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Article

Probing Families of Optical Soliton Solutions in Fractional Perturbed Radhakrishnan–Kundu–Lakshmanan Model with Improved Versions of Extended Direct Algebraic Method

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Abstract: In this investigation, we utilize advanced versions of the Extended Direct Algebraic Method (EDAM), namely the modified EDAM (mEDAM) and r+mEDAM, to explore families of optical soliton solutions in the Fractional Perturbed Radhakrishnan–Kundu–Lakshmanan Model (FPRKLM). Our study stands out due to its in-depth investigation and the identification of multiple localized and stable soliton families, illuminating their complex behavior. We offer visual validation via carefully designed 3D graphics that capture the complex behaviors of these solitons. The implications of our research extend to fiber optics, communication systems, and nonlinear optics, with the potential for driving developments in optical devices and information processing technologies. This study conveys an important contribution to the field of nonlinear optics, paving the way for future advancements and a greater comprehension of optical solitons and their applications.

Keywords: Fractional Perturbed Radhakrishnan Kundu Lakshmanan Model; Extended Direct Algebraic Method; Nonlinear Ordinary Differential Equation; optical soliton solutions; variable transformation; generalized trigonometric functions

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1. Introduction

Fractional Partial Differential Equations (FPDEs) have received great attention in different fields of science due to their ability to accurately model complex physical phenomena [1–4]. This encourages researchers to dedicate their efforts to studying, examining, and analyzing FPDEs. Researchers have used numerical and analytical techniques to understand and analyze the behavior of FPDEs. Numerical methods are based on discretization techniques that approximate the solution through iterative calculations [5–7]. These numerical methods are powerful and widely used but often have limitations, such as computational expenses and the inability to provide exact solutions. In contrast, analytic techniques aim to obtain exact solutions using mathematical techniques and transformations. Researchers often prefer closed formulas and analytical techniques that can provide greater insight into the underlying mathematical structure of a problem. Analytic solutions provide a comprehensive understanding of system behavior, facilitating further theoretical analysis and investigation of physical effects. Therefore, different analytical approaches, such as the Variational Iteration Method (VIM) [8], the Fractional Differential Transform Method (FDTM) [9], the (G′/G)-expansion method [10], the exp-function method [11], the tan-expansion method [12], the Adomian Decomposition Method (ADM) [13], the Laplace Transform Method (LTM) [14] and the EDAM [15,16], etc., are introduced to tackle FPDEs.

The EDAM is a particularly efficient and reliable approach among these analytical techniques. The method first transforms a complex nonlinear FPDE into a Nonlinear Ordinary Differential Equation (NODE) by fitly choosing variable transformations. Then, using another ODE, the EDAM assumes a series-form solution. Substituting this solution into NODE adeptly transforms the NODE into a system of algebraic equations. By solving this system of equations skillfully, the EDAM allows us to construct different families of soliton solutions, each with profound implications in different scientific fields. This amazing ability of the EDAM enriches our understanding and exploration of FPDEs and opens the door to groundbreaking discoveries and major advances in the scientific field.

Our study's main goal is to investigate the variety of optical soliton solutions for the FPRKLM using two upgraded versions of the EDAM, namely mEDAM and r+EDAM. The FPRKLM is a special type of FPDE incorporating perturbations into the Radhakrishnan–Kundu–Lakshmanan Model (RKLM), a well-known equation governing soliton dynamics. The FPRKLM exhibits rich dynamics and can be applied to various physical systems, such as nonlinear optics, Bose–Einstein condensation, and plasma physics. The FPRKLM provides a valuable theoretical framework for studying wave phenomena and has practical implications. Optical solitons, which are self-amplifying single waves, have attracted much attention due to their potential applications in high-speed communication systems, optical fibers, and optical signal processing. Analyzing the family of optical soliton solutions in the FPRKLM provides important insights into the behavior and manipulation of optical pulses and enables their advancement. With this analytical approach, we hope to decipher the complex wave phenomena of soliton solutions and provide valuable insights into the behavior of optical solitons in the FPRKLM. This investigation's results are important for understanding the FPRKLM and further developing nonlinear optics and related fields. The proposed complex structural FPRKLM under the Kerr law nonlinearity is given by [17]:

$$iD_t^{\alpha}u + a_1 D_{xx}^{2\beta}u + b_1|u|^2u - i\delta D_x^{\beta}u - i\mu_1 D_x^{\beta}(|u|^2u) - i\sigma u D_x^{\beta}(|u|^2) - i\gamma D_{xxx}^{3\beta}u = 0, \quad (1)$$

where $0 < \alpha, \beta \leq 1$ and u represents the complex-valued wave-function in space, x, and time, t. $D_t^{\alpha}u$ denotes the fractional time evolution of the nonlinear wave, while $D_x^{\beta}u$, $D_{xx}^{2\beta}u$ and $D_{xxx}^{3\beta}u$ denote spatial fractional derivatives. In this study, both time and spatial fractional derivatives are defined in Caputo's derivative sense given in (2). The proposed model was described in terms of time-fractional derivatives in [17]. The goal of this study is to solve the problem using a more thorough model that includes complete fractional derivatives. As a consequence, we generalise the model from [17] by substituting a fractional derivative, D_x^{β}, for the traditional spatial derivative. The inclusion of spatial fractional derivatives captures genuine occurrences and improves the description of the system by taking fractional diffusion and the interaction of temporal and spatial dynamics into consideration. It also generalises the issue, allowing for fascinating mathematical analysis, and broadens the study's application to complex systems with temporal and spatial fractional dynamics. The coefficient a_1 denotes Group velocity dispersion (GVD), b_1 denotes the nonlinearity coefficient, δ represents Inter-Modal Dispersion (IMD), μ_1 corresponds to short-pulse self-tilt coefficient, and σ denotes higher-order dispersion. In contrast, the coefficient γ corresponds to third-order dispersion terms.

Prior to this work, many mathematicians have studied the optical wave phenomena of the proposed model in both integer and fractional forms for exploring optical soliton solutions using various analytical approaches. In their work [17], Sulaiman et al. delved into the study of dark, bright, and dark-light mixtures; single, mixed singular optical solitons; and singular periodic wave solutions in time-fractional FPRKLM. Similarly, Saima et al. focused on PRKLM for scattered light solitons of bright, dark, singular and dark singular combinations using (G'/G^2)-expansion and sine-Gordon expansion methods [18]. Tukur and Hasan [19] used the extended rational sine-cosine/sinh-cosh method to tackle local M-fractional RKL equations. Finally, Kudryashov [20] studied complex RKL equations using the fourth-power polynomial law of nonlinearity, especially for solitary wave construction.

The fractional derivatives presented in (1) are defined in the Caputo's derivative sense. The operator for this differentiation is defined as [21]

$$D_y^{\gamma} u(x,y) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \int_0^y \frac{\partial}{\partial \omega} z(x,\omega)(\omega - y)^{-\gamma} d\omega, & \gamma \in (0,1) \\ \frac{\partial u(x,y)}{\partial y}, & \gamma = 1 \end{cases} \tag{2}$$

where the function $u(x,y)$ is fairly smooth. We rely on the application of the subsequent two operator's properties to convert the FPDE indicated in (1) into NODEs:

$$D_{\varphi}^{\kappa} \varphi^{r} = \frac{\Gamma(1+r)}{\Gamma(1+r-\kappa)} \varphi^{r-\kappa}, \tag{3}$$

$$D_{\varphi}^{\gamma} y[x(\varphi)] = y'_x(x(\varphi)) D_{\varphi}^{\gamma} x(\varphi), \tag{4}$$

Here, we presume that $x(\varphi)$ & $y(\varphi)$ symbolises functions that maintain differentiability, whereas r is a real number.

2. Method and Materials

This section outlines the EDAM's operational procedures. Take into account the general FPDE listed below [16]:

$$M(h, \partial_t^{\alpha} h, \partial_{v_1}^{\beta} h, \partial_{v_2}^{\gamma} h, h\partial_{v_1}^{\beta} h, \dots) = 0, \;\; 0 < \alpha, \beta, \gamma \leq 1, \tag{5}$$

where $h = h(t, v_1, v_2, v_3, \dots, v_i)$.

Following these steps allows us to solve problem (5):

1. First, $h(t, v_1, v_2, v_3, \dots, v_i) = H(\zeta), \zeta = \zeta(t, v_1, v_2, v_3, \dots, v_i)$ (ζ can be written in many ways) is executed to turn (5) into a NODE of the form:

$$T(H, H', H'H, \dots) = 0, \tag{6}$$

   where H in (6) has derivatives with respect to ζ. (6) may occasionally be integrated once or more to obtain the integration's constant.
2. We assume one of the following solutions for (6) based on the version of EDAM:
   (a) The following series form solution is suggested by the mEDAM:

$$H(\zeta) = \sum_{m=-j}^{j} C_m (G(\zeta))^m, \tag{7}$$

   (b) While the r+mEDAM offers the subsequent solution:

$$H(\zeta) = \sum_{m=-j}^{j} C_m (r + G(\zeta))^m, \tag{8}$$

   where $C_m (m = -j, \dots, 0, \dots, j)$ are arbitrary parameters that will be found later and $G(\varphi)$ satisfies the subsequent nonlinear ODE:

$$G'(\zeta) = (c(G(\zeta))^2 + bG(\zeta) + a) ln(\mu), \tag{9}$$

   Here, it ought to be pointed out that μ presumes a value different from 0 and 1, whereas a, b, and c remain constant during the investigation.
3. The positive integer symbolised as j in (7) and (8) is often referred to as the balance number. It is calculated by applying homogeneous balancing between the greatest nonlinear component in Equation (6) and the highest order derivative.

4. Following that, we insert (7) or (8) into (6) or into the equation created by integrating (6), and we then compile all of the terms of $G(\zeta)$ that are in the same order and produce an expression in $G(\zeta)$. A system of algebraic equations in $C_m(m = -j, \ldots, 0, \ldots, j)$ and other parameters is produced by equating all the coefficients of the expression to zero using the concept of comparison of coefficients.
5. We use the Maple programme for resolving this set of algebraic equations.
6. The next step is to determine the coefficients and extra parameters, which we then include into Equation (7) or (8) along with the general solution of Equation (9), denoted as $G(\zeta)$, in order to study the optical soliton solutions for Equation (5). We may produce several families of soliton solutions by using the general solution given in Equation (10), as shown below.

Family 1. In the case when Q is below 0 and c is not equal to 0. the use of the general solutions if nonlinear ODE provided in Equation (9) results into the development of the given family of travelling soliton solutions:

$$G_1(\zeta) = -\frac{b}{2c} + \frac{\sqrt{-Q}\tan_\mu\left(1/2\sqrt{-R}\zeta\right)}{2c},$$

$$G_2(\zeta) = -\frac{b}{2c} - \frac{\sqrt{-Q}\cot_\mu\left(1/2\sqrt{-R}\zeta\right)}{2c},$$

$$G_3(\zeta) = -\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(\sqrt{-Q}\zeta\right) \pm \left(\sqrt{pq}\sec_\mu\left(\sqrt{-Q}\zeta\right)\right)\right)}{2c},$$

$$G_4(\zeta) = -\frac{b}{2c} - \frac{\sqrt{-Q}\left(\cot_\mu\left(\sqrt{-Q}\zeta\right) \pm \left(\sqrt{pq}\csc_\mu\left(\sqrt{-Q}\zeta\right)\right)\right)}{2c},$$

and

$$G_5(\zeta) = -\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(1/4\sqrt{-Q}\zeta\right) - \cot_\mu\left(1/4\sqrt{-Q}\zeta\right)\right)}{4c}.$$

Family 2. The generic solutions derived from Equation (9) lead to the following family of traveling soliton solutions when Q is larger than zero and c is not equal to zero:

$$G_6(\zeta) = -\frac{b}{2c} - \frac{\sqrt{Q}\tanh_\mu\left(1/2\sqrt{Q}\zeta\right)}{2c},$$

$$G_7(\zeta) = -\frac{b}{2c} - \frac{\sqrt{Q}\coth_\mu\left(1/2\sqrt{Q}\zeta\right)}{2c},$$

$$G_8(\zeta) = -\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(\sqrt{Q}\zeta\right) \pm \left(\sqrt{pq}sech_\mu\left(\sqrt{Q}\zeta\right)\right)\right)}{2c},$$

$$G_9(\zeta) = -\frac{b}{2c} - \frac{\sqrt{Q}\left(\coth_\mu\left(\sqrt{Q}\zeta\right) \pm \left(\sqrt{pq}csch_\mu\left(\sqrt{Q}\zeta\right)\right)\right)}{2c},$$

and

$$G_{10}(\zeta) = -\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(1/4\sqrt{Q}\zeta\right) - \coth_\mu\left(1/4\sqrt{Q}\zeta\right)\right)}{4c}.$$

Family 3. The generic solutions stated in Equation (9) are applied in the case where the product ac is higher than 0 and b is equal to 0, producing the required family of traveling soliton solutions:

$$G_{11}(\zeta) = \sqrt{\frac{a}{c}}\tan_\mu\left(\sqrt{ac}\zeta\right),$$

$$G_{12}(\zeta) = -\sqrt{\frac{a}{c}}\cot_\mu\left(\sqrt{ac}\zeta\right),$$

$$G_{13}(\zeta) = \sqrt{\frac{a}{c}}\left(\tan_{\mu}\left(2\sqrt{ac}\zeta\right) \pm \left(\sqrt{pq}\sec_{\mu}\left(2\sqrt{ac}\zeta\right)\right)\right),$$

$$G_{14}(\zeta) = -\sqrt{\frac{a}{c}}\left(\cot_{\mu}\left(2\sqrt{AC}\zeta\right) \pm \left(\sqrt{pq}\csc_{\mu}\left(2\sqrt{ac}\zeta\right)\right)\right),$$

and

$$G_{15}(\zeta) = \frac{1}{2}\sqrt{\frac{a}{c}}\left(\tan_{\mu}\left(1/2\sqrt{ac}\zeta\right) - \cot_{\mu}\left(1/2\sqrt{ac}\zeta\right)\right).$$

Family 4. The generic solutions of (9) provide the following family of traveling soliton solutions for $ac > 0$ & $b = 0$:

$$G_{16}(\zeta) = -\sqrt{-\frac{a}{c}}\tanh_{\mu}\left(\sqrt{-ac}\zeta\right),$$

$$G_{17}(\zeta) = -\sqrt{-\frac{a}{c}}\coth_{\mu}\left(\sqrt{-ac}\zeta\right),$$

$$G_{18}(\zeta) = -\sqrt{-\frac{a}{c}}\left(\tanh_{\mu}\left(2\sqrt{-ac}\zeta\right) \pm \left(i\sqrt{pq}sech_{A}\left(2\sqrt{-ac}\zeta\right)\right)\right),$$

$$G_{19}(\zeta) = -\sqrt{-\frac{a}{c}}\left(\coth_{\mu}\left(2\sqrt{-ac}\zeta\right) \pm \left(\sqrt{pq}csch_{\mu}\left(2\sqrt{-ac}\zeta\right)\right)\right),$$

and

$$G_{20}(\zeta) = -\frac{1}{2}\sqrt{-\frac{a}{c}}\left(\tanh_{\mu}\left(1/2\sqrt{-ac}\zeta\right) + \coth_{\mu}\left(1/2\sqrt{-ac}\zeta\right)\right).$$

Family 5. The general solutions derived from Equation (9) give birth to the following specific family of travelling soliton solutions as follows when c equals a and b equals 0:

$$G_{21}(\zeta) = \tan_{\mu}(a\zeta),$$

$$G_{22}(\zeta) = -\cot_{\mu}(a\zeta),$$

$$G_{23}(\zeta) = \tan_{\mu}(2A\zeta) \pm \left(\sqrt{pq}\sec_{\mu}(2a\zeta)\right),$$

$$G_{24}(\zeta) = -\cot_{\mu}(2a\zeta) \pm \left(\sqrt{pq}\csc_{\mu}(2a\zeta)\right),$$

and

$$G_{25}(\zeta) = \frac{1}{2}\tan_{\mu}(1/2\,a\zeta) - 1/2\,\cot_{\mu}(1/2\,a\zeta).$$

Family 6. The following family of traveling soliton solutions is produced when the general solutions derived from Equation (9) are used in the situation when c is equal to $-a$ and b is equal to zero:

$$G_{26}(\zeta) = -\tanh_{\mu}(a\zeta),$$

$$G_{27}(\zeta) = -\coth_{\mu}(a\zeta),$$

$$G_{28}(\zeta) = -\tanh_{\mu}(2a\zeta) \pm \left(i\sqrt{pq}sech_{\mu}(2a\zeta)\right),$$

$$G_{29}(\zeta) = -\coth_{\mu}(2a\zeta) \pm \left(\sqrt{pq}csch\text{-}(2a\zeta)\right),$$

and

$$G_{30}(\zeta) = -\frac{1}{2}\tanh_{\mu}(1/2\,a\zeta) - 1/2\,\coth_{\mu}(1/2\,a\zeta).$$

Family 7. The application of the general solutions derived from Equation (9) yields the specified family of traveling soliton solutions when Q is equal to zero:

$$G_{31}(\zeta) = \frac{-2a(b\zeta\, ln(\mu) + 2)}{b^2\zeta\, ln(\mu)}.$$

Family 8. The generic solutions derived from Equation (9) produce the following family of traveling soliton solutions where b is equal to ν, a is equal to $N\nu$ (where N is a non-zero number), and c is equal to zero:

$$G_{32}(\zeta) = \mu^{\nu\,\zeta} - N.$$

Family 9. The generic solutions derived from Equation (9) give rise to the specified family of traveling soliton solutions when both b and c are equal to zero:

$$G_{33}(\zeta) = a\zeta\, ln(\mu).$$

Family 10. The general solutions derived from Equation (9) result in the stated set of traveling soliton solutions when both b and a are zero:

$$G_{34}(\zeta) = -\frac{1}{c\zeta\, ln(\mu)}.$$

Family 11. The general solutions resulting from Equation (9) result in the stated family of traveling soliton solutions when a is zero, b is not equal to zero, and c is not equal to zero:

$$G_{35}(\zeta) = -\frac{pb}{c\big(\cosh_\mu(b\zeta) - \sinh_\mu(b\zeta) + p\big)},$$

and

$$G_{36}(\zeta) = -\frac{b\big(\cosh_\mu(b\zeta) + \sinh_\mu(b\zeta)\big)}{\big(c\cosh_\mu(b\zeta) + c\sinh_\mu(b\zeta) + cq\big)},$$

Family 12. The general solutions derived from Equation (9) result in the following set of traveling soliton solutions where b is equal to ν, c is equal to $N\nu$ (where N is a non-zero number), and a is equal to zero:

$$G_{37}(\zeta) = \frac{p\mu^{\nu\,\zeta}}{p - Nq\mu^{\nu\,\zeta}}.$$

Here, p and q are both greater than zero, which are known as the deformation parameters. In addition, Q is defined as $b^2 - 4ac$. Our solutions contain generalised trigonometric and hyperbolic functions that may be represented as follows:

$$\begin{aligned}
\sin_\mu(\zeta) &= \frac{p\mu^{i\zeta} - q\mu^{-i\zeta}}{2i}, \quad \cos_\mu(\zeta) = \frac{p\mu^{-i\zeta} + q\mu^{i\zeta}}{2},\\
\sec_\mu(\zeta) &= \frac{1}{\cos_\mu(\zeta)}, \quad \csc_\mu(\zeta) = \frac{1}{\sin_\mu(\zeta)},\\
\cot_\mu(\zeta) &= \frac{\cos_\mu(\zeta)}{\sin_\mu(\zeta)}, \quad \tan_\mu(\zeta) = \frac{\sin_\mu(\zeta)}{\cos_\mu(\zeta)}.
\end{aligned}$$

Similarly,

$$\begin{aligned} \sinh_\mu(\zeta) &= \frac{p\mu^{\zeta} - q\mu^{-\zeta}}{2}, \quad \cosh_\mu(\zeta) = \frac{p\mu^{-\zeta} + q\mu^{\zeta}}{2}, \\ sech\text{-}(\imath) &= \frac{1}{\cosh\text{-}(\imath)}, \quad csch\text{-}(\imath) = \frac{1}{\sinh\text{-}(\imath)}, \\ \coth_\mu(\zeta) &= \frac{\cosh_\mu(\zeta)}{\sinh_\mu(\zeta)}, \quad \tanh_\mu(\zeta) = \frac{\sinh_\mu(\zeta)}{\cosh_\mu(\zeta)}. \end{aligned}$$

3. Results

In this section, the targeted problem is addressed with improved versions of the EDAM. We debut with the following traveling wave transformation:

$$\begin{aligned} &u(x,t) = U(\zeta)e^{i\theta}, \quad where \\ &\zeta = \lambda\left(\frac{x^\beta}{\Gamma(\beta+1)} - \frac{c_1 t^\alpha}{\Gamma(\alpha+1)}\right), \quad and \quad \theta = -\frac{kx^\beta}{\Gamma(\beta+1)} + \frac{\omega t^\alpha}{\Gamma(\alpha+1)} + \vartheta, \end{aligned} \tag{10}$$

substituting (10) in (1) yields:

$$\lambda^2(a_1 + 3k\gamma)U'' + (b_1 - k\mu)U^3 - (\omega + \delta k + a_1 k^2 + \gamma k^3)U = 0, \tag{11}$$

from the real part while the imaginary part gives:

$$\lambda^2\gamma U''' - (c_1 + 2a_1 k + 3k^2\gamma + \delta)U' - (2\sigma + 3\mu)U^2U' = 0. \tag{12}$$

By integrating (12) with respect to ζ once and setting constant of integration to zero, we have:

$$3\lambda^2\gamma U'' - 3(c_1 + 2a_1 k + 3k^2\gamma + \delta)U - 3(2\sigma + 3\mu)U^3 = 0. \tag{13}$$

(11) and (13) have the same forms under the following constraint condition:

$$\frac{a_1 + 3k\gamma}{3\lambda^2} = -\frac{b_1 - k\mu}{(2\sigma + 3\mu)} = \frac{\omega + \delta k + a_1 k^2 + \gamma k^3}{3(c_1 + 2a_1 k + 3k^2\gamma + \delta)} \tag{14}$$

Solving (14) for c_1 and k yields:

$$k = -\frac{2a_1\sigma + 3b_1\gamma + 3a_1\mu}{6(\sigma + \mu)\gamma}, \tag{15}$$

$$c_1 = \frac{(\omega + \delta k + a_1 k^2 + \gamma k^3)\gamma}{a_1 + 3\gamma k} - (2ka_1 + \delta + 3k^2\gamma). \tag{16}$$

The constraints in (14)–(16) reduces the FPRKLM to a single ODE given in (11). The next goal is to solve (11) using the proposed versions of the EDAM for generating families of optical soliton solutions for (1). Balancing the highest order nonlinear term U^3 and highest order derivative U'' gives $m = 1$.

3.1. Application of the mEDAM

First we wish to use M to solve Equation (11). The following series-based solution for problem (11) is obtained by inserting $j = 1$ in Equation (7):

$$U(\zeta) = \sum_{m=-1}^{1} C_m(G(\zeta))^m = C_{-1}(G(\zeta))^{-1} + C_0 + C_1(G(\zeta))^1, \tag{17}$$

where C_{-1}, C_0 & C_1 are unknown constants. A system of nonlinear algebraic equations is produced by substituting Equation (17) into Equation (11). We use the Maple software to solve this problem, and it offers us the following two sets of solutions:

Case 1.

$$\begin{aligned}&C_1 = 0, C_{-1} = 2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}, C_0 = -\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}b,\\&\lambda = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}},\end{aligned} \tag{18}$$

Case 2.

$$\begin{aligned}&C_1 = 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}, C_{-1} = 0, C_0 = -\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}b,\\&\lambda = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}},\end{aligned} \tag{19}$$

where

$$\begin{aligned}m_1 &= a_1 + 3k\gamma\\m_2 &= \omega + \delta k + a_1k^2 + \gamma k^3\\m_3 &= b_1 - k\mu\end{aligned} \tag{20}$$

Taking Case 1 into consideration, we arrive at the following families of optical soliton solutions:

Family 1. When Q is less than 0 and a, b, and c are all non-zero, Equations (17) and (10), and the generic solutions obtained from Equation (9), together, give birth to a specific family of optical soliton solutions, which may be stated as follows:

$$\begin{aligned}&u_1(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c}+\frac{\sqrt{-Q}\tan_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)^{-1}\\&-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b),\end{aligned} \tag{21}$$

$$\begin{aligned}&u_2(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c}-\frac{\sqrt{-Q}\cot_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)^{-1}\\&-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b),\end{aligned} \tag{22}$$

$$\begin{aligned}&u_3(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times\\&\left(-\frac{b}{2c}+\frac{\sqrt{-Q}\left(\tan_\mu\left(\sqrt{-Q}(\zeta)\right)\pm\left(\sqrt{pq}\sec_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1}\\&-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b),\end{aligned} \tag{23}$$

$$u_4(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \left(-\frac{b}{2c}-\frac{\sqrt{-Q}\left(\cot_{\mu}\left(\sqrt{-Q}(\zeta)\right)\pm\left(\sqrt{pq}\,\csc_{\mu}\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b), \tag{24}$$

and

$$u_5(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \left(-\frac{b}{2c}+\frac{\sqrt{-Q}\left(\tan_{\mu}\left(1/4\,\sqrt{-Q}(\zeta)\right)-\cot_{\mu}\left(1/4\,\sqrt{-Q}(\zeta)\right)\right)}{4c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b). \tag{25}$$

Family 2. When Q is greater than 0 and a, b, and c are all non-zero, Equations (17) and (10), and the generic solutions obtained from Equation (9), together, give birth to a specific family of optical soliton solutions, which may be stated as follows:

$$u_6(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c}-\frac{\sqrt{Q}\tanh_{\mu}\left(1/2\,\sqrt{Q}(\zeta)\right)}{2c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b), \tag{26}$$

$$u_7(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c}-\frac{\sqrt{Q}\coth_{\mu}\left(1/2\,\sqrt{Q}(\zeta)\right)}{2c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b), \tag{27}$$

$$u_8(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \left(-\frac{b}{2c}-\frac{\sqrt{Q}\left(\tanh_{\mu}\left(\sqrt{Z}(\zeta)\right)\pm\left(\sqrt{pq}sech_{\mu}\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b), \tag{28}$$

$$u_9(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \left(-\frac{b}{2c}-\frac{\sqrt{Q}\left(\coth_{\mu}\left(\sqrt{Q}(\zeta)\right)\pm\left(\sqrt{pq}csch_{\mu}\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} - \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b), \tag{29}$$

and

$$u_{10}(x,t) = e^{i\theta}(2\,a\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times$$
$$\left(-\frac{b}{2c} - \frac{\sqrt{Q}\big(\tanh_\mu\big(1/4\,\sqrt{Q}(\zeta)\big) - \coth_\mu\big(1/4\,\sqrt{Q}(\zeta)\big)\big)}{4c}\right)^{-1} \tag{30}$$
$$-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b).$$

Family 3. Equations (17) and (10), and the related general solutions obtained from Equation (9), when used in conjunction, produce a particular family of optical soliton solutions where the product ac is larger than zero and b is equal to zero, which may be written as follows:

$$u_{11}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(\tan_\mu\big(\sqrt{ac}(\zeta)\big)\big)^{-1}), \tag{31}$$

$$u_{12}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\big(\cot_\mu\big(\sqrt{ac}(\zeta)\big)\big)^{-1}), \tag{32}$$

$$u_{13}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(\tan_\mu\big(2\,\sqrt{ac}(\zeta)\big) \pm \big(\sqrt{pq}\,\sec_\mu\big(2\,\sqrt{ac}(\zeta)\big)\big)\big)^{-1}), \tag{33}$$

$$u_{14}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\big(\cot_\mu\big(2\,\sqrt{ac}(\zeta)\big) \pm \big(\sqrt{pq}\,\csc_\mu\big(2\,\sqrt{ac}(\zeta)\big)\big)\big)^{-1}), \tag{34}$$

and

$$u_{15}(x,t) = e^{i\theta}(2\sqrt{\frac{-m_2}{m_3}}\big(\tan_\mu\big(1/2\,\sqrt{ac}(\zeta)\big) - \cot_\mu\big(1/2\,\sqrt{ac}(\zeta)\big)\big)^{-1}). \tag{35}$$

Family 4. Combining the use of Equations (17) and (10), and the related general solutions obtained from Equation (9) results in a unique family of optical soliton solutions in the situation when the product ac is higher than zero and b is equal to zero. These solutions are represented as follows:

$$u_{16}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\Big(\tanh_\mu\Big(\sqrt{-ac}(\zeta)\Big)\Big)^{-1}), \tag{36}$$

$$u_{17}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\Big(\coth_\mu\Big(\sqrt{-ac}(\zeta)\Big)\Big)^{-1}), \tag{37}$$

$$u_{18}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}(\tanh_\mu\Big(2\,\sqrt{-ac}(\zeta)\Big)$$
$$\pm\Big(i\sqrt{pq}sech_\mu\Big(2\,\sqrt{-ac}(\zeta)\Big)\Big))^{-1}), \tag{38}$$

$$u_{19}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}(\coth_\mu\left(2\sqrt{-ac}(\zeta)\right) \\ \pm\left(\sqrt{pq}csch_\mu\left(2\sqrt{-ac}(\zeta)\right)\right))^{-1}), \tag{39}$$

and

$$u_{20}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}(\tanh_\mu\left(1/2\sqrt{-ac}(\zeta)\right) \\ +\coth_\mu\left(1/2\sqrt{-ac}(\zeta)\right))^{-1}). \tag{40}$$

Family 5. Equations (17) and (10), and the related general solutions obtained from Equation (9) are used to construct a specific family of optical soliton solutions where c is equal to a and b is equal to zero. These solutions are represented as follows:

$$u_{21}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\left(\tan_\mu(a(\zeta))\right)^{-1}), \tag{41}$$

$$u_{22}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\left(\cot_\mu(a(\zeta))\right)^{-1}), \tag{42}$$

$$u_{23}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\left(\tan_\mu(2\,a(\zeta)) \pm \left(\sqrt{pq}\sec_\mu(2\,a(\zeta))\right)\right)^{-1}), \tag{43}$$

$$u_{24}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\left(-\cot_\mu(2\,a(\zeta)) \mp \left(\sqrt{pq}\csc_\mu(2\,a(\zeta))\right)\right)^{-1}), \tag{44}$$

and

$$u_{25}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\left(1/2\,\tan_\mu(1/2\,a(\zeta)) - 1/2\,\cot_\mu(1/2\,a(\zeta))\right)^{-1}). \tag{45}$$

Family 6. Equations (17) and (10), and the related general solutions obtained from Equation (9) are used to construct a specific family of optical soliton solutions where c is equal to $-a$ and b is equal to zero. These solutions are represented as follows:

$$u_{26}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu(a(\zeta))\right)^{-1}), \tag{46}$$

$$u_{27}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\left(\coth_\mu(a(\zeta))\right)^{-1}), \tag{47}$$

$$u_{28}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\left(-\tanh_\mu(2\,a(\zeta)) \mp \left(i\sqrt{pq}sech_\mu(2\,a(\zeta))\right)\right)^{-1}), \tag{48}$$

$$u_{29}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}(-\coth_\mu(2\,a(\zeta)) \mp (\sqrt{pq}csch_\mu(2\,a(\zeta))))^{-1}), \tag{49}$$

and

$$u_{30}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}(-1/2\,\tanh_\mu(1/2\,a(\zeta)) - 1/2\,\coth_\mu(1/2\,a(\zeta)))^{-1}). \tag{50}$$

Family 7. Equations (17) and (10), and the associated general solutions derived from Equation (9) are used to produce a specific family of optical soliton solutions in the case where b is equal to ν, a is equal to $n\nu$ (where n is a non-zero value), and c is equal to zero. These solutions are expressed as follows:

$$u_{31}(x,t) = e^{i\theta}(2\,n\sqrt{\frac{m_2}{m_3}}\left(\mu^{\nu\,(\zeta)} - n\right)^{-1} - \sqrt{\frac{m_2}{m_3}}). \tag{51}$$

where $\zeta = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}}(-\frac{c_1t^\alpha}{\Gamma(\alpha+1)} + \frac{x^\beta}{\Gamma(\beta+1)}), \quad \& \quad \theta = -\frac{kx^\beta}{\Gamma(\beta+1)} + \frac{\omega t^\alpha}{\Gamma(\alpha+1)} + \vartheta.$

Now assuming Case 2, we get the following cluster of optical soliton solutions:

Family 8. Equations (17) and (10), and the equivalent general solutions obtained from Equation (9) when Q is less than zero and a, b, and c are all non-zero, result in a particular family of optical soliton solutions, which may be written as follows:

$$\begin{aligned} u_{32}(x,t) = e^{i\theta}(&-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b \\ &+ 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c} + \frac{\sqrt{-Q}\tan_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)), \end{aligned} \tag{52}$$

$$\begin{aligned} u_{33}(x,t) = e^{i\theta}(&-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b \\ &+ 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c} - \frac{\sqrt{-Q}\cot_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)), \end{aligned} \tag{53}$$

$$\begin{aligned} &u_{34}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\ &\left(-\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\sec_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)), \end{aligned} \tag{54}$$

$$\begin{aligned} &u_{35}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\ &\left(-\frac{b}{2c} - \frac{\sqrt{-Q}\left(\cot_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\csc_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)), \end{aligned} \tag{55}$$

and

$$u_{36}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}} \times \left(-\frac{b}{2c} + \frac{\sqrt{-Q}\big(\tan_\mu\big(1/4\,\sqrt{-Q}(\zeta)\big) - \cot_\mu\big(1/4\,\sqrt{-Q}(\zeta)\big)\big)}{4c}\right)). \tag{56}$$

Family 9. Equations (17) and (10), and the related general solutions obtained from Equation (9) are all applied in the case when Q is higher than zero and a, b, and c are all non-zero, leading to a specific family of optical soliton solutions, which may be stated as follows:

$$u_{37}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c} - \frac{\sqrt{Q}\tanh_\mu\big(1/2\,\sqrt{Q}(\zeta)\big)}{2c}\right)), \tag{57}$$

$$u_{38}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\left(-\frac{b}{2c} - \frac{\sqrt{Q}\coth_\mu\big(1/2\,\sqrt{Q}(\zeta)\big)}{2c}\right)), \tag{58}$$

$$u_{39}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}} \times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\Big(\tanh_\mu\Big(\sqrt{Z}(\zeta)\Big) \pm \big(\sqrt{pq}sech_\mu\big(\sqrt{Q}(\zeta)\big)\big)\Big)}{2c}\right)), \tag{59}$$

$$u_{40}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}} \times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\big(\coth_\mu\big(\sqrt{Q}(\zeta)\big) \pm \big(\sqrt{pq}csch_\mu\big(\sqrt{Q}(\zeta)\big)\big)\big)}{2c}\right)), \tag{60}$$

and

$$u_{41}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}b + 2\,c\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}} \times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\big(\tanh_\mu\big(1/4\,\sqrt{Q}(\zeta)\big) - \coth_\mu\big(1/4\,\sqrt{Q}(\zeta)\big)\big)}{4c}\right)), \tag{61}$$

Family 10. Equations (17) and (10), and the related general solutions obtained from Equation (9) are used to provide a particular family of optical soliton solutions where the product ac is larger than zero and b is equal to zero. These solutions are represented as follows:

$$u_{42}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\tan_\mu\big(\sqrt{ac}(\zeta)\big)), \tag{62}$$

$$u_{43}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\cot_\mu(\sqrt{ac}(\zeta))), \tag{63}$$

$$u_{44}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}(\tan_\mu(2\sqrt{ac}(\zeta)) \pm (\sqrt{pq}\sec_\mu(2\sqrt{ac}(\zeta))))), \tag{64}$$

$$u_{45}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}(\cot_\mu(2\sqrt{ac}(\zeta)) \pm (\sqrt{pq}\csc_\mu(2\sqrt{ac}(\zeta))))), \tag{65}$$

and

$$u_{46}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}(\tan_\mu(1/2\sqrt{ac}(\zeta)) - \cot_\mu(1/2\sqrt{ac}(\zeta)))). \tag{66}$$

Family 11. Equations (17) and (10), and the related general solutions obtained from Equation (9) are used to provide a particular family of optical soliton solutions where the product ac is less than zero and b is equal to zero. These solutions are represented as follows:

$$u_{47}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\tanh_\mu\left(\sqrt{-ac}(\zeta)\right)), \tag{67}$$

$$u_{48}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\coth_\mu\left(\sqrt{-ac}(\zeta)\right)), \tag{68}$$

$$u_{49}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(i\sqrt{pq}sech_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)), \tag{69}$$

$$u_{50}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\left(\coth_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(\sqrt{pq}csch_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)), \tag{70}$$

and

$$u_{51}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(1/2\sqrt{-ac}(\zeta)\right) + \coth_\mu\left(1/2\sqrt{-ac}(\zeta)\right)\right)). \tag{71}$$

Family 12. The use of Equations (17) and (10), and the related general solutions obtained from Equation (9) results in a different family of optical soliton solutions in the situation when c is equal to a and b is equal to zero. These solutions are represented as follows:

$$u_{52}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\tan_\mu(a(\zeta))), \tag{72}$$

$$u_{53}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\cot_\mu(a(\zeta))), \tag{73}$$

$$u_{54}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(\tan_\mu(2\,a(\zeta)) \pm \big(\sqrt{pq}\sec_\mu(2\,a(\zeta))\big)\big)), \tag{74}$$

$$u_{55}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(-\cot_\mu(2\,a(\zeta)) \mp \big(\sqrt{pq}\csc_\mu(2\,a(\zeta))\big)\big)), \tag{75}$$

and

$$u_{56}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(1/2\,\tan_\mu(1/2\,a(\zeta)) - 1/2\,\cot_\mu(1/2\,a(\zeta))\big)). \tag{76}$$

Family 13. The use of Equations (17) and (10), and the related general solutions obtained from Equation (9) results in a different family of optical soliton solutions in the situation when c is equal to $-a$ and b is equal to zero. These solutions are represented as follows:

$$u_{57}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\tanh_\mu(a(\zeta))), \tag{77}$$

$$u_{58}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\coth_\mu(a(\zeta))), \tag{78}$$

$$u_{59}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\big(-\tanh_\mu(2\,a(\zeta)) \mp \big(i\sqrt{pq}sech_\mu(2\,a(\zeta))\big)\big)), \tag{79}$$

$$u_{60}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\big(-\coth_\mu(2\,a(\zeta)) \mp \big(\sqrt{pq}csch_\mu(2\,a(\zeta))\big)\big)), \tag{80}$$

and

$$u_{61}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\big(-1/2\,\tanh_\mu(1/2\,a(\zeta)) - 1/2\,\coth_\mu(1/2\,a(\zeta))\big)). \tag{81}$$

Family 14. Equations (17) and (10), and the equivalent general solutions obtained from Equation (9) when a is equal to zero, b is not equal to zero, and c is not equal to zero, produce a particular family of optical soliton solutions, which may be written as follows:

$$u_{62}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}} - 2\sqrt{\frac{m_2}{m_3}}p\big(\cosh_\mu(b(\zeta)) - \sinh_\mu(b(\zeta)) + p\big)^{-1}), \tag{82}$$

and

$$u_{63}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}} - 2\sqrt{\frac{m_2}{m_3}}\frac{\cosh_\mu(b(\zeta)) + \sinh_\mu(b(\zeta))}{\cosh_\mu(b(\zeta)) + \sinh_\mu(b(\zeta)) + q}). \tag{83}$$

Family 15. Equations (17) and (10) & the associated general solutions derived from Equation (9) produce a particular set of optical soliton solutions in the case where b is equal to ν, c is equal to $n\nu$ (where n is a non-zero value), and a is equal to zero. These solutions are expressed as follows:

$$u_{64}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}} + 2\,n\sqrt{\frac{m_2}{m_3}}p\mu^{\nu\,(\zeta)}\left(p - nq\mu^{\nu\,(\zeta)}\right)^{-1}). \tag{84}$$

where $\zeta = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}}(\frac{x^\beta}{\Gamma(\beta+1)} - \frac{c_1 t^\alpha}{\Gamma(\alpha+1)})$, *and* $\theta = -\frac{kx^\beta}{\Gamma(\beta+1)} + \frac{\omega t^\alpha}{\Gamma(\alpha+1)} + \vartheta$.

3.2. Application of the r+mEDAM

Now we wish to address (11) using the r+mEDAM. Putting $m = 1$ in (8) gives the subsequent series-based solution for (11):

$$U(\zeta) = \sum_{m=-1}^{1} C_m(r + G(\zeta))^m = C_{-1}(r + G(\zeta))^{-1} + C_0 + C_1(r + G(\zeta))^1. \tag{85}$$

The coefficients C_{-1}, C_0, and C_1 are referred to as unknown parameters. A system of nonlinear algebraic equations is produced by putting Equation (85) into Equation (11). We use the Maple programme to address this problem, and it offers us the following two sets of solutions:

Case 1

$$\begin{aligned} &C_1 = 0, C_{-1} = 2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}, \\ &C_0 = \sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr), \lambda = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2 + 4\,ac)}} \end{aligned} \tag{86}$$

Case 2

$$\begin{aligned} &C_1 = 2\,c\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}, C_{-1} = 0, C_0 = -\sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr), \\ &\lambda = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2 + 4\,ac)}} \end{aligned} \tag{87}$$

In light of Case 1, we discover the families of optical soliton solutions shown below:

Family 16. Equations (85) and (10), and the related general solutions obtained from Equation (9) together produce a particular family of optical soliton solutions in the case when Q is less than zero and a, b, and c are all non-zero:

$$\begin{aligned} &u_{65}(x,t) = e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}\times \\ &\left(-\frac{b}{2c} + \frac{\sqrt{-Q}\tan_\mu(1/2\sqrt{-Q}(\zeta))}{2c}\right)^{-1} + \sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr)), \end{aligned} \tag{88}$$

$$\begin{aligned} &u_{66}(x,t) = e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}\times \\ &\left(-\frac{b}{2c} - \frac{\sqrt{-Q}\cot_\mu(1/2\sqrt{-Q}(\zeta))}{2c}\right)^{-1} + \sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr)), \end{aligned} \tag{89}$$

$$
\begin{aligned}
u_{67}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\sec_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} \\
&+ \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)),
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
u_{68}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} - \frac{\sqrt{-Q}\left(\cot_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\csc_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} \\
&+ \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)),
\end{aligned} \tag{91}
$$

and

$$
\begin{aligned}
u_{69}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(1/4\sqrt{-Q}(\zeta)\right) - \cot_\mu\left(1/4\sqrt{-Q}(\zeta)\right)\right)}{4c}\right)^{-1} \\
&+ \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)).
\end{aligned} \tag{92}
$$

Family 17. Equations (85) and (10), and the related general solutions obtained from Equation (9) together produce a particular family of optical soliton solutions in the case when Q is greater than zero and a, b, and c are all non-zero:

$$
\begin{aligned}
u_{70}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} - \frac{\sqrt{Q}\tanh_\mu\left(1/2\sqrt{Q}(\zeta)\right)}{2c}\right)^{-1} + \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)),
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
u_{71}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} - \frac{\sqrt{Q}\coth_\mu\left(1/2\sqrt{Q}(\zeta)\right)}{2c}\right)^{-1} + \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)),
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
u_{72}(x,t) &= e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2-4\,ac)}}\times \\
&\left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(\sqrt{Z}(\zeta)\right) \pm \left(\sqrt{pq}sech_\mu\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} \\
&+ \sqrt{-\frac{m_2}{m_3(-b^2+4\,ac)}}(b - 2\,cr)),
\end{aligned} \tag{95}
$$

$$u_{73}(x,t) = e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}\times$$
$$\left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\coth_\mu\left(\sqrt{Q}(\zeta)\right) \pm \left(\sqrt{pq}csch_\mu\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)^{-1} \quad (96)$$
$$+\sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr)),$$

and

$$u_{74}(x,t) = e^{i\theta}(2\left(a - rb + r^2c\right)\sqrt{\frac{m_2}{m_3(b^2 - 4\,ac)}}\times$$
$$\left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(1/4\,\sqrt{Q}(\zeta)\right) - \coth_\mu\left(1/4\,\sqrt{Q}(\zeta)\right)\right)}{4c}\right)^{-1} \quad (97)$$
$$+\sqrt{-\frac{m_2}{m_3(-b^2 + 4\,ac)}}(b - 2\,cr)).$$

Family 18. Equations (85) and (10), and the related general solutions obtained from Equation (9) result in a specific family of optical soliton solutions where the product ac is larger than zero and b is equal to zero. These solutions are represented as follows:

$$u_{75}(x,t) = e^{i\theta}((1 + \frac{cr^2}{a})\sqrt{\frac{-m_2}{m_3}}\left(\tan_\mu\left(\sqrt{ac}(\zeta)\right)\right)^{-1} - \sqrt{-\frac{m_2c}{m_3a}}r), \quad (98)$$

$$u_{76}(x,t) = e^{i\theta}(-(1 + \frac{cr^2}{a})\sqrt{\frac{-m_2}{m_3}}\left(\cot_\mu\left(\sqrt{ac}(\zeta)\right)\right)^{-1} - \sqrt{-\frac{m_2c}{m_3a}}r), \quad (99)$$

$$u_{77}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2c}{m_3a}}r$$
$$(1 + \frac{cr^2}{a})\sqrt{\frac{-m_2}{m_3}}\left(\tan_\mu\left(2\,\sqrt{ac}(\zeta)\right) \pm \left(\sqrt{pq}\,\sec_\mu\left(2\,\sqrt{ac}(\zeta)\right)\right)\right)^{-1}), \quad (100)$$

$$u_{78}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2c}{m_3a}}r$$
$$-(1 + \frac{cr^2}{a})\sqrt{\frac{-m_2}{m_3}}\left(\cot_\mu\left(2\,\sqrt{ac}(\zeta)\right) \pm \left(\sqrt{pq}\,\csc_\mu\left(2\,\sqrt{ac}(\zeta)\right)\right)\right)^{-1}), \quad (101)$$

and

$$u_{79}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2c}{m_3a}}r$$
$$2(1 + \frac{cr^2}{a})\sqrt{\frac{-m_2}{m_3}}\left(\tan_\mu\left(1/2\,\sqrt{ac}(\zeta)\right) - \cot_\mu\left(1/2\,\sqrt{ac}(\zeta)\right)\right)^{-1}). \quad (102)$$

Family 19. Equations (85) and (10), and the related general solutions obtained from Equation (9) result in a specific family of optical soliton solutions where ac is less than zero and b is equal to zero. These solutions are represented as follows:

$$u_{80}(x,t) = e^{i\theta}(-(1+\frac{cr^2}{a})\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(\sqrt{-ac}(\zeta)\right)\right)^{-1} - \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{103}$$

$$u_{81}(x,t) = e^{i\theta}(-(1+\frac{cr^2}{a})\sqrt{\frac{m_2}{m_3}}\left(\coth_\mu\left(\sqrt{-ac}(\zeta)\right)\right)^{-1} - \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{104}$$

$$\begin{aligned} u_{82}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2 c}{m_3 a}}r \\ &- (1+\frac{cr^2}{a})\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(i\sqrt{pq}sech_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)^{-1}), \end{aligned} \tag{105}$$

$$\begin{aligned} u_{83}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2 c}{m_3 a}}r \\ &- (1+\frac{cr^2}{a})\sqrt{\frac{m_2}{m_3}}\left(\coth_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(\sqrt{pq}csch_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)^{-1}), \end{aligned} \tag{106}$$

and

$$\begin{aligned} u_{84}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2 c}{m_3 a}}r \\ &- 2(1+\frac{cr^2}{a})\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(1/2\sqrt{-ac}(\zeta)\right) + \coth_\mu\left(1/2\sqrt{-ac}(\zeta)\right)\right)^{-1}). \end{aligned} \tag{107}$$

Family 20. Equations (85) and (10), and the related general solutions obtained from Equation (9) are used to produce a unique family of optical soliton solutions in the situation when c is equal to a and b is equal to zero. These solutions are written as follows:

$$u_{85}(x,t) = e^{i\theta}((1+r^2)\sqrt{\frac{-m_2}{m_3}}(\tan_\mu(a(\zeta)))^{-1} - \sqrt{-\frac{m_2}{m_3}}r), \tag{108}$$

$$u_{86}(x,t) = e^{i\theta}(-(1+r^2)\sqrt{\frac{-m_2}{m_3}}(\cot_\mu(a(\zeta)))^{-1} - \sqrt{-\frac{m_2}{m_3}}r), \tag{109}$$

$$\begin{aligned} u_{87}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2}{m_3}}r \\ &(1+r^2)\sqrt{\frac{-m_2}{m_3}}(\tan_\mu(2\,a(\zeta)) \pm (\sqrt{pq}\sec_\mu(2\,a(\zeta))))^{-1}), \end{aligned} \tag{110}$$

$$\begin{aligned} u_{88}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2}{m_3}}r \\ &(1+r^2)\sqrt{\frac{-m_2}{m_3}}(-\cot_\mu(2\,a(\zeta)) \mp (\sqrt{pq}\csc_\mu(2\,a(\zeta))))^{-1}), \end{aligned} \tag{111}$$

and

$$u_{89}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3}}r \\ (1+r^2)\sqrt{\frac{-m_2}{m_3}}\left(1/2\,\tan_\mu(1/2\,a(\zeta)) - 1/2\,\cot_\mu(1/2\,a(\zeta))\right)^{-1}). \tag{112}$$

Family 21. Equations (85) and (10), and the related general solutions obtained from Equation (9) are used to produce a unique family of optical soliton solutions in the situation when c is equal to $-a$ and b is equal to zero. These solutions are written as follows:

$$u_{90}(x,t) = e^{i\theta}(-(1-r^2)\sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu(a(\zeta))\right)^{-1} + \sqrt{\frac{m_2}{m_3}}r), \tag{113}$$

$$u_{91}(x,t) = e^{i\theta}(-(1-r^2)\sqrt{\frac{m_2}{m_3}}\left(\coth_\mu(a(\zeta))\right)^{-1} + \sqrt{\frac{m_2}{m_3}}r), \tag{114}$$

$$u_{92}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}r \\ +(1-r^2)\sqrt{\frac{m_2}{m_3}}\left(-\tanh_\mu(2\,a(\zeta)) \mp \left(i\sqrt{pq}sech_\mu(2\,a(\zeta))\right)\right)^{-1}), \tag{115}$$

$$u_{93}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}r \\ +(1-r^2)\sqrt{\frac{m_2}{m_3}}\left(-\coth_\mu(2\,a(\zeta)) \mp \left(\sqrt{pq}csch_\mu(2\,a(\zeta))\right)\right)^{-1}), \tag{116}$$

and

$$u_{94}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}r \\ +(1-r^2)\sqrt{\frac{m_2}{m_3}}\left(-1/2\,\tanh_\mu(1/2\,a(\zeta)) - 1/2\,\coth_\mu(1/2\,a(\zeta))\right)^{-1}). \tag{117}$$

Family 22. Equations (85) and (10), and the associated general solutions derived from Equation (9) produce a specific family of optical soliton solutions when b is equal to ν, a is equal to $n\nu$ (where n is a non-zero value), and c is equal to zero. These solutions are expressed as follows:

$$u_{95}(x,t) = e^{i\theta}(2\,(n-r)\sqrt{\frac{m_2}{m_3}}\left(\mu^{\nu\,(\zeta)} - n\right)^{-1} + \sqrt{\frac{m_2}{m_3}}). \tag{118}$$

Family 23. Equations (85) and (10), and the associated general solutions derived from Equation (9) are used to produce a specific family of optical soliton solutions in the case where a is equal to zero, b is not equal to zero, and c is not equal to zero. These solutions are expressed as follows:

$$\begin{aligned} u_{96}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2}{m_3 b^2}}(b-2\,cr) \\ &- 2\left(-rb+r^2c\right)\sqrt{\frac{m_2}{m_3}}\frac{c\left(\cosh_\mu(b(\zeta))-\sinh_\mu(b(\zeta))+p\right)}{pb^2}), \end{aligned} \tag{119}$$

and

$$\begin{aligned} u_{97}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2}{m_3 b^2}}(b-2\,cr) \\ &- 2\left(-rb+r^2c\right)\sqrt{\frac{m_2}{m_3}}\frac{c\left(\cosh_\mu(b(\zeta))+\sinh_\mu(b(\zeta))+q\right)}{b\left(\cosh_\mu(b(\zeta))+\sinh_\mu(b(\zeta))\right)}). \end{aligned} \tag{120}$$

Family 24. Equations (85) and (10), and the associated general solutions derived from Equation (9) produce a specific family of optical soliton solutions when b is equal to ν, c is equal to $n\nu$ (where n is a non-zero value), and a is equal to zero. These solutions are expressed as follows

$$u_{98}(x,t) = e^{i\theta}(2\left(-r+r^2n\right)\sqrt{\frac{m_2}{m_3}}\frac{\left(p-nq\mu^{\nu\,(\zeta)}\right)}{p\left(\mu^{\nu\,(\zeta)}\right)} + \sqrt{\frac{m_2}{m_3}}(1-2\,nr)). \tag{121}$$

where $\zeta = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}}(-\frac{c_1 t^\alpha}{\Gamma(\alpha+1)} + \frac{x^\beta}{\Gamma(\beta+1)})$, & $\theta = -\frac{kx^\beta}{\Gamma(\beta+1)} + \frac{\omega t^\alpha}{\Gamma(\alpha+1)} + \vartheta$.

Now, assuming Case 2, we obtain the subsequent families of optical soliiton solutions:

Family 25. Equations (85) and (10), and the related general solutions deriving from Equation (9) result in a specific family of optical soliton solutions in the situation when Q is less than zero and a, b, and c are all non-zero:

$$\begin{aligned} u_{99}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) \\ &+ 2\,c\sqrt{\frac{m_2}{m_3 Q}}\left(-\frac{b}{2c}+\frac{\sqrt{-Q}\tan_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)), \end{aligned} \tag{122}$$

$$\begin{aligned} u_{100}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) \\ &+ 2\,c\sqrt{\frac{m_2}{m_3 Q}}\left(-\frac{b}{2c}-\frac{\sqrt{-Q}\cot_\mu\left(1/2\sqrt{-Q}(\zeta)\right)}{2c}\right)), \end{aligned} \tag{123}$$

$$\begin{aligned} u_{101}(x,t) &= e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3 Q}}\times \\ &\left(-\frac{b}{2c}+\frac{\sqrt{-Q}\left(\tan_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\sec_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)), \end{aligned} \tag{124}$$

$$u_{102}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3 Q}}\times \left(-\frac{b}{2c} - \frac{\sqrt{-Q}\left(\cot_\mu\left(\sqrt{-Q}(\zeta)\right) \pm \left(\sqrt{pq}\csc_\mu\left(\sqrt{-Q}(\zeta)\right)\right)\right)}{2c}\right)), \tag{125}$$

and

$$u_{103}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\times \left(-\frac{b}{2c} + \frac{\sqrt{-Q}\left(\tan_\mu\left(1/4\sqrt{-Q}(\zeta)\right) - \cot_\mu\left(1/4\sqrt{-Q}(\zeta)\right)\right)}{4c}\right)). \tag{126}$$

Family 26. Equations (85) and (10), and the related general solutions deriving from Equation (9) result in a specific family of optical soliton solutions in the situation when Q is greater than zero and a, b, and c are all non-zero:

$$u_{104}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\left(-\frac{b}{2c} - \frac{\sqrt{Q}\tanh_\mu\left(1/2\sqrt{Q}(\zeta)\right)}{2c}\right)), \tag{127}$$

$$u_{105}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\left(-\frac{b}{2c} - \frac{\sqrt{Q}\coth_\mu\left(1/2\sqrt{Q}(\zeta)\right)}{2c}\right)), \tag{128}$$

$$u_{106}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(\sqrt{Z}(\zeta)\right) \pm \left(\sqrt{pq}sech_\mu\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)), \tag{129}$$

$$u_{107}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\coth_\mu\left(\sqrt{Q}(\zeta)\right) \pm \left(\sqrt{pq}csch_\mu\left(\sqrt{Q}(\zeta)\right)\right)\right)}{2c}\right)), \tag{130}$$

and

$$u_{108}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3(-Q)}}(b-2\,cr) + 2\,c\sqrt{\frac{m_2}{m_3(Q)}}\times \left(-\frac{b}{2c} - \frac{\sqrt{Q}\left(\tanh_\mu\left(1/4\sqrt{Q}(\zeta)\right) - \coth_\mu\left(1/4\sqrt{Q}(\zeta)\right)\right)}{4c}\right)). \tag{131}$$

Family 27. When the product ac is larger than 0 and b is equal to 0, the application of Equations (85) and (10), and the associated general solutions obtained from Equation (9) results in a different family of optical soliton solutions, which may be represented as follows:

$$u_{109}(x,t) = e^{i\theta}(\sqrt{-\frac{m_2}{m_3}}\tan_\mu(\sqrt{ac}(\zeta)) + \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{132}$$

$$u_{110}(x,t) = e^{i\theta}(-\sqrt{-\frac{m_2}{m_3}}\cot_\mu(\sqrt{ac}(\zeta)) + \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{133}$$

$$\begin{aligned} u_{111}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2}{m_3}}(\tan_\mu(2\sqrt{ac}(\zeta)) \pm (\sqrt{pq}\sec_\mu(2\sqrt{ac}(\zeta)))) \\ &+ \sqrt{-\frac{m_2 c}{m_3 a}}r), \end{aligned} \tag{134}$$

$$\begin{aligned} u_{112}(x,t) &= e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}(\cot_\mu(2\sqrt{ac}(\zeta)) \pm (\sqrt{pq}\csc_\mu(2\sqrt{ac}(\zeta)))) \\ &+ \sqrt{\frac{-m_2 c}{m_3 a}}r), \end{aligned} \tag{135}$$

and

$$\begin{aligned} u_{113}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2}{4m_3}}(\tan_\mu(1/2\sqrt{ac}(\zeta)) - \cot_\mu(1/2\sqrt{ac}(\zeta))) \\ &+ \sqrt{-\frac{m_2 c}{m_3 a}}r). \end{aligned} \tag{136}$$

Family 28. When the product ac is less than 0 and b is equal to 0, the application of Equations (85) and (10), and the associated general solutions obtained from Equation (9) results in a different family of optical soliton solutions, which may be represented as follows:

$$u_{114}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\tanh_\mu\left(\sqrt{-ac}(\zeta)\right) + \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{137}$$

$$u_{115}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}\coth_\mu\left(\sqrt{-ac}(\zeta)\right) + \sqrt{-\frac{m_2 c}{m_3 a}}r), \tag{138}$$

$$\begin{aligned} u_{116}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2 c}{m_3 a}}r \\ &- \sqrt{\frac{m_2}{m_3}}\left(\tanh_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(i\sqrt{pq}sech_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)), \end{aligned} \tag{139}$$

$$\begin{aligned} u_{117}(x,t) &= e^{i\theta}(\sqrt{-\frac{m_2 c}{m_3 a}}r \\ &- \sqrt{\frac{m_2}{m_3}}\left(\coth_\mu\left(2\sqrt{-ac}(\zeta)\right) \pm \left(\sqrt{pq}csch_\mu\left(2\sqrt{-ac}(\zeta)\right)\right)\right)), \end{aligned} \tag{140}$$

and

$$u_{118}(x,t) = e^{i\theta}(\sqrt{-\frac{m_2 c}{m_3 a}}r - \sqrt{\frac{m_2}{4m_3}}\Big(\tanh_\mu\Big(1/2\sqrt{-ac}(\zeta)\Big) + \coth_\mu\Big(1/2\sqrt{-ac}(\zeta)\Big)\Big)). \tag{141}$$

Family 29. Equations (85) and (10), and the related general solutions obtained from Equation (9) result in a specific family of optical soliton solutions in the case when c is equal to a and b is equal to zero. These solutions are written as follows:

$$u_{119}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\tan_\mu(a(\zeta)) + \sqrt{-\frac{m_2}{m_3}}r), \tag{142}$$

$$u_{120}(x,t) = e^{i\theta}(-\sqrt{\frac{-m_2}{m_3}}\cot_\mu(a(\zeta)) + \sqrt{-\frac{m_2}{m_3}}r), \tag{143}$$

$$u_{121}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}\big(\tan_\mu(2\,a(\zeta)) \pm \big(\sqrt{pq}\sec_\mu(2\,a(\zeta))\big)\big) + \sqrt{-\frac{m_2}{m_3}}r), \tag{144}$$

$$u_{122}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}(-\cot_\mu(2\,a(\zeta)) \mp \big(\sqrt{pq}\csc_\mu(2\,a(\zeta))\big)) + \sqrt{-\frac{m_2}{m_3}}r), \tag{145}$$

and

$$u_{123}(x,t) = e^{i\theta}(\sqrt{\frac{-m_2}{m_3}}(1/2\,\tan_\mu(1/2\,a(\zeta)) - 1/2\,\cot_\mu(1/2\,a(\zeta))) + \sqrt{-\frac{m_2}{m_3}}r). \tag{146}$$

Family 30. Equations (85) and (10), and the related general solutions obtained from Equation (9) result in a specific family of optical soliton solutions in the case when c is equal to $-a$ and b is equal to zero. These solutions are written as follows:

$$u_{124}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\tanh_\mu(a(\zeta)) + \sqrt{\frac{m_2}{m_3}}r), \tag{147}$$

$$u_{125}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\coth_\mu(a(\zeta)) + \sqrt{\frac{m_2}{m_3}}r), \tag{148}$$

$$u_{126}(x,t) = e^{i\theta}(\sqrt{\frac{m_2}{m_3}}\big(\tanh_\mu(2\,a(\zeta)) \pm \big(i\sqrt{pq}sech_\mu(2\,a(\zeta))\big)\big) + \sqrt{\frac{m_2}{m_3}}r), \tag{149}$$

$$\begin{aligned} u_{127}(x,t) &= e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}(-\coth_\mu(2\,a(\zeta)) \\ &\mp (\sqrt{pq}csch_\mu(2\,a(\zeta)))) + \sqrt{\frac{m_2}{m_3}}r), \end{aligned} \tag{150}$$

and

$$\begin{aligned} u_{128}(x,t) &= e^{i\theta}(\sqrt{\frac{m_2}{m_3}}(\frac{1}{2}\tanh_\mu\left(\frac{1}{2}a(\zeta)\right) \\ &+ \frac{1}{2}\coth_\mu\left(\frac{1}{2}a(\zeta)\right)) + \sqrt{\frac{m_2}{m_3}}r). \end{aligned} \tag{151}$$

Family 31. Equations (85) and (10), and the related general solutions obtained from Equation (9) are used to provide a particular family of optical soliton solutions where a is equal to zero, b is not equal to zero, and c is not equal to zero. These solutions are represented as follows:

$$u_{129}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3 b}}(b - 2\,cr) - 2\,\frac{\sqrt{\frac{m_2}{m_3}}p}{\cosh_\mu(b(\zeta)) - \sinh_\mu(b(\zeta)) + p}), \tag{152}$$

and

$$\begin{aligned} u_{130}(x,t) &= e^{i\theta}(-\sqrt{\frac{m_2}{m_3 b^2}}(b - 2\,cr) \\ &- 2\,\frac{\sqrt{\frac{m_2}{m_3}}(\cosh_\mu(b(\zeta)) + \sinh_\mu(b(\zeta)))}{\cosh_\mu(b(\zeta)) + \sinh_\mu(b(\zeta)) + q}). \end{aligned} \tag{153}$$

Family 32. Equations (85) and (10), and the associated general solutions derived from Equation (9) produce a specific family of optical soliton solutions when b is equal to ν, c is equal to $n\nu$ (where n is a non-zero value), and a is equal to zero. These solutions are expressed as follows:

$$u_{131}(x,t) = e^{i\theta}(-\sqrt{\frac{m_2}{m_3}}(1 - 2\,rn) + 2\,n\sqrt{\frac{m_2}{m_3}}\frac{p\mu^{\nu(\zeta)}}{p - nq\mu^{\nu(\zeta)}}). \tag{154}$$

where $\zeta = \frac{\sqrt{2}}{ln(\mu)}\sqrt{\frac{m_2}{m_1(-b^2+4\,ac)}}(\frac{x^\beta}{\Gamma(\beta+1)} - \frac{c_1 t^\alpha}{\Gamma(\alpha+1)}), \quad and \quad \theta = -\frac{kx^\beta}{\Gamma(\beta+1)} + \frac{\omega t^\alpha}{\Gamma(\alpha+1)} + \vartheta.$

4. Discussion and Graphs

The present study used two improved versions of the EDAM approach, especially the mEDAM and r+mEDAM, to successfully build families of optical soliton solutions for the FPRKLM. These findings contribute to further development of the field related to the FPRKLM and enable a deeper understanding of complex waves in nonlinear optical systems. Our obtained results also determine the cogency of the mEDAM and r+mEDAM approaches in obtaining analytical solutions for the FPRKLM. Both techniques offer a systematic approach for solving complex FPDEs and provide explicit formulations for optical soliton solutions.

By assigning different values to the model's parameters, several figures have been plotted to show the wave behavior of the designed optical solution. These plots represent the relationship between wave amplitudes and spatial variables, showing the different profiles observed in the solution. The resulting wave profiles include periodic waves, kink waves, solitary waves, lump waves, and more. The presence of these different wave profiles in the optical soliton solution of the FPRKLM highlights the rich dynamics of the model. Each profile produces a different peculiar behavior of the system and provides valuable insight into the underlying physics. Periodic waves indicate the presence of oscillatory motion, kink waves indicate the presence of local disturbances or sudden changes in wave behavior, solitary waves represent self-supporting local structures, and lump waves indicate local concentrations of energy.

The relationship between these waveform profiles and the proposed model is attributed to the nonlinear terms present in the equations and the particular shape of the fractional perturbations. These features introduce nonlinearity and complexity into the system, leading to the emergence of various wave phenomena. The mEDAM and r+mEDAM techniques provide powerful tools to capture and understand these phenomena, allowing us to study the complex dynamics of the FPRKLM.

Remark 1. *Figure 1 indicates a captivating M-shaped periodic wave structure in the optical soliton solution for the FPRKLM. This wave pattern is governed by the nonlinear behaviour of the system and the fractional perturbations it involves. The parameters in the model, such as the GVD coefficient (a_1), nonlinearity coefficient (b_1), IMD δ, μ_1, σ, and γ, considerably impact the wave profile. GVD plays a role in the formation of distinctive peaks and troughs in the M-shaped pattern, whereas the non-linearity coefficient determines soliton intensity and stability. The relationship between fractional perturbations and σ sets forth complexities and modulations, further shaping the M-shaped wave. Furthermore, taking into account the wave velocities (k and ω) permits for an analysis of soliton spreading characteristics, figuring out the speed and phase that influence the M-shaped periodic wave.*

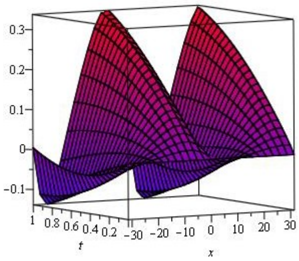

Figure 1. A three-dimensional graph of the function u_4 that appears in Equation (24) for $a = 2, b = 1, c = 2, \mu = e, k = 0, \omega = 1, \vartheta = 0, a_1 = 3, b_1 = 3, c_1 = 0, \gamma = 2, \delta = 2, p = 3, q = 2, \alpha = 0.9, \beta = 1$.

Remark 2. *Figure 2 shows an asymmetric kink wave that was seen in the FPRKLM's optical soliton solutions. These kink waves, which are distinguished by their unique characteristics, are caused by the existence of nonlinearity inside the model. The soliton solution generally experiences quick transitions between stable states at these locations, causing abrupt changes or discontinuities in the wave pattern. The development and behaviour of these asymmetric kink waves are significantly influenced by the precise parameters regulating the FPRKLM, such as the nonlinearity coefficient (b_1). Understanding the underlying processes and how they interact with the nonlinear dynamics of the model helps us better understand how such kink wave occurrences in optical solitons develop.*

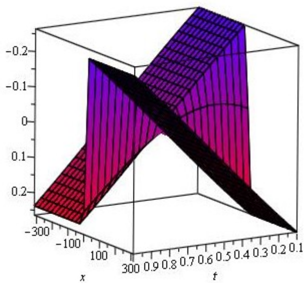

Figure 2. A three-dimensional graph of the function u_{52} that appears in Equation (72) for $a = 3, b = 0, c = 3, \mu = e, k = 0, \omega = -1, \vartheta = 10, a_1 = 3, b_1 = 3, c_1 = -4, \gamma = 2, \delta = 2, p = 3, q = 2, \alpha = \beta = 1$.

Remark 3. *A rogue wave seen in the optical soliton solutions of the FPRKLM is shown graphically in Figure 3. The model's innate nonlinearity and dispersiveness can be used to explain the appearance of rogue waves in the data. These waves, which have amplitudes that are noticeably bigger than those of their neighbours, result from the constructive interference of smaller waves that are modulated and interacted with by nonlinear processes and dispersion effects. Rogue waves display variable amplitudes during propagation as a result of the complex interaction between dispersion* (a_1)*, nonlinear effects* (b_1)*, and the underlying dynamics of the FPRKLM system. Understanding the processes that cause rogue waves to form and behave in the FPRKLM might help one better understand the intricate wave phenomena brought on by nonlinear interactions and dispersion in optical solitons.*

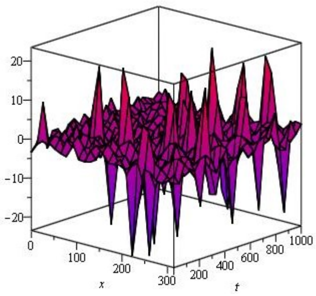

Figure 3. A three-dimensional graph of the function u_{72} that appears in Equation (95) for $a = 3, b = 10, c = 3, \mu = e, k = 1, \omega = 1, \vartheta = 0, r = 6, a_1 = 3, b_1 = 3, c_1 = 0, \gamma = 2, \delta = 2, p = 3, q = 2, \alpha = 0.5, \beta = 0.9$.

Remark 4. *The profile in Figure 4 demonstrates another rogue wave that travels smoothly until it reaches the domain's limit before abruptly changing in amplitude. The occurrence of smooth rogue waves in the optical soliton solutions of the FPRKLM that undergo abrupt changes at certain domain borders may be explained by a combination of components, including critical points, bifurcations, and nonlinear interactions within the system. When the parameters of the FPRKLM system approach their critical values, a transition occurs that results in rapid changes in wave behaviour and the formation of rogue waves. The development of nonlinear interactions within the system may be aided by nonlinear effects and instabilities, which may ultimately lead to abrupt changes in the wave profile. Through analysis and numerical simulations, the specifics of these phenomena may be further investigated. The specifics of these phenomena rely on the system's characteristics and beginning circumstances.*

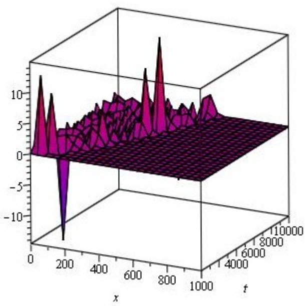

Figure 4. A three-dimensional graph of the function u_{131} that appears in Equation (154) for $a = 3, b = 10, c = 3, \mu = 2, k = 1, \omega = 1, \vartheta = 0, r = 6, a_1 = 3, b_1 = 3, c_1 = 0, \gamma = 2, \delta = 2, p = 3, q = 2, \alpha = 0.5, \beta = 0.9, \nu = i, n = 1/12$.

5. Conclusions

In the present investigation, we used the mEDAM and r+mEDAM methods to explore optical soliton solutions in the FPRKLM. Our study was concentrated on discovering the complex structure of the FPRKLM as well as comprehending the wave behavior of the system via exact analytical formulations obtained by means of these sophisticated methods. The obtained wave profiles, provided graphically, displayed a diverse range of behaviors, which include periodic waves, lump waves, kink waves, solitary waves, and more, demonstrating the complex nature of the FPRKLM. The research we conducted revealed the association between these wave profiles and the nonlinear terms and fractional perturbation of the model, showing the efficacy of the mEDAM and r+mEDAM methods in studying these kinds of phenomena. The novelty of our study is rooted in improving the comprehension of non linear optical systems by offering an outline for future studies and potential applications in this field.

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