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On a System of Hadamard Fractional Differential Equations with Nonlocal Boundary Conditions on an Infinite Interval

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Abstract: Our research focuses on investigating the existence of positive solutions for a system of nonlinear Hadamard fractional differential equations. These equations are defined on an infinite interval and involve non-negative nonlinear terms. Additionally, they are subject to nonlocal coupled boundary conditions, incorporating Riemann–Stieltjes integrals and Hadamard fractional derivatives. To establish the main theorems, we employ the Guo–Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem.

Keywords: Hadamard fractional differential equations; coupled nonlinear boundary conditions; positive solutions

MSC: 34A08; 34B15; 34B18; 34B10

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1. Introduction

We shall examine a system comprising nonlinear Hadamard fractional differential equations

$$\begin{cases} {}^{H}D^{\alpha}_{1+}\Phi(v) + \mathfrak{c}(v)\mathfrak{L}(\Phi(v),\Psi(v)) = 0, \ \ v \in (1,\infty), \\ {}^{H}D^{\beta}_{1+}\Psi(v) + \mathfrak{d}(v)\mathfrak{M}(\Phi(v),\Psi(v)) = 0, \ \ v \in (1,\infty), \end{cases} \tag{1}$$

supplemented with the nonlocal coupled boundary conditions

$$\begin{cases} \Phi(1) = \Phi'(1) = \cdots = \Phi^{(n-2)}(1) = 0, \\ {}^{H}D^{\alpha-1}_{1+}\Phi(\infty) = \int_1^\infty \Phi(\omega)\, d\mathfrak{I}_1(\omega) + \int_1^\infty \Psi(\omega)\, d\mathfrak{I}_2(\omega), \\ \Psi(1) = \Psi'(1) = \cdots = \Psi^{(m-2)}(1) = 0, \\ {}^{H}D^{\beta-1}_{1+}\Psi(\infty) = \int_1^\infty \Phi(\omega)\, d\mathfrak{J}_1(\omega) + \int_1^\infty \Psi(\omega)\, d\mathfrak{J}_2(\omega), \end{cases} \tag{2}$$

where $\alpha \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 2$, $\beta \in (m-1,m]$, $m \in \mathbb{N}$, $m \geq 2$, ${}^{H}D^{\kappa}_{1+}$ denotes the Hadamard fractional derivative of order p (for $p = \alpha, \alpha-1, \beta, \beta-1$), the functions $\mathfrak{c}, \mathfrak{d} : (1,\infty) \to \mathbb{R}_+$ and $\mathfrak{L}, \mathfrak{M} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ verify some assumptions, ($\mathbb{R}_+ = [0,\infty)$), and the functions $\mathfrak{I}_i, \mathfrak{J}_i : [1,\infty) \to \mathbb{R}$, $i = 1,2$ from the Riemann–Stieltjes integrals of (2) have bounded variations.

By employing the Guo–Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem, we will establish the existence of positive solutions of the problem (1),(2) subject to certain conditions on the problem's data. A positive solution of (1),(2) is represented by a pair of functions $(\Phi(v),\Psi(v))$, $v \in [1,\infty)$ satisfying (1) and (2), with $\Phi(\zeta) \geq 0$, $\Psi(\zeta) \geq 0$ for all $\zeta \in [1,\infty)$, and $\Phi(\zeta) > 0$ for all $\zeta \in (1,\infty)$ or $\Psi(\zeta) > 0$ for all

$\zeta \in (1,\infty)$. This problem generalizes the problem from [1]. In the paper [1], the authors studied the system (1) with α, $\beta \in (1,2]$ ($n = m = 2$), with the boundary conditions

$$\begin{cases} \Phi(1) = 0, \ \ {}^{H}D_{1+}^{\alpha-1}\Phi(\infty) = \sum_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\alpha_i}\Psi(\eta), \\ \Psi(1) = 0, \ \ {}^{H}D_{1+}^{\beta-1}\Psi(\infty) = \sum_{j=1}^{n} \sigma_j \, {}^{H}I_{1+}^{\beta_j}\Phi(\xi), \end{cases} \tag{3}$$

where λ_i, $\sigma_j > 0$ for $i = 1,\ldots,m$, $j = 1,\ldots,n$, $\eta > 1$, $\xi > 1$, and ${}^{H}I_{1+}^{k}$ is the Hadamard fractional integral of order k with lower limit 1 for $k = \alpha_1,\ldots,\alpha_m$, $\beta_1,\ldots,\beta_n$, defined by

$${}^{H}I_{1+}^{k}\mathrm{h}(v) = \frac{1}{\Gamma(k)} \int_1^v \left(\ln\frac{v}{\omega}\right)^{k-1} \frac{\mathrm{h}(\omega)}{\omega}\, d\omega, \ \ v > 0,$$

(for a function $\mathrm{h} : [1,\infty) \to \mathbb{R}$), (see [2]). The last conditions for ∞ from (3) are particular cases of our conditions from (2). Indeed, one finds

$$\sum_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\alpha_i}\Psi(\eta) = \int_1^\infty \Psi(\omega)\, d\mathfrak{J}_2(\omega), \ \ \text{and} \ \ \sum_{j=1}^{n} \sigma_j \, {}^{H}I_{1+}^{\beta_j}\Phi(\xi) = \int_1^\infty \Phi(\omega)\, d\mathfrak{J}_1(\omega),$$

with $\mathfrak{J}_2(\omega) = \sum_{i=1}^{m} \lambda_i \widetilde{H}_i(\omega)$, ($\mathfrak{J}_1 \equiv 0$), and $\mathfrak{J}_1(\omega) = \sum_{j=1}^{n} \sigma_j \widetilde{K}_j(\omega)$, ($\mathfrak{J}_2 \equiv 0$), where

$$\widetilde{H}_i(\omega) = \begin{cases} \dfrac{1}{\Gamma(\alpha_i+1)}\left((\ln\eta)^{\alpha_i} - \left(\ln\dfrac{\eta}{\omega}\right)^{\alpha_i}\right), & \text{if } 0 \le \omega \le \eta, \\ \dfrac{1}{\Gamma(\alpha_i+1)}(\ln\eta)^{\alpha_i}, & \text{if } \omega \ge \eta, \end{cases}$$

and

$$\widetilde{K}_j(\omega) = \begin{cases} \dfrac{1}{\Gamma(\beta_j+1)}\left((\ln\xi)^{\beta_j} - \left(\ln\dfrac{\xi}{\omega}\right)^{\beta_j}\right), & \text{if } 0 \le \omega \le \xi, \\ \dfrac{1}{\Gamma(\beta_j+1)}(\ln\xi)^{\beta_j}, & \text{if } \omega \ge \xi, \end{cases}$$

for $i = 1,\ldots,m$, and $j = 1,\ldots,n$. Therefore, in our paper, we consider general orders for the fractional derivatives in the equations of system (1). Furthermore, in the boundary conditions (2), the fractional derivatives ${}^{H}D_{1+}^{\alpha-1}\Phi$ and ${}^{H}D_{1+}^{\beta-1}\Psi$ for ∞ are dependent on both functions Φ and Ψ, in comparison with the condition (3) from [1], where the derivative of Φ is dependent only on Ψ, and the derivative of Ψ is dependent only on Φ. In addition, the conditions (2) with Riemann–Stieltjes integrals generalize, as we saw before, the conditions (3). These aspects represent the novelties for our problem (1),(2).

In the upcoming discussion, we will introduce additional fractional boundary value problems that have been investigated in recent years. These problems are connected to the main problem (1),(2) and involve Hadamard fractional differential equations accompanied by diverse nonlocal boundary conditions. In [3], the authors studied the existence of non-negative multiple solutions for the Hadamard fractional differential equation with integro-differential boundary conditions

$$\begin{cases} {}^{H}D_{1+}^{\alpha}\mathfrak{U}(v) + \mathrm{a}(v)\mathrm{f}(\mathfrak{U}(v)) = 0, \ \ v \in (0,\infty), \\ \mathfrak{U}(1) = 0, \ \ {}^{H}D_{1+}^{\alpha-1}\mathfrak{U}(\infty) = \sum_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\beta_i}\mathfrak{U}(\eta), \end{cases}$$

where $\alpha \in (1,2]$, $\eta \in (1,\infty)$, $\beta_i > 0$ and $\lambda_i \ge 0$ for all $i = 1,\ldots,m$. In the proofs of the main results of [3], they applied the Leggett–Williams and Guo–Krasnosel'skii fixed point theorems. In [4], by using the Banach fixed point theorem, the authors investigated the existence of the unique solution for the Hadamard fractional integro-differential equation subject to Hadamard fractional integral boundary conditions

$$\begin{cases} {}^{H}D^{\gamma}_{1+}\mathfrak{U}(v) + \mathrm{f}(v, \mathfrak{U}(v), {}^{H}I^{q}_{1+}\mathfrak{U}(v)) = 0, \ \ v \in (1, \infty), \\ \mathfrak{U}(1) = \mathfrak{U}'(1) = 0, \ \ {}^{H}D^{\gamma-1}_{1+}\mathfrak{U}(\infty) = \sum_{i=1}^{m} \lambda_i \, {}^{H}I^{\beta_i}_{1+}\mathfrak{U}(\eta), \end{cases}$$

where $\gamma \in (2,3)$, $\eta \in (1,\infty)$, $q > 0$, $\beta_i > 0$ and $\lambda_i \geq 0$ for all $i = 1, \ldots, m$. In [5], the authors studied the existence of positive solutions for the Hadamard fractional differential equation supplemented with Hadamard integral and multi-point boundary conditions

$$\begin{cases} {}^{H}D^{q}_{1+}\Phi(v) + \sigma(v)\mathrm{f}(v, \Phi(v)) = 0, \ \ v \in (1, \infty), \\ \Phi(1) = \Phi'(1) = 0, \ \ {}^{H}D^{q-1}_{1+}\Phi(\infty) = \mathrm{a}\, {}^{H}I^{\beta}_{1+}\Phi(\xi) + \mathrm{b} \sum_{j=1}^{m-2} \alpha_j \Phi(\eta_j), \end{cases}$$

where $q \in (2,3]$, $1 < \xi < \eta_1 < \cdots < \eta_{m-2} < \infty$, $\mathrm{a}, \mathrm{b} \in \mathbb{R}$, and $\alpha_i \geq 0$ for all $i = 1, \ldots, m-2$. In the main theorems, they used the monotone iterative method to find two "twin" positive solutions of the problem, and then presented monotone iterative schemes convergent to a unique positive solution. In [6], the authors investigated the nonlinear Hadamard fractional differential equation with nonlocal boundary conditions

$$\begin{cases} {}^{H}D^{\alpha}_{1+}\Phi(v) + \mathfrak{c}(v)\mathfrak{L}(v, \Phi(v)) = 0, \ \ v \in (1, \infty), \\ \Phi(1) = \Phi'(1) = 0, \ \ {}^{H}D^{\alpha-1}_{1+}\Phi(\infty) = \sum_{i=1}^{m} \alpha_i \, {}^{H}I^{\beta_i}_{1+}\Phi(\eta) + b \sum_{j=1}^{n} \sigma_j \Phi(\xi_j), \end{cases} \tag{4}$$

with $\alpha \in (2,3)$, $\beta_i > 0$ and $\alpha_i \geq 0$ for all $i = 1, \ldots, m$, $\sigma_j \geq 0$ for all $j = 1, \ldots, n$, $1 < \eta < \xi_1 < \ldots < \xi_n < \infty$. They studied the existence, uniqueness and multiplicity of positive solutions for problem (4), by applying the Schauder fixed point theorem, the Banach fixed point theorem, the monotone iterative method, and a fixed point theorem due to Avery and Peterson. In [7], the authors proved a generalization of a fixed point theorem due to Avery and Henderson, and then they applied it to problem (4) and proved the existence of at least three positive solutions of (4). To explore further developments in the realm of Hadamard fractional differential equations, we encourage readers to refer to the following papers: [8–15]. Additionally, we would like to highlight the significance of the monographs [2,16–25], which delve into various aspects of fractional differential equations and systems, encompassing diverse boundary conditions and their wide-ranging applications across different fields.

The paper is structured as follows: Section 2 presents a collection of preliminary results that are crucial for the subsequent sections. This includes the solution to the corresponding linear problem, an examination of the properties of Green functions, and other relevant aspects. Section 3 focuses on the main existence theorems for the problem (1),(2). Section 4 provides a detailed analysis of an example that serves to illustrate the obtained results. Finally, in Section 5, we draw conclusions based on the findings presented in our paper.

2. Auxiliary Results

Some preliminary results that will be used in our further considerations are presented in this section.

Definition 1 ([2])**.** *The Hadamard integral of fractional order $p > 0$ with lower limit* a, *(*$\mathrm{a} \geq 0$*) of a function* $\Xi : [\mathrm{a}, \infty) \to \mathbb{R}$ *is defined by*

$$({}^{H}I^{p}_{\mathrm{a}+}\Xi)(\phi) = \frac{1}{\Gamma(p)} \int_{\mathrm{a}}^{\phi} \left(\ln \frac{\phi}{v}\right)^{p-1} \frac{\Xi(v)}{v}\, dv, \ \ \phi > 0.$$

Definition 2 ([2]). *The Hadamard fractional derivative of order $p \geq 0$ with lower limit* a, *(*a ≥ 0*) of a function $\Xi : [\mathrm{a}, \infty) \to \mathbb{R}$ is defined by*

$$ {}^{H}D^{p}_{\mathrm{a}+}\Xi(\phi) = \left(\phi \frac{d}{d\phi}\right)^{n} {}^{H}I^{n-p}_{a+}\Xi(\phi) = \frac{1}{\Gamma(n-p)} \left(\phi \frac{d}{d\phi}\right)^{n} \int_{\mathrm{a}}^{\phi} \left(\ln \frac{\phi}{v}\right)^{n-p-1} \frac{\Xi(v)}{v} \, dv, $$

with $n = [p] + 1$.

For $p = m \in \mathbb{N}$, then ${}^{H}D^{m}_{\mathrm{a}+}\Xi(\phi) = (\delta^{m}\Xi)(\phi)$, $\phi > \mathrm{a}$, where $\delta = \phi \frac{d}{d\phi}$ is the δ-derivative.

Lemma 1 ([2]). *If α, $\beta > 0$, and* a > 0*, then*

$$ \begin{aligned} \left({}^{H}I^{\alpha}_{\mathrm{a}+}\left(\ln \frac{v}{\mathrm{a}}\right)^{\beta-1}\right)(\phi) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{\phi}{\mathrm{a}}\right)^{\beta+\alpha-1}, \\ \left({}^{H}D^{\alpha}_{\mathrm{a}+}\left(\ln \frac{v}{\mathrm{a}}\right)^{\beta-1}\right)(\phi) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{\phi}{\mathrm{a}}\right)^{\beta-\alpha-1}, \end{aligned} $$

and in particular, ${}^{H}D^{p}_{\mathrm{a}+}(\ln \frac{v}{\mathrm{a}})^{p-j}(\phi) = 0$, for $j = 1, \ldots, [p] + 1$.

Lemma 2 ([2]). *Let $p > 0$ and $\Xi, \mathfrak{U} \in C[1, \infty) \cap L^{1}(0, \infty)$. The solutions of the Hadamard fractional differential equation ${}^{H}D^{p}_{1+}\Xi(v) = 0$ are*

$$ \Xi(v) = \sum_{i=1}^{n} c_i (\ln v)^{p-i}, $$

and the next formula is satisfied

$$ {}^{H}I^{p}_{1+} \, {}^{H}D^{p}_{1+}\mathfrak{U}(v) = \mathfrak{U}(v) + \sum_{i=1}^{n} d_i (\ln v)^{p-i}, $$

with $c_i, d_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $n = [p] + 1$.

Let us now examine the system comprising fractional differential equations

$$ \begin{cases} {}^{H}D^{\alpha}_{1+}\Phi(v) + \mathrm{h}(v) = 0, & v \in (1, \infty), \\ {}^{H}D^{\beta}_{1+}\Psi(v) + \mathrm{k}(v) = 0, & v \in (1, \infty), \end{cases} \tag{5} $$

where $\mathrm{h}, \mathrm{k} \in C([1, \infty), \mathbb{R}_{+})$, with the boundary conditions (2). Let us define the constants:

$$ \begin{aligned} & a = \Gamma(\alpha) - \int_{1}^{\infty} (\ln \xi)^{\alpha-1} \, d\mathfrak{I}_1(\xi), \;\; b = \int_{1}^{\infty} (\ln \xi)^{\beta-1} \, d\mathfrak{I}_2(\xi), \\ & c = \int_{1}^{\infty} (\ln \xi)^{\alpha-1} \, d\mathfrak{I}_1(\xi), \;\; d = \Gamma(\beta) - \int_{1}^{\infty} (\ln \zeta)^{\beta-1} \, d\mathfrak{I}_2(\zeta), \\ & \Delta = ad - bc. \end{aligned} \tag{6} $$

Lemma 3. *Assume that a, b, c, $d \in \mathbb{R}$ and the functions $\mathrm{h}, \mathrm{k} \in C([1,\infty), \mathbb{R}_+)$ satisfy the conditions $\int_1^\infty \mathrm{h}(\xi)\,\frac{d\xi}{\xi} < \infty$ and $\int_1^\infty \mathrm{k}(\xi)\,\frac{d\xi}{\xi} < \infty$. If $\Delta \neq 0$, then the solution of problem (5),(2) is given by*

$$\begin{aligned}
\Phi(v) &= -\frac{1}{\Gamma(\alpha)}\int_1^v \left(\ln\frac{v}{\omega}\right)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad+\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg[d\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{d}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\
&\quad-\frac{d}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega) \\
&\quad+b\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{b}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\
&\quad-\frac{b}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega)\Bigg],\ \ v\in[1,\infty), \\
\Psi(v) &= -\frac{1}{\Gamma(\beta)}\int_1^v \left(\ln\frac{v}{\omega}\right)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega} \\
&\quad+\frac{(\ln v)^{\beta-1}}{\Delta}\Bigg[a\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{a}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\
&\quad-\frac{a}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega) \\
&\quad+c\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{c}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\
&\quad-\frac{c}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega)\Bigg],\ \ v\in[1,\infty).
\end{aligned} \tag{7}$$

Proof. By using Lemma 2, the solutions of system (5) are given by

$$\begin{aligned}
\Phi(v) &= -\frac{1}{\Gamma(\alpha)}\int_1^v\left(\ln\frac{v}{\omega}\right)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad+a_1(\ln v)^{\alpha-1}+a_2(\ln v)^{\alpha-2}+\cdots+a_n(\ln v)^{\alpha-n},\ \ v\in[1,\infty), \\
\Psi(v) &= -\frac{1}{\Gamma(\beta)}\int_1^v\left(\ln\frac{v}{\omega}\right)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega} \\
&\quad+b_1(\ln v)^{\beta-1}+b_2(\ln v)^{\beta-2}+\cdots+b_m(\ln v)^{\beta-m},\ \ v\in[1,\infty),
\end{aligned} \tag{8}$$

with a_j, $b_k \in \mathbb{R}$ for $j = 1,\ldots,n$, $k = 1,\ldots,m$. Because $\Phi(1) = \Phi'(1) = \cdots = \Phi^{(n-2)}(1) = 0$ and $\Psi(1) = \Psi'(1) = \cdots = \Psi^{(m-2)}(1) = 0$, we deduce that $a_j = 0$ for $j = 2,\ldots,n$ and $b_k = 0$ for $k = 2,\ldots,m$. So, by (8), one finds

$$\begin{aligned}
\Phi(v) &= -\frac{1}{\Gamma(\alpha)}\int_1^v\left(\ln\frac{v}{\omega}\right)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega}+a_1(\ln v)^{\alpha-1},\ \ v\in[1,\infty), \\
\Psi(v) &= -\frac{1}{\Gamma(\beta)}\int_1^v\left(\ln\frac{v}{\omega}\right)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}+b_1(\ln v)^{\beta-1},\ \ v\in[1,\infty).
\end{aligned} \tag{9}$$

We have ${}^{H}D_{1+}^{\alpha-1}\Phi(v) = -\int_1^v \mathrm{h}(\omega)\frac{d\omega}{\omega} + a_1\Gamma(\alpha)$ and ${}^{H}D_{1+}^{\beta-1}\Psi(v) = -\int_1^v \mathrm{k}(\omega)\frac{d\omega}{\omega} + b_1\Gamma(\beta)$. Then, the last conditions from (2), namely ${}^{H}D_{1+}^{\alpha-1}\Phi(\infty) = \int_1^\infty \Phi(\omega)\,d\mathfrak{I}_1(\omega) + \int_1^\infty \Psi(\omega)\,d\mathfrak{I}_2(\omega)$ and ${}^{H}D_{1+}^{\beta-1}\Psi(\infty) = \int_1^\infty \Phi(\omega)\,d\mathfrak{I}_1(\omega) + \int_1^\infty \Psi(\omega)\,d\mathfrak{I}_2(\omega)$, for the solution (9), give us

$$\begin{cases} -\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} + a_1\Gamma(\alpha) = \int_1^\infty \left[-\frac{1}{\Gamma(\alpha)}\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi} + a_1(\ln\omega)^{\alpha-1}\right] d\Im_1(\omega) \\ \quad + \int_1^\infty \left[-\frac{1}{\Gamma(\beta)}\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\frac{d\xi}{\xi} + b_1(\ln\omega)^{\beta-1}\right] d\Im_2(\omega), \\ -\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} + b_1\Gamma(\beta) = \int_1^\infty \left[-\frac{1}{\Gamma(\alpha)}\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi} + a_1(\ln\omega)^{\alpha-1}\right] d\Im_1(\omega) \\ \quad + \int_1^\infty \left[-\frac{1}{\Gamma(\beta)}\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\frac{d\xi}{\xi} + b_1(\ln\omega)^{\beta-1}\right] d\Im_2(\omega), \end{cases}$$

or

$$\begin{cases} a_1\left[\Gamma(\alpha) - \int_1^\infty (\ln\omega)^{\alpha-1}\,d\Im_1(\omega)\right] - b_1\int_1^\infty (\ln\omega)^{\beta-1}\,d\Im_2(\omega) \\ \quad = \int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{1}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ \quad -\frac{1}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{s}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega), \\ -a_1\int_1^\infty (\ln\omega)^{\alpha-1}\,d\Im_1(\omega) + b_1\left[\Gamma(\beta) - \int_1^\infty (\ln\omega)^{\beta-1}\,d\Im_2(\omega)\right] \\ \quad = \int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{1}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ \quad -\frac{1}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega). \end{cases}$$

By assuming the non-zero determinant Δ of the aforementioned system (with respect to the unknowns a_1 and b_1), we establish the uniqueness of the solution for the system as follows:

$$\begin{aligned} a_1 = \frac{1}{\Delta}\Bigg[& d\int_1^\infty \mathrm{h}(\omega)\frac{d\omega}{\omega} - \frac{d}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ & -\frac{d}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega) \\ & +b\int_1^\infty \mathrm{k}(\omega)\frac{d\omega}{\omega} - \frac{b}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ & -\frac{b}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega)\Bigg], \\ b_1 = \frac{1}{\Delta}\Bigg[& a\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{a}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ & -\frac{a}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega) \\ & +c\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{c}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_1(\omega) \\ & -\frac{c}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\Im_2(\omega)\Bigg]. \end{aligned}$$

By replacing the above formulas for a_1 and b_1 in (9), we obtain the solution of problem (5),(2) given by (7). The converse follows by direct computation. □

Lemma 4. *Given the conditions stated in Lemma 3, the solution to problem (5),(2) is determined as follows:*

$$\begin{cases} \Phi(v) = \int_1^\infty \mathfrak{E}_1(v,\omega)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \int_1^\infty \mathfrak{E}_2(v,\omega)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}, \ \ v \in [1,\infty), \\ \Psi(v) = \int_1^\infty \mathfrak{E}_3(v,\omega)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \int_1^\infty \mathfrak{E}_4(v,\omega)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}, \ \ v \in [1,\infty), \end{cases} \tag{10}$$

with the Green functions $\mathfrak{E}_i, \ i = 1,\ldots,4$ *given by*

$$\begin{aligned} \mathfrak{E}_1(v,\omega) &= g_\alpha(v,\omega) + \frac{(\ln v)^{\alpha-1}}{\Delta}\left(d\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi) + b\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\right), \\ \mathfrak{E}_2(v,\omega) &= \frac{(\ln v)^{\alpha-1}}{\Delta}\left(d\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi) + b\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\right), \\ \mathfrak{E}_3(v,\omega) &= \frac{(\ln v)^{\beta-1}}{\Delta}\left(c\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi) + a\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\right), \\ \mathfrak{E}_4(v,\omega) &= g_\beta(v,\omega) + \frac{(\ln v)^{\beta-1}}{\Delta}\left(c\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi) + a\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\right), \end{aligned} \tag{11}$$

and

$$\begin{aligned} g_\alpha(v,\omega) &= \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln v)^{\alpha-1} - \left(\ln\frac{v}{\omega}\right)^{\alpha-1}, \ 1 \le \omega \le v, \\ (\ln v)^{\alpha-1}, \ 1 \le v \le \omega, \end{cases} \\ g_\beta(v,\omega) &= \frac{1}{\Gamma(\beta)}\begin{cases} (\ln v)^{\beta-1} - \left(\ln\frac{v}{\omega}\right)^{\beta-1}, \ 1 \le \omega \le v, \\ (\ln v)^{\beta-1}, \ 1 \le v \le \omega. \end{cases} \end{aligned} \tag{12}$$

Proof. By (7), one obtains

$$\begin{aligned} \Phi(v) = & \frac{1}{\Gamma(\alpha)}\int_1^\infty (\ln v)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{\Delta}{\Gamma(\alpha)\Delta}\int_1^\infty (\ln v)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\ & -\frac{1}{\Gamma(\alpha)}\int_1^v \left(\ln\frac{v}{\omega}\right)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\ & +\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg[d\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{d}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\ & -\frac{d}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega) \\ & +b\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{b}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\ & -\frac{b}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega)\Bigg] \\ = & \underbrace{\frac{1}{\Gamma(\alpha)}\int_1^v\left[(\ln v)^{\alpha-1} - \left(\ln\frac{v}{\omega}\right)^{\alpha-1}\right]\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \frac{1}{\Gamma(\alpha)}\int_v^\infty (\ln v)^{\alpha-1}\mathrm{h}(\omega)\,\frac{d\omega}{\omega}}_{A_0} \\ & +\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg[-\frac{ad-bc}{\Gamma(\alpha)}\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} + d\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\ & -\frac{d}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \\ & -\frac{d}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega) + b\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} \\ & -\frac{b}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_1(\omega) \end{aligned}$$

$$
\begin{aligned}
&-\frac{b}{\Gamma(\beta)}\int_1^\infty\left(\int_1^\omega\left(\ln\frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right)d\mathfrak{I}_2(\omega)\Bigg]\\
&=A_0+\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg\{-\frac{1}{\Gamma(\alpha)}\Bigg[\Gamma(\alpha)\Gamma(\beta)-\Gamma(\beta)\int_1^\infty(\ln\omega)^{\alpha-1}\,d\mathfrak{I}_1(\omega)\\
&-\Gamma(\alpha)\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)+\left(\int_1^\infty(\ln\omega)^{\alpha-1}\,d\mathfrak{I}_1(\omega)\right)\left(\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right)\\
&-\left(\int_1^\infty(\ln\omega)^{\alpha-1}\,d\mathfrak{I}_1(\omega)\right)\left(\int_1^\infty(\ln\omega)^{\beta-1}\,\mathfrak{I}_2(\omega)\right)\Bigg]\int_1^\infty\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+\left[\Gamma(\beta)-\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right]\int_1^\infty\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&-\frac{1}{\Gamma(\alpha)}\left[\Gamma(\beta)-\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right]\int_1^\infty\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&-\frac{1}{\Gamma(\beta)}\left[\Gamma(\beta)-\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right]\int_1^\infty\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\\
&+\left(\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right)\int_1^\infty\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\\
&-\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right)\left(\int_1^\infty\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\right)\\
&-\frac{1}{\Gamma(\beta)}\left(\int_1^\infty(\ln\omega)^{\beta-1}\,d\mathfrak{I}_2(\omega)\right)\left(\int_1^\infty\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\right)\Bigg\}\\
&=A_0+\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg\{\int_1^\infty\Bigg[\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\\
&-\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\\
&-\frac{\Gamma(\beta)}{\Gamma(\alpha)}\int_\omega^\infty\left(\ln\frac{\xi}{s}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\\
&-\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right)\Bigg]\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+\int_1^\infty\Bigg[-\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\\
&+\frac{1}{\Gamma(\beta)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\\
&+\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\\
&-\frac{1}{\Gamma(\beta)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\Bigg]\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg\}\\
&=A_0+\frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg\{\int_1^\infty\Bigg\{\frac{\Gamma(\beta)}{\Gamma(\alpha)}\left[\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)-\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right]\\
&-\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\left[\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)-\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right]\\
&+\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\\
&\times\left[\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)-\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\alpha-1}d\mathfrak{I}_1(\xi)\right]\Bigg\}\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+\int_1^\infty\Bigg\{\left[\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)-\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\right]\\
&+\frac{1}{\Gamma(\beta)}\Bigg[\left(\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\left(\int_\omega^\infty\left(\ln\frac{\xi}{\omega}\right)^{\beta-1}d\mathfrak{I}_2(\xi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi)\right)\left(\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi)\right)\\
&+\left(\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi)\right)\left(\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi)\right)\\
&-\left(\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi)\right)\left(\int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{I}_2(\xi)\right)\Bigg]\Bigg\}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg\}\\
=\;& A_0 + \frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg\{\int_1^\infty \Bigg\{\frac{d}{\Gamma(\alpha)}\left[\int_1^\infty (\ln \xi)^{\alpha-1}\, d\mathfrak{I}_1(\xi) - \int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{I}_1(\xi)\right]\\
&+\frac{b}{\Gamma(\alpha)}\left[\int_1^\infty (\ln \xi)^{\alpha-1}\, d\mathfrak{I}_1(\xi) - \int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{I}_1(\xi)\right]\Bigg\}\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+\int_1^\infty \Bigg\{\frac{d}{\Gamma(\beta)}\left[\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi) - \int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{I}_2(\xi)\right]\\
&+\frac{b}{\Gamma(\beta)}\left[\int_1^\infty (\ln \xi)^{\beta-1}\, d\mathfrak{I}_2(\xi) - \int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{I}_2(\xi)\right]\Bigg\}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg\}\\
=\;& \int_1^\infty g_\alpha(v,\omega)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \frac{(\ln v)^{\alpha-1}}{\Delta}\Bigg[d\int_1^\infty \left(\int_1^\infty g_\alpha(\xi,\omega)\, d\mathfrak{I}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+b\int_1^\infty \left(\int_1^\infty g_\alpha(\xi,\omega)\, d\mathfrak{I}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega}\\
&+d\int_1^\infty \left(\int_1^\infty g_\beta(\xi,\omega)\, d\mathfrak{I}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\\
&+b\int_1^\infty \left(\int_1^\infty g_\beta(\xi,\omega)\, d\mathfrak{I}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg]\\
=\;& \int_1^\infty \mathfrak{E}_1(v,\omega)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \int_1^\infty \mathfrak{E}_2(v,\omega)\mathrm{k}(\omega)\,\frac{d\omega}{\omega},\;\; t\in[1,\infty),
\end{aligned}
$$

where g_α and g_β are given by (12), and $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are given by (11). In the last relations above, we used the equality

$$
\begin{aligned}
&\int_1^\infty (\ln \xi)^{\alpha-1}\, d\mathfrak{I}_1(\xi) - \int_\omega^\infty \left(\ln \frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{I}_1(\xi)\\
&\quad= \int_1^\omega (\ln \xi)^{\alpha-1}\, d\mathfrak{I}_1(\xi) + \int_\omega^\infty \left[(\ln \xi)^{\alpha-1} - \left(\ln \frac{\xi}{\omega}\right)^{\alpha-1}\right] d\mathfrak{I}_1(\xi)\\
&\quad= \Gamma(\alpha)\int_1^\infty g_\alpha(\xi,\omega)\, d\mathfrak{I}_1(\xi),
\end{aligned}
$$

and similar equalities for $\mathfrak{I}_2(\xi)$, $\mathfrak{J}_1(\xi)$ and $\mathfrak{J}_2(\xi)$.

In a similar manner, by (7), one finds

$$
\begin{aligned}
\Psi(v) =\;& \frac{1}{\Gamma(\beta)}\int_1^\infty (\ln v)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{\Delta}{\Gamma(\beta)\Delta}\int_1^\infty (\ln v)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\\
&-\frac{1}{\Gamma(\beta)}\int_1^v \left(\ln \frac{v}{\omega}\right)^{\beta-1}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\\
&+\frac{(\ln v)^{\beta-1}}{\Delta}\Bigg[a\int_1^\infty \mathrm{k}(\omega)\,\frac{d\omega}{\omega} - \frac{a}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln \frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\mathfrak{I}_1(\omega)\\
&-\frac{a}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln \frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\mathfrak{I}_2(\omega)\\
&+c\int_1^\infty \mathrm{h}(\omega)\,\frac{d\omega}{\omega} - \frac{c}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^\omega \left(\ln \frac{\omega}{\xi}\right)^{\alpha-1}\mathrm{h}(\xi)\,\frac{d\xi}{\xi}\right) d\mathfrak{J}_1(\omega)\\
&-\frac{c}{\Gamma(\beta)}\int_1^\infty \left(\int_1^\omega \left(\ln \frac{\omega}{\xi}\right)^{\beta-1}\mathrm{k}(\xi)\,\frac{d\xi}{\xi}\right) d\mathfrak{J}_2(\omega)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{\frac{1}{\Gamma(\beta)} \int_1^{v} \left[(\ln v)^{\beta-1} - \left(\ln \frac{v}{\omega} \right)^{\beta-1} \right] \mathrm{k}(\omega) \frac{d\omega}{\omega} + \frac{1}{\Gamma(\beta)} \int_v^{\infty} (\ln v)^{\beta-1} \mathrm{k}(\omega) \frac{d\omega}{\omega}}_{B_0} \\
&\quad + \frac{(\ln v)^{\beta-1}}{\Delta} \Bigg[-\frac{ad-bc}{\Gamma(\beta)} \int_1^{\infty} \mathrm{k}(\omega) \frac{d\omega}{\omega} + a \int_1^{\infty} \mathrm{k}(\omega) \frac{d\omega}{\omega} \\
&\quad - \frac{a}{\Gamma(\alpha)} \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\alpha-1} \mathrm{h}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_1(\omega) \\
&\quad - \frac{a}{\Gamma(\beta)} \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\beta-1} \mathrm{k}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_2(\omega) \\
&\quad + c \int_1^{\infty} \mathrm{h}(\omega) \frac{d\omega}{\omega} - \frac{c}{\Gamma(\alpha)} \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\alpha-1} \mathrm{h}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_1(\omega) \\
&\quad - \frac{c}{\Gamma(\beta)} \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\beta-1} \mathrm{k}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_2(\omega) \Bigg] \\
&= B_0 + \frac{(\ln v)^{\beta-1}}{\Delta} \Bigg\{ -\frac{1}{\Gamma(\beta)} \Bigg[\Gamma(\alpha)\Gamma(\beta) - \Gamma(\beta) \int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \\
&\quad - \Gamma(\alpha) \int_1^{\infty} (\ln \omega)^{\beta-1} d\mathfrak{I}_2(\omega) + \left(\int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right) \left(\int_1^{\infty} (\ln \omega)^{\beta-1} d\mathfrak{I}_2(\omega) \right) \\
&\quad - \left(\int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right) \left(\int_1^{\infty} (\ln \omega)^{\beta-1} d\mathfrak{I}_2(\omega) \right) \Bigg] \int_1^{\infty} \mathrm{k}(\omega) \frac{d\omega}{\omega} \\
&\quad + \left[\Gamma(\alpha) - \int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right] \int_1^{\infty} \mathrm{k}(\omega) \frac{d\omega}{\omega} \\
&\quad - \frac{1}{\Gamma(\alpha)} \left[\Gamma(\alpha) - \int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right] \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\alpha-1} \mathrm{h}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_1(\omega) \\
&\quad - \frac{1}{\Gamma(\beta)} \left[\Gamma(\alpha) - \int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right] \int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\beta-1} \mathrm{k}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_2(\omega) \\
&\quad + \left(\int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right) \int_1^{\infty} \mathrm{h}(\omega) \frac{d\omega}{\omega} \\
&\quad - \frac{1}{\Gamma(\alpha)} \left(\int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right) \left(\int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\alpha-1} \mathrm{h}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_1(\omega) \right) \\
&\quad - \frac{1}{\Gamma(\beta)} \left(\int_1^{\infty} (\ln \omega)^{\alpha-1} d\mathfrak{I}_1(\omega) \right) \left(\int_1^{\infty} \left(\int_1^{\omega} \left(\ln \frac{\omega}{\xi} \right)^{\beta-1} \mathrm{k}(\xi) \frac{d\xi}{\xi} \right) d\mathfrak{I}_2(\omega) \right) \Bigg\} \\
&= B_0 + \frac{(\ln v)^{\beta-1}}{\Delta} \Bigg\{ \int_1^{\infty} \Bigg[- \int_{\omega}^{\infty} \left(\ln \frac{\xi}{\omega} \right)^{\alpha-1} d\mathfrak{I}_1(\xi) \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \int_{\omega}^{\infty} \left(\ln \frac{\xi}{s} \right)^{\alpha-1} d\mathfrak{I}_1(\xi) \\
&\quad + \int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \\
&\quad - \frac{1}{\Gamma(\alpha)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \int_{\omega}^{\infty} \left(\ln \frac{\xi}{\omega} \right)^{\alpha-1} d\mathfrak{I}_1(\xi) \Bigg] \mathrm{h}(\omega) \frac{d\omega}{\omega} \\
&\quad + \int_1^{\infty} \Bigg[\frac{\Gamma(\alpha)}{\Gamma(\beta)} \int_1^{\infty} (\ln \xi)^{\beta-1} d\mathfrak{I}_2(\xi) \\
&\quad - \frac{1}{\Gamma(\beta)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \left(\int_1^{\infty} (\ln \xi)^{\beta-1} d\mathfrak{I}_2(\xi) \right) \\
&\quad + \frac{1}{\Gamma(\beta)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \left(\int_1^{\infty} (\ln \xi)^{\beta-1} d\mathfrak{I}_2(\xi) \right) \\
&\quad - \frac{\Gamma(\alpha)}{\Gamma(\beta)} \int_{\omega}^{\infty} \left(\ln \frac{\xi}{\omega} \right)^{\beta-1} d\mathfrak{I}_2(\xi) \\
&\quad + \frac{1}{\Gamma(\beta)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \int_{\omega}^{\infty} \left(\ln \frac{\xi}{\omega} \right)^{\beta-1} d\mathfrak{I}_2(\xi) \\
&\quad - \frac{1}{\Gamma(\beta)} \left(\int_1^{\infty} (\ln \xi)^{\alpha-1} d\mathfrak{I}_1(\xi) \right) \int_{\omega}^{\infty} \left(\ln \frac{\xi}{\omega} \right)^{\beta-1} d\mathfrak{I}_2(\xi) \Bigg] \mathrm{k}(\omega) \frac{d\omega}{\omega} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= B_0 + \frac{(\ln v)^{\beta-1}}{\Delta}\Bigg\{\int_1^\infty \Bigg\{\Bigg[\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{J}_1(\xi)\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{J}_1(\xi) \\
&\quad - \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi)\right) \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi)\right)\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right) \\
&\quad - \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi)\right)\left(\int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{I}_1(\xi)\right)\Bigg\}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \Bigg\{\frac{\Gamma(\alpha)}{\Gamma(\beta)}\Bigg[\int_1^\infty (\ln\xi)^{\beta-1}\,d\mathfrak{J}_2(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{J}_2(\xi)\Bigg] \\
&\quad - \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\Bigg[\int_1^\infty (\ln\xi)^{\beta-1}\,d\mathfrak{J}_2(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{J}_2(\xi)\Bigg] \\
&\quad + \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi)\right)\Bigg[\int_1^\infty (\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi) \\
&\quad - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{I}_2(\xi)\Bigg]\Bigg\}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg\} \\
&= B_0 + \frac{(\ln v)^{\beta-1}}{\Delta}\Bigg\{\int_1^\infty \Bigg\{\frac{a}{\Gamma(\alpha)}\Bigg[\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{J}_1(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{J}_1(\xi)\Bigg] \\
&\quad + \frac{c}{\Gamma(\alpha)}\Bigg[\int_1^\infty (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\alpha-1} d\mathfrak{I}_1(\xi)\Bigg]\Bigg\}\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \Bigg\{\frac{a}{\Gamma(\beta)}\Bigg[\int_1^\infty (\ln\xi)^{\beta-1}\,d\mathfrak{J}_2(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{J}_2(\xi)\Bigg] \\
&\quad + \frac{c}{\Gamma(\beta)}\Bigg[\int_1^\infty (\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi) - \int_\omega^\infty \left(\ln\frac{\xi}{\omega}\right)^{\beta-1} d\mathfrak{I}_2(\xi)\Bigg]\Bigg\}\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg\} \\
&= \int_1^\infty g_\beta(v,\omega)\mathrm{k}(\omega)\,\frac{d\omega}{\omega} + \frac{(\ln v)^{\beta-1}}{\Delta}\Bigg[a\int_1^\infty\left(\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{J}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + c\int_1^\infty\left(\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\right)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + a\int_1^\infty\left(\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{J}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + c\int_1^\infty\left(\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\right)\mathrm{k}(\omega)\,\frac{d\omega}{\omega}\Bigg] \\
&= \int_1^\infty \mathfrak{E}_3(v,\omega)\mathrm{h}(\omega)\,\frac{d\omega}{\omega} + \int_1^\infty \mathfrak{E}_4(v,\omega)\mathrm{k}(\omega)\,\frac{d\omega}{\omega},\ \ t\in[1,\infty),
\end{aligned}
$$

where $\mathfrak{E}_3$ and $\mathfrak{E}_4$ are given by (11). So we deduced the solution $(\Phi(v),\Psi(v)),\ v\in[1,\infty)$ given by (10). □

The next lemma can be easily obtained by utilizing the definitions of the functions g_α, g_β, and $\mathfrak{E}_i$, where $i=1,\ldots,4$.

Lemma 5. *Assume that the functions $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{J}_1$, $\mathfrak{J}_2$ are nondecreasing functions, a, $d\in\mathbb{R}_+$, b, $c\in\mathbb{R}$, and $\Delta>0$, and let $\theta>1$. Then, the functions g_α, g_β, and $\mathfrak{E}_i$, $i=1,\ldots,4$ have the following properties:*

(a) The functions g_α and g_β are continuous on $[1,\infty)\times[1,\infty)$;

(b) $0\le g_\alpha(v,\omega)\le\dfrac{1}{\Gamma(\alpha)}(\ln v)^{\alpha-1},\ \ 0\le g_\beta(v,\omega)\le\dfrac{1}{\Gamma(\beta)}(\ln v)^{\beta-1},\ \ \forall\, v,\omega\in[1,\infty)$;

(c) $0\le\dfrac{g_\alpha(v,\omega)}{1+(\ln v)^{\alpha-1}}\le\dfrac{1}{\Gamma(\alpha)},\ \ 0\le\dfrac{g_\beta(v,\omega)}{1+(\ln v)^{\beta-1}}\le\dfrac{1}{\Gamma(\beta)},\ \ \forall\, v,\omega\in[1,\infty)$;

(d) The functions $\mathfrak{E}_i,\ i=1,\ldots,4$ are continuous on $[1,\infty)\times[1,\infty)$;

(e) $\mathfrak{E}_i(v,\omega)\ge 0$ for all $(v,\omega)\in[1,\infty)\times[1,\infty)$ and $i=1,\ldots,4$;

(f) $\dfrac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}} \le \dfrac{1}{\Gamma(\alpha)} + \dfrac{1}{\Delta\Gamma(\alpha)}\left(d\int_1^\infty (\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta) + b\int_1^\infty (\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta)\right)$
$= \dfrac{d}{\Delta} =: \Lambda_1 > 0,\ \ \forall\, v,\omega \in [1,\infty);$

(g) $\dfrac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}} \le \dfrac{1}{\Delta\Gamma(\beta)}\left(d\int_1^\infty (\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta) + b\int_1^\infty (\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta)\right)$
$= \dfrac{b}{\Delta} =: \Lambda_2 \ge 0,\ \ \forall\, v,\omega \in [1,\infty);$

(h) $\dfrac{\mathfrak{E}_3(v,\omega)}{1+(\ln v)^{\beta-1}} \le \dfrac{1}{\Delta\Gamma(\alpha)}\left(c\int_1^\infty (\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta) + a\int_1^\infty (\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta)\right)$
$= \dfrac{c}{\Delta} =: \Lambda_3 \ge 0,\ \ \forall\, v,\omega \in [1,\infty);$

(i) $\dfrac{\mathfrak{E}_4(v,\omega)}{1+(\ln v)^{\beta-1}} \le \dfrac{1}{\Gamma(\beta)} + \dfrac{1}{\Delta\Gamma(\beta)}\left(c\int_1^\infty (\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta) + a\int_1^\infty (\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta)\right)$
$= \dfrac{a}{\Delta} =: \Lambda_4 > 0,\ \ \forall\, v,\omega \in [1,\infty);$

(j) $\min\limits_{v\in[\theta,\infty)} \dfrac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}} \ge \dfrac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$
$\times\left(d\int_1^\infty g_\alpha(\zeta,s)\,d\mathfrak{I}_1(\zeta) + b\int_1^\infty g_\alpha(\zeta,s)\,d\mathfrak{I}_1(\zeta)\right), \forall\,\omega \in [1,\infty);$

(k) $\min\limits_{v\in[\theta,\infty)} \dfrac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}} = \dfrac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$
$\times\left(d\int_1^\infty g_\beta(\zeta,s)\,d\mathfrak{I}_2(\zeta) + b\int_1^\infty g_\beta(\zeta,s)\,d\mathfrak{I}_2(\zeta)\right), \forall\,\omega \in [1,\infty);$

(l) $\min\limits_{v\in[\theta,\infty)} \dfrac{\mathfrak{E}_3(v,\omega)}{1+(\ln v)^{\beta-1}} = \dfrac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$
$\times\left(c\int_1^\infty g_\alpha(\zeta,s)\,d\mathfrak{I}_1(\zeta) + a\int_1^\infty g_\alpha(\zeta,s)\,d\mathfrak{I}_1(\zeta)\right), \forall\,\omega \in [1,\infty);$

(m) $\min\limits_{v\in[\theta,\infty)} \dfrac{\mathfrak{E}_4(v,\omega)}{1+(\ln v)^{\beta-1}} \ge \dfrac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$
$\times\left(c\int_1^\infty g_\beta(\zeta,s)\,d\mathfrak{I}_2(\zeta) + a\int_1^\infty g_\beta(\zeta,s)\,d\mathfrak{I}_2(\zeta)\right), \forall\,\omega \in [1,\infty).$

Remark 1. *Under the assumptions of Lemma 5, one finds that $a, d > 0$ and $b, c \ge 0$, so $\Lambda_1, \Lambda_4 > 0$ and $\Lambda_2, \Lambda_3 \ge 0$.*

3. Main Results

We introduce the space

$$X_1 = \left\{\Phi \in C(I,\mathbb{R}),\ \sup_{v\in I} \frac{|\Phi(v)|}{1+(\ln v)^{\alpha-1}} < \infty\right\},$$

where $I = [1,\infty)$, with the norm $\|\Phi\|_1 = \sup_{v\in I} \frac{|\Phi(v)|}{1+(\ln v)^{\alpha-1}}$, the space

$$X_2 = \left\{\Psi \in C(I,\mathbb{R}),\ \sup_{v\in I} \frac{|\Psi(v)|}{1+(\ln v)^{\beta-1}} < \infty\right\},$$

with the norm $\|\Psi\|_2 = \sup_{v\in I} \frac{|\Psi(v)|}{1+(\ln v)^{\beta-1}}$, and the space $X = X_1 \times X_2$ with the norm $\|(\Phi,\Psi)\| = \|\Phi\|_1 + \|\Psi\|_2$. The spaces $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ and $(X, \|(\cdot,\cdot)\|)$ are Banach spaces (see [3], Lemma 2.7).

Lemma 6 ([3], Lemma 2.8)**.** *Let $\Omega \subset X_1$ be a bounded set, which satisfies the following conditions:*

(i) The functions $\dfrac{\Phi(v)}{1+(\ln v)^{\alpha-1}}$, $\Phi \in \Omega$ are equicontinuous on any compact interval of I;

(ii) For any $\epsilon > 0$, there exists a constant $T = T(\epsilon) > 0$ such that

$$\left| \frac{\Phi(v_1)}{1 + (\ln v_1)^{\alpha-1}} - \frac{\Phi(v_2)}{1 + (\ln v_2)^{\alpha-1}} \right| < \epsilon, \ \forall\, v_1, v_2 \geq T, \ \forall\, \Phi \in \Omega.$$

Then, Ω is relatively compact in X_1.

Let us now define the positive cone $\mathcal{P} \subset X$ by

$$\mathcal{P} = \{(\Phi, \Psi) \in X, \ \Phi(v) \geq 0, \ \Psi(v) \geq 0, \ \forall\, v \in [1, \infty)\},$$

and the operator $\mathcal{A} : \mathcal{P} \to X$ by $\mathcal{A}(\Phi, \Psi) = (\mathcal{A}_1(\Phi, \Psi), \mathcal{A}_2(\Phi, \Psi)), \ (\Phi, \Psi) \in \mathcal{P}$, where the operators $\mathcal{A}_1 : \mathcal{P} \to X_1$ and $\mathcal{A}_2 : \mathcal{P} \to X_2$ are defined by

$$\begin{aligned}
\mathcal{A}_1(\Phi, \Psi)(v) &= \int_1^\infty \mathfrak{E}_1(v, \omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \mathfrak{E}_2(v, \omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega}, \ \ v \in I, \\
\mathcal{A}_2(\Phi, \Psi)(v) &= \int_1^\infty \mathfrak{E}_3(v, \omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \mathfrak{E}_4(v, \omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega}, \ \ v \in I,
\end{aligned}$$

for $(\Phi, \Psi) \in \mathcal{P}$.

Subsequently, we outline the fundamental assumptions that will serve as the foundation for our main results.

$(H1)$ $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 2$, $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{J}_1, \mathfrak{J}_2 : [1, \infty) \to \mathbb{R}$ are nondecreasing functions, $a, d \in \mathbb{R}_+$, $b, c \in \mathbb{R}$, and $\Delta > 0$ (given by (6)).

$(H2)$ The functions $\mathfrak{L}$ and $\mathfrak{M}$ belong to the set of continuous functions $C(\mathbb{R}+ \times \mathbb{R}+, \mathbb{R}+)$, and they are non-zero on every subinterval of $(0, \infty) \times (0, \infty)$. Furthermore, both $\mathfrak{L}$ and $\mathfrak{M}$ are bounded on the entire domain $\mathbb{R}+ \times \mathbb{R}_+$.

$(H3)$ The functions $\mathfrak{c}, \mathfrak{d} : [1, \infty) \to \mathbb{R}_+$ are non-zero on any subinterval of $[1, \infty)$, and $0 < \int_1^\infty \frac{\mathfrak{c}(\omega)}{\omega} d\omega < \infty, 0 < \int_1^\infty \frac{\mathfrak{d}(\omega)}{\omega} d\omega < \infty$.

Lemma 7. *If $(H1)$–$(H3)$ hold, then the operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Under the assumptions of this lemma, one obtains $\mathcal{A}(\Phi, \Psi) \in \mathcal{P}$ for all $(\Phi, \Psi) \in \mathcal{P}$, that is, $\mathcal{A} : \mathcal{P} \to \mathcal{P}$. We will prove this lemma in four steps.

(I) We show firstly that the operator $\mathcal{A}$ is uniformly bounded on $\mathcal{P}$. Let $\mathcal{S}$ be a bounded set of $\mathcal{P}$. Then, there exists $r > 0$ such that $\|(\Phi, \Psi)\| \leq r$, and so $\|\Phi\|_1 \leq r$ and $\|\Psi\|_2 \leq r$ for all $(\Phi, \Psi) \in \mathcal{S}$. By $(H2)$, there exist $M_1 > 0$ and $M_2 > 0$ such that $\mathfrak{L}(\Phi, \Psi) \leq M_1$ and $\mathfrak{M}(\Phi, \Psi) \leq M_2$ for all Φ, $\Psi \in \mathbb{R}_+$. Then, by $(H3)$, for any $(\Phi, \Psi) \in \mathcal{S}$, one obtains

$$\begin{aligned}
\|\mathcal{A}_1(\Phi, \Psi)\|_1 &= \sup_{v \in I} \frac{|\mathcal{A}_1(\Phi, \Psi)(v)|}{1 + (\ln v)^{\alpha-1}} \\
&= \sup_{v \in I} \frac{1}{1 + (\ln v)^{\alpha-1}} \left(\int_1^\infty \mathfrak{E}_1(v, \omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega} \right. \\
&\quad \left. + \int_1^\infty \mathfrak{E}_2(v, \omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega} \right) \\
&\leq M_1 \Lambda_1 \int_1^\infty \mathfrak{c}(\omega)\, \frac{d\omega}{\omega} + M_2 \Lambda_2 \int_1^\infty \mathfrak{d}(\omega)\, \frac{d\omega}{\omega} =: M_3 < \infty, \\
\|\mathcal{A}_2(\Phi, \Psi)\|_2 &= \sup_{v \in I} \frac{|\mathcal{A}_2(\Phi, \Psi)(v)|}{1 + (\ln v)^{\beta-1}} \\
&= \sup_{v \in I} \frac{1}{1 + (\ln v)^{\beta-1}} \left(\int_1^\infty \mathfrak{E}_3(v, \omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega))\, \frac{d\omega}{\omega} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_1^\infty \mathfrak{E}_4(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\Big)\\
&\le M_1\Lambda_3\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega}+M_2\Lambda_4\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega}=: M_4<\infty.
\end{aligned}
$$

So,

$$
\|\mathcal{A}(\Phi,\Psi)\|=\|\mathcal{A}_1(\Phi,\Psi)\|_1+\|\mathcal{A}_2(\Phi,\Psi)\|_2\le M_3+M_4<\infty,\ \ \forall\,(\Phi,\Psi)\in\mathcal{S},
$$

that is, operator $\mathcal{A}$ is uniformly bounded.

(II) We will now demonstrate the equicontinuity of $\mathcal{A}$ on every compact interval of I. Let us consider the interval $[1,T]$, where $T>1$. Then, for any $v_1,v_2\in[1,T]$, with $v_1<v_2$ and $(\Phi,\Psi)\in\mathcal{S}$, we have

$$
\begin{aligned}
\Xi_1&=\left|\frac{\mathcal{A}_1(\Phi,\Psi)(v_2)}{1+(\ln v_2)^{\alpha-1}}-\frac{\mathcal{A}_1(\Phi,\Psi)(v_1)}{1+(\ln v_1)^{\alpha-1}}\right|\\
&=\Big|\int_1^\infty \frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad+\int_1^\infty \frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad-\int_1^\infty \frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad-\int_1^\infty \frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\Big|\\
&\le\left|\int_1^\infty\left(\frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\right|\\
&\quad+\left|\int_1^\infty\left(\frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\right|\\
&\le\int_1^\infty\left|\frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right|\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad+\int_1^\infty\left|\frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right|\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\le\int_1^\infty\Bigg[\left|\frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right|\\
&\quad+\frac{1}{\Delta}\left(d\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)+b\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\right)\\
&\quad\times\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\Bigg]\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad+\int_1^\infty\frac{1}{\Delta}\left(d\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)+b\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\right)\\
&\quad\times\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\le M_1\underbrace{\int_1^\infty\left|\frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}}-\frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right|\mathfrak{c}(\omega)\,\frac{d\omega}{\omega}}_{\Xi_0}\\
&\quad+\frac{M_1}{\Delta\Gamma(\alpha)}\left(d\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)+b\int_1^\infty(\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\\
&\quad\times\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\int_1^\infty\mathfrak{c}(\omega)\,\frac{d\omega}{\omega}\\
&\quad+\frac{M_2}{\Delta\Gamma(\beta)}\left(d\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)+b\int_1^\infty(\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\\
&\quad\times\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\int_1^\infty\mathfrak{d}(\omega)\,\frac{d\omega}{\omega}\\
&=M_1\Xi_0+\frac{M_1(d\Gamma(\alpha)-\Delta)}{\Delta\Gamma(\alpha)}\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\int_1^\infty\mathfrak{c}(\omega)\,\frac{d\omega}{\omega}\\
&\quad+\frac{M_2 b}{\Delta}\left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}}-\frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right)\int_1^\infty\mathfrak{d}(\omega)\,\frac{d\omega}{\omega}.
\end{aligned}
$$

For Ξ_0, one deduces

$$
\begin{aligned}
\Xi_0 &= \int_1^{v_1} \left| \frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}} \right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \int_{v_1}^{v_2} \left| \frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}} \right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \int_{t_2}^{\infty} \left| \frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}} \right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&= \frac{1}{\Gamma(\alpha)} \int_1^{v_1} \left| \frac{(\ln v_2)^{\alpha-1} - \left(\ln \frac{v_2}{\omega}\right)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1} - \left(\ln \frac{v_1}{s}\right)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{v_1}^{v_2} \left| \frac{(\ln v_2)^{\alpha-1} - \left(\ln \frac{v_2}{\omega}\right)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{v_2}^{\infty} \left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right) \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\le \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right) \int_1^{v_1} \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^{v_1} \left(\frac{\left(\ln \frac{v_2}{\omega}\right)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{\left(\ln \frac{v_1}{\omega}\right)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right) \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right) \int_{v_1}^{v_2} \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{v_1}^{v_2} \frac{\left(\ln \frac{v_2}{s}\right)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\quad + \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right) \int_{v_2}^{\infty} \mathfrak{c}(\omega)\,\frac{d\omega}{\omega}.
\end{aligned}
$$

Because the functions $\frac{(\ln v)^{\alpha-1}}{1+(\ln v)^{\alpha-1}}$ and $\frac{(\ln(v/\omega))^{\alpha-1}}{1+(\ln v)^{\alpha-1}}$ are uniformly continuous on $[1,T]$, respectively on $[1,T]\times[1,T]$, and the integrals $\int_0^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega}$, $\int_0^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega}$ are convergent, we conclude that $\Xi_0 \to 0$ and $\Xi_1 \to 0$ as $v_2 \to v_1$ uniformly with respect to $(\Phi,\Psi)\in\mathcal{S}$.

In a similar manner, one obtains

$$
\left| \frac{\mathcal{A}_2(\Phi,\Psi)(v_2)}{1+(\ln v_2)^{\beta-1}} - \frac{\mathcal{A}_2(\Phi,\Psi)(v_1)}{1+(\ln v_1)^{\beta-1}} \right| \to 0, \text{ as } v_2 \to v_1,
$$

uniformly with respect to $(\Phi,\Psi)\in\mathcal{S}$. Then, $\mathcal{A}(\mathcal{S})$ is equicontinuous on $[1,T]$.

(III) In what follows, we will show that $\mathcal{A}$ is equiconvergent at ∞. We will prove firstly that $\mathcal{A}_1$ is equiconvergent at ∞, that is, for any $\epsilon>0$ there exists $T_0>0$ such that for all $v_1, v_2 \ge T_0$ and $(\Phi,\Psi)\in\mathcal{S}$, one finds

$$
\left| \frac{\mathcal{A}_1(\Phi,\Psi)(v_2)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathcal{A}_1(\Phi,\Psi)(v_1)}{1+(\ln v_1)^{\alpha-1}} \right| < \Lambda_0\epsilon, \ (\Lambda_0>0).
$$

For this, let $\epsilon>0$. Then, there exists $\delta_1>1$ such that $\int_{\delta_1}^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} < \epsilon$ and $\int_{\delta_1}^\infty \mathfrak{d}(\omega)\frac{d\omega}{\omega} < \epsilon$. Because $\lim_{v\to\infty} \frac{(\ln v)^{\alpha-1}}{1+(\ln v)^{\alpha-1}} = 1$ and $\lim_{v\to\infty} \frac{g_\alpha(v,\delta_1)}{1+(\ln v)^{\alpha-1}} = 0$, one deduces that there exist $\delta_2>0$ and $\delta_3>\delta_1$ such that for any $v_1, v_2 > \delta_2$ we have

$$
\left| \frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}} \right| < \epsilon,
$$

and for any $v_1, v_2 > \delta_3$ and $1 \le \omega \le \delta_1$, one finds

$$
\begin{aligned}
\left| \frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}} \right| &\le \left| \frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} \right| + \left| \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}} \right| \\
&\le \left| \frac{g_\alpha(v_2,\delta_1)}{1+(\ln v_2)^{\alpha-1}} \right| + \left| \frac{g_\alpha(v_1,\delta_1)}{1+(\ln v_1)^{\alpha-1}} \right| < \epsilon.
\end{aligned}
$$

Let us choose $T_0 > \max\{\delta_2, \delta_3\}$. Then, for any $(\Phi, \Psi) \in \mathcal{S}$ and $v_1, v_2 \geq T_0$, one obtains

$$\begin{aligned}
&\left|\frac{\mathcal{A}_1(\Phi,\Psi)(v_2)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathcal{A}_1(\Phi,\Psi)(v_1)}{1+(\ln v_1)^{\alpha-1}}\right| \\
&\quad \leq \int_1^\infty \left|\frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\qquad + \int_1^\infty \left|\frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad \leq \int_1^{\delta_1} \left|\frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\qquad + \int_{\delta_1}^\infty \left|\frac{\mathfrak{E}_1(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_1(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{c}(\omega)\mathfrak{L}(x(\omega),y(\omega))\,\frac{d\omega}{\omega} \\
&\qquad + \int_1^{\delta_1} \left|\frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\qquad + \int_{\delta_1}^\infty \left|\frac{\mathfrak{E}_2(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{\mathfrak{E}_2(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad \leq M_1 \int_1^{\delta_1} \left|\frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\qquad + M_1 \int_{\delta_1}^\infty \left|\frac{g_\alpha(v_2,\omega)}{1+(\ln v_2)^{\alpha-1}} - \frac{g_\alpha(v_1,\omega)}{1+(\ln v_1)^{\alpha-1}}\right| \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\qquad + \frac{M_1}{\Delta}\left|\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right| \\
&\qquad \times \left(d\int_1^\infty \frac{1}{\Gamma(\alpha)}\xi^{\alpha-1}\,d\mathfrak{I}_1(\xi) + b\int_1^\infty \frac{1}{\Gamma(\alpha)}\xi^{\alpha-1}\,d\mathfrak{I}_1(\xi)\right)\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\qquad + \frac{M_2}{\Delta}\left|\frac{(\ln v_2)^{\alpha-1}}{1+(\ln v_2)^{\alpha-1}} - \frac{(\ln v_1)^{\alpha-1}}{1+(\ln v_1)^{\alpha-1}}\right| \\
&\qquad \times \left(d\int_1^\infty \frac{1}{\Gamma(\beta)}\xi^{\beta-1}\,d\mathfrak{I}_2(\xi) + b\int_1^\infty \frac{1}{\Gamma(\beta)}\xi^{\beta-1}\,d\mathfrak{I}_2(\xi)\right)\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} \\
&\quad \leq \epsilon M_1 \int_1^{\delta_1} \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} + \frac{2\epsilon M_1}{\Gamma(\alpha)} + \frac{\epsilon M_1(d\Gamma(\alpha)-\Delta)}{\Delta\Gamma(\alpha)}\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} \\
&\qquad + \frac{\epsilon M_2 b}{\Delta}\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} = \Lambda_0\epsilon, \quad (\Lambda_0 > 0).
\end{aligned}$$

So, $\mathcal{A}_1$ is equiconvergent at ∞. In a similar manner, we show that $\mathcal{A}_2$ is equiconvergent at ∞, and then one deduces that $\mathcal{A}$ is equiconvergent at ∞.

(IV) In the final part of the proof, we will establish the continuity of the operator $\mathcal{A}$. Let $((\Phi_n, \Psi_n))_{n\in\mathbb{N}}$, $(\Phi, \Psi) \in X$, $(\Phi_n, \Psi_n) \to (\Phi, \Psi)$ in $X = X_1 \times X_2$, for $n \to \infty$, that is,

$$\sup_{v\in I} \frac{|\Phi_n(v) - \Phi(v)|}{1+(\ln v)^{\alpha-1}} \to 0, \text{ and } \sup_{v\in I} \frac{|\Psi_n(v) - \Psi(v)|}{1+(\ln v)^{\beta-1}} \to 0, \text{ as } n \to \infty.$$

Then, one deduces that for any $\omega \in I$, $\Phi_n(\omega) - \Phi(\omega) \to 0$ and $\Psi_n(\omega) - \Psi(\omega) \to 0$, as $n \to \infty$.

Because

$$\begin{aligned}
&|\mathfrak{L}(\Phi_n(\omega),\Psi_n(\omega)) - \mathfrak{L}(\Phi(\omega),\Psi(\omega))| \leq 2M_1, \ \forall\,\omega \in I, \ n \in \mathbb{N}, \\
&|\mathfrak{M}(\Phi_n(\omega),\Psi_n(\omega)) - \mathfrak{M}(\Phi(\omega),\Psi(\omega))| \leq 2M_2, \ \forall\,\omega \in I, \ n \in \mathbb{N},
\end{aligned}$$

one obtains by the Lebesgue convergence theorem that

$$\begin{aligned}
&\int_1^\infty \mathfrak{c}(\omega)|\mathfrak{L}(\Phi_n(\omega),\Psi_n(\omega)) - \mathfrak{L}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega} \to 0, \text{ as } n \to \infty, \\
&\int_1^\infty \mathfrak{d}(\omega)|\mathfrak{M}(\Phi_n(\omega),\Psi_n(\omega)) - \mathfrak{M}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega} \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{13}$$

By using Lemma 5 and (13), one finds

$$
\begin{aligned}
&\|\mathcal{A}_1(\Phi_n,\Psi_n)-\mathcal{A}_1(\Phi,\Psi)\|_1 = \sup_{v\in I}\frac{|\mathcal{A}_1(\Phi_n,\Psi_n)(v)-\mathcal{A}_1(\Phi,\Psi)(v)|}{1+(\ln v)^{\alpha-1}}\\
&= \sup_{v\in I}\bigg|\int_1^\infty \frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi_n(\omega),\Psi_n(\omega))\,\frac{d\omega}{\omega}\\
&\quad +\int_1^\infty \frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi_n(\omega),\Psi_n(\omega))\,\frac{d\omega}{\omega}\\
&\quad -\int_1^\infty \frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\\
&\quad -\int_1^\infty \frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\bigg|\\
&\le \sup_{v\in I}\bigg(\int_1^\infty \frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{c}(\omega)|\mathfrak{L}(\Phi_n(\omega),\Psi_n(\omega))-\mathfrak{L}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega}\\
&\quad +\int_1^\infty \frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{d}(\omega)|\mathfrak{M}(\Phi_n(\omega),\Psi_n(\omega))-\mathfrak{M}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega}\bigg)\\
&\le \Lambda_1\int_1^\infty \mathfrak{c}(\omega)|\mathfrak{L}(\Phi_n(\omega),\Psi_n(\omega))-\mathfrak{L}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega}\\
&\quad +\Lambda_2\int_1^\infty \mathfrak{d}(\omega)|\mathfrak{M}(\Phi_n(\omega),\Psi_n(\omega))-\mathfrak{M}(\Phi(\omega),\Psi(\omega))|\,\frac{d\omega}{\omega}\to 0,\ \text{as}\ n\to\infty.
\end{aligned}
$$

Then, we deduce that $\|\mathcal{A}_1(\Phi_n,\Psi_n)-\mathcal{A}_1(\Phi,\Psi)\|_1\to 0$, as $n\to\infty$. Using a similar approach, we can demonstrate that $\|\mathcal{A}_2(\Phi_n,\Psi_n)-\mathcal{A}_2(\Phi,\Psi)\|_2\to 0$, as $n\to\infty$. Therefore, one obtains $\|\mathcal{A}(\Phi_n,\Psi_n)-\mathcal{A}(\Phi,\Psi)\|\to 0$ as $n\to\infty$. So, the operator $\mathcal{A}$ is continuous.

By combining Lemma 6 with the aforementioned steps, one deduces that the operator $\mathcal{A}$ is indeed completely continuous. □

For $\theta>1$, let us now introduce the following constants:

$$
\begin{aligned}
&Q_1=\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega},\ Q_2=\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega},\ Q_3=\int_\theta^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega},\ Q_4=\int_\theta^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega},\\
&\widetilde{L}_1=\frac{d}{\Gamma(\alpha)}\int_1^\theta(\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta)+\frac{b}{\Gamma(\alpha)}\int_1^\theta(\ln\zeta)^{\alpha-1}\,d\mathfrak{J}_1(\zeta),\\
&\widetilde{L}_2=\frac{d}{\Gamma(\beta)}\int_1^\theta(\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta)+\frac{b}{\Gamma(\beta)}\int_1^\theta(\ln\zeta)^{\beta-1}\,d\mathfrak{J}_2(\zeta),\\
&\widetilde{L}_3=\frac{c}{\Gamma(\alpha)}\int_1^\theta(\ln\zeta)^{\alpha-1}\,d\mathfrak{I}_1(\zeta)+\frac{a}{\Gamma(\alpha)}\int_1^\theta(\ln\zeta)^{\alpha-1}\,d\mathfrak{J}_1(\zeta),\\
&\widetilde{L}_4=\frac{c}{\Gamma(\beta)}\int_1^\theta(\ln\zeta)^{\beta-1}\,d\mathfrak{I}_2(\zeta)+\frac{a}{\Gamma(\beta)}\int_1^\theta(\ln\zeta)^{\beta-1}\,d\mathfrak{J}_2(\zeta),\\
&L_1=\frac{\widetilde{L}_1(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\ L_2=\frac{\widetilde{L}_2(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\\
&L_3=\frac{\widetilde{L}_3(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\ L_4=\frac{\widetilde{L}_4(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\\
&\Upsilon_1=Q_3(L_1+L_3),\ \Upsilon_2=Q_4(L_2+L_4),\ \Upsilon_3=Q_1(\Lambda_1+\Lambda_3),\ \Upsilon_4=Q_2(\Lambda_2+\Lambda_4).
\end{aligned}
\tag{14}
$$

Our first main theorem establishes the existence of at least one positive solution to the problem (1),(2) by employing the Guo–Krasnosel'skii fixed point theorem.

Theorem 1. *Assume that assumptions* $(H1)$–$(H3)$ *hold, and there exists* $\theta>1$ *such that* $Q_j>0$, $j=3,4$, *and* $\widetilde{L}_i>0$, $i=1,\ldots,4$. *In addition, suppose that there exist positive constants* r_1,r_2 *with* $r_1<r_2$, *and* $\sigma_1\in[\Upsilon_1^{-1},\infty)$, $\sigma_2\in[\Upsilon_2^{-1},\infty)$, $\sigma_3\in(0,\Upsilon_3^{-1}]$ *and* $\sigma_4\in(0,\Upsilon_4^{-1}]$ *such that*

$(H4)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\ge\frac{\sigma_1 r_1}{2},\ \forall\,v\in[\theta,\infty),\ (\phi,\psi)\in[0,r_1]\times[0,r_1]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\ge\frac{\sigma_2 r_1}{2},\ \forall\,v\in[\theta,\infty),\ (\phi,\psi)\in[0,r_1]\times[0,r_1]$;

$(H5)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\le\frac{\sigma_3 r_2}{2},\ \forall\,v\in I,\ (\phi,\psi)\in[0,r_2]\times[0,r_2]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\le\frac{\sigma_4 r_2}{2},\ \forall\,v\in I,\ (\phi,\psi)\in[0,r_2]\times[0,r_2]$.

Then, the problem (1),(2) has at least one positive solution (Φ,Ψ) *such that* $r_1\le\|(\Phi,\Psi)\|\le r_2$.

Proof. The operator $\mathcal{A}$ is proven to be completely continuous according to Lemma 7. We introduce the set $\Omega_1 = \{(\Phi, \Psi) \in X,\ \|(\Phi, \Psi)\| < r_1\}$. Then, for $(\Phi, \Psi) \in \mathcal{P} \cap \partial\Omega_1$, one obtains $\|\Phi\|_1 + \|\Psi\|_2 = r_1$, so $\|\Phi\|_1 \le r_1$ and $\|\Psi\|_2 \le r_1$, that is, $0 \le \frac{\Phi(v)}{1+(\ln v)^{\alpha-1}} \le r_1$ and $0 \le \frac{\Psi(v)}{1+(\ln v)^{\beta-1}} \le r_1$ for all $v \in I$.

Therefore, by $(H4)$, Lemma 5 and (14), one finds

$$\begin{aligned}
\|\mathcal{A}_1(\Phi,\Psi)\|_1 &= \sup_{v\in I} \frac{1}{1+(\ln v)^{\alpha-1}} \bigg(\int_1^\infty \mathfrak{E}_1(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \mathfrak{E}_2(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\bigg) \\
&\ge \inf_{v\in[\theta,\infty)} \frac{1}{1+(\ln v)^{\alpha-1}} \bigg(\int_1^\infty \mathfrak{E}_1(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \mathfrak{E}_2(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega}\bigg) \\
&\ge \inf_{v\in[\theta,\infty)} \frac{1}{1+(\ln v)^{\alpha-1}} \int_1^\infty \mathfrak{E}_1(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \inf_{v\in[\theta,\infty)} \frac{1}{1+(\ln v)^{\alpha-1}} \int_1^\infty \mathfrak{E}_2(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\ge \int_1^\infty \inf_{v\in[\theta,\infty)} \frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \inf_{v\in[\theta,\infty)} \frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\ge \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \bigg(d\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi) \\
&\quad + b\int_1^\infty g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\bigg)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \bigg(d\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi) \\
&\quad + b\int_1^\infty g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\bigg)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\ge \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \bigg(d\int_1^\theta g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi) \\
&\quad + b\int_1^\theta g_\alpha(\xi,\omega)\,d\mathfrak{I}_1(\xi)\bigg)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \bigg(d\int_1^\theta g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi) \\
&\quad + b\int_1^\theta g_\beta(\xi,\omega)\,d\mathfrak{I}_2(\xi)\bigg)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&= \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad \times \bigg(\frac{d}{\Gamma(\alpha)}\int_1^\theta (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi) + \frac{b}{\Gamma(\alpha)}\int_1^\theta (\ln\xi)^{\alpha-1}\,d\mathfrak{I}_1(\xi)\bigg) \\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad \times \bigg(\frac{d}{\Gamma(\beta)}\int_1^\theta (\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi) + \frac{b}{\Gamma(\beta)}\int_1^\theta (\ln\xi)^{\beta-1}\,d\mathfrak{I}_2(\xi)\bigg) \\
&= \frac{(\ln\theta)^{\alpha-1}\widetilde{L}_1}{\Delta(1+(\ln\theta)^{\alpha-1})}\frac{\sigma_1 r_1}{2}\int_\theta^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} + \frac{(\ln\theta)^{\alpha-1}\widetilde{L}_2}{\Delta(1+(\ln\theta)^{\alpha-1})}\frac{\sigma_2 r_1}{2}\int_\theta^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} \\
&= \frac{\sigma_1 r_1 L_1 Q_3}{2} + \frac{\sigma_2 r_1 L_2 Q_4}{2} = \bigg(\frac{\sigma_1 L_1 Q_3}{2} + \frac{\sigma_2 L_2 Q_4}{2}\bigg) r_1.
\end{aligned}$$

In a similar manner, one deduces

$$\begin{aligned}
\|\mathcal{A}_2(\Phi,\Psi)\|_2 &\geq \frac{(\ln\theta)^{\beta-1}\widetilde{L}_3}{\Delta(1+(\ln\theta)^{\beta-1})}\frac{\sigma_1 r_1}{2}\int_\theta^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} + \frac{(\ln\theta)^{\beta-1}\widetilde{L}_4}{\Delta(1+(\ln\theta)^{\beta-1})}\frac{\sigma_2 r_1}{2}\int_\theta^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} \\
&= \frac{\sigma_1 r_1 L_3 Q_3}{2} + \frac{\sigma_2 r_1 L_4 Q_4}{2} = \left(\frac{\sigma_1 L_3 Q_3}{2} + \frac{\sigma_2 L_4 Q_4}{2}\right) r_1.
\end{aligned}$$

Therefore, one finds

$$\begin{aligned}
\|\mathcal{A}(\Phi,\Psi)\| &= \|\mathcal{A}_1(\Phi,\Psi)\|_1 + \|\mathcal{A}_2(\Phi,\Psi)\|_2 \\
&\geq \left[\frac{(L_1+L_3)\sigma_1 Q_3}{2} + \frac{(L_2+L_4)\sigma_2 Q_4}{2}\right] r_1 = \left(\frac{\sigma_1 \Upsilon_1}{2} + \frac{\sigma_2 \Upsilon_2}{2}\right) r_1 \geq r_1,
\end{aligned}$$

which gives us

$$\|\mathcal{A}(\Phi,\Psi)\| \geq \|(\Phi,\Psi)\|, \ \ \forall\, (\Phi,\Psi) \in \mathcal{P}\cap\partial\Omega_1. \tag{15}$$

Now, let us introduce the set $\Omega_2 = \{(\Phi,\Psi) \in X,\ \|(\Phi,\Psi)\| < r_2\}$. For any $(\Phi,\Psi) \in \mathcal{P}\cap\partial\Omega_2$, we have $\|\Phi\|_1 + \|\Psi\|_2 = r_2$, and then $\|\Phi\|_1 \leq r_2$ and $\|\Psi\|_2 \leq r_2$, that is, $0 \leq \frac{\Phi(v)}{1+(\ln v)^{\alpha-1}} \leq r_2$ and $0 \leq \frac{\Psi(v)}{1+(\ln v)^{\beta-1}} \leq r_2$ for all $v \in I$.

Then, by $(H5)$ and Lemma 5, one obtains

$$\begin{aligned}
\|\mathcal{A}_1(\Phi,\Psi)\|_1 &= \sup_{v\in I}\frac{\mathcal{A}_1(\Phi,\Psi)(v)}{1+(\ln v)^{\alpha-1}} \\
&\leq \sup_{v\in I}\frac{1}{1+(\ln v)^{\alpha-1}}\int_1^\infty \mathfrak{E}_1(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \sup_{v\in I}\frac{1}{1+(\ln v)^{\alpha-1}}\int_1^\infty \mathfrak{E}_2(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \int_1^\infty \sup_{v\in I}\frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \sup_{v\in I}\frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \Lambda_1\int_1^\infty \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} + \Lambda_2\int_1^\infty \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \frac{\sigma_3 r_2 \Lambda_1}{2}\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} + \frac{\sigma_4 r_2 \Lambda_2}{2}\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} \\
&= \left(\frac{\sigma_3 Q_1 \Lambda_1}{2} + \frac{\sigma_4 Q_2 \Lambda_2}{2}\right) r_2, \\
\|\mathcal{A}_2(\Phi,\Psi)\|_2 &= \sup_{v\in I}\frac{\mathcal{A}_2(\Phi,\Psi)(v)}{1+(\ln v)^{\beta-1}} \\
&\leq \sup_{v\in I}\frac{1}{1+(\ln v)^{\beta-1}}\int_1^\infty \mathfrak{E}_3(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \sup_{v\in I}\frac{1}{1+(\ln v)^{\beta-1}}\int_1^\infty \mathfrak{E}_4(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \int_1^\infty \sup_{v\in I}\frac{\mathfrak{E}_3(v,\omega)}{1+(\ln v)^{\beta-1}}\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \sup_{v\in I}\frac{\mathfrak{E}_4(v,\omega)}{1+(\ln v)^{\beta-1}}\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \Lambda_3\int_1^\infty \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} + \Lambda_4\int_1^\infty \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega),\Psi(\omega))\,\frac{d\omega}{\omega} \\
&\leq \frac{\sigma_3 r_2 \Lambda_3}{2}\int_1^\infty \mathfrak{c}(\omega)\,\frac{d\omega}{\omega} + \frac{\sigma_4 r_2 \Lambda_4}{2}\int_1^\infty \mathfrak{d}(\omega)\,\frac{d\omega}{\omega} \\
&= \left(\frac{\sigma_3 Q_1 \Lambda_3}{2} + \frac{\sigma_4 Q_2 \Lambda_4}{2}\right) r_2.
\end{aligned}$$

Then, one deduces

$$\begin{aligned}\|\mathcal{A}(\Phi,\Psi)\| &\leq \left[\frac{(\Lambda_1+\Lambda_3)\sigma_3 Q_1}{2}+\frac{(\Lambda_2+\Lambda_4)\sigma_4 Q_2}{2}\right]r_2\\ &=\left(\frac{\sigma_3\Upsilon_3}{2}+\frac{\sigma_4\Upsilon_4}{2}\right)r_2\leq r_2,\end{aligned}$$

that is,

$$\|\mathcal{A}(\Phi,\Psi)\|\leq\|(\Phi,\Psi)\|,\ \ \forall\,(\Phi,\Psi)\in\mathcal{P}\cap\partial\Omega_2. \tag{16}$$

Hence, based on relations (15) and (16), as well as the Guo–Krasnosels'kii fixed point theorem (see [26]), it can be deduced that the operator $\mathcal{A}$ possesses a fixed point $(\Phi,\Psi)\in\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$, where $r_1\leq|(\Phi,\Psi)|\leq r_2$. So, $\Phi(v)\geq 0$ and $\Psi(v)\geq 0$ for all $v\in I$, and by $(H2)$,$(H3)$, one obtains that $\Phi(v)>0$ for all $v\in(1,\infty)$ or $\Psi(v)>0$ for all $v\in(1,\infty)$. Consequently, the pair (Φ,Ψ) serves as a positive solution to the problem (1),(2). □

By employing analogous reasoning as in the proof of Theorem 1, we can derive the following theorem.

Theorem 2. *Assume that assumptions* $(H1)$*–*$(H3)$ *hold, and there exists* $\theta>1$ *such that* $Q_j>0$, $j=3,4$, *and* $\widetilde{L}_i>0$, $i=1,\ldots,4$. *In addition, suppose that there exist positive constants* r_1,r_2 *with* $r_1<r_2$, *and* $\sigma_1\in[\Upsilon_1^{-1},\infty)$, $\sigma_2\in[\Upsilon_2^{-1},\infty)$, $\sigma_3\in(0,\Upsilon_3^{-1}]$ *and* $\sigma_4\in(0,\Upsilon_4^{-1}]$ *such that*

$(H6)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\leq\frac{\sigma_3 r_1}{2},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,r_1]\times[0,r_1]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\leq\frac{\sigma_4 r_1}{2},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,r_1]\times[0,r_1]$;

$(H7)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\geq\frac{\sigma_1 r_2}{2},\ \ \forall\,v\in[\theta,\infty),\ \ (\phi,\psi)\in[0,r_2]\times[0,r_2]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\geq\frac{\sigma_2 r_2}{2},\ \ \forall\,v\in[\theta,\infty),\ \ (\phi,\psi)\in[0,r_2]\times[0,r_2]$.

Then, the problem (1),(2) has at least one positive solution (Φ,Ψ) *such that* $r_1\leq\|(\Phi,\Psi)\|\leq r_2$.

In what follows, we will prove the existence of at least three positive solutions for the problem (1),(2) by applying the Leggett–Williams fixed point theorem (Theorem 3.3 from [27]).

Theorem 3. *Assume that assumptions* $(H1)$*–*$(H3)$ *hold, and there exists* $\theta>1$ *such that* $Q_j>0$, $j=3,4$, *and* $\widetilde{L}_i>0$, $i=1,\ldots,4$. *In addition, suppose that there exist positive constants* $a_0<b_0<c_0$ *such that*

$(H8)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)<\frac{a_0}{2\Upsilon_3},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,a_0]\times[0,a_0]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)<\frac{a_0}{2\Upsilon_4},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,a_0]\times[0,a_0]$;

$(H9)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)>\frac{b_0}{2\Upsilon_1},\ \ \forall\,v\in[\theta,\infty),\ \ \phi,\psi\geq 0,\ \ b_0\leq\phi+\psi\leq c_0$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)>\frac{b_0}{2\Upsilon_2},\ \ \forall\,v\in[\theta,\infty),\ \ \phi,\psi\geq 0,\ \ b_0\leq\phi+\psi\leq c_0$;

$(H10)$ $\mathfrak{L}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\leq\frac{c_0}{2\Upsilon_3},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,c_0]\times[0,c_0]$,
$\mathfrak{M}((1+(\ln v)^{\alpha-1})\phi,(1+(\ln v)^{\beta-1})\psi)\leq\frac{c_0}{2\Upsilon_4},\ \ \forall\,v\in I,\ \ (\phi,\psi)\in[0,c_0]\times[0,c_0]$.

Then, the problem (1),(2) has at least three positive solutions, (Φ_1,Ψ_1), (Φ_2,Ψ_2) *and* (Φ_3,Ψ_3), *such that* $\|(\Phi_1,\Psi_1)\|<a_0$, $\|(\Phi_3,\Psi_3)\|>a_0$, *and*

$$\inf_{v\in[\theta,\infty)}\left(\frac{\Phi_2(v)}{1+(\ln v)^{\alpha-1}}+\frac{\Psi_2(v)}{1+(\ln v)^{\beta-1}}\right)>b_0,\ \inf_{v\in[\theta,\infty)}\left(\frac{\Phi_3(v)}{1+(\ln v)^{\alpha-1}}+\frac{\Psi_3(v)}{1+(\ln v)^{\beta-1}}\right)<b_0.$$

Proof. We show firstly that operator $\mathcal{A} : \overline{\mathcal{P}}_{c_0} \to \overline{\mathcal{P}}_{c_0}$. For any $(\Phi, \Psi) \in \overline{\mathcal{P}}_{c_0}$, we have $\|(\Phi, \Psi)\| \le c_0$, and so $\|\Phi\|_1 \le c_0$, $\|\Psi\|_2 \le c_0$. Using the assumption $(H10)$ and Lemma 5, one finds

$$\begin{aligned}
\|\mathcal{A}_1(\Phi, \Psi)\|_1 &\le \sup_{v\in I} \frac{1}{1+(\ln v)^{\alpha-1}} \int_1^\infty \mathfrak{E}_1(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\quad + \sup_{v\in I} \frac{1}{1+(\ln v)^{\alpha-1}} \int_1^\infty \mathfrak{E}_2(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \int_1^\infty \sup_{v\in I} \frac{\mathfrak{E}_1(v,\omega)}{1+(\ln v)^{\alpha-1}} \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \sup_{v\in I} \frac{\mathfrak{E}_2(v,\omega)}{1+(\ln v)^{\alpha-1}} \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \Lambda_1 \int_1^\infty \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} + \Lambda_2 \int_1^\infty \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \Lambda_1 \frac{c_0}{2\Upsilon_3} \int_1^\infty \mathfrak{c}(\omega) \frac{d\omega}{\omega} + \Lambda_2 \frac{c_0}{2\Upsilon_4} \int_1^\infty \mathfrak{d}(\omega) \frac{d\omega}{\omega} \\
&= \left(\frac{\Lambda_1 Q_1}{\Upsilon_3} + \frac{\Lambda_2 Q_2}{\Upsilon_4} \right) \frac{c_0}{2}, \\
\|\mathcal{A}_2(\Phi, \Psi)\|_2 &\le \sup_{v\in I} \frac{1}{1+(\ln v)^{\beta-1}} \int_1^\infty \mathfrak{E}_3(v,\omega)\mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\quad + \sup_{v\in I} \frac{1}{1+(\ln v)^{\beta-1}} \int_1^\infty \mathfrak{E}_4(v,\omega)\mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \int_1^\infty \sup_{v\in I} \frac{\mathfrak{E}_3(v,\omega)}{1+(\ln v)^{\beta-1}} \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\quad + \int_1^\infty \sup_{v\in I} \frac{\mathfrak{E}_4(v,\omega)}{1+(\ln v)^{\beta-1}} \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \Lambda_3 \int_1^\infty \mathfrak{c}(\omega)\mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} + \Lambda_4 \int_1^\infty \mathfrak{d}(\omega)\mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\le \Lambda_3 \frac{c_0}{2\Upsilon_3} \int_1^\infty \mathfrak{c}(\omega) \frac{d\omega}{\omega} + \Lambda_4 \frac{c_0}{2\Upsilon_4} \int_1^\infty \mathfrak{d}(\omega) \frac{d\omega}{\omega} \\
&= \left(\frac{\Lambda_3 Q_1}{\Upsilon_3} + \frac{\Lambda_4 Q_2}{\Upsilon_4} \right) \frac{c_0}{2}.
\end{aligned}$$

Then, one obtains

$$\|\mathcal{A}(\Phi, \Psi)\| \le \left[\frac{(\Lambda_1 + \Lambda_3) Q_1}{\Upsilon_3} + \frac{(\Lambda_2 + \Lambda_4) Q_2}{\Upsilon_4} \right] \frac{c_0}{2} = c_0,$$

so $\mathcal{A} : \overline{\mathcal{P}}_{c_0} \to \overline{\mathcal{P}}_{c_0}$.

Let $d_0 \in (b_0, c_0)$ be chosen, and let us define the concave non-negative continuous functional w on $\mathcal{P}$ as follows:

$$w(\Phi, \Psi) = \inf_{v\in[\theta,\infty)} \left(\frac{\Phi(v)}{1+(\ln v)^{\alpha-1}} + \frac{\Psi(v)}{1+(\ln v)^{\beta-1}} \right), \ (\Phi, \Psi) \in \mathcal{P}.$$

One can easily see that $w(\Phi, \Psi) \le \|(\Phi, \Psi)\|$ for all $(\Phi, \Psi) \in \overline{\mathcal{P}}_{c_0}$.

Next, we will verify the conditions (i)–(iii) of Theorem 3.3 from [27]. We verify condition (ii) first. For $(\Phi, \Psi) \in \overline{\mathcal{P}}_{a_0}$, we will show that $\|\mathcal{A}(\Phi, \Psi)\| < a_0$. For this, let $(\Phi, \Psi) \in \overline{\mathcal{P}}_{a_0}$. As in the above inequalities, one obtains

$$\|\mathcal{A}_1(\Phi, \Psi)\|_1 < \left(\frac{\Lambda_1 Q_1}{\Upsilon_3} + \frac{\Lambda_2 Q_2}{\Upsilon_4} \right) \frac{a_0}{2}, \ \|\mathcal{A}_2(\Phi, \Psi)\|_2 < \left(\frac{\Lambda_3 Q_1}{\Upsilon_3} + \frac{\Lambda_4 Q_2}{\Upsilon_4} \right) \frac{a_0}{2},$$

and so $\|\mathcal{A}(\Phi, \Psi)\| < a_0$. Then, we have assumption (ii).

We verify condition (i) from Theorem 3.3 from [27]. Let us choose the element

$$(\Phi_0(v), \Psi_0(v)) = \left(\frac{b_0 + d_0}{4} \left(1 + (\ln v)^{\alpha-1} \right), \frac{b_0 + d_0}{4} (1 + (\ln v)^{\beta-1}) \right), \ v \in I.$$

Because $\Phi_0(v) \geq 0, \Psi_0(v) \geq 0$ for all $v \in I$, $\|(\Phi_0, \Psi_0)\| = \frac{b_0+d_0}{2} < d_0$ and $w(\Phi_0, \Psi_0) = \frac{b_0+d_0}{2} > b_0$, one deduces that $(\Phi_0, \Psi_0) \in \{(\Phi, \Psi) \mid (\Phi, \Psi) \in \mathcal{P}(w, b_0, d_0),\ w(\Phi, \Psi) > b_0\}$. Now, let $(\Phi, \Psi) \in \mathcal{P}(w, b_0, d_0)$, that is, $(\Phi, \Psi) \in \mathcal{P}$, $w(\Phi, \Psi) \geq b_0$ and $\|(\Phi, \Psi)\| \leq d_0$. So, one finds $\frac{\Phi(v)}{1+(\ln v)^{\alpha-1}} + \frac{\Psi(v)}{1+(\ln v)^{\beta-1}} \leq d_0$ for all $v \in I$, and $\inf_{v\in[\theta,\infty)} \left(\frac{\Phi(v)}{1+(\ln v)^{\alpha-1}} + \frac{\Psi(v)}{1+(\ln v)^{\beta-1}} \right) \geq b_0$. Then, by $(H9)$ and Lemma 5, and similar computations to those from the first part of the proof of Theorem 1, one deduces

$$
\begin{aligned}
w(\mathcal{A}(\Phi, \Psi)) &= \inf_{v\in[\theta,\infty)} \left(\frac{\mathcal{A}_1(\Phi, \Psi)(v)}{1 + (\ln v)^{\alpha-1}} + \frac{\mathcal{A}_2(\Phi, \Psi)(v)}{1 + (\ln v)^{\beta-1}} \right) \\
&\geq L_1 \int_\theta^\infty \mathfrak{c}(\omega) \mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} + L_2 \int_\theta^\infty \mathfrak{d}(\omega) \mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&\quad + L_3 \int_\theta^\infty \mathfrak{c}(\omega) \mathfrak{L}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} + L_4 \int_\theta^\infty \mathfrak{d}(\omega) \mathfrak{M}(\Phi(\omega), \Psi(\omega)) \frac{d\omega}{\omega} \\
&> L_1 \frac{b_0 Q_3}{2\Upsilon_1} + L_2 \frac{b_0 Q_4}{2\Upsilon_2} + L_3 \frac{b_0 Q_3}{2\Upsilon_1} + L_4 \frac{b_0 Q_4}{2\Upsilon_2} \\
&= (L_1 + L_3) \frac{b_0 Q_3}{2\Upsilon_1} + (L_2 + L_4) \frac{b_0 Q_4}{2\Upsilon_2} = b_0.
\end{aligned}
$$

Therefore, $w(\mathcal{A}(\Phi, \Psi)) > b_0$, and we have assumption (i).

Now, we verify condition (iii) of Theorem 3.3 from [27], namely $w(\mathcal{A}(\Phi, \Psi)) > b_0$ for $(\Phi, \Psi) \in \mathcal{P}(w, b_0, c_0)$ and $\|\mathcal{A}(\Phi, \Psi)\| > d_0$. So, let $(\Phi, \Psi) \in \mathcal{P}(w, b_0, c_0)$ and $\|\mathcal{A}(\Phi, \Psi)\| > d_0$. By arguments similar to those used earlier, one deduces that $w(\mathcal{A}(\Phi, \Psi)) > b_0$, that is, assumption (iii) is satisfied.

Applying Theorem 3.3 from [27] along with assumptions $(H2)$ and $(H3)$, we can infer that the problem (1),(2) possesses at least three positive solutions: (Φ_1, Ψ_1), (Φ_2, Ψ_2), and (Φ_3, Ψ_3). Moreover, these solutions satisfy the following properties: $|(\Phi_1, \Psi_1)| < a_0$, $|(\Phi_3, \Psi_3)| > a_0$, $w(\Phi_2, \Psi_2) > b_0$, and $w(\Phi_3, \Psi_3) < b_0$. $\square$

4. An Example

Let $\alpha = \frac{5}{2}$, $\beta = \frac{10}{3}$, $n = 3$, $m = 4$, $\mathfrak{c}(v) = \frac{1}{(v-1/2)^{1/3}}$, $v \in [1, \infty)$, $\mathfrak{d}(v) = \frac{v+2}{(v-1/3)^{5/4}}$, $v \in [1, \infty)$, $\mathfrak{L}(\phi, \psi) = 6 + e^{-|\phi-\psi|} \sin^2(\phi\psi)$, $\phi, \psi \geq 0$, $\mathfrak{M}(\phi, \psi) = \frac{2}{3} + 95e^{-\phi\psi} \cos^4(\phi + \psi)$, $\phi, \psi \geq 0$, $\mathfrak{I}_1(\omega) = \{\frac{19}{110},\ \omega \in [1, 3);\ \frac{\omega}{11} - \frac{1}{10},\ \omega \in [3, 7);\ \frac{59}{110},\ \omega \in [7, 10);\ \frac{116}{165},\ \omega \in [10, \infty)\}$, $\mathfrak{I}_2(\omega) = \{\frac{5}{136}(\omega - 1)^{17/5},\ \omega \in [1, 2);\ \frac{5}{136},\ \omega \in (2, \infty)\}$, $\mathfrak{J}_1(\omega) = \{1,\ \omega \in [1, \frac{5}{3});\ \frac{14}{13},\ \omega \in [\frac{5}{3}, 2);\ \frac{7}{290}(\omega - 2)^{29/7} + \frac{14}{13},\ \omega \in [2, \frac{15}{6});\ \frac{7}{290}(\frac{1}{2})^{29/7} + \frac{14}{13},\ \omega \in [\frac{15}{6}, \infty)\}$, $\mathfrak{J}_2(\omega) = \{3,\ \omega \in [1, 4);\ \frac{286}{95},\ \omega \in [4, 9);\ \frac{\omega^2}{146} + \frac{34061}{13870},\ \omega \in [9, 11);\ \frac{45556}{13870},\ \omega \in [11, \infty)\}$.

Let us consider the system of fractional differential equations represented by

$$
\begin{cases}
{}^{H}D_{1+}^{5/2}\Phi(v) + \dfrac{1}{(v - 1/2)^{1/3}} \left(6 + e^{-|\Phi(v)-\Psi(v)|} \sin^2(\Phi(v)\Psi(v)) \right) = 0, \ \ v \in (1, \infty), \\
{}^{H}D_{1+}^{10/3}\Psi(v) + \dfrac{v + 2}{(v - 1/3)^{5/4}} \left(\dfrac{2}{3} + 95e^{-\Phi(v)\Psi(v)} \cos^4(\Phi(v) + \Psi(v)) \right) = 0, \ \ v \in (1, \infty),
\end{cases} \tag{17}
$$

subject to the boundary conditions

$$
\begin{cases}
\Phi(1) = \Phi'(1) = 0, \ \ \Psi(1) = \Psi'(1) = \Psi''(1) = 0, \\
{}^{H}D_{1+}^{3/2}\Phi(\infty) = \dfrac{1}{11} \displaystyle\int_3^7 \Phi(\omega)\, d\omega + \dfrac{1}{6}\Phi(10) + \dfrac{1}{8} \int_1^2 (\omega - 1)^{12/5} \Psi(\omega)\, d\omega, \\
{}^{H}D_{1+}^{7/3}\Psi(\infty) = \dfrac{1}{13}\Phi\left(\dfrac{5}{3}\right) + \dfrac{1}{10} \displaystyle\int_2^{15/6} (\omega - 2)^{22/7} \Phi(\omega)\, d\omega + \dfrac{1}{95}\Psi(4) + \dfrac{1}{73} \int_9^{11} \omega\Psi(\omega)\, d\omega.
\end{cases} \tag{18}
$$

By using the Mathematica program, one obtains $a \approx 0.01753901$, $b \approx 0.01031779$, $c \approx 0.02920547$, $d \approx 0.83235873$, $\Delta \approx 0.01429742 > 0$, $\int_1^\infty \mathfrak{c}(\omega) \frac{d\omega}{\omega} \approx 3.15855472$, $\int_1^\infty \mathfrak{d}(\omega) \frac{d\omega}{\omega} \approx 6.53061854$. So, assumptions $(H1) - (H3)$ are satisfied. In addition, we take $\theta = 2$, and we deduce $\Lambda_1 = \frac{d}{\Delta} \approx 58.21742336$, $\Lambda_2 = \frac{b}{\Delta} \approx 0.72165469$, $\Lambda_3 = \frac{c}{\Delta} \approx 2.04270988$, $\Lambda_4 = \frac{a}{\Delta} \approx 1.22672621$, $\widetilde{L}_1 \approx 0.00021797$, $\widetilde{L}_2 \approx 0.00309129$, $\widetilde{L}_3 \approx 0.00037053$,

$\widetilde{L}_4 \approx 0.00010846$, $L_1 \approx 0.00557882$, $L_2 \approx 0.07911642$, $L_3 \approx 0.00773204$, $L_4 \approx 0.00226336$, $Q_1 \approx 3.15855472$, $Q_2 \approx 6.53061854$, $Q_3 \approx 2.43620073$, $Q_4 \approx 4.28276191$, $\Upsilon_1 \approx 0.03242794$, $\Upsilon_2 \approx 0.34853023$, $\Upsilon_3 \approx 190.33492843$, $\Upsilon_4 \approx 12.72413242$.

Let us consider $r_1 = \frac{1}{3}$, $r_2 = 2800$, $\sigma_1 = 31 > \Upsilon_1^{-1}$, $\sigma_2 = 3 > \Upsilon_2^{-1}$, $\sigma_3 = 0.005 < \Upsilon_3^{-1}$ and $\sigma_4 = 0.07 < \Upsilon_4^{-1}$. Then, one obtains the inequalities

$$\begin{aligned}&\mathfrak{L}((1+(\ln v)^{3/2})\phi,(1+(\ln v)^{7/3})\psi) \geq 6 > \tfrac{\sigma_1 r_1}{2} = \tfrac{31}{6}, \ \ \forall\, v \in [2,\infty),\ \ \phi,\psi \in [0,\tfrac{1}{3}],\\ &\mathfrak{M}((1+(\ln v)^{3/2})\phi,(1+(\ln v)^{7/3})\psi) \geq \tfrac{2}{3} > \tfrac{\sigma_2 r_1}{2} = \tfrac{1}{2}, \ \ \forall\, v \in [2,\infty),\ \ \phi,\psi \in [0,\tfrac{1}{3}],\end{aligned}$$

so assumption $(H4)$ is satisfied. In addition, one finds

$$\begin{aligned}&\mathfrak{L}((1+(\ln v)^{3/2})\phi,(1+(\ln v)^{7/3})\psi) \leq 7 = \tfrac{\sigma_3 r_2}{2}, \ \ \forall\, v \in [1,\infty),\ \ \phi,\psi \in [0,2800],\\ &\mathfrak{M}((1+(\ln v)^{3/2})\phi,(1+(\ln v)^{7/3})\psi) \leq \tfrac{287}{3} < \tfrac{\sigma_4 r_2}{2} = 98, \ \ \forall\, v \in [1,\infty),\ \ \phi,\psi \in [0,2800],\end{aligned}$$

hence, assumption $(H5)$ is also satisfied. Therefore, by Theorem 1 we conclude that the problem (17),(18) has at least one positive solution $(\Phi(v),\Psi(v))$, $v \in [1,\infty)$, with $\frac{1}{3} \leq \sup_{v\in[1,\infty)} \frac{\Phi(v)}{1+(\ln v)^{3/2}} + \sup_{v\in[1,\infty)} \frac{\Psi(v)}{1+(\ln v)^{7/3}} \leq 2800$.

5. Conclusions

In this paper, we investigated the system of nonlinear fractional differential equations (1) with Hadamard derivatives of various orders $\alpha \in (n-1,n]$ and $\beta \in (m-1,m]$, respectively, and non-negative nonlinearities, on the infinite interval $(1,\infty)$. The system (1) is supplemented with general nonlocal boundary conditions (2), where the unknown functions Φ and Ψ in the point 1 and their derivatives until orders $n-2$ and $m-2$, respectively, are all 0, and the Hadamard derivatives of Φ and Ψ of order $n-1$ and $m-1$ at ∞ are dependent on both Riemann–Liouville integrals of Φ and Ψ. Our problem generalizes the problem studied in [1], by considering here different orders for the fractional derivatives in the equations of system (1), and also a general form of the boundary conditions from (2) at ∞. Under some assumptions on the data of this problem, we gave firstly the solution of the associated linear boundary value problem, and the corresponding Green functions with their properties. Then, in the main section of the paper, we proved the existence of positive solutions of (1),(2) by applying the Guo–Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem. An example which illustrates the main theorems is finally presented. Our results can be generalized in the future for other fractional derivatives in the system (1), and for some complex networks, such as those from paper [28].

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