fractal and fractional

Article

Fractional Factor Model for Data Transmission in Real-Time Monitoring Network

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Abstract: Modeling data transmission problems in graph theory is internalized to the existence of fractional flows, and thus can be surrogated to be characterized by a fractional factor in diversified settings. We study the fractional factor framework in the network environment when some sites are damaged. The setting we focus on refers to the lower and upper fractional degrees described by two functions on the vertex set. It is determined that G is fractional (g, f, n) critical if $\delta(G) \geq \lfloor \frac{a^2+b^2+2ab+2a+2b-3}{4a} \rfloor + n$ and $I(G) > \frac{n+\lfloor \frac{(a+b-1)^2}{2a} + \frac{2b-1}{a} \rfloor}{2}$, where $1 \leq a \leq b$ and $b \geq 2$.

Keywords: graph; isolated toughness; fractional (g, f)-factor; fractional (g, f, n)-critical graph

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Citation: Zhou, J.; Gao, W.; Zhang, H. Fractional Factor Model for Data Transmission in Real-Time Monitoring Network. *Fractal Fract.* **2023**, *7*, 493. https://doi.org/10.3390/fractalfract7070493

Academic Editor: António Lopes

Received: 26 May 2023
Revised: 18 June 2023
Accepted: 20 June 2023
Published: 22 June 2023

1. Introduction

The graphs we discuss in this article are undirected, finite, and simple. We denote $i(G)$ as the cardinality of the isolated vertex set in G. The operator $\vee$ in $H_1 \vee H_2$ means adding edges for all pairs of vertices between H_1 and H_2. For a subgraph H of graph G, $*_H(\cdot)$ denotes the graph function, which is restricted to H. Throughout the entire paper, $a, b, k \in \mathbb{N}$ with $1 \leq a \leq b$ and $b \geq 2$, and h: $E(G) \rightarrow [0, 1]$ is the fractional indicator function (FIF). The standard graph notations and terminologies can be referred to in [1].

1.1. Data Transmission in Real-Time Monitoring Network

In traditional graph theory, data transmission (DT) between vertices is accomplished through the shortest path. However, in communication networks, this naive idea is not feasible due to the conflict between large data size and the limit of channel capacity. In real data transmission networks, a large data packet is divided into several small data packets, transmitted via different channels, and finally assembled at the target site. The feasibility of DT is quantified by fractional flow, which is represented by the existence of the fractional factor (FF) in the idealized state. We say a graph G (corresponding to a specific network) admits a fractional k-factor (FkF) if FIF exists such that $d_G^h(x) = \sum_{x' \in N(x)} h(xx') = k$ for each $x \in V(G)$, where $d_G^h(x)$ is the fractional degree (FD) of vertex x. Obviously, the traditional k-factor is a specific setting of FkF if h becomes binary.

It is noteworthy that the constraint condition on the fractional degree is too strict for the communication network application, which inquires its value to be equal to k for each vertex. The fractional $[a, b]$ factor (FabF) is introduced to relax the original fractional k factor, which requires the fractional degree of each vertex to fall into the interval $[a, b]$, i.e., $a \leq d_G^h(x) \leq b$. FkF is apparently a special version of FabF when interval $[a, b]$ collapses into a real number k.

However, FabF is not applicable to real networks since it treats all sites equally. In a real network, each site has its own computability and throughput based on its status, which leads to the fractional degree of each site not being the same. To cope with this problem, fractional (g, f) factors are utilized. Let g and f be two integer-valued functions on the vertex set satisfying $0 \leq g(x) \leq f(x)$ for arbitrary vertex x. A *fractional (g, f)-factor* (FgfF) is a spanning subgraph consisting of edge set $E_h = \{e \in E(G)|h(e) > 0\}$ such that

$$g(x) \leq d_G^h(x) = \sum_{x' \in N(x)} h(xx') \leq f(x) \tag{1}$$

for any $x \in V(G)$. A graph G admits a FgfF if FIF satisfies (1). Again, FabF is an extreme situation of FgfF if both g and f are constant functions.

Unfortunately, even FgfF cannot be directly applied to real networks. Imagine that many stations send data to external stations simultaneously. Due to the limitation of the channel capacity, parts of stations and channels will be congested. Or in the scenario of a network attack, parts of the sites under attack will be unable to be used. In this spirit, the entire network needs to have a monitoring ability to promptly label sites in congested, damaged, or maintenance status. In real-time data transmission, it is necessary to delete these specially annotated sites and query the availability of DT in the remaining subnetworks, that is, to determine the existence of FFs in the remaining subgraph. The network with real-time monitoring action is called the self-definition network (SDN), and its difference from traditional networks is similar to the difference between buses and private cars. In traditional networks, the transmission path of segmented data packets is determined by algorithm implementation in advance, which cannot be changed. In self-definition networks, in terms of the real-time monitoring of data, algorithms can temporarily change transmission paths to avoid congested or damaged sites. The foregoing circumstances can be characterized by fractional critical graphs. For $n \in \mathbb{N}$, G is a *fractional (g, f, n)-critical graph* (FgfnCG) if deleting any n vertices from G, and the resulting subgraph still admits a FgfF. Liu [2] determined that G is a FgfnCG if and only if

$$f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) \geq \max\{f(U) : U \subseteq S, |U| = n\} \tag{2}$$

for any disjoint subsets S and T of $V(G)$ with $|S| \geq n$.

The fractional (a, b, n)-critical graph (FabnCG) and fractional (k, n)-critical graph (FknCG) can be regarded as the special case of FgfnCG when $(g(x), f(x)) = (a, b)$ and $(g(x), f(x)) = (k, k)$ for all vertices in G, respectively.

1.2. Network Stability Based on Graph Parameter

As a salient graph-theoretic variable, *isolated toughness* (IT) is introduced by [3] which is formulated by

$$I(G) = \min\left\{\frac{|S|}{i(G-S)}\Big|S \subset V(G), i(G-S) > 1\right\},$$

and specifically $I(G) = +\infty$ for complete graphs since no S satisfies the constraint condition. In computer networks, isolated toughness is utilized to quantify the stability of the network. The greater the isolated toughness, the more robust the network, and the network is more vulnerable to intrusion, on the contrary. For the network corresponding to the non-complete graph, the target sites of the network attack are stated by S, which minimizes the ratio $|S|/i(G-S)$. From this perspective, analyzing the isolated toughness of the network is beneficial to identify the target sites for network attacks.

1.3. Motivation

It is well known that the more robust the network, the higher the construction costs. In the real network, it is not necessary to build a fully connected network topology, and actually, most networks present sparse structures. Network designers often struggle with the issue of finding a balance between budget and network performance, and it is a crux to simultaneously balance the stability and the feasibility of DT in networks. Thankfully, the theoretical results show isolated toughness pertinent to the fractional factor, and it becomes an intriguing topic both in graph theory and computer networks.

In the early years, Ma and Liu [4] determined the sharp IT bound for a graph G, admitting a FkF. Nevertheless, the study has stagnated for a long time since 2006. Gao and Wang [5] argued that G is a FgfnCG if $I(G) \geq \frac{b^2+bn-b+a}{a}$ and $\delta(G) \geq \frac{bn}{a} + \frac{(b+2)^2}{4a} + b - 1$ (i.e., $\delta(G) \geq \lceil \frac{bn}{a} + \frac{(b+2)^2}{4a} + b - 1 \rceil$), where a and b are the lower bound of g and upper bound of f, respectively. However, the counterexample only shows that the $I(G)$ bound is tight in a very special case (i.e., $g(x) = f(x)$, which equals to a constant for each vertex x). Recently, there has been a dramatic breakthrough in the isolation toughness bound of fractional critical graphs. Gao et al. [6] proposed the sharp IT bound for FknCGs. This conclusion was extended by Gao et al. [7], i.e., G is a FabnCG if $\delta(G) \geq a + n$ and $I(G) > a - 1 + \frac{n+1}{n_{a,b}}$, where $2 \leq a \leq b$ and $n_{a,b} \geq 2$ is an integer satisfying $(n_{a,b} - 1)a \leq b \leq n_{a,b}a - 1$.

Wei et al. [8] investigated the existence of FF from high-dimensional space, and the results for the corresponding $\mathcal{H}$-factors are characterized in surface forms. Dimitrov and Hosam [9] proposed the independent set neighborhood union bound for FF in a specific setting. Zhou [10] determined a generalized binding number bound for fractional ID-factor-critical graphs, where the deleted vertex subset is an independent set of the graph. More recent results on factor in graphs can be referred to in [11–13].

The optimal isolated toughness condition in the fractional critical setting has cardinal engineering significance. It tells network designers that in order to ensure that the remaining subnetwork data transmission is still feasible under a certain degree of network attack (where n sites are simultaneously destroyed), large isolated toughness parameter values are needed. At the same time, this threshold value also provides the lowest budget for building the corresponding network.

Since the fractional $[a, b]$-factor is only an extreme circumstance of FgfF when both g and f are constant functions on $V(G)$, it is natural to ask the following question (raised by Gao et al. [7]):

- What is the sharp IT bound for FgfnCGs?

In addition, to pursue the optimal topology structure parameter, the following intuitive problem on minimum degree is proposed:

- Can the minimum degree bound $\delta(G) \geq \frac{bn}{a} + \frac{(b+2)^2}{4a} + b - 1$ of Gao and Wang [5] be strengthened or not?

1.4. Main Result and Counterexamples

Motivated by the aforementioned questions, in this contribution, we determine the tight IT bound for FgfnCGs, which both improves the previous bound in [5] and answers the open question raised by [7].

Theorem 1. *Let G be a graph, and a, b, n be positive integers with $1 \leq a \leq b$ and $b \geq 2$. Let g and f be integer-valued functions on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for all $x \in V(G)$. If $\delta(G) \geq \lfloor \frac{a^2+b^2+2ab+2a+2b-3}{4a} \rfloor + n$ and $I(G) > \frac{n+\lfloor \frac{(a+b-1)^2}{2a} + \frac{2b-1}{a} \rfloor}{2}$, then G is a FgfnCG.*

Theorem 1 has explicitly guiding significance in network designing, where the tight IT bound provides the characteristics that a network topology needs to possess under specific network vulnerability conditions. Specifically, the network is not necessary to be fully connected. As long as the IT parameter meets the lower bound requirement, theoretically, data transmission services can be maintained when a specific number of sites are attacked.

For shorthand notation, we define

$$\Theta := a^2 + b^2 + 2ab + 2a + 2b - 3,$$

and
$$\Phi := (a+b-1)^2.$$

In the following four counterexamples (from G_1 to G_4), for given S and T, we assume $f(x) = g(x) = a$ for all $x \in S$ and $f(x) = g(x) = b$ for all $x \in T$, and these graphs are not FgfnCGs, which are assessed by (2).

It is identified that $\frac{bn}{a} + \frac{(b+2)^2}{4a} + b - 1 > \frac{\Theta}{4a} + n$, which reveals that the minimum degree condition improves the previous bound stated by Gao and Wang [5]. We now show that $\delta(G) \geq \lfloor\frac{\Theta}{4a}\rfloor + n$ is tight to ensure a meaningful $I(G)$ bound. If $a \not\equiv b(\mathrm{mod}2)$, then focusing on $G_1 = K_{\lfloor\frac{b^2-a^2+2b-3}{4a}\rfloor+n} \vee (K_{\frac{a+b+1}{2}} \cup K_t)$ with $\delta(G_1) = \lfloor\frac{a^2+b^2+2ab-2a+2b-3}{4a}\rfloor + n$, and $t \in \mathbb{N}$ is a large number. Clearly, the value of $I(G_1)$ can be taken to be arbitrarily large since t is a large number. Set $S = V(K_{\lfloor\frac{b^2-a^2+2b-3}{4a}\rfloor+n})$ and $T = V(K_{\frac{a+b+1}{2}})$, then

$$\begin{aligned}
& f(S) - g(T) + \sum_{x\in T} d_{G_1-S}(x) - \max\{f(U) : U \subseteq S, |U| = n\} \\
= \quad & a\lfloor\frac{b^2-a^2+2b-3}{4a}\rfloor - \frac{(a+b+1)b}{2} + \frac{a+b+1}{2}\frac{a+b-1}{2} \\
\leq \quad & a\frac{b^2-a^2+2b-3}{4a} - \frac{a+b+1}{2}\frac{b-a+1}{2} = -1,
\end{aligned}$$

which implies that G_1 is not a FgfnCG. If $a \equiv b(\mathrm{mod}2)$, then considering $G_2 = (K_{\frac{a+b}{2}} \cup K_t) \vee K_{\lfloor\frac{b^2-a^2+2a+2b-3}{4a}\rfloor+n}$ with $\delta(G_2) = \lfloor\frac{a^2+b^2+2ab-2a+2b-3}{4a}\rfloor + n$, and $t \in \mathbb{N}$ is a large number. Obviously, the value of $I(G_2)$ can be taken to be arbitrarily large since t is a large number. Set $S = V(K_{\lfloor\frac{b^2-a^2+2a+2b-3}{4a}\rfloor+n})$ and $T = V(K_{\frac{a+b}{2}})$, then

$$\begin{aligned}
& f(S) - g(T) + \sum_{x\in T} d_{G_2-S}(x) - \max\{f(U) : U \subseteq S, |U| = n\} \\
= \quad & a\lfloor\frac{b^2-a^2+2a+2b-3}{4a}\rfloor - \frac{(a+b)b}{2} + \frac{a+b}{2}\frac{a+b-2}{2} \\
\leq \quad & a\frac{b^2-a^2+2a+2b-3}{4a} - \frac{a+b}{2}\frac{b-a+2}{2} = -\frac{3}{4},
\end{aligned}$$

which reveals that G_2 is not a FgfnCG.

The sharpness of the IT bound in Theorem 1 is showcased by considering the following instances. If $b \not\equiv a(\mathrm{mod}2)$, then consider $G_3 = K_{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-1)} \vee (2K_{\frac{a+b+1}{2}})$, where $b \geq a \geq 1$ and $b \geq 2$ are integers. We directly obtain (for $\delta(G_3)$ part, it can be identified by checking $b = a+1$ and $b \geq a+2$, respectively)

$$\begin{aligned}
\delta(G_3) \quad &= \quad n + \lfloor\frac{\Phi}{2a} + \frac{2b-1}{a}\rfloor - (a+b-1) + \frac{a+b-1}{2} \\
&\geq \quad \lfloor\frac{\Theta}{4a}\rfloor + n
\end{aligned}$$

and

$$\begin{aligned}
I(G_3) \quad &= \quad \frac{n + \lfloor\frac{\Phi}{2a} + \frac{2b-1}{a}\rfloor - (a+b-1) + 2\frac{a+b-1}{2}}{2} \\
&= \quad \frac{n + \lfloor\frac{\Phi}{2a} + \frac{2b-1}{a}\rfloor}{2}.
\end{aligned}$$

Set $S = V(K_{n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-1)})$ and $T = V(2K_{\frac{a+b+1}{2}})$, then

$$
\begin{aligned}
& f(S)-g(T)+\sum_{x\in T} d_{G_3-S}(x)-\max\{f(U): U\subseteq S, |U|=n\} \\
= \ & a\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor - a(a+b-1) - 2b\frac{a+b+1}{2} + 2\frac{a+b+1}{2}(\frac{a+b+1}{2}-1) \\
\leq \ & a(\frac{\Phi}{2a}+\frac{2b-1}{a}) - a(a+b-1) - \frac{(a+b+1)(b+1-a)}{2} \\
= \ & -1.
\end{aligned}
$$

Thus, G_3 is not a FgfnCG.

If $b \equiv a(\text{mod}2)$, then consider $G_4 = K_{n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-2)} \vee (2K_{\frac{a+b}{2}})$ where $b \geq a \geq 1$ and $b \geq 2$ are integers. We directly obtain

$$
\begin{aligned}
\delta(G_4) &= n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-2)+\frac{a+b-2}{2} \\
&\geq \lfloor \frac{\Theta}{4a}\rfloor + n
\end{aligned}
$$

and

$$
\begin{aligned}
I(G_4) &= \frac{n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-2)+2\frac{a+b-2}{2}}{2} \\
&= \frac{n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}.
\end{aligned}
$$

Set $S = V(K_{n+\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor-(a+b-2)})$ and $T = V(2K_{\frac{a+b}{2}})$, then

$$
\begin{aligned}
& f(S)-g(T)+\sum_{x\in T} d_{G_4-S}(x)-\max\{f(U): U\subseteq S, |U|=n\} \\
= \ & a\lfloor \frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor - a(a+b-2) - 2b\frac{a+b}{2} + 2\frac{a+b}{2}(\frac{a+b}{2}-1) \\
\leq \ & a(\frac{\Phi}{2a}+\frac{2b-1}{a}) - a(a+b-2) - (a+b)(\frac{b-a}{2}+1) \\
= \ & -\frac{1}{2}.
\end{aligned}
$$

Thus, G_4 is not a FgfnCG.

Remark 1. *By observation, it can be seen that the sharp IT bound for FgfnCGs cannot directly derive the tight IT condition for FabnCGs determined by Gao et al. [7]. We explain its intrinsic mechanism. In the setting of the FabnCG, the feasible interval of FD for each vertex is $[a,b]$. Nevertheless, from the above counterexample, in the extreme FgfnCG setting, $g(x)=f(x)=a$ for some vertices, while $g(x)=f(x)=b$ for other parts of vertices. It is well-known that FgfF degenerates to FabF iff $(g(x), f(x)) = (a,b)$ for all vertices. The extreme setting of the g and f functions mentioned in the above counterexample makes it impossible to bespeak the characteristic in a fractional $[a,b]$ factor setting.*

Remark 2. *The celebrated Lemma 2.2 in Liu and Zhang [14] has by now become an invaluable tool in the study of subgraphs in special divisions, and its slightly improved version is used in the next section.*

The detailed proof of Theorem 1 is presented in the subsequent section.

2. Proof of Theorem 1

We only consider that G is not complete because the result for the complete graph can be immediately obtained by $\delta(G) \geq \lfloor \frac{\Theta}{4a} \rfloor + n$. Assume that G satisfies the hypothesis of Theorem 1 but is not a FgfnCG. According to the sufficient and necessary condition stated in Liu [2], there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n$ satisfying

$$\begin{aligned} & a(|S|-n) - b|T| + \sum_{x\in T} d_{G-S}(x) \\ \leq \quad & f(S) - g(T) + \sum_{x\in T} d_{G-S}(x) - \max\{f(U) : U \subseteq S, |U| = n\} \leq -1. \end{aligned} \tag{3}$$

We choose S and T with the smallest value of $|T|$. Thus, $|T| \geq 1$, and $d_{G-S}(x) \leq b-1$ for any $x \in T$.

Let l be the number of the K_b components in $H' = G[T]$ and let T_0 be the set of vertices in H', whose degree is zero in $G - S$. Let H be the subgraph obtained from H' by deleting T_0 and K_b components. Let S' be a set of vertices that contains exactly $b-1$ vertices in each component of K_b in H'.

If $|V(H)| = 0$, then in view of (3), $\delta(G) \geq \lfloor \frac{\Theta}{4a} \rfloor + n \geq b + n + \lfloor \frac{b-1}{a} \rfloor$ (verify separately for three scenarios: $b = a$, $b = a+1$, and $b \geq a+2$), we yield

$$\begin{aligned} & n + 1 + \lfloor \frac{b-1}{a} \rfloor \\ = \quad & b + n + \lfloor \frac{b-1}{a} \rfloor - (b-1) \\ \leq \quad & \lfloor \frac{\Theta}{4a} \rfloor + n - (b-1) \\ \leq \quad & |S| \leq \frac{b(|T_0|+l)-1}{a} + n, \end{aligned}$$

i.e., $b = a + a(\frac{b-1}{a} - \frac{a-1}{a}) \leq a + a\lfloor \frac{b-1}{a} \rfloor \leq b(|T_0|+l) - 1$ which implies $i(G - S \cup S') = |T_0| + l \geq 2$. With the aid of the definition of IT, we deduce

$$\begin{aligned} I(G) \quad & \leq \quad \frac{|S \cup S'|}{i(G-S-S')} \leq \frac{\lfloor \frac{b(|T_0|+l)-1}{a} + n + l(b-1) \rfloor}{|T_0|+l} \\ & \leq \quad \frac{\lfloor (b-1+\frac{b}{a})(|T_0|+l) + n - \frac{1}{a} \rfloor}{|T_0|+l} \\ & = \quad b - 1 + \frac{n + \lfloor \frac{b(|T_0|+l)-1}{a} \rfloor}{|T_0|+l}. \end{aligned}$$

Let $b(|T_0|+l) - 1 = ma + c$, where $m \in \mathbb{N}$ and $c \in \{0, \cdots, a-1\}$. Then, by $\frac{1}{a} \leq \frac{c+1}{a} \leq 1$ and $n \geq 1$, we infer

$$\begin{aligned} & b - 1 + \max\{\frac{n + \lfloor \frac{b(|T_0|+l)-1}{a} \rfloor}{|T_0|+l}\} \\ = \quad & b - 1 + \max\{\frac{n + \frac{b(|T_0|+l)-1}{a} - \frac{c}{a}}{|T_0|+l}\} \\ = \quad & b - 1 + \frac{b}{a} + \max\{\frac{n - \frac{c+1}{a}}{|T_0|+l}\}, \end{aligned}$$

which implies that $b - 1 + \frac{n + \lfloor \frac{b(|T_0|+l)-1}{a} \rfloor}{|T_0|+l}$ reaches the maximum value when $|T_0| + l$ reaches its lower bound 2. Hence,

$$I(G) \leq b - 1 + \frac{n + \lfloor \frac{2b-1}{a} \rfloor}{2},$$

which contradicts $I(G) > \frac{n + \lfloor \frac{\Phi}{2a} + \frac{2b-1}{a} \rfloor}{2}$ and $b - 1 \leq \frac{\Phi}{4a}$.

Hence, we have $V(H) \neq \emptyset$. Let $H = H_1 \cup H_2$, where H_1 is the union of components of H, which satisfies that $d_{G-S}(v) = b-1$ for each vertex $v \in V(H_1)$ and $H_2 = H - H_1$. Assume that I_i and $C_i = V(H_i) - I_i$ are the maximum independent set and the covering set of H_i, where $i \in \{1,2\}$. Furthermore, denote $I^{(i)} = \{v \in I_1, d_{H_1}(v) = b-i\}$ for $1 \leq i \leq b$ and $\sum\limits_{i=1}^{b} |I^{(i)}| = |I_1|$. Using the definition of H and H_2, we verify that each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. Clearly, if $H_2 \neq \emptyset$, then $b \geq 3$, and I_2 can be selected by the algorithm in [6].

Set $W = V(G) - S - T$ and $U = S \cup S' \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$. The subsequent derivation is divided into two situations.

Case 1. $|T_0| + l \geq 1$**.**

First, we analyze the circumstances if $H_1 = \emptyset$ or $H_2 = \emptyset$.

Claim 1. *If* $|T_0| + l \geq 1$, *then* $|I_2| \neq 0$.

Proof. Suppose $|I_2| = 0$. Then $|I_1| \neq 0$ by $|V(H)| > 0$. Hence, $|T_0| + l + |I_1| \geq 2$.

We partition I_1 into two subsets.

I_{11}: $v \in I_{11}$ if there exists $v' \in I_1 \setminus \{v\}$ such that $N_{G-S}(v) \cap N_{G-S}(v') \neq \emptyset$;

I_{12}: $v \in I_{12}$ if there is no intersection between $N_{G-S}(v)$ and $N_{G-S}(I_1 \setminus \{v\})$.

In light of Lemma 2.2 in Liu and Zhang [14], we yield

$$a(|S| - n) \leq |V(H_1)| + b|T_0| + lb - 1 \leq (b - \frac{1}{2})|I_{11}| + (b-1)|I_{12}| + b(|T_0| + l) - 1$$

and

$$|S| \leq (\frac{b}{a} - \frac{1}{2a})|I_{11}| + \frac{b-1}{a}|I_{12}| - \frac{1}{a} + \frac{b(|T_0| + l)}{a} + n.$$

Moreover,

$$\begin{aligned}
& |S \cup S' \cup N_{G-S}(I_1)| \\
\leq\ & (\frac{b}{a} - \frac{1}{2a})|I_{11}| + \frac{b-1}{a}|I_{12}| - \frac{1}{a} + \frac{b(|T_0| + l)}{a} + n \\
& + l(b-1) + (b - 1 - \frac{1}{2})|I_{11}| + (b-1)|I_{12}| \\
=\ & (b - 1 + \frac{b}{a} - \frac{a+1}{2a})|I_{11}| + (b - 1 + \frac{b-1}{a})|I_{12}| + \frac{b}{a}|T_0| \\
& + (b - 1 + \frac{b}{a})l + n - \frac{1}{a} \\
\leq\ & (b - 1 + \frac{b}{a})(|T_0| + l) + (b - 1 + \frac{b}{a} - \frac{1}{a})|I_1| + n - \frac{1}{a} \\
\leq\ & (b - 1 + \frac{b}{a})(|I_1| + |T_0| + l) + n - \frac{2}{a},
\end{aligned}$$

and thus

$$\begin{aligned}
I(G) &\leq \frac{|S \cup S' \cup N_{G-S}(I_1)|}{i(G - S \cup S' \cup N_{G-S}(I_1))} \\
&\leq \frac{\lfloor (b - 1 + \frac{b}{a})(|I_1| + |T_0| + l) + n - \frac{2}{a} \rfloor}{|I_1| + l + |T_0|} \\
&= b - 1 + \frac{\lfloor \frac{b(|I_1| + l + |T_0|) - 2}{a} \rfloor + n}{|I_1| + l + |T_0|}.
\end{aligned}$$

Let $b(|I_1| + l + |T_0|) - 2 = m_1 a + c_1$, where $m_1 \in \mathbb{N}$ and $c_1 \in \{0, \cdots, a-1\}$. Then, we obtain

$$\begin{aligned} & b-1+\max\{\frac{\lfloor\frac{b(|I_1|+l+|T_0|)-2}{a}\rfloor+n}{|I_1|+l+|T_0|}\} \\ = \;& b-1+\max\{\frac{\frac{b(|I_1|+l+|T_0|)-2-c_1}{a}+n}{|I_1|+l+|T_0|}\} \\ = \;& b-1+\frac{b}{a}+\max\{\frac{n-\frac{2+c_1}{a}}{|I_1|+l+|T_0|}\}. \end{aligned}$$

If $n=1$ and $c_1=a-1$, then $\frac{n-\frac{2+c_1}{a}}{|I_1|+l+|T_0|}$ is negative, thus $I(G)<b-1+\frac{b}{a}$, which contradicts $I(G)>\frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}\geq b-1+\frac{n+\lfloor\frac{2b-1}{a}\rfloor}{2}\geq b-1+\frac{1+(\frac{2b-1}{a}-\frac{a-1}{a})}{2}=b-1+\frac{b}{a}$. Hence, $\frac{n-\frac{2+c_1}{a}}{|I_1|+l+|T_0|}$ is a non-negative term and it reaches the maximum value when $|T_0|+l+|I_1|$ reaches its lower bound. Therefore,

$$I(G)\leq b-1+\frac{\lfloor\frac{2b-2}{a}\rfloor+n}{2},$$

which contradicts the hypothesis of $I(G)>\frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}$ and the truth that $b-1\leq\frac{\Phi}{4a}$. □

Remark 3. *It can be concluded that we always check that $I(G)$ reaches the maximum value when $|T_0|+l+|I_1|+|I_2|$ (some terms may become zero in a special setting) arrives at the lower bound. In the following arguments, the trick is the same as the aforementioned discussion, and we will skip the illustration of these details.*

Claim 2. *If $|T_0|+l\geq 1$, then $|I_1|\neq 0$.*

Proof. If $|I_1|=0$. We yield $|I_2|\neq 0$ by $V(H)\neq\emptyset$, and hence $b\geq 3$ and $|T_0|+l+|I_2|\geq 2$.

Let $v_1,v_2,\cdots,v_{|I_2|}$ be vertices in I_2 such that $d_{G-S}(v_1)\leq b-2$ and $d_{G-S}(v_1)\leq d_{G-S}(v_2)\leq\cdots\leq d_{G-S}(v_{|I_2|})$. Then $|V(T)|=|V(H_2)|+|T_0|+bl$,

$$\begin{aligned} a(|S|-n) &\leq b|T|-d_{G-S}(T)-1 \\ &\leq b|T_0|+bl+\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))-1, \end{aligned}$$

and

$$|S|\leq\frac{b(|T_0|+l)}{a}+\frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))}{a}+n-\frac{1}{a}.$$

We infer $i(G-U)\geq 2$ where $U=S\cup S'\cup N_{G-S}(I_2)$ and

$$\begin{aligned} |U| &\leq |S|+|S'|+|N_{G-S}(I_2)| \\ &\leq \frac{b(|T_0|+l)}{a}+\frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))}{a}+n-\frac{1}{a}+l(b-1)+\sum_{i=1}^{|I_2|}d_{G-S}(v_i) \\ &= \frac{b|T_0|}{a}+l(b-1+\frac{b}{a})+\sum_{i=1}^{|I_2|}(-\frac{d^2_{G-S}(v_i)}{a}+\frac{a+b-1}{a}d_{G-S}(v_i)+\frac{b}{a})+n-\frac{1}{a} \\ &\leq \frac{b|T_0|}{a}+l(b-1+\frac{b}{a})+|I_2|(-\frac{(\frac{a+b-1}{2})^2}{a}+\frac{a+b-1}{a}\frac{a+b-1}{2}+\frac{b}{a})+n-\frac{1}{a} \\ &= \frac{b|T_0|}{a}+l(b-1+\frac{b}{a})+(\frac{\Phi}{4a}+\frac{b}{a})|I_2|+n-\frac{1}{a} \\ &\leq (\frac{\Phi}{4a}+\frac{b}{a})(|T_0|+l+|I_2|)+n-\frac{1}{a}. \end{aligned}$$

According to the definition of IT, we verify

$$\begin{aligned} I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{\lfloor(\frac{\Phi}{4a}+\frac{b}{a})(|T_0|+l+|I_2|)+n-\frac{1}{a}\rfloor}{|I_2|+|T_0|+l} \\ &\leq \frac{\lfloor 2(\frac{\Phi}{4a}+\frac{b}{a})+n-\frac{1}{a}\rfloor}{2} \\ &= \frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}, \end{aligned}$$

a contradiction. □

From Claims 1 and 2, we check $|I_1|>0$, $|I_2|>0$ and $b\geq 3$.

Denote $v_1, v_2, \cdots, v_{|I_2|}$ as vertices in I_2. Then $|V(T)| = |V(H_1)| + |V(H_2)| + |T_0| + bl$,

$$a(|S|-n) \leq b(|T_0|+l) + (b-\frac{1}{2})|I_{11}| + (b-1)|I_{12}| + \sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i)) - 1,$$

and

$$\begin{aligned} |S| &\leq \frac{b}{a}(|T_0|+l) + (\frac{b}{a}-\frac{1}{2a})|I_{11}| + \frac{b-1}{a}|I_{12}| \\ &\quad + \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))}{a} + n - \frac{1}{a}. \end{aligned}$$

We acquire $i(G-U)\geq 3$, where $U = S\cup S'\cup N_{G-S}(I_1)\cup N_{G-S}(I_2)$,

$$\begin{aligned} |U| &\leq |S| + |S'| + |C_1| + |N_G(I_1)\cap W| + |N_{G-S}(I_2)| \\ &\leq \frac{b}{a}(|T_0|+l) + (\frac{b}{a}-\frac{1}{2a})|I_{11}| + \frac{b-1}{a}|I_{12}| + \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))}{a} \\ &\quad + n - \frac{1}{a} + l(b-1) + (b-\frac{1}{2}-1)|I_{11}| + (b-1)|I_{12}| + \sum_{i=1}^{|I_2|} d_{G-S}(v_i) \\ &\leq \frac{b}{a}|T_0| + l(b-1+\frac{b}{a}) + (b-1+\frac{b}{a}-\frac{1}{a})|I_1| + (\frac{\Phi}{4a}+\frac{b}{a})|I_2| + n - \frac{1}{a} \\ &\leq (|T_0|+l+|I_1|+|I_2|)(\frac{\Phi}{4a}+\frac{b}{a}) + n - \frac{1}{a}, \end{aligned}$$

and hence

$$\begin{aligned} I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{\lfloor(\frac{\Phi}{4a}+\frac{b}{a})(|T_0|+l+|I_1|+|I_2|)+n-\frac{1}{a}\rfloor}{|I_1|+|I_2|+|T_0|+l} \\ &\leq \frac{\lfloor 2(\frac{\Phi}{4a}+\frac{b}{a})+n-\frac{1}{a}\rfloor}{2} \\ &= \frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}, \end{aligned}$$

which contradicts $I(G) > \frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}$ and $n\geq 1$.

Case 2. $|T_0|+l=0$.

Similar to Case 1, we cope with the circumstances if one of H_1 and H_2 is empty.

Claim 3. *If* $|T_0|+l=0$, *then* $|I_2|\neq 0$.

Proof. Suppose $|I_2|=0$, then we obtain $|I_1|\neq 0$, $|V(T)|=|V(H_1)|$ and $a(|S|-n)\leq b|T|-d_{G-S}(T)-1=|T|-1$.

If $|I_1| = 1$, then $|T| \le b-1$, $|S| \le \frac{|T|+an-1}{a} \le n + \frac{b-2}{a}$, and

$$\begin{aligned} & \frac{a^2+b^2+2ab-2a+2b-2}{4a}+n \\ = & \frac{\Theta}{4a} - \frac{4a-1}{4a} + n \\ \le & \lfloor \frac{\Theta}{4a} \rfloor + n \\ \le & \delta(G) \le |S| + b - 1 \\ \le & n + \frac{b-2}{a} + b - 1, \end{aligned}$$

a contradiction (identified by three cases: $b = a$, $b = a+1$ and $b \ge a+2$). Hence, $|I_1| \ge 2$.

In this case, $i(G-U) \ge |I_1| \ge 2$, where $U = S \cup N_{G-S}(I_1)$, and using the deduction in Claim 1, we obtain

$$|S| \le (\frac{b}{a} - \frac{1}{2a})|I_{11}| + \frac{b-1}{a}|I_{12}| - \frac{1}{a} + n < \frac{b|I_1|}{a} + n,$$

$$|U| \le |S| + |C_1| + \sum_{i=1}^{k}(i-1)|I^{(i)}| \le (b-1+\frac{b}{a})|I_1| + n - \frac{3}{a}.$$

Hence,

$$I(G) \le \frac{|U|}{i(G-U)} \le \frac{\lfloor (b-1+\frac{b}{a})|I_1| + n - \frac{3}{a} \rfloor}{|I_1|} \le b-1+\frac{n+\lfloor \frac{2b-3}{a} \rfloor}{2},$$

which contradicts the hypothesis of isolated toughness and $b-1 \le \frac{\Phi}{4a}$. □

Claim 4. *If $|T_0| + l = 0$, then $|I_1| \ne 0$.*

Proof. Suppose $|I_1| = 0$, then $|I_2| \ne 0$ and hence $b \ge 3$.

If $|I_2| = 1$, then we set $d_{min} = \min\{d_{G-S}(v) | v \in H_2\}$ and $z \in V(H_2)$ such that $d_{G-S}(z) = d_{min}$, thus $d_{min} \in \{1, \cdots, b-2\}$. Hence, we deduce

$$a(|S|-n) \le b|T| - d_{G-S}(T) - 1 \le |T|(b-d_{min}) - 1,$$

$$|S| \le \frac{|T|(b-d_{min}) + an - 1}{a} \le \frac{(d_{min}+1)(b-d_{min})-1}{a} + n,$$

and

$$\begin{aligned} \delta(G) & \le d_{min} + |S| \le d_{min} + \frac{(d_{min}+1)(b-d_{min})-1}{a} + n \\ & = -\frac{d_{min}^2}{a} + \frac{(b+a-1)d_{min}}{a} + \frac{b-1}{a} + n \\ & \le -\frac{(\frac{a+b-1}{2})^2}{a} + \frac{(b+a-1)\frac{a+b-1}{2}}{a} + \frac{b-1}{a} + n \\ & = \frac{a^2+b^2+2ab-2a+2b-3}{4a} + n, \end{aligned}$$

which reveals

$$\delta(G) \le \lfloor \frac{a^2+b^2+2ab-2a+2b-3}{4a} \rfloor + n = \lfloor \frac{\Theta}{4a} \rfloor + n - 1.$$

A contradiction to the hypothesis of $\delta(G)$.

Hence, we obtain $|I_2| \geq 2$. Let $v_1, v_2, \cdots, v_{|I_2|}$ be vertices in I_2. We have $|V(T)| = |V(H_2)|$, and

$$|S| \leq \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(b-d_{G-S}(v_i))}{a} + n - \frac{1}{a}.$$

We infer $i(G-U) \geq |I_2| \geq 2$ where $U = S \cup N_{G-S}(I_2)$, and in view of the discussion in Claim 2, we yield

$$|U| \leq |S| + |N_{G-S}(I_2)| \leq (\frac{\Phi}{4a} + \frac{b}{a})|I_2| + n - \frac{1}{a}.$$

Using the same fashion as presented before, we obtain

$$\begin{aligned} I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{\lfloor(\frac{\Phi}{4a}+\frac{b}{a})|I_2| + n - \frac{1}{a}\rfloor}{|I_2|} \\ &\leq \frac{\lfloor 2(\frac{\Phi}{4a}+\frac{b}{a}) + n - \frac{1}{a}\rfloor}{2} \\ &= \frac{n + \lfloor\frac{\Phi}{2a} + \frac{2b-1}{a}\rfloor}{2}, \end{aligned}$$

which contradicts $I(G) > \frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}$. □

It is verified from Claims 3 and 4 that $|I_1| \geq 1$, $|I_2| \geq 1$ and $b \geq 3$. We verify that $i(G-U) \geq 2$, where $U = S \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$,

$$|U| \leq (\frac{\Phi}{4a} + \frac{b}{a})(|I_1| + |I_2|) + n - \frac{1}{a},$$

and

$$\begin{aligned} I(G) &\leq \frac{|U|}{i(G-U)} \leq \frac{\lfloor(\frac{\Phi}{4a}+\frac{b}{a})(|I_1|+|I_2|) + n - \frac{1}{a}\rfloor}{|I_1|+|I_2|} \\ &\leq \frac{\lfloor 2(\frac{\Phi}{4a}+\frac{b}{a}) + n - \frac{1}{a}\rfloor}{2} \\ &= \frac{n + \lfloor\frac{\Phi}{2a} + \frac{2b-1}{a}\rfloor}{2}, \end{aligned}$$

which contradicts $I(G) > \frac{n+\lfloor\frac{\Phi}{2a}+\frac{2b-1}{a}\rfloor}{2}$.

Therefore, contradictions are derived in all situations, and Theorem 1 follows. □

3. Future Works

The main contribution of our article provides an IT bound for a graph to be a FgfnCG and demonstrates its optimality in light of counterexamples. However, there is still a significant gap between the theoretical results and practical applications. The following intriguing issues are related to the network applications and can be considered future research topics:

1. Is there a linear (quadratic) complexity approximation algorithm to calculate the IT value of a given graph?
2. The existence proof in this article cannot elaborate on FF. However, in practical applications, assume the network graph G is known to have a FF; to pave the way for application, an effective algorithm needs to be designed to obtain the specific FF, i.e., input (sub)graph G and output the FIF h.

3. It is imperative to demystify the mechanism that determines the specific forms of functions g and f. For instance, assuming that we obtain the computing ability and throughput of each site in the network, how can we reasonably identify g and f?

Author Contributions: Writing—original draft preparation, J.Z.; investigation, W.G.; writing—review and editing, H.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work has been partially supported by National Science Foundation of China (No. 12161094).

Data Availability Statement: No data were generated during this study.

Acknowledgments: We thank the reviewers for their constructive comments in improving the quality of this paper.

Conflicts of Interest: The authors declare no competing interest.

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