau))d\tau\right)^{q-1} ds, \quad \forall x \in E.$$

Then, it follows from Lemma 3 that the solution of the boundary value problem (31) is equivalent to the fixed point of B.

Notice that $\xi \in K^*$, and there exists a constant $0 < h_\xi < 1$ such that

$$h_\xi e^{-\lambda t}t^{\beta-1} \leq \xi(t) \leq h_\xi^{-1}e^{-\lambda t}t^{\beta-1},\ t \in [0,1].$$

Consequently, for all $x \in E$, by (30), Hölder inequality and Lemma 6, we obtain

$$\begin{aligned}
(Bx)(t) &= \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,x(\tau))d\tau\right)^{q-1} ds \\
&\le e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,x(\tau))d\tau\right)^{q-1} ds \\
&\le e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,\xi(\tau))d\tau\right)^{q-1} ds \\
&\le e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1} ds \\
&\le e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)} \\
&\quad\times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} ds \\
&\le M_1^* e^{-\lambda t}t^{\beta-1}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} \\
&< +\infty,
\end{aligned} \tag{32}$$

which implies that B is bounded. Thus, it follows from the continuity of F and H that $B : E \to E$ is a continuous operator.

Assume that $\Omega \subset E$ is a bounded set; then, for all $x \in \Omega$, there exists some positive constant $N > 0$ such that $||x|| \le N$. Now, let

$$L = \max_{0\le t\le 1, 0\le x\le N} |F(t,x)| + 1.$$

Since $H(t,s)$ is uniformly continuous in $[0,1]\times[0,1]$, for any $\epsilon > 0$ and $s \in [0,1]$, there exists $\sigma > 0$ such that for any $t_1, t_2 \in [0,1]$ and $\mid t_1 - t_2 \mid < \sigma$, one has

$$\mid H(t_1,s) - H(t_2,s) \mid < \frac{1}{L^q}\left(\frac{e^\lambda}{\alpha\Gamma(\alpha)}\right)^{1-q}\epsilon.$$

As

$$\begin{aligned}
&\left|\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,x(\tau))d\tau\right)^{q-1}\right| \\
&\le \left|\left(\int_0^1 \frac{Le^\lambda}{\Gamma(\alpha)}(1-\tau)^{\alpha-1}d\tau\right)^{q-1}\right| \\
&= \left(\frac{Le^\lambda}{\alpha\Gamma(\alpha)}\right)^{q-1},
\end{aligned} \tag{33}$$

we have

$$\mid Bx(t_1) - Bx(t_2) \mid \le \int_0^1 \mid H(t_1,s) - H(t_2,s) \mid \left|\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,x(\tau))d\tau\right)^{q-1}\right| ds < \epsilon,$$

which implies that $B(\Omega)$ is equicontinuous.

Thus, by the Arzela–Ascoli theorem, $B : E \to E$ is a completely continuous operator. Hence, the Schauder fixed point theorem guarantees that B has a fixed point w such that $w = Bw$.

In the following, we shall prove that the fixed point w of the operator B is also the fixed point of the operator S. In fact, from the definition of F, it is sufficient to prove

$$\xi(t) \leq w(t) \leq \eta(t), \quad t \in [0,1]. \tag{34}$$

Firstly, we show $w(t) \leq \eta(t)$. If not, we have $w(t) > \eta(t)$; thus, according to the definition of F, we have

$${}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} w(t)\right)\right) = F(t,w(t)) = f(t,\eta(t)), \quad t \in [0,1]. \tag{35}$$

Noticing $\eta(t)$ is an upper solution of (1), we have

$${}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} \eta(t)\right)\right) \geq f(t,\eta(t)), \quad t \in [0,1]. \tag{36}$$

Let $x(t) = \varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right) - \varphi_p\left({}_0^R D_t^{\beta,\lambda} w(t)\right)$, $t \in [0,1]$. Since w is a fixed point of B, by (31) and (29), we have

$$\begin{aligned} w(0) &= {}_0^R D_t^{\beta,\lambda} w(0) = 0,\ w(1) = \int_0^1 e^{-\lambda(1-t)} w(t)dt, \\ \eta(0) &= {}_0^R D_t^{\beta,\lambda} \eta(0) = 0,\ \eta(1) = \int_0^1 e^{-\lambda(1-t)} \eta(t)dt, \end{aligned} \tag{37}$$

which imply that

$$x(0) = \varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(0)\right) - \varphi_p\left({}_0^R D_t^{\beta,\lambda} w(0)\right) = 0,\ t \in [0,1] \tag{38}$$

It follows from (35) and (36) that

$$\begin{aligned} {}_0^R D_t^{\alpha,\lambda} x(t) &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right)\right) - {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda} w(t)\right)\right) \\ &\geq f(t,\eta(t)) - F(t,\eta(t)) = 0. \end{aligned} \tag{39}$$

Thus, (38), (39) and Lemma 5 guarantee

$$x(t) \geq 0, \quad t \in [0,1],$$

i.e.,

$$\varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right) \geq \varphi_p\left({}_0^R D_t^{\beta,\lambda} w(t)\right),\ t \in [0,1].$$

Since φ_p is monotone increasing and (10) implies that ${}_0^R D_t^{\beta,\lambda}\eta(t)$ and ${}_0^R D_t^{\beta,\lambda} w(t)$ are non-negative, we have

$${}_0^R D_t^{\beta,\lambda}\eta(t) \geq {}_0^R D_t^{\beta,\lambda} w(t), \ \text{ i.e., } \ {}_0^R D_t^{\beta,\lambda}(\eta - w)(t) \geq 0. \tag{40}$$

Thus, by (37), (40) and Lemma 5, we have

$$\eta(t) - w(t) \geq 0,$$

that is $w(t) \leq \eta(t)$ on $[0,1]$, which contradicts $w(t) > \eta(t)$. Thus, $w(t) \leq \eta(t)$ on $[0,1]$.

Following the same strategy, one has $w(t) \geq \xi(t)$ on $[0,1]$. Hence,

$$\xi(t) \leq w(t) \leq \eta(t), \quad t \in [0,1],$$

which yields

$$F(t,w(t)) = f(t,w(t)),\ t \in [0,1].$$

Thus, the fixed point of B is also the fixed point of S. Consequently, $w(t)$ is a positive solution of the tempered fractional Equation (1).

Next, we focus on the estimation and asymptotic behavior of the solution of the tempered fractional Equation (1). In view of $\xi \in K^*$ and (34), there exists $0 < h_\xi < 1$ such that

$$w(t) \geq \xi(t) \geq h_\xi e^{-\lambda t} t^{\beta-1}. \tag{41}$$

Thus, from (41) and (18) and the Hölder inequality, we have

$$\begin{aligned}
w(t) &= \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}F(\tau,w(\tau))d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,w(\tau))d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{q-1} ds \\
&\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)} \\
&\quad \times \left(\int_0^s e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} ds \\
&\leq M_1^*\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)} e^{-\lambda t}t^{\beta-1} \\
&= k_2 e^{-\lambda t}t^{\beta-1}.
\end{aligned} \tag{42}$$

Hence, it follows from (41) and (42) that

$$k_1 e^{-\lambda t}t^{\beta-1} \leq w(t) \leq k_2 e^{-\lambda t}t^{\beta-1},$$

where

$$\begin{aligned}
k_1 &= h_\xi, \\
k_2 &= M_1^*\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau,l_\xi e^{-\lambda\tau}\tau^{\beta-1})d\tau\right)^{\sigma(q-1)}.
\end{aligned}$$

□

Theorem 2. *Suppose the following conditions hold*

$(B1)$ $f \in C((0,1)\times[0,\infty),[0,+\infty))$, *and* $f(t,z)$ *is decreasing in* $z > 0$.

$(B2)$ $f(t,0) \not\equiv 0$ *for any* $t \in (0,1)$, *and there exists a constant* $0 < \sigma < \alpha$ *such that*

$$0 < \int_0^1 e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau,0)d\tau < +\infty. \tag{43}$$

Then, Equation (1) *has at least one positive solution* $w(t)$, *and there exists a constant* $\mathcal{N}^* > 0$ *such that*

$$0 \leq w(t) \leq \mathcal{N}^* e^{-\lambda t}t^{\beta-1}.$$

Proof. We replace the set K^* by

$$K = \{x \in E : x(t) \geq 0,\, t \in [0,1]\},$$

and let

$$\kappa(t) = \min\{0, S0\}, \;\; \theta(t) = \max\{0, S0\}.$$

Noting that $S0 \geq 0$, we have

$$\kappa(t) = 0, \ \ \theta(t) = S0.$$

Let

$$\xi(t) = S\theta(t), \ \ \eta(t) = S\kappa(t),$$

then, we have $\eta(t), \xi(t) \in K$ and

$$0 \leq \eta(t) = S\kappa(t) = S0 \ \ and \ \ 0 \leq \xi(t) = S\theta(t) = S(S0) = (S\eta)(t) \leq S0 = \eta(t). \tag{44}$$

On the other hand, we also have

$$\begin{aligned} &{}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\xi(t)\right)\right) - f(t,\xi(t)) \\ &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\theta)(t)\right)\right) - f(t,(S\theta)(t)) \\ &\leq {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\theta)(t)\right)\right) - f(t,(S0)(t)) \\ &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\theta)(t)\right)\right) - f(t,\theta(t)) = 0, \ \ t \in (0,1), \end{aligned} \tag{45}$$

and

$$\begin{aligned} &{}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}\eta(t)\right)\right) - f(t,\eta(t)) \\ &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\kappa)(t)\right)\right) - f(t,(S\kappa)(t)) \\ &\geq {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\kappa)(t)\right)\right) - f(t,(S0)(t)) \\ &= {}_0^R D_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R D_t^{\beta,\lambda}(S\kappa)(t)\right)\right) - f(t,\kappa(t)) = 0. \end{aligned} \tag{46}$$

Thus, from (44)–(46), $\xi(t)$ and $\eta(t)$ are the lower and upper solutions of the boundary value problem (1), respectively. Hence, it follows from the proof of Theorem 1 that S has a fixed point $w \in K$ and

$$\begin{aligned} w(t) = (Sw)(t) &= \int_0^1 H(t,s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,w(\tau))d\tau\right)^{q-1} ds \\ &\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,w(\tau))d\tau\right)^{q-1} ds \\ &\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\int_0^s \frac{e^{-\lambda s}}{\Gamma(\alpha)}(s-\tau)^{\alpha-1}e^{\lambda\tau}f(\tau,0)d\tau\right)^{q-1} ds \\ &\leq e^{-\lambda t}t^{\beta-1}\int_0^1 M_1(s)\left(\frac{e^{-\lambda s}}{\Gamma(\alpha)}\right)^{q-1}\left(\int_0^s (s-\tau)^{\frac{\alpha-1}{1-\sigma}}d\tau\right)^{(1-\sigma)(q-1)} \\ &\quad \times\left(\int_0^s e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,0)d\tau\right)^{\sigma(q-1)} ds \\ &\leq M_1^* e^{-\lambda t}t^{\beta-1}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,0)d\tau\right)^{\sigma(q-1)} \\ &\leq \mathcal{N}^* e^{-\lambda t}t^{\beta-1}, \end{aligned} \tag{47}$$

where

$$\mathcal{N}^* = M_1^*\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{(1-\sigma)(q-1)}\left(\int_0^1 e^{\frac{\lambda\tau}{\sigma}}f^{\frac{1}{\sigma}}(\tau,0)d\tau\right)^{\sigma(q-1)}.$$

Thus, the conclusion of Theorem 2 is true. □

Theorem 3. *Suppose*
$(C1)$ *$f \in C([0,1] \times [0,\infty),[0,+\infty))$ satisfies $f(t,0) \not\equiv 0$ for any $t \in [0,1]$, and $f(t,z)$ is decreasing in $z > 0$.*

Then, Equation (1) has at least one positive solution $w(t)$, and there exists a constant $\mathcal{N}^* > 0$ such that

$$0 \le w(t) \le \mathcal{N}^* e^{-\lambda t} t^{\beta-1}.$$

Proof. Clearly, $(C1)$ implies that $(B2)$ holds. Following the proof of Theorem 2, we can then obtain the conclusion. □

4. Example

Example 1. *Take*

$$\alpha = \frac{1}{2},\ \beta = \frac{3}{2},\ \lambda = 2,\ \ p = \frac{3}{2}.$$

Consider the tempered fractional equation with integral boundary conditions

$$\begin{cases} {}_0^R D_t^{\frac{1}{2},2}\left(\varphi_{\frac{3}{2}}\left({}_0^R D_t^{\frac{3}{2},2} x(t)\right)\right) = \dfrac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}} x^{\frac{1}{6}}(t)}, \\ x(0) = 0,\ {}_0^R D_t^{\frac{3}{2},2} x(0) = 0,\ \ x(1) = \displaystyle\int_0^1 e^{-2(1-t)} x(t) dt. \end{cases} \tag{48}$$

Conclusion 1. *The tempered fractional Equation (48) has at least one positive solution $w(t)$. Moreover, there exist two constants $k_1, k_2 > 0$ such that*

$$k_1 e^{-2t} t^{\frac{1}{2}} \le w(t)) \le k_2 e^{-2t} t^{\frac{1}{2}}.$$

Proof. Denote

$$f(t,z) = \frac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}} z^{\frac{1}{6}}}.$$

It is clear that $f \in C((0,1) \times (0,\infty),[0,+\infty))$, and $f(t,z)$ is decreasing in $z > 0$, so $(A1)$ is satisfied.

Now, choose $\sigma = \frac{1}{3} < \alpha = \frac{1}{2}$, for any $\rho > 0$; then, we have

$$f(t,\rho) = \frac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}} \rho^{\frac{1}{6}}} \not\equiv 0, t \in (0,1),$$

and

$$\begin{aligned} 0 < \int_0^1 e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau, \rho e^{-\lambda\tau}\tau^{\beta-1}) d\tau &= \int_0^1 e^{6\tau} \left[\frac{1}{(1-\tau)^{\frac{1}{6}} \tau^{\frac{1}{12}} \rho^{\frac{1}{6}} \left(e^{-2\tau}\tau^{\frac{1}{2}}\right)^{\frac{1}{6}}} \right]^3 d\tau \\ = \rho^{-\frac{1}{6}} \int_0^1 \frac{e^{7\tau}}{(1-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} d\tau \le \rho^{-\frac{1}{6}} e^7 \pi < +\infty. \end{aligned} \tag{49}$$

Thus, $(A2)$ also holds.

By Theorem 1, the tempered fractional Equation (48) has at least one positive solution $W(t)$, and there exist two constants $k_1, k_2 > 0$ such that

$$k_1 e^{-2t} t^{\frac{1}{2}} \le w(t)) \le k_2 e^{-2t} t^{\frac{1}{2}}.$$

□

Example 2. *Consider the tempered fractional Equation (48) with* $f(t,z) = \frac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}} (z+1)^{\frac{1}{6}}}$,

$$\begin{cases} {}_0^R D_t^{\frac{1}{2},2}\left(\varphi_{\frac{3}{2}}\left({}_0^R D_t^{\frac{3}{2},2} x(t)\right)\right) = \dfrac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}} (x(t)+1)^{\frac{1}{6}}}, \\ x(0) = 0, {}_0^R D_t^{\frac{3}{2},2} x(0) = 0,\ x(1) = \displaystyle\int_0^1 e^{-2(1-t)} x(t) dt. \end{cases} \tag{50}$$

Conclusion 2. *The tempered fractional Equation (50) has at least one positive solution* $w(t)$, *and there exists a constant* $\mathcal{N}^* > 0$ *such that*

$$0 \le w(t) \le \mathcal{N}^* e^{-2t} t^{\frac{1}{2}}.$$

Proof. Obviously, $f \in C((0,1) \times [0,\infty), [0,+\infty))$, and $f(t,z)$ is decreasing in $z > 0$, and thus, $(B1)$ holds. In addition, we have

$$f(t,0) = \frac{1}{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}}} \not\equiv 0$$

for any $t \in (0,1)$. Take $0 < \sigma = \frac{1}{3} < \alpha = \frac{1}{2}$; then, we have

$$\begin{aligned} 0 &< \int_0^1 e^{\frac{\lambda\tau}{\sigma}} f^{\frac{1}{\sigma}}(\tau,0) d\tau = \int_0^1 e^{6\tau} \left[\frac{1}{(1-\tau)^{\frac{1}{6}} \tau^{\frac{1}{12}}}\right]^3 d\tau \\ &= \int_0^1 \frac{e^{6\tau}}{(1-\tau)^{\frac{1}{2}} \tau^{\frac{1}{4}}} d\tau \le e^7 \mathbf{B}(\frac{1}{2}, \frac{3}{4}) \approx 2.3963 e^7 < +\infty, \end{aligned} \tag{51}$$

that is, $(B2)$ holds.

Then, by Theorem 2, Equation (50) has at least one positive solution $x(t)$, and there exists a constant $\mathcal{N}^* > 0$ such that

$$0 \le x(t) \le \mathcal{N}^* e^{-2t} t^{\frac{1}{2}}.$$

□

Example 3. *Consider the tempered fractional Equation (48) with* $f(t,z) = \frac{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}}}{(z+1)^{\frac{1}{6}}}$,

$$\begin{cases} {}_0^R D_t^{\frac{1}{2},2}\left(\varphi_{\frac{3}{2}}\left({}_0^R D_t^{\frac{3}{2},2} x(t)\right)\right) = \dfrac{(1-t)^{\frac{1}{6}} t^{\frac{1}{12}}}{(x(t)+1)^{\frac{1}{6}}}, \\ x(0) = 0, {}_0^R D_t^{\frac{3}{2},2} x(0) = 0,\ x(1) = \displaystyle\int_0^1 e^{-2(1-t)} x(t) dt. \end{cases} \tag{52}$$

Conclusion 3. *The tempered fractional Equation (50) has at least one positive solution* $w(t)$, *and there exists a constant* $\mathcal{N}^* > 0$ *such that*

$$0 \le w(t) \le \mathcal{N}^* e^{-2t} t^{\frac{1}{2}}.$$

Proof. Clearly, $f \in C([0,1] \times [0,\infty), [0,+\infty))$ satisfies $f(t,0) \not\equiv 0$ for any $t \in [0,1]$, and $f(t,z)$ is decreasing in $z > 0$. Then, by Theorem 3, Equation (52) has at least one positive solution $x(t)$, and there exists a constant $\mathcal{N}^* > 0$ such that

$$0 \le x(t) \le \mathcal{N}^* e^{-\lambda t} t^{\beta-1}.$$

□

5. Conclusions

In this paper, we establish some new results for the existence of positive solutions of a class of singular tempered fractional equations with a p-Laplacian operator. The main contribution is the construction of a pair of suitable upper and lower solutions to solve the difficulty of singularities, which includes two aspects:

(i). The nonlinearity of the equation allows having singularities at time and space variables;

(ii). The order of the fractional derivative can be less than 1, and the corresponding operator allows having a singular kernel.

In addition, our results are also comprehensive, which contain three different cases, i.e., all singular and nonsingular cases are discussed,

Case 1. $f(t,z)$ may be singular at $t=0, t=1$ and $z=0$ in Theorem 1;

Case 2. $f(t,z)$ may be singular at $t=0, t=1$ and has no singularity at $z=0$ in Theorem 2;

Case 3. $f(t,z)$ has no singularity at $t=0, t=1$ and $z=0$ in Theorem 3.

In this paper, we only consider the existence of positive solution for tempered fractional Equation (1), so some further work can continue to be considered such as the uniqueness and multiplicity of positive solutions, the case where the nonlinearity is changing sign or the p-Laplacian operator becomes a nonlinear operator, etc.

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