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Article
Nonlinear Inverse Problems for Equations with Dzhrbashyan–Nersesyan Derivatives

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Abstract: The unique solvability in the sense of classical solutions for nonlinear inverse problems to differential equations, solved for the oldest Dzhrbashyan–Nersesyan fractional derivative, is studied. The linear part of the equation contains a bounded operator, a continuous nonlinear operator that depends on lower-order Dzhrbashyan–Nersesyan derivatives, and an unknown element. The inverse problem is given by an equation, special initial value conditions for lower Dzhrbashyan–Nersesyan derivatives, and an overdetermination condition, which is defined by a linear continuous operator. Applying the fixed-point method for contraction mapping a theorem on the existence of a local unique solution is proved under the condition of local Lipschitz continuity of the nonlinear mapping. Analogous nonlocal results were obtained for the case of the nonlocally Lipschitz continuous nonlinear operator in the equation. The obtained results for the problem in arbitrary Banach spaces were used for the research of nonlinear inverse problems with time-dependent unknown coefficients at lower-order Dzhrbashyan–Nersesyan time-fractional derivatives for integro-differential equations and for a linearized system of dynamics of fractional Kelvin–Voigt viscoelastic media.

Keywords: fractional differential equation; Dzhrbashyan–Nersesyan fractional derivative; inverse problem; identification problem; equation with unknown coefficients; initial boundary value problem

MSC: 34G20; 35R30; 35R11; 34K29

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1. Introduction

Fractional integro-differential calculus is actively used for problems of mathematical modeling of the dynamics of various processes and phenomena. Therefore, it attracts more and more interest both for the theory development [1–4] and in terms of use in applied problems (see, e.g., [5–8]). In recent years, active research has been conducted on various problems for equations with fractional derivatives, including direct and inverse problems, optimal control problems [9], controllability problems [10], and others. In these problems, researchers have considered Riemann–Liouville and Gerasimov–Caputo derivatives, which have already become a traditional object of research, as well as Hilfer derivatives [5], various φ-derivatives [11], new Caputo–Fabrizio [12] and Atangana–Baleanu [13] derivatives with integral kernels without singularity, stochastic differential equations [11], etc.

At the same time, inverse problems for differential equations [14–18], i.e., problems for differential equations with unknown parameters and additional overdetermination conditions, arise in physics, astronomy, geophysics, chemistry, and biology, when conducting research of processes, some parameters of which are not available for direct measurements. In recent years, linear inverse problems to equations with fractional derivatives have been researched in the works [19–25].

Consider the nonlinear inverse problem

$$D^{\sigma_n} z(t) = Az(t) + B(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \dots, D^{\sigma_{n-1}} z(t), u(t)), \quad t \in [t_0, T], \tag{1}$$

$$D^{\sigma_k}z(t_0) = z_k, \quad k = 0, 1, \ldots, n-1, \tag{2}$$

$$\Phi z(t) = \Psi(t), \quad t \in [t_0, T]. \tag{3}$$

Here, $D^{\sigma_k}z(t)$ are Dzhrbashyan–Nersesyan fractional derivatives [26], which correspond to $\alpha_k \in (0,1]$, $k = 0, 1, \ldots, n$ (see Section 2 for the exact definition), A is a linear-bounded operator on some Banach space $\mathcal{Z}$, Φ is a linear-bounded mapping from $\mathcal{Z}$ to another Banach space $\mathcal{U}$, and $B : [t_0, T] \times \mathcal{Z}^n \times \mathcal{U} \to \mathcal{Z}$, $z_k \in \mathcal{Z}$, $k = 0, 1, \ldots, n-1$, $\Psi \in C([t_0, T]; \mathcal{U})$ are given. The overdetermination condition has the form (3); unknown functions are $z(t)$ and $u(t)$. We investigate Problem (1)–(3) by applying the methods used in [15] to study unique solvability issues for the analogous nonlinear inverse problem to the first-order equation, which contains a nonlinear mapping $B = B(t, z(t), u(t))$. In [27], a similar approach has been applied to prove the existence of a unique local solution of an analogous nonlinear inverse problem to a differential equation in a Banach space with Gerasimov–Caputo derivatives and an unbounded sectorial operator A in the linear part of the equation.

The unique solvability of Problem (2) (so-called direct problem) for an inhomogenous linear equation independent of $u(t)$ (1) (when $B = f(t)$ is a continuous function) was studied in [28]. Results for the direct problem to the quasilinear equation were obtained in work [29]. Problem (2) to the inhomogeneous linear Equation (1) with a sectorial unbounded linear operator A was studied in [30]. There are other works on various differential equations with Dzhrbashyan–Nersesyan fractional derivatives [31–37].

The issues of the existence and uniqueness of a solution are very important, since they are the first to arise for any researcher when considering a new problem, applied or theoretical. These issues for nonlinear inverse problems are traditionally difficult to study. At the same time, such problems are often found in applied research. This work is aimed at the study of a new class of such problems. The method of the investigation of a nonlinear inverse problem in this work consists in its reduction to a system of nonlinear equations of the form $y_k = H_k(y_0, y_1, \ldots, y_n)$, $k = 0, 1, \ldots, n$, with the subsequent application of the Banach fixed point theorem to it. The possibility for such reduction is provided by special conditions for the form of the nonlinear operator B and its connection with the overdetermination operator Φ (Conditions (A) and (C) in Section 6). The possibility of fulfilling such conditions in applications is demonstrated on examples in Sections 5 and 6. There, when considering the initial boundary value problems for an equation or a system of partial differential equations, the fulfillment of these conditions is ensured by the fact that the overdetermination operator acts on spatial variables, and the nonlinear operator with respect to all unknown elements in the equation is linear with respect to unknown coefficients at the derivatives of the unknown function; these coefficients do not depend on spatial variables. The obtained results on the solvability of a class of nonlinear inverse problems, as can be seen from the examples in Sections 5 and 6, allow us to prove the existence of a unique solution for complicate inverse coefficient problems to partial differential equations or systems of such equations.

The second section of this work contains the definition of Dzhrbashyan–Nersesyan fractional derivatives and the theorem obtained in [28] on the unique solvability of initial value problem (2) for linear inhomogeneous equation (1). A theorem on the existence of a unique local solution to Problem (1)–(3) has been proved in Section 3. Section 4 contains an analogous theorem on a nonlocal solution to Problem (1)–(3). In both cases, the methods of contraction mappings were used. In Section 5, a problem for an integro-differential equation with unknown coefficients at lower-order Dzhrbashyan–Nersesyan time-fractional derivatives have been investigated by the application of the obtained abstract results. In Section 6, the problem for a linearized system of the dynamics of Kelvin–Voigt fractional viscoelastic media depending on time-unknown coefficients and with overdetermination conditions in the form of integrals over the considered spatial region have also been studied by the application of the abstract results .

2. Preliminaries

For $0 < \alpha_k \leq 1$, $k = 0, 1, \ldots, n \in \mathbb{N}$, define differential operators

$$D^{\sigma_0} z(t) = D_t^{\alpha_0 - 1} z(t), \quad D^{\sigma_k} z(t) = D_t^{\alpha_k - 1} D_t^{\alpha_{k-1}} D_t^{\alpha_{k-2}} \ldots D_t^{\alpha_0} z(t), \quad k = 1, 2, \ldots, n. \tag{4}$$

Here, $D_t^{\beta} := J_t^{-\beta}$ is the Riemann–Liouville fractional integral of an order $-\beta > 0$. If $\beta < 0$, D_t^0 is the identical operator, $D_t^{\beta} := D_t^m J_t^{m-\beta}$ is the Riemann–Liouville fractional derivative of an order $\beta \in (m-1, m]$, $m \in \mathbb{N}$. The Dzhrbashyan–Nersesyan fractional derivatives of [26] of the orders σ_k, $k = 0, 1, \ldots, n$, which correspond to the sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $0 < \alpha_k \leq 1$, $k = 0, 1, \ldots, n \in \mathbb{N}$, are determined by equalities (4). The Dzhrbashyan–Nersesian derivative is a natural general construction of a fractional integro-differential operator (see [26,31,32]); note that its partial cases are the Riemann–Liouville fractional derivative ($\alpha_0 \in (0,1)$, $\alpha_k = 1$, $k = 1, 2, \ldots, n$) and the Gerasimov–Caputo fractional derivative ($\alpha_k = 1$, $k = 0, 1, \ldots, n-1$, $\alpha_n \in (0,1)$). Denote also

$$\sigma_k = \sum_{j=0}^{k} \alpha_j - 1, \quad k = 0, 1, \ldots, n.$$

Consider a Banach space $\mathcal{Z}$. Denote by $\mathcal{L}(\mathcal{Z})$ the Banach algebra of all linear-bounded operators in $\mathcal{Z}$. For $\alpha, \beta > 0$, we will use the Mittag–Leffler function

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad z \in \mathcal{L}(\mathcal{Z}).$$

Consider an initial problem

$$D^{\sigma_k} z(t_0) = z_k, \quad k = 0, 1, \ldots, n-1, \tag{5}$$

to a linear inhomogeneous equation

$$D^{\sigma_n} z(t) = Az(t) + f(t). \tag{6}$$

A function $z \in C((t_0, T]; \mathcal{Z})$ is a solution to Problem (5) and (6) on a segment $[t_0, T]$. If $D^{\sigma_k} z \in C([t_0, T]; \mathcal{Z})$, $k = 0, 1, \ldots, n-1$, $D^{\sigma_n} z \in C((t_0, T]; \mathcal{Z})$, Condition (5) is fulfilled, and for all $t \in (t_0, T]$, Equality (6) holds.

Theorem 1 ([28]). *Let $A \in \mathcal{L}(\mathcal{Z})$, $z_k \in \mathcal{Z}$, $0 < \alpha_k \leq 1$, $k = 0, 1 \ldots, n$, $\alpha_0 + \alpha_n > 1$, $f \in C([t_0, T]; \mathcal{Z})$. Then, there exists a unique solution to Problem* (5) *and* (6)*. Moreover, the solution has the form*

$$z(t) = \sum_{k=0}^{n-1} (t - t_0)^{\sigma_k} E_{\sigma_n, \sigma_k + 1}((t - t_0)^{\sigma_n} A) z_k + \int_{t_0}^{t} (t - s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t - s)^{\sigma_n} A) f(s) ds.$$

3. Local Solvability of Nonlinear Inverse Problem

Let $\mathcal{Z}$ and $\mathcal{U}$ be Banach spaces, subset Z be open in $\mathbb{R} \times \mathcal{Z}^n$, and $B : Z \times \mathcal{U} \to \mathcal{Z}$ be a nonlinear mapping. Consider a nonlinear inverse problem

$$D^{\sigma_n} z(t) = Az(t) + B(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \ldots, D^{\sigma_{n-1}} z(t), u(t)), \tag{7}$$

$$D^{\sigma_k} z(t_0) = z_k, \quad k = 0, 1, \ldots, n-1, \tag{8}$$

$$\Phi z(t) = \Psi(t), \quad t \in [t_0, T]. \tag{9}$$

A solution to Problem (7)–(9) on segment $[t_0, T]$ is a pair $(z, u) \in C([t_0, T]; \mathcal{Z}) \times C([t_0, T]; \mathcal{U})$, such that for $k = 0, 1, \dots, n$ $D^{\sigma_k} z \in C([t_0, T]; \mathcal{Z})$, Equality (8) is valid for all $t \in [t_0, T]$ $(t, D^{\sigma_0} z(t), D^{\sigma_1} z(t), \dots, D^{\sigma_{n-1}} z(t)) \in Z$, and Equalities (7) and (9) hold.

Remark 1. *Here, we require the continuity of solution z at point $t = t_0$, in contrast to a general situation* [29,30]*. This is required by the specifics of the research method.*

Denote $\bar{y} = (y_0, y_1, \dots, y_{n-1})$. Furthermore, we will use the following conditions (A)–(F).

(A) The mapping $B : Z \times \mathcal{U} \to \mathcal{Z}$ has the form

$$B(t, \bar{y}, u) = B_1(t, \bar{y}) + B_2(t, \bar{y}, u), \quad (t, \bar{y}, u) \in Z \times \mathcal{U}.$$

Take $\bar{a} = (a_0, a_1, \dots, a_{n-1}) \in \mathcal{Z}^n$, for $R, T > 0$ denote

$$S_{\mathcal{Z}^n}(\bar{a}, R) = \{\bar{y} \in \mathcal{Z}^n : \ \|y_k - a_k\|_{\mathcal{Z}} < R, \, k = 0, 1, \dots, n-1\},$$

$$S_{\mathcal{Z}^n}(\bar{a}, R, T) = [t_0, T] \times S_{\mathcal{Z}^n}(\bar{a}, R).$$

For $(t_0, z_0, z_1, \dots, z_{n-1}) \in Z$ and a sufficiently smooth function Ψ, take

$$v_0 = D^{\sigma_n} \Psi(t_0) - \Phi A z_0 - \Phi B_1(t_0, z_0, z_1, \dots, z_{n-1}).$$

Moreover, the following conditions will be used:

(B) The equation $\Phi B_2(t_0, z_0, z_1, \dots, z_{n-1}, u) = v_0$ with respect to the variable u has a unique solution $u_0 \in \mathcal{U}$;

(C) The operator $B_3 : [t_0, T] \times \mathcal{U}^{n+1} \to \mathcal{U}$ satisfies the identity

$$\Phi B_2(t, \bar{y}, u) = B_3(t, \Phi y_0, \Phi y_1, \dots, \Phi y_{n-1}, u), \quad (t, \bar{y}, u) \in Z \times \mathcal{U};$$

(D) For some $R > 0$ and for all $t \in [t_0, T]$, the mapping

$$v = B_3(t, D^{\sigma_0} \Psi(t), D^{\sigma_1} \Psi(t), \dots, D^{\sigma_{n-1}} \Psi(t), u)$$

in the ball $S_{\mathcal{U}}(u_0, R)$ has an inverse mapping $u = F(t, v)$;

(E) The operator F is continuous in the totality of the variables (t, v) on the set $S_{\mathcal{U}}(u_0, R, T)$ for some $R > 0$ and Lipschitz continuous in v on this set;

(F) Both mappings $B_1(t, \bar{y})$ and $B_2(t, \bar{y}, u)$ are continuous in the totality of the variables on $S_{\mathcal{Z}^n \times \mathcal{U}}((z_0, z_1, \dots, z_{n-1}, u_0), R, T) \subset Z \times \mathcal{U}$ for some $R > 0$; moreover, they satisfy the Lipschitz condition in $(\bar{y}, u)$ on this set.

Theorem 2. *Let $A \in \mathcal{L}(\mathcal{Z})$, $n \in \mathbb{N}$, Z be an open set in $\mathbb{R} \times \mathcal{Z}^n$, $B : Z \times \mathcal{U} \to \mathcal{Z}$, $z_k \in \mathcal{Z}$, $k = 0, 1, \dots, n-1$, $(t_0, z_0, z_1, \dots, z_{n-1}) \in Z$, $\alpha_0 = 1$, $0 < \alpha_k \leq 1$, $k = 1, 2, \dots, n$, $\Phi \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$, $D^{\sigma_k} \Psi \in C([t_0, T]; \mathcal{U})$, $k = 0, 1, \dots, n$, $\Phi z_0 = \Psi(t_0)$, assumptions (A)–(F) hold. Then there exists $T_1 \in (t_0, T]$, such that nonlinear inverse problem* (7)–(9) *have a unique solution to $[t_0, T_1]$.*

Proof. First, note that due to Conditions $n \in \mathbb{N}$ and $\alpha_0 = 1$, we have $\sigma_n > 0$, $\alpha_0 + \alpha_n > 1$.

For $k, l = 0, 1, \dots, n-1$, in Lemma 1 [28], it was proved that

$$D^{\sigma_k} t^{\sigma_l} E_{\sigma_n, \sigma_l + 1}(t^{\sigma_n} A) = t^{\sigma_l - \sigma_k} E_{\sigma_n, \sigma_l - \sigma_k + 1}(t^{\sigma_n} A) \in C([0, T]; \mathcal{L}(\mathcal{Z})), \quad k \leq l,$$

$$D^{\sigma_k} t^{\sigma_k} E_{\sigma_n, \sigma_k + 1}(t^{\sigma_n} A) = A t^{\sigma_l - \sigma_k + \sigma_n} E_{\sigma_n, \sigma_l - \sigma_k + \sigma_n + 1}(t^{\sigma_n} A) \in C([0, T]; \mathcal{L}(\mathcal{Z})), \quad k > l.$$

The equality

$$D^{\sigma_k}\int_{t_0}^{t}(t-s)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-s)^{\sigma_n}A)f(s)ds=\int_{t_0}^{t}(t-s)^{\sigma_n-\sigma_k-1}E_{\sigma_n,\sigma_n-\sigma_k}((t-s)^{\sigma_n}A)f(s)ds$$

for $k=0,1,\dots,n$, $f\in C([t_0,T];\mathcal{Z})$ was obtained in the proof of Lemma 2 in [28]. Therefore, for $k=0,1,\dots,n-1$

$$D^{\sigma_k}z(t)=\sum_{l=0}^{k-1}(t-t_0)^{\sigma_l-\sigma_k+\sigma_n}E_{\sigma_n,\sigma_l-\sigma_k+\sigma_n+1}((t-t_0)^{\sigma_n}A)Az_l+$$

$$+\sum_{l=k}^{n-1}(t-t_0)^{\sigma_l-\sigma_k}E_{\sigma_n,\sigma_l-\sigma_k+1}((t-t_0)^{\sigma_n}A)z_l+$$

$$+\int_{t_0}^{t}(t-s)^{\sigma_n-\sigma_k-1}E_{\sigma_n,\sigma_n-\sigma_k}((t-s)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),\dots,D^{\sigma_{n-1}}z(s),u(s))ds. \tag{10}$$

If Problem (7)–(9) have a solution $(z,u)\in C([t_0,T];\mathcal{Z})\times C([t_0,T];\mathcal{U})$, then for $k=0,1,\dots,n-1$, the inclusions $D^{\sigma_k}z\in C([t_0,T];\mathcal{Z})$ are true, and for every $t\in[t_0,T]$, Equalities (9) and

$$z(t)=\sum_{k=0}^{n-1}(t-t_0)^{\sigma_k}E_{\sigma_n,\sigma_k+1}((t-t_0)^{\sigma_n}A)z_k+$$

$$\int_{t_0}^{t}(t-s)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-s)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),\dots,D^{\sigma_{n-1}}z(s),u(s))ds$$

are satisfied. Substitute the function $z(t)$ into (9) and obtain

$$\Phi\sum_{k=0}^{n-1}(t-t_0)^{\sigma_k}E_{\sigma_n,\sigma_k+1}((t-t_0)^{\sigma_n}A)z_k+$$

$$+\Phi\int_{t_0}^{t}(t-s)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-s)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(s))ds=\Psi(t).$$

Acting as the derivative D^{σ_n} on both parts of this equality and taking into account Theorem 1, we obtain

$$\Phi A\sum_{k=0}^{n-1}(t-t_0)^{\sigma_k}E_{\sigma_n,\sigma_k+1}((t-t_0)^{\sigma_n}A)z_k+$$

$$+\Phi A\int_{t_0}^{t}(t-s)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-s)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(s),u(s))ds+$$

$$+\Phi B(t,D^{\sigma_0}z(t),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(t),u(t))=D^{\sigma_n}\Psi(t).$$

Using Conditions (A) and (C), rewrite this equality as

$$\Phi B_2(t,D^{\sigma_0}z(t),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(t),u(t))=$$

$$=B_3(t,\Phi D^{\sigma_0}z(t),\Phi D^{\sigma_1}z(t),\dots,\Phi D^{\sigma_{n-1}}z(t),u(t))=$$

$$=B_3(t,D^{\sigma_0}\Psi(t),D^{\sigma_1}\Psi(t),\dots,D^{\sigma_{n-1}}\Psi(t),u(t))=D^{\sigma_n}\Psi(t)-$$

$$-\Phi A\sum_{k=0}^{n-1}(t-t_0)^{\sigma_k}E_{\sigma_n,\sigma_k+1}((t-t_0)^{\sigma_n}A)z_k-$$

$$-\Phi A\int_{t_0}^{t}(t-t_0)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-t_0)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),\dots,D^{\sigma_{n-1}}z(s),u(s))ds-$$

$$-\Phi B_1(t,D^{\sigma_0}z(t),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(t)). \tag{11}$$

Condition (D) and Equation (11) imply that

$$u(t)=F(t,v(t)) \tag{12}$$

with

$$v(t)=D^{\sigma_n}\Psi(t)-\Phi A\sum_{l=0}^{n-1}(t-t_0)^{\sigma_l}E_{\sigma_n,\sigma_l+1}((t-t_0)^{\sigma_n}A)z_l-$$

$$-\Phi A\int_{t_0}^{t}(t-t_0)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-t_0)^{\sigma_n}A)B(s,D^{\sigma_0}z(s),\dots,D^{\sigma_{n-1}}z(s),u(s))ds-$$

$$-\Phi B_1(t,D^{\sigma_0}z(t),D^{\sigma_1}z(t),\dots,D^{\sigma_{n-1}}z(t)). \tag{13}$$

Note that $v(t_0)=D^{\sigma_n}\Psi(t_0)-\Phi Az_0-\Phi B_1(t_0,z_0,z_1,\dots,z_{n-1})=v_0$.

Then, the system of equations for $n-1$ functions $y_0:=D^{\sigma_0}z$, $y_1:=D^{\sigma_1}z,\dots,y_{n-1}:=D^{\sigma_{n-1}}z$ and for the n-th function u consists of Equations (10) for $k=0,1,\dots,n-1$ and (12).

Consider the set

$$\mathfrak{M}_T=\{(\overline{y},u)\in C([t_0,T];\mathcal{Z}^n\times\mathcal{U}):\|y_k(t)-z_k\|_{\mathcal{Z}}\le R,k=0,1,\dots,n-1,$$
$$\|u(t)-u_0\|_{\mathcal{U}}\le R,\ t\in[t_0,T]\},$$

with the metrics $d((\overline{y},u),(\overline{x},v))=\|(\overline{y}-\overline{x},u-v))\|_{C([0,T];\mathcal{Z}^n\times\mathcal{U})}$. This metric space is evidently complete. Define the mapping H with components H_0, H_1, …, H_n: for $k=0,1,\dots,n-1$

$$H_k(y_0,y_1\dots,y_{n-1},u)=\sum_{l=0}^{k-1}(t-t_0)^{\sigma_l-\sigma_k+\sigma_n}E_{\sigma_n,\sigma_l-\sigma_k+\sigma_n+1}((t-t_0)^{\sigma_n}A)Az_l+$$

$$+\sum_{l=k}^{n-1}(t-t_0)^{\sigma_l-\sigma_k}E_{\sigma_n,\sigma_l-\sigma_k+1}((t-t_0)^{\sigma_n}A)z_l+$$

$$+\int_{t_0}^{t}(t-s)^{\sigma_n-\sigma_k-1}E_{\sigma_n,\sigma_n-\sigma_k}((t-s)^{\sigma_n}A)B(s,y_0(s),y_1(s),\dots,y_{n-1}(s),u(s))ds,$$

$$H_n(y_0,y_1,\dots,y_{n-1},u)=F\Bigg(t,D^{\sigma_n}\Psi(t)-\Phi A\sum_{l=0}^{n-1}(t-t_0)^{\sigma_l}E_{\sigma_n,\sigma_l+1}((t-t_0)^{\sigma_n}A)z_l-$$

$$-\Phi A\int_{t_0}^{t}(t-t_0)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-t_0)^{\sigma_n}A)B(s,y_0(s),y_1(s),\dots,y_{n-1}(s),u(s))ds-$$

$$-\Phi B_1(t,y_0(t),y_1(t),\dots,y_{n-1}(t))\Bigg).$$

Consequently, inverse problem (7)–(9) are equivalent to the system

$$\begin{gathered}
y_0(t) = H_0(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\
y_1(t) = H_1(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\
\ldots, \\
y_{n-1}(t) = H_{n-1}(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\
u(t) = H_n(H_0(\overline{y}(t), u(t)), H_1(\overline{y}(t), u(t)), \ldots, H_{n-1}(\overline{y}(t), u(t)), u(t)).
\end{gathered} \tag{14}$$

We have for $k = 0, 1, \ldots, n-1$ $H_k(y_0(t_0), y_1(t_0), \ldots, y_{n-1}(t_0), u(t_0)) = y_k(t_0) = z_k$, $H_n(y_0(t_0), y_1(t_0), \ldots, y_{n-1}(t_0), u(t_0)) = F(t_0, v(t_0)) = F(t_0, v_0) = u_0$. Due to the conditions of this theorem the vector function $H(y_0(t), \ldots, y_{n-1}(t), u(t))$ is continuous in t on $[t_0, T]$ for $(y_0, y_1, \ldots, y_{n-1}, u) \in \mathfrak{M}_T$; therefore, for sufficiently small T_1, we have that $H[\mathfrak{M}_{T_1}] \subset \mathfrak{M}_{T_1}$.

Take for $j = 1, 2$, $k = 0, 1, \ldots, n-1$ $x_k^j(t) = H_k(y_0^j(t), y_1^j(t), \ldots, y_{n-1}^j(t), u^j(t))$, $v^j(t) = H_n(H_0(\overline{y}^j(t), u^j(t)), H_1(\overline{y}^j(t), u^j(t)), \ldots, H_{n-1}(\overline{y}^j(t), u^j(t)), u^j(t))$. Since B_1, B_2, F are Lipschitz continuous in the phase variables, for $k = 0, 1, \ldots, n-1$,

$$\|x_k^1(t) - x_k^2(t)\|_{\mathcal{Z}} \leq C_1 (T_1 - t_0)^{\sigma_n - \sigma_k} E_{\sigma_n, \sigma_n - \sigma_k + 1}((T_1 - t_0)^{\sigma_n} \|A\|_{\mathcal{L}(\mathcal{Z})}) \times$$

$$\times \left(\sum_{l=0}^{n-1} \sup_{s \in [t_0, T_1]} \|y_l^1(s) - y_l^2(s)\|_{\mathcal{Z}} + \sup_{s \in [t_0, T_1]} \|u^1(s) - u^2(s)\|_{\mathcal{U}} \right) =$$

$$= C(T_1) \left(\sum_{l=0}^{n-1} \|y_l^1 - y_l^2\|_{C([t_0, T_1]; \mathcal{Z})} + \|u^1 - u^2\|_{C([t_0, T_1]; \mathcal{U})} \right),$$

$$\|v^1(t) - v^2(t)\|_{\mathcal{U}} \leq C(T_1) \left(\sum_{l=0}^{n-1} \|y_l^1 - y_l^2\|_{C([t_0, T_1]; \mathcal{Z})} + \|u^1 - u^2\|_{C([t_0, T_1]; \mathcal{U})} \right) +$$

$$+ C_2 \sum_{k=0}^{n-1} \sup_{s \in [t_0, T_1]} \|x_k^1(s) - x_k^2(s)\|_{\mathcal{Z}} \leq$$

$$\leq (1 + nC_2) C(T_1) \left(\sum_{l=0}^{n-1} \|y_l^1 - y_l^2\|_{C([t_0, T_1]; \mathcal{Z})} + \|u^1 - u^2\|_{C([t_0, T_1]; \mathcal{U})} \right).$$

Therefore, for a sufficiently small $T_1 > 0$, such that $(n + 1 + nC_2) C(T_1) < 1$, the mapping H has a unique fixed point $(y_0^0, y_1^0, \ldots, y_{n-1}^0, u^0) \in \mathfrak{M}_{T_1}$.

Since $\alpha_0 = 1$ and by the construction of system of Equations (14) $y_k^0 = D^{\sigma_k} y_0^0$, $k = 0, 1, \ldots, n-1$, we have

$$D^{\sigma_0} y_0^0(t) = y_0^0(t) = \sum_{l=0}^{n-1} (t - t_0)^{\sigma_l} E_{\sigma_n, \sigma_l + 1}((t - t_0)^{\sigma_n} A) z_l +$$

$$+ \int_{t_0}^{t} (t - s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t - s)^{\sigma_n} A) B(s, y_0^0(s), y_1^0(s), \ldots, y_{n-1}^0(s), u^0(s)) ds =$$

$$= \sum_{l=0}^{n-1} (t - t_0)^{\sigma_l} E_{\sigma_n, \sigma_l + 1}((t - t_0)^{\sigma_n} A) z_l +$$

$$+ \int_{t_0}^{t} (t - s)^{\sigma_n - 1} E_{\sigma_n, \sigma_n}((t - s)^{\sigma_n} A) B(s, D^{\sigma_0} y_0^0(s), D^{\sigma_1} y_0^0(s), \ldots, D^{\sigma_{n-1}} y_0^0(s), u^0(s)) ds.$$

The last equality in (14) implies the fulfillment of overdetermination condition (9). Since the mapping

$$t \to B(t, D^{\sigma_0}y_0^0(t), D^{\sigma_1}y_0^0(t), \ldots, D^{\sigma_{n-1}}y_0^0(t), u^0(t))$$

is continuous, due to Theorem 1, the pair $(y_0^0(t), u^0(t))$ is a solution to (7)–(9).

If there exist two solutions (z^1, u^1) and (z^2, u^2), then by Theorem 1, every solution corresponds to a fixed point $(D^{\sigma_0}z^1, \ldots, D^{\sigma_{n-1}}z^1, u^1)$ and $(D^{\sigma_0}z^2, \ldots, D^{\sigma_{n-1}}z^2, u^2)$ of the operator H, respectively. Since a fixed point is unique, $u^1(t) = u^2(t)$, $z^1(t) = D^{\sigma_0}z^1(t) = D^{\sigma_0}z^2(t) = z^2(t)$ for $t \in [t_0, T_1]$ at some $T_1 > 0$. □

Remark 2. *The condition $\alpha_0 = 1$ in Theorem 2 may be replaced by a more general condition of the form:*

$$\sigma_0, \sigma_1, \ldots, \sigma_{k-1} < 0, \quad \sigma_k \geq 0, \quad z_0 = z_1 = \cdots = z_{k-1} = 0$$

for some $k \in \{0, 1, \ldots, n-1\}$. Indeed, under this condition, the function $v(t)$ from (13) has a finite limit as $t \to t_0+$ (see Remark 1).

Remark 3. *In Paper [27], local solutions of similar inverse problem are considered for an equation with Gerasimov–Caputo fractional derivatives and with an unbounded sectorial operator A in the linear part. Due to the unboundedness of A, it makes sense to consider the case of generalized and smooth solutions. In this paper, we consider only smooth solutions, since the operator A is bounded, so the generalized solutions are smooth, which is shown in the proof of the theorem.*

4. Nonlocal Solvability of Nonlinear Inverse Problem

Let $\mathcal{Z}$ and $\mathcal{U}$ be Banach spaces, $T > t_0$, and $B : [t_0, T] \times \mathcal{Z}^n \times \mathcal{U} \to \mathcal{Z}$ be a nonlinear operator. Consider problem (7)–(9) on $[t_0, T]$.

Let Assumptions (A)–(C) with $Z = \mathcal{Z}^n \times \mathcal{U}$ and the next conditions (D_1)–(F_1) be satisfied:

(D_1) For $t \in [t_0, T]$, the mapping $v = B_3(t, D^{\alpha_1}\Psi(t), D^{\alpha_2}\Psi(t), \ldots, D^{\alpha_n}\Psi(t), u)$ of the variable u has an inverse function $u = F(t, v)$ on $\mathcal{U}$;

(E_1) The function F is continuous in the totality of the variables (t, v) on $[t_0, T] \times \mathcal{U}$, and Lipschitz is continuous with respect to v on it;

(F_1) Both mappings $B_1(t, \overline{y})$ and $B_2(t, \overline{y}, u)$ are continuous in the totality of the variables on the set $[t_0, T] \times \mathcal{Z}^n \times \mathcal{U}$, and Lipschitz is continuous with respect to $(\overline{y}, u)$ on it.

Theorem 3. *Let $A \in \mathcal{L}(\mathcal{Z})$, $n \in \mathbb{N}$, $z_k \in \mathcal{Z}$, $\alpha_0 = 1$, $0 < \alpha_k \leq 1$, $k = 1, 2, \ldots, n$, $\Phi \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$, $D^{\sigma_k}\Psi \in C([t_0, T]; \mathcal{U})$, $k = 0, 1, \ldots, n$, $\Phi z_0 = \Psi(t_0)$, Conditions (A)–(C) with $Z = \mathcal{Z}^n \times \mathcal{U}$, and Conditions (D_1)–(F_1) be satisfied. Then, there exists a unique solution on $[t_0, T]$ of nonlinear inverse problem (7)–(9) .*

Proof. Similarly to the proof of Theorem 2, consider the system of equations

$$\begin{aligned} y_0(t) &= H_0(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ y_1(t) &= H_1(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ &\ldots, \\ y_{n-1}(t) &= H_{n-1}(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ u(t) &= H_n(H_0(\overline{y}(t), u(t)), H_1(\overline{y}(t), u(t)), \ldots, H_{n-1}(\overline{y}(t), u(t)), u(t)), \end{aligned} \tag{15}$$

but here in the complete metric space $C([t_0, T]; \mathcal{X}^n \times \mathcal{U})$. As before, for $k = 0, 1, \ldots, n-1$,

$$H_k(y_0, y_1 \ldots, y_{n-1}, u) = \sum_{l=0}^{k-1} (t - t_0)^{\sigma_l - \sigma_k + \sigma_n} E_{\sigma_n, \sigma_l - \sigma_k + \sigma_n + 1}((t - t_0)^{\sigma_n} A) A z_l +$$

$$+\sum_{l=k}^{n-1}(t-t_0)^{\sigma_l-\sigma_k}E_{\sigma_n,\sigma_l-\sigma_k+1}((t-t_0)^{\sigma_n}A)z_l+$$

$$+\int_{t_0}^{t}(t-s)^{\sigma_n-\sigma_k-1}E_{\sigma_n,\sigma_n-\sigma_k}((t-s)^{\sigma_n}A)B(s,y_0(s),y_1(s),\dots,y_{n-1}(s),u(s))ds,$$

$$H_n(y_0,y_1,\dots,y_{n-1},u)=F\Bigg(t,D^{\sigma_n}\Psi(t)-\Phi A\sum_{l=0}^{n-1}(t-t_0)^{\sigma_l}E_{\sigma_n,\sigma_l+1}((t-t_0)^{\sigma_n}A)z_l-$$

$$-\Phi A\int_{t_0}^{t}(t-t_0)^{\sigma_n-1}E_{\sigma_n,\sigma_n}((t-t_0)^{\sigma_n}A)B(s,y_0(s),y_1(s),\dots,y_{n-1}(s),u(s))ds-$$

$$-\Phi B_1(t,y_0(t),y_1(t),\dots,y_{n-1}(t))\Bigg).$$

It was proved that for $(y_0,y_1,\dots,y_{n-1},u)\in C([t_0,T];\mathcal{Z}^n\times\mathcal{U})$, we have

$$H(y_0(t),y_1(t),\dots,y_{n-1}(t),u(t))\in C([t_0,T];\mathcal{Z}^n\times\mathcal{U}),$$

$$H_k(y_0(t_0),y_1(t_0),\dots,y_{n-1}(t_0),u(t_0))=y_k(t_0)=z_k,\quad k=0,1,\dots,n-1,$$

$$H_n(y_0(t_0),y_1(t_0),\dots,y_{n-1}(t_0),u(t_0))=F(t_0,v(t_0))=F(t_0,v_0)=u_0.$$

For $(\overline{y}^j,u^j)\in C([t_0,T];\mathcal{Z}^n\times\mathcal{U})$, $j=1,2$, $k=0,1,\dots,n-1$, due to the Lipschitz continuity in the phase variables of the mappings B_1,B_2,F, we have

$$\|H_k(\overline{y}^1,u^1)-H_k(\overline{y}^2,u^2)\|_{\mathcal{Z}}\le C_1(t-t_0)^{\sigma_n-\sigma_k}\left(\sum_{l=0}^{n-1}\|y_l^1-y_l^2\|_{C([t_0,t];\mathcal{Z})}+\right.$$

$$\left.+\|u^1-u^2\|_{C([t_0,t];\mathcal{U})}\right),\quad k=0,1,\dots,n-1,$$

$$\|H_n(\overline{y}^1,u^1)-H_n(\overline{y}^2,u^2)\|_{\mathcal{U}}\le C_1(t-t_0)^{\sigma_n}\left(\sum_{l=0}^{n-1}\|y_l^1-y_l^2\|_{C([t_0,t];\mathcal{Z})}+\right.$$

$$\left.+\|u^1-u^2\|_{C([t_0,t];\mathcal{U})}\right)+C_2\sum_{k=0}^{n-1}\sup_{s\in[t_0,t]}\|H_k(\overline{y}^1,u^1)-H_k(\overline{y}^2,u^2))\|_{\mathcal{Z}}\le$$

$$\le(1+nC_2)C_1(t-t_0)^{\alpha_n}\left(\sum_{l=0}^{n-1}\|y_l^1-y_l^2\|_{C([t_0,t];\mathcal{Z})}+\|u^1-u^2\|_{C([t_0,t];\mathcal{U})}\right),$$

$$\|H(\overline{y}^1,u^1)-H(\overline{y}^2,u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})}\le C_3(t-t_0)^{\alpha_n}\|(\overline{y}^1,u^1)-(\overline{y}^2,u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})}.$$

Denote $H^2(\overline{y},u)=H(H(\overline{y}^1,u^1))$ $H^{l+1}(\overline{y},u)=H(H^l(\overline{y},u))$,

$$H^l(\overline{y},u)=(H_0^l(\overline{y},u),H_1^l(\overline{y},u),\dots,H_n^l(\overline{y},u)),\quad l\in\mathbb{N}.$$

Therefore, for $k=0,1,\dots,n-1$

$$\|H_k^2(\overline{y}^1,u^1)-H_k^2(\overline{y}^2,u^2)\|_{\mathcal{Z}}\le C_1(t-t_0)^{\alpha_n}\int_{t_0}^{t}\sum_{l=0}^{n}\|H_l(\overline{y}^1,u^1)-H_l(\overline{y}^2,u^2)\|_{C([t_0,s];\mathcal{Z})}ds,$$

$$\|H_n^2(\overline{y}^1,u^1)-H_n^2(\overline{y}^2,u^2)\|_{\mathcal{U}}\le C_1(t-t_0)^{\alpha_n}\int_{t_0}^{t}\sum_{l=0}^{n}\|H_l(\overline{y}^1,u^1)-H_l(\overline{y}^2,u^2)\|_{C([t_0,s];\mathcal{Z})}ds+$$

$$+C_2\sum_{k=0}^{n-1}\|H_k^2(\overline{y}^1,u^1)-H_k^{[2]}(\overline{y}^2,u^2)\|_{C([t_0,t];\mathcal{Z})}\le$$

$$\leq (1 + nC_2)C_1(t - t_0)^{\alpha_n} \int\limits_{t_0}^{t} \sum_{l=0}^{n} \|H_l(\overline{y}^1, u^1) - H_l(\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z})} \leq$$

$$\leq (1 + nC_2)C_1C_3(T - t_0)^{\alpha_n} \frac{(t - t_0)^{\alpha_n+1}}{\alpha_n + 1} \|(\overline{y}^1, u^1) - (\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})},$$

$$\|H^2(\overline{y}^1, u^1) - H^2(\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})} \leq$$

$$\leq C_3^2(T - t_0)^{\alpha_n} \frac{(t - t_0)^{\alpha_n+1}}{\alpha_n + 1} \|(\overline{y}^1, u^1) - (\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})}.$$

Analogously, we obtain

$$\|H^3(\overline{y}^1, u^1) - H^3(\overline{y}^2, u^2)\|_{\mathcal{Z}} \leq$$

$$\leq C_3^3(T - t_0)^{2\alpha_n} \frac{(t - t_0)^{\alpha_n+2}}{(\alpha_n + 1)(\alpha_n + 2)} \|(\overline{y}^1, u^1) - (\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})} \leq$$

$$\leq C_3^3(T - t_0)^{2\alpha_n} \frac{(t - t_0)^{\alpha_n+2}}{2!} \|(\overline{y}^1, u^1) - (\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})}, \dots,$$

$$\|H^l(\overline{y}^1, u^1) - H^l(\overline{y}^2, u^2)\|_{\mathcal{Z}} \leq$$

$$\leq C_3^l(T - t_0)^{(l-1)\alpha_n} \frac{(t - t_0)^{\alpha_n+l-1}}{(l - 1)!} \|(\overline{y}^1, u^1) - (\overline{y}^2, u^2)\|_{C([t_0,t];\mathcal{Z}^n\times\mathcal{U})}, \quad l \in \mathbb{N}.$$

Thus, for sufficiently large $l \in \mathbb{N}$, the mapping H^l has a unique fixed point

$$(y_0^0, y_1^0, \dots, y_{n-1}^0, u^0) \in C([t_0, T]; \mathcal{Z}^n \times \mathcal{U}).$$

As for a local solution, we can prove that the pair $(y_0^0(t), u^0(t))$ is a solution to (7)–(9) and can show the uniqueness of a solution. □

5. Nonlinear Inverse Problem for an Integro-Differential Equation

Consider an inverse problem

$$D_t^{\sigma_k} w(\xi, 0) = w_k(\xi), \quad k = 0, 1, \dots, n - 1, \quad \xi \in \Omega, \tag{16}$$

$$D_t^{\sigma_n} w(\xi, t) = \int\limits_{\Omega} K(\xi, \eta) w(\eta, t) d\eta + \sum_{k=0}^{n-1} u_k(t) D_t^{\sigma_k} w(\xi, t) +$$

$$+ u_n(t) g(\xi, t) + h(\xi, t), \quad (\xi, t) \in \Omega \times [t_0, T], \tag{17}$$

$$\langle \eta_j(\cdot), w(\cdot, t) \rangle_{L_2(\Omega)} = \psi_j(t), \quad t \in [t_0, T], \quad j = 0, 1, \dots, n. \tag{18}$$

Here, Ω is a measurable region in $\mathbb{R}^d$, and $D_t^{\sigma_k}$ are the partial Dzhrbashyan–Nersesyan derivatives in t, which correspond to $\alpha_k \in (0, 1]$, $k = 0, 1 \dots, n$, $\alpha_0 = 1$; $w_k \in L_2(\Omega)$, $k = 0, 1, \dots, n - 1$, $g, h \in C([t_0, T]; L_2(\Omega))$, $\eta_j \in L_2(\Omega)$, $j = 0, 1, \dots, n$, and $\langle \cdot, \cdot \rangle_{L_2(\Omega)}$ the inner product in $L_2(\Omega)$ is denoted.

Theorem 4. *Let $w_k \in L_2(\Omega)$, $k = 0, 1, \dots, n - 1$, $\alpha_0 = 1$, $0 < \alpha_k \leq 1$, $k = 1, 2, \dots, n$, $K \in L_2(\Omega \times \Omega)$, $w_k \in L_2(\Omega)$, $D^{\sigma_k} \psi_j \in C([t_0, T]; \mathcal{U})$, $k = 0, 1, \dots, n$, $\eta_j \in L_2(\Omega)$, $\langle \eta_j(\cdot), w_0(\cdot) \rangle_{L_2(\Omega)} = \psi_j(0)$, $j = 0, 1, \dots, n$, $g, h \in C([0, T]; L_2(\Omega))$,* det $\|a_{jk}\|_{j,k=0}^n \neq 0$, det $\|b_{jk}(t)\|_{j,k=0}^n \neq 0$ *for all $t \in [t_0, T]$, where $a_{jk} := \langle \eta_j(\cdot), w_k(\cdot) \rangle_{L_2(\Omega)}$, $k = 0, 1, \dots, n - 1$, $j = 0, 1, \dots, n$, $a_{jn} := \langle \eta_j(\cdot), g(\cdot, 0) \rangle_{L_2(\Omega)}$, $j = 0, 1, \dots, n$, $b_{jk}(t) := D_t^{\sigma_k} \psi_j(t)$, $k = 0, 1, \dots, n - 1$, $b_{jn}(t) := \langle \eta_j(\cdot), g(\cdot, t) \rangle_{L_2(\Omega)}$ for $j = 0, 1, \dots, n$. Then, there exists $T_1 \in (t_0, T]$, such that there is a unique solution on the segment $[t_0, T_1]$ of nonlinear inverse problem (16)–(18).*

Proof. Set $\mathcal{Z} := L_2(\Omega)$; since $K \in L_2(\Omega \times \Omega)$, for $v \in L_2(\Omega)$, we have

$$(Av)(\xi) := \int_{\Omega} K(\xi, \eta) v(\eta) d\eta, \ A \in \mathcal{L}(\mathcal{Z}).$$

Therefore, Problem (16)–(18) are representable in Forms (7)–(9), where $Z = \mathbb{R} \times L_2(\Omega)^n, \mathcal{U} = \mathbb{R}^{n+1}, B : \mathbb{R} \times L_2(\Omega)^n \times \mathbb{R}^{n+1} \to L_2(\Omega)$,

$$B(t, \overline{y}, u) = \sum_{k=0}^{n-1} u_k y_k + u_n g(\cdot, t) + h(\cdot, t), \quad \overline{y} = (y_0, y_1, \dots, y_{n-1}), \quad u = (u_0, u_1, \dots, u_n),$$

$$B_1(t, \overline{y}) = h(\cdot, t), \quad B_2(t, \overline{y}, u) = \sum_{k=0}^{n-1} u_k y_k + u_n g(\cdot, t),$$

$z_k = w_k(\cdot)$, $k = 0, 1, \dots, n-1$. Check the conditions of Theorem 2.

The local Lipschitz continuity of B with respect to $\overline{y}$, u is evident, and we have $\Phi \in \mathcal{L}(L_2(\Omega); \mathbb{R}^{n+1})$, since

$$\Phi = (\Phi_0, \Phi_1, \dots, \Phi_n) : L_2(\Omega) \to \mathbb{R}^{n+1}, \ \Phi_j(z) = \int_{\Omega} \eta_j(\xi) z(\xi) d\xi, \ j = 0, 1, \dots, n.$$

Set $v_0 = (v_{00}, v_{01}, \dots, v_{0n})$,

$$v_{0j} = D^{\sigma_n} \psi_j(0) - \left\langle \eta_j(\cdot), \int_{\Omega} K(\cdot, \eta) w_0(\eta) d\eta + h(\cdot, 0) \right\rangle_{L_2(\Omega)},$$

then the system of equations

$$\left\langle \eta_j(\cdot), \sum_{k=0}^{n-1} u_k w_k(\cdot) + u_n g(\cdot, 0) \right\rangle_{L_2(\Omega)} = v_{0j}, \quad j = 0, 1, \dots, n,$$

is uniquely resolved with respect to $(u_0, u_1, \dots, u_n)$, since $\det \|a_{jk}\|_{j,k=0}^n \neq 0$. Hence, Condition (B) is true. Denote the unique solution by $(u_0^0, u_1^0, \dots, u_n^0)$. Further, we have for $j = 0, 1, \dots, n$

$$\left\langle \eta_j(\cdot), \sum_{k=0}^{n-1} u_k y_k(\cdot) + u_n g(\cdot, t) \right\rangle_{L_2(\Omega)} = \sum_{k=0}^{n-1} u_k \langle \eta_j(\cdot), y_k(\cdot) \rangle_{L_2(\Omega)} + u_n \langle \eta_j(\cdot), g(\cdot, t) \rangle_{L_2(\Omega)}.$$

Hence, Condition (C) is satisfied with $B_3 = (B_{30}, B_{31}, \dots, B_{3n})$, where for $j = 0, 1, \dots, n$

$$B_{3j}(t, u_{00}, u_{01}, \dots, u_{n\,n}) = \sum_{k=0}^{n-1} u_{j\,k} u_{n\,k} + u_{nn} \langle \eta_j(\cdot), g(\cdot, t) \rangle_{L_2(\Omega)}.$$

The vector function

$$v_j = \sum_{k=0}^{n-1} D_t^{\sigma_k} \psi_j(t) u_k + u_n \langle \eta_j(\cdot), g(\cdot, t) \rangle_{L_2(\Omega)}$$

has an inverse vector function $u = b(t)^{-1} v$, which is continuous on $[t_0, T] \times \mathbb{R}^{n+1}$ and satisfies the local Lipschitz condition in v, since the matrix $b(t) = \|b_{jk}(t)\|_{j,k=0}^n$ is invertible for all $t \in [t_0, T]$. As before $u = (u_0, u_1, \dots, u_n)$, $v = (v_0, v_1, \dots, v_n)$.

Thus, Conditions (A)–(F) are fulfilled, and by Theorem 2, inverse problem (16)–(18) have a unique local solution. □

6. Nonlinear Inverse Problem for a Kelvin–Voigt Time-Fractional System

Consider the problem

$$(\lambda - \Delta)D_t^{\sigma_n} w(\xi, t) = \nu\Delta w(\xi, t) - r(\xi, t) +$$

$$+(\lambda - \Delta)\sum_{k=0}^{n-1} u_k(t)D_t^{\sigma_k} w(\xi, t) + u_n(t)g(\xi, t) + h(\xi, t), \quad (\xi, t) \in \Omega \times [0, T], \tag{19}$$

$$\nabla \cdot w(\xi, t) = 0, \quad (\xi, t) \in \Omega \times [0, T], \tag{20}$$

$$w(\xi, t) = 0, \quad (\xi, t) \in \partial\Omega \times [0, T], \tag{21}$$

$$D_t^{\sigma_k} w(\xi, 0) = w_k(\xi), \quad \xi \in \Omega, \quad k = 0, 1, \dots, n-1, \tag{22}$$

$$\int_\Omega \langle \eta_j(\xi), w(\xi, t)\rangle_{\mathbb{R}^d} d\xi = \psi_j(t), \quad t \in [0, T], \quad j = 0, 1, \dots, n, \tag{23}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded region with a smooth boundary $\partial\Omega$, $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is the inner product in $\mathbb{R}^d$, $D_t^{\sigma_k}$ are the partial Dzhrbashyan–Nersesyan derivatives in t associated with $\alpha_k \in (0, 1]$, $k = 0, 1 \dots, n$, $\alpha_0 = 1$; $\lambda, \nu \in \mathbb{R}$, $\Delta = D_{\xi_1}^2 + D_{\xi_2}^2 + \dots + D_{\xi_d}^2$ is the Laplace operator, and $g, h : \Omega \times [0, T] \to \mathbb{R}^d$, $w_k : \Omega \to \mathbb{R}^d$, $k = 0, 1, \dots, n-1$, $\eta_j : \Omega \to \mathbb{R}^d$, $\psi_j : [0, T] \to \mathbb{R}$, $j = 0, 1, \dots, n$, are set. Vector functions of the velocity $w = (w^1, w^2, \dots, w^d)$ and of the pressure gradient $r = (r^1, r^2, \dots, r^d) = \nabla p$ for some pressure $p \in H^1(\Omega)$ and functions $u_j(t)$, $j = 0, 1, \dots, n$, are unknown.

Remark 4. *In the case of* $u_j \equiv 0$, $j = 0, 1, \dots, n-1$, *System* (19) *and* (20) *are linearized in a neighborhood of zero point system, which describe the dynamics of the Kelvin–Voigt time-fractional viscoelastic medium.*

Let $\mathbb{L}_2 := (L_2(\Omega))^d$, $\mathbb{H}^1 := (W_2^1(\Omega))^d$, $\mathbb{H}^2 := (W_2^2(\Omega))^d$. The closure of the subspace $\mathfrak{L} := \{v \in (C_0^\infty(\Omega))^d : \nabla \cdot v = 0\}$ in the norm of the space $\mathbb{L}_2$ will be denoted by $\mathbb{H}_\sigma$ and in the norm of $\mathbb{H}^1$ by $\mathbb{H}_\sigma^1$. We will also use the notations: $\mathbb{H}_\sigma^2 := \mathbb{H}_\sigma^1 \cap \mathbb{H}^2$, $\mathbb{H}_\pi$ as the orthogonal complement for $\mathbb{H}_\sigma$ in $\mathbb{L}_2$, $\Sigma : \mathbb{L}_2 \to \mathbb{H}_\sigma$, $\Pi := I - \Sigma$ as the appropriate orthoprojectors.

The operator $\Lambda := \Sigma\Delta$, extended to a closed operator in $\mathbb{H}_\sigma$ with the domain $\mathbb{H}_\sigma^2$, has a real negative discrete spectrum with finite eigenvalues multiplicities; it is condensed at $-\infty$ only [38]. Eigenfunctions $\{\lambda_l\}$ of Λ are numbered in non-increasing order, taking into account their multiplicities. An orthonormal system of respective eigenfunctions $\{\varphi_l\}$ forms a basis in $\mathbb{H}_\sigma$ [38].

Under the assumption $\lambda \notin \{\lambda_l\}$ using the projectors Σ and Π, we obtain restrictions of (19) and (20) into subspaces $\mathbb{H}_\sigma$ and $\mathbb{H}_\pi$ accordingly:

$$D_t^{\sigma_n} w(\xi, t) = \nu(\lambda - \Lambda)^{-1}\Lambda w(\xi, t) + \sum_{j=0}^{n-1} u_j(t)D_t^{\sigma_j} w(\xi, t) +$$

$$+u_n(t)(\lambda - \Lambda)^{-1}\Sigma g(\xi, t) + (\lambda - \Lambda)^{-1}\Sigma h(\xi, t), \ (\xi, t) \in \Omega \times [0, T], \tag{24}$$

$$r(\xi, t) = \Pi\Delta D_t^{\sigma_n} w(\xi, t) + \nu\Pi\Delta w(\xi, t) +$$

$$+u_n(t)\Pi g(\xi, t) + \Pi h(\xi, t), \ (\xi, t) \in \Omega \times [0, T]. \tag{25}$$

Therefore, we can find $w : [0,T] \to \mathbb{H}_\sigma$, $u_0, u_1, \ldots, u_n : [0,T] \to \mathbb{R}$ from Equation (24), then a solution $r : [0,T] \to \mathbb{H}_\pi$ to Equation (25) can be found. Therefore, we will consider inverse problem (20)–(24).

Theorem 5. *Let* $w_k \in \mathbb{H}_\sigma$, $k = 0,1,\ldots,n-1$, $\alpha_0 = 1$, $0 < \alpha_k \le 1$, $k = 1,2,\ldots,n$, $D^{\sigma_k}\psi_j \in C([t_0,T];\mathcal{U})$, $k = 0,1,\ldots,n$, $\eta_j \in \mathbb{L}_2$, $\langle \eta_j(\cdot), w_0(\cdot)\rangle_{\mathbb{L}_2} = \psi_j(0)$, $j = 0,1,\ldots,n$, $g, h \in C([0,T];\mathbb{L}_2)$, $\det \|a_{jk}\|_{j,k=0}^n \ne 0$, $\det \|b_{jk}(t)\|_{j,k=0}^n \ne 0$ *for* $t \in [0,T]$, *where* $a_{jk} := \langle \eta_j(\cdot), w_k(\cdot)\rangle_{\mathbb{L}_2}$, $k = 0,1,\ldots,n-1$, $a_{jn} := \langle \eta_j(\cdot), (\lambda-\Lambda)^{-1}\Sigma g(\cdot,0)\rangle_{\mathbb{L}_2}$, $j = 0,1,\ldots,n$, $b_{jk}(t) := D_t^{\sigma_k}\psi_j$, $k = 0,1,\ldots,n-1$, $b_{jn}(t) := \left\langle \eta_j(\cdot), (\lambda-\Lambda)^{-1}\Sigma g(\cdot,t)\right\rangle_{\mathbb{L}_2}$, $j = 0,1,\ldots,n$. *Then, there exists* $T_1 \in (0,T]$, *such that inverse problem* (20)–(24) *have a solution to the segment* $[0,T_1]$.

Proof. Due to the incompressibility Equation (20), we reduce Problem (20)–(24) to Problem (7)–(9) with $\mathcal{Z} = \mathbb{H}_\sigma$, $A = (\lambda-\Lambda)^{-1}\Lambda \in \mathcal{L}(\mathbb{H}_\sigma)$, $\mathcal{U} = \mathbb{R}^{n+1}$, $Z = \mathbb{R} \times \mathbb{H}_\sigma^n$, $B : \mathbb{R} \times \mathbb{H}_\sigma^n \times \mathbb{R}^{n+1} \to \mathbb{H}_\sigma$,

$$B(t,\overline{y},u) = \sum_{k=0}^{n-1} u_k y_k(\cdot) + u_n(\lambda-\Lambda)^{-1}\Sigma g(\cdot,t) + (\lambda-\Lambda)^{-1}\Sigma h(\cdot,t),$$

$$B_1(t,\overline{y}) = (\lambda-\Lambda)^{-1}\Sigma h(\cdot,t), \quad B_2(t,\overline{y},u) = \sum_{k=0}^{n-1} u_k y_k(\cdot) + u_n(\lambda-\Lambda)^{-1}\Sigma g(\cdot,t),$$

where $\overline{y} = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{H}_\sigma^n$, $u = (u_0, u_1, \ldots, u_n) \in \mathbb{R}^{n+1}$;

$$\Phi = (\Phi_0, \Phi_1, \ldots, \Phi_n) \in \mathcal{L}(\mathbb{H}_\sigma;\mathbb{R}^{n+1}), \;\; \Phi_j(y) = \langle \eta_j(\cdot), y(\cdot)\rangle_{\mathbb{L}_2}, \; j = 0,1,\ldots,n,$$

$\langle\cdot,\cdot\rangle_{\mathbb{L}_2}$ is the inner product in $\mathbb{L}_2$, $z_k = w_k(\cdot)$, $k = 0,1,\ldots,n-1$, $z(t) = w(\cdot,t) \in \mathbb{H}_\sigma$, $u(t) = (u_0(t), u_1(t), \ldots, u_n(t))$, $\Psi(t) = (\psi_0(t), \psi_1(t), \ldots, \psi_n(t)) \in \mathbb{R}^{n+1}$ for $t \in [0,T]$. Check the conditions of Theorem 2.

The mapping B is continuous on $[0,T] \times \mathbb{H}_\sigma^n \times \mathbb{R}^{n+1}$ and is locally Lipschitz continuous in $\overline{y}, u$. Define $v_0 = (v_{00}, v_{01}, \ldots, v_{0n})$,

$$v_{0j} = D^{\sigma_n}\psi_j(0) - \langle \eta_j(\xi), (\lambda-\Lambda)^{-1}\Lambda w_0(\xi) + (\lambda-\Lambda)^{-1}\Sigma h(\cdot,0)\rangle_{\mathbb{L}_2},$$

then the system of equations

$$\left\langle \eta_j(\cdot), \sum_{k=0}^{n-1} u_k w_k(\cdot) + u_n(\lambda-\Lambda)^{-1}\Sigma g(\cdot,0)\right\rangle_{\mathbb{L}_2} = v_{0j}, \quad j = 0,1,\ldots,n,$$

has a unique solution $(u_0^0, u_1^0, \ldots, u_n^0)$, since $\det \|a_{jk}\|_{j,k=0}^n \ne 0$. Therefore, Condition (B) is fulfilled. Further, for $j = 0,1,\ldots,n$,

$$\left\langle \eta_j(\cdot), \sum_{k=0}^{n-1} u_k y_k(\cdot) + u_n(\lambda-\Lambda)^{-1}\Sigma g(\cdot,t)\right\rangle_{\mathbb{L}_2} =$$

$$= \sum_{k=0}^{n-1} u_k \Phi_j(y_k) + u_n\left\langle \eta_j(\cdot), (\lambda-\Lambda)^{-1}\Sigma g(\cdot,t)\right\rangle_{\mathbb{L}_2}.$$

Therefore, Condition (C) holds with $B_3 = (B_{30}, B_{31}, \ldots, B_{3n})$, where for $j = 0,1,\ldots,n$

$$B_{3j}(u_{00}, u_{01}, \ldots, u_{n\,n}) = \sum_{k=0}^{n-1} u_{j\,k} u_{n\,k} + u_{nn}\left\langle \eta_j(\xi), (\lambda-\Lambda)^{-1}\Sigma g(\xi,t)\right\rangle_{\mathbb{L}_2}.$$

Since the matrix $b(t) = \|b_{jk}(t)\|_{j,k=0}^{n}$ is invertible for $t \in [0,T]$, the vector function

$$v_j = \sum_{k=0}^{n-1} D_t^{\sigma_k} \psi_j(t) u_k + u_n \Big\langle \eta_j(\xi), (\lambda - \Lambda)^{-1} \Sigma g(\xi, t) \Big\rangle_{\mathbb{L}_2}$$

has the inverse mapping $u = b(t)^{-1} v$, which is continuous on $[0,T] \times \mathbb{R}^{n+1}$ and locally Lipschitz continuous in $v = (v_0, v_1, \dots, v_n)$.

Thus, due to Theorem 2, nonlinear inverse problem (20)–(24) have a unique local solution. □

7. Conclusions

A new class of inverse problems for differential equations is investigated. The abstract results on the existence of a unique solution allowed us to study the unique solvability of new coefficient inverse problems for partial differential equations with Dzhrbashyan–Nersesyan time-fractional derivatives, in other words, inverse problems for partial differential equations with unknown coefficients at derivatives of the unknown function. We plan to use the methods applied in this paper and the results obtained as the basis for studies of wider classes of nonlinear inverse problems to equations with an unbounded operator in the linear part, as well as for so-called degenerate equations containing a degenerate operator at the highest Dzhrbashyan–Nersesyan fractional derivative.

Note that we used the fixed-point theorem, the proof of which contains the construction of a sequence of approximate solutions converging to the exact solution. Therefore, the approach proposed in this paper can become the basis for a numerical study of this class of inverse problems.

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References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science: Philadelphia, PA, USA, 1993.
2. Podlubny, I. *Fractional Differential Equations*; Academic: Boston, MA, USA, 1999.
3. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka: Moscow, Russia, 2005. (In Russian)
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Publishing: Amsterdam, The Netherlands, 2006.
5. Hilfer, R. *Applications of Fractional Calculus in Physics*; WSPC: Singapore, 2000.
6. Nakhushev, A.M. *Fractional Calculus ant Its Applications*; Fizmatlit: Moscow, Russia, 2003. (In Russian)
7. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles*; Springer: New York, NY, USA, 2011.
8. Uchaykin, V.V. *Fractional Derivatives for Physicists and Engineers*; Higher Education Press: Beijing, China, 2012.
9. Guechi, S.; Dhayal, R.; Debbouche, A.; Malik, M. Analysis and optimal control of φ-Hilfer fractional semilinear equations involving nonlocal impulsive conditions. *Symmetry* **2021**, *13*, 2084. [CrossRef]
10. Vijayakumar, V.; Nisar, K.S.; Chalishajar, D.; Shukla, A.; Malik, M.; Alsaadi, A.; Aldosary, S.F. A note on approximate controllability of fractional semilinear integrodifferential control systems via resolvent operators. *Fractal Fract.* **2022**, *6*, 73. [CrossRef]
11. Hakkar, N.; Dhayal, R.; Debbouche, A.; Torres, D.F.M. Approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects. *Fractal Fract.* **2023**, *7*, 104. [CrossRef]
12. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.

13. Dokuyucu, M.A.; Baleanu, D.; Çelik, E. Analysis of Keller-Segel Model with Atangana–Baleanu Fractional Derivative. *Filomat* **2018**, *32*, 5633–5643. [CrossRef]
14. Kozhanov, A.I. *Composite Type Equations and Inverse Problems*; VSP: Utrecht, The Netherlands, 1999.
15. Prilepko, A.I.; Orlovsky, D.G.; Vasin, I.A. *Methods for Solving Inverse Problems in Mathematical Physics*; Marcel Dekker, Inc.: New York, NY, USA, 2000.
16. Belov, Y.Y. *Inverse Problems for Parabolic Equations*; VSP: Utrecht, The Netherlands, 2002.
17. Favini, A.; Lorenzi, A. *Differential Equations. Inverse and Direct Problems*; Chapman and Hall/CRC: New York, NY, USA, 2006.
18. Kabanikhin, S.I.; Krivorot'ko, O.I. Optimization methods for solving inverse immunology and epidemiology problems. *Comput. Math. Math. Phys.* **2020**, *60*, 580–589. [CrossRef]
19. Orlovsky, D.G. Parameter determination in a differential equation of fractional order with Riemann–Liouville fractional derivative in a Hilbert space. *J. Sib. Fed. Univ. Math. Phys.* **2015**, *8*, 55–63. [CrossRef]
20. Fedorov, V.E.; Ivanova, N.D. Identification problem for degenerate evolution equations of fractional order. *Fract. Calc. Appl. Anal.* **2017**, *20*, 706–721. [CrossRef]
21. Orlovsky, D.G. Determination of the parameter of the differential equation of fractional order with the Caputo derivative in Hilbert space. *J. Physics Conf. Ser.* **2019**, *1205*, 012042. [CrossRef]
22. Fedorov, V.E.; Kostić, M. Identification problem for strongly degenerate evolution equations with the Gerasimov–Caputo derivative. *Differ. Equations* **2020**, *56*, 1613–1627. [CrossRef]
23. Fedorov, V.E.; Nagumanova, A.V.; Avilovich, A.S. A class of inverse problems for evolution equations with the Riemann–Liouville derivative in the sectorial case. *Math. Methods Appl. Sci.* **2021**, *44*, 11961–11969. [CrossRef]
24. Kostin, A.B.; Piskarev, S.I. Inverse source problem for the abstract fractional differential equation. *J. Inverse III Posed Probl.* **2021**, *29*, 267–281. [CrossRef]
25. Orlovsky, D.; Piskarev, S. Inverse problem with final overdetermination for time-fractional differential equation in a Banach space. *J. Inverse III Posed Probl.* **2022**, *30*, 221–237. [CrossRef]
26. Dzhrbashyan, M.M.; Nersesyan, A.B. Fractional derivatives and the Cauchy problem for differential equations of fractional order. *Izv. Akad. Nauk. Armyanskoy Ssr. Mat.* **1968**, *3*, 3–28. (In Russian) [CrossRef]
27. Fedorov, V.E.; Ivanova, N.D.; Borel, L.V.; Avilovich, A.S. Nonlinear inverse problems for fractional differential equations with sectorial operators. *Lobachevskii J. Math.* **2022**, *43*, 3125–3141. [CrossRef]
28. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Initial value problem for linear equations with the Dzhrbashyan–Nersesyan derivative in Banach spaces. *Symmetry* **2021**, *13*, 1058. [CrossRef]
29. Plekhanova, M.V.; Izhberdeeva, E.M. Local unique solvability of a quasilinear equation with the Dzhrbashyan–Nersesyan derivatives. *Lobachevskii J. Math.* **2022**, *43*, 1379–1388. [CrossRef]
30. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Analytic resolving families for equations with the Dzhrbashyan–Nersesyan fractional derivative. *Fractal Fract.* **2022**, *6*, 541. [CrossRef]
31. Pskhu, A.V. The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math.* **2009**, *73*, 351–392. [CrossRef]
32. Pskhu, A.V. Fractional diffusion equation with discretely distributed differentiation operator. *Sib. Electron. Math. Rep.* **2016**, *13*, 1078–1098.
33. Pskhu, A.V. Boundary value problem for a first-order partial differential equation with a fractional discretely distributed differentiation operator. *Differ. Equations* **2016**, *52*, 1610–1623. [CrossRef]
34. Pskhu, A.V. Stabilization of solutions to the Cauchy problem for fractional diffusion-wave equation. *J. Math. Sci.* **2020**, *250*, 800–810. [CrossRef]
35. Mamchuev, M.O. Cauchy problem for a linear system of ordinary differential equations of the fractional order. *Mathematics* **2020**, *8*, 1475.
36. Bogatyreva, F.T. On representation of a solution for first-order partial differential equation with Dzhrbashyan–Nersesyan operator of fractional differentiation. *Dokl. Adyg. Mezhdunarodnoy Akad. Nauk.* **2020**, *20*, 6–11. (In Russian) [CrossRef]
37. Bogatyreva, F.T. Boundary value problems for first order partial differential equation with the Dzhrbashyan–Nersesyan operators. *Chelyabinsk Phys. Math. J.* **2021**, *6*, 403–416.
38. Ladyzhenskaya, O.A. *The Mathematical Theory of Viscous Incompressible Flow*; Gordon and Breach: New York, NY, USA, 1969.