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A Time-Fractional Schrödinger Equation with Singular Potentials on the Boundary

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Abstract: A Schrödinger equation with a time-fractional derivative, posed in $(0,\infty)\times\mathbb{I}$, where $\mathbb{I}=]a,b]$, is investigated in this paper. The equation involves a singular Hardy potential of the form $\frac{\lambda}{(x-a)^2}$, where the parameter λ belongs to a certain range, and a nonlinearity of the form $\mu(x-a)^{-\rho}|u|^p$, where $\rho\geq 0$. Using some a priori estimates, necessary conditions for the existence of weak solutions are obtained.

Keywords: time-fractional Schrödinger equation; Caputo fractional derivative; singular potentials; nonexistence

MSC: 35A01; 35B44; 26A33

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1. Introduction

We consider the fractional Schrödinger equation

$$i^\alpha\partial_t^\alpha u+\partial_{xx}u+\frac{\lambda}{(x-a)^2}u=\mu(x-a)^{-\rho}|u|^p\quad\text{in } (0,\infty)\times\mathbb{I}, \tag{1}$$

where $u=u(t,x)$ is a complex-valued function, $\mathbb{I}=]a,b]$, $a,b\in\mathbb{R}$, $a<b$, $0<\alpha<1$, $i^\alpha=e^{\frac{i\alpha\pi}{2}}$, ∂_t^α is the Caputo derivative of order α, with respect to the variable t, $\lambda\leq\lambda^*=\frac{1}{4}$ ($\lambda\in\mathbb{R}$), μ is a nonzero complex number, $\rho\geq 0$ and $p>1$. Notice that λ^* is the sharp constant for the Hardy inequality involving the distance to the boundary (see e.g., [1]). We study (1) subject to

$$u(0,x)=g(x)\quad\text{in } \mathbb{I} \tag{2}$$

and

$$u(t,b)=\delta(t+1)^\tau\quad\text{in } (0,\infty), \tag{3}$$

where g is a complex-valued function, $\delta\in\mathbb{C}$ and $\tau\in\mathbb{R}$. More precisely, our goal is to obtain sufficient conditions so that the set of weak solutions is empty.

Elliptic operators with inverse square potentials play a key role in many problems of physics. For instance, the heat and Schrödinger flows for the elliptic operator $\Delta+\frac{\lambda}{|x|^2}$ have been studied in the theory of combustion (see e.g., [2]) and in quantum mechanics (see [3]). Nonlinear Schrödinger equations with inverse square potentials appear in a variety of physical settings, such as quantum field equations or black hole solutions of the Einstein's equations (see e.g., [3,4]).

The classical Schrödinger equation with non-gauge power nonlinearity

$$\begin{cases} i\partial_t u+\Delta u=\mu|u|^p & \text{in} \quad (0,\infty)\times\mathbb{R}^N,\\ u(0,x)=u_0(x) & \text{in} \quad \mathbb{R}^N\end{cases} \tag{4}$$

has been investigated by many authors, whose finding leads to the consideration of certain special exponents' values. For $0 \le s < \frac{N}{2}$ and $1 < p < 1 + \frac{4}{N-2s}$, it is well known that local well-posedness for (4) holds in $H^s(\mathbb{R}^N)$ (see e.g., [5,6]). In the special case $N = 1$ and $p = 2$, it was proven in [7] that for $s > -\frac{1}{4}$, (4) is locally well posed in $H^s(\mathbb{R})$. For an arbitrary dimension N, when $u_0 \in L^2(\mathbb{R}^N) \cap L^{1+\frac{1}{p}}(\mathbb{R}^N)$ and $p_S < p < 1 + \frac{4}{N}$, where p_S is the Strauss exponent, the global existence for (4) for small initial data holds (see [8]). In [9], Ikeda and Wakasugi studied the existence of a blow-up solution to (4). Namely, when $1 < p \le 1 + \frac{2}{N}$, they proved the blow-up of the L^2-norm of solutions with suitable initial data. Later, Ikeda and Inui [10] established a small data blow-up result of L^2 and H^1-solution for (4) when $1 < p < 1 + \frac{4}{N}$. Moreover, they obtained an upper bound of the lifespan.

In recent years, it was shown that fractional differential equations have many applications in various problems from physics, biology, chemistry, and others (see e.g., [11–13]). Due to this fact, the study of fractional differential equations and fractional partial differential equations received a great attention from the mathematical community. In particular, several contributions have been focused on studying fractional Schrödinger equations in both theoretical and numerical aspects. For numerical studies, we reefer to [14], where numerical schemes based on Fourier–Galerkin/Legendre–Galerkin spectral methods have been implemented for solving the time-fractional Schrödinger equation with Caputo or Riemann–Liouville fractional derivative. We also refer to [15], where second-order and linear finite element schemes for solving multi-dimensional nonlinear time-fractional Schrödinger equations have been used. Other references related to numerical solutions to time-fractional Schrödinger equations can be found in [16–18].

Moreover, some fractional versions of (4) have been considered in certain papers. In [19], Fino et al. studied the space-fractional Schrödinger equation

$$\begin{cases} i\partial_t u - (-\Delta)^{\frac{s}{2}} u = \mu |u|^p & \text{in} \quad (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{5}$$

where $0 < s < 2$, $u_0 \in H^{\frac{s}{2}}(\mathbb{R}^N)$ and $(-\Delta)^{\frac{s}{2}}$ is the fractional Laplacian operator of order $\frac{s}{2}$. Namely, they investigated the local well-posedness of solutions to (5) in $H^{\frac{s}{2}}(\mathbb{R}^N)$ and derived a finite-time blow-up result, under suitable conditions on the initial data.

Kirane and Nabti [20] considered problem (4) with a nonlinear memory term. More precisely, they investigated the nonlocal in time nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \dfrac{\mu}{\Gamma(\sigma)} \displaystyle\int_0^t (t-s)^{\sigma-1} |u(s,x)|^p \, ds & \text{in} \quad (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{6}$$

where $0 < \sigma < 1$. Namely, they derived a blow-up exponent and obtained an estimate of the life span of blowing-up solutions to (6).

Zhang et al. [21] considered the time-fractional version of (4), that is, the Cauchy problem

$$\begin{cases} i^\alpha \partial_t^\alpha u + \Delta u = \mu |u|^p & \text{in} \quad (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{7}$$

where $0 < \alpha < 1$. Let us mention that the fractional model (7) has been introduced by Naber [22]. In [21], it was shown that (7) admits no global weak solution with suitable initial data when $1 < p < 1 + \frac{1}{2N}$. Moreover, the authors derived sufficient conditions for which (7) admits no global weak solution for every $p > 1$.

Very recently, Kirane and Fino [23] studied the space-time fractional Schrödinger equation

$$\begin{cases} i^\alpha \partial_t^\alpha u - (-\Delta)^{\frac{s}{2}} u = \mu |u|^p & \text{in} \quad (0,T) \times \mathbb{R}^N, \\ u(0,x) = \varepsilon u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{8}$$

where $0 < \alpha < 1$, $0 < s < 2$, $\varepsilon > 0$ and $T > 0$. Namely, they investigated the nonexistence of local L^1 or L^2 weak solutions as well as the global L^1 or L^2 weak solutions to (8), under some conditions on the initial data and the nonlinear term.

As in [21,23], in order to study the nonexistence of weak solutions to (1)–(3), we use the nonlinear capacity method, which is based on a judicious choice of test functions. The main interest of problem (1) is the presence of the so-called "Hardy potential" (or "inverse-square potential") $\frac{\lambda}{(x-a)^2}$, which is singular at the extremity a of the bounded interval $\mathbb{I}$. This causes new difficulties for the construction of adequate test functions.

The originality of this work lies on the following facts:

- In all the above mentioned contributions, the Schrödinger equation has been investigated in $\mathbb{R}^N$ or unbounded domains of $\mathbb{R}^N$ (see e.g., [24] in the case of exterior domains). In this paper, the considered problem (1) is posed in the bounded domain $\mathbb{I}$ of $\mathbb{R}$.
- To the best of our knowledge, the study of nonexistence of solutions to Schrödinger equation (time-Schrödinger equation) with a Hardy potential has not been considered in previous works.
- The Hardy potential $\frac{\lambda}{(x-a)^2}$ (as well as the potential function $(x-a)^{-\rho}$) involved in (1) is singular on the extremity a.
- The boundary condition (3) involves the variable time.

In Section 2, we recall some basic notions and results from fractional calculus. In Section 3, we define weak solutions to problem (1) under the initial condition (2) and the boundary condition (3), and state our main results. In Section 4, we prove some lemmas that will be useful in Section 5, where we establish the main results.

In all the paper, we denote by C (or C_j) a generic positive constant, whose value could be changed from one line to another.

2. Basics from Fractional Calculus and Notations

For more details about the above notions, we refer to [25].

Let $\sigma > 0$ and $g \in L^1([0,\sigma],\mathbb{R})$. For $\omega > 0$, we consider the operators $I_0^\omega g$ (the left-sided Riemann–Liouville fractional integral of order ω of g) and $I_\sigma^\omega g$ (the right-sided Riemann–Liouville fractional integral of order ω of g) defined by

$$(I_0^\omega g)(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t-\iota)^{\omega-1} g(\iota)\, d\iota$$

and

$$(I_\sigma^\omega g)(t) = \frac{1}{\Gamma(\omega)} \int_t^\sigma (\iota-t)^{\omega-1} g(\iota)\, d\iota$$

a.e. in $[0,\sigma]$. It can be easily seen that, if g is continuous, then

$$\lim_{t\to 0^+} (I_0^\omega g)(t) = \lim_{t\to\sigma^-} (I_\sigma^\omega g)(t) = 0. \tag{9}$$

The following result can be found in [25] [Lemma 2.7].

Lemma 1. *Assume that $g, h : [0,\sigma] \to \mathbb{R}$ are continuous. Then for all $\omega > 0$, we have*

$$\int_0^\sigma (I_0^\omega g)(t) h(t)\, dt = \int_0^\sigma g(t)(I_\sigma^\omega h)(t)\, dt.$$

Let $g \in AC([0,\sigma],\mathbb{R})$ (i.e., g is absolutely continuous in $[0,\sigma]$) and $0 < \omega < 1$. The Caputo fractional derivative of order ω of g, is defined by

$${}^C D_0^\omega g(t) = (I_0^{1-\omega} g')(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\iota)^{-\omega} g'(\iota)\, d\iota$$

for a.e. $t \in [0, \sigma]$.

Given $z \in \mathbb{C}$, the real and imaginary parts of z, are denoted respectively by z_1 and z_2, that is,

$$z = z_1 + iz_2, \quad (z_1, z_2) \in \mathbb{R}^2.$$

Similarly, for a complex-valued function g, the real and imaginary parts of g, are denote respectively by g_1 and g_2, that is,

$$g(t) = g_1(t) + ig_2(t), \quad (g_1(t), g_2(t)) \in \mathbb{R}^2.$$

Let $g \in L^1([0, \sigma], \mathbb{C})$. The left-sided and right-sided Riemann–Liouville fractional integrals of order ω of g, are defined respectively by

$$(I_0^\omega g)(t) = (I_0^\omega g_1)(t) + i(I_0^\omega g_2)(t)$$

and

$$(I_\sigma^\omega g)(t) = (I_\sigma^\omega g_1)(t) + i(I_\sigma^\omega g_2)(t)$$

for a.e. $t \in [0, \sigma]$.

Similarly, the Caputo fractional derivative of order $\omega \in (0, 1)$ of g, is defined by

$${}^C D_0^\omega g(t) = {}^C D_0^\omega g_1(t) + i\, {}^C D_0^\omega g_2(t)$$

for a.e. $t \in [0, \sigma]$.

Let $G = G(t, x) : [0, \sigma] \times J \to \mathbb{C}$, where $J \subset \mathbb{R}$. We shall use the notations:

$$\begin{aligned} (I_0^\omega G)(t, x) &= (I_0^\omega G(\cdot, x))(t), \\ (I_\sigma^\omega G)(t, x) &= (I_\sigma^\omega G(\cdot, x))(t) \end{aligned}$$

and

$$\partial_t^\omega G(t, x) = {}^C D_0^\omega G(\cdot, x)(t) = (I_0^{1-\omega} \partial_t G)(t, x), \quad 0 < \omega < 1.$$

3. Main Results

We first define weak solutions to (1)–(3). For $T > 0$, let $Q_T = [0, T] \times \mathbb{I}$.

Definition 1. *By Ψ_T ($T > 0$), we mean the st of functions $\psi = \psi(t, x)$ satisfying:*

(i) $\psi \in C^2(Q_T, \mathbb{R})$, $\mathrm{supp}_x(\psi) \subset\subset Q_T$, $\psi \geq 0$;

(ii) $\psi(\cdot, b) = 0$.

Definition 2 (Weak solution). *Let $g \in L^1_{\mathrm{loc}}(\mathbb{I}, \mathbb{C})$.A function $u \in L^p_{\mathrm{loc}}([0, \infty) \times \mathbb{I}, \mathbb{C})$ is a weak solution to* (1)–(3)*, if*

$$\begin{aligned} &\int_{Q_T} \mu (x-a)^{-\rho} |u|^p \psi \, dx \, dt + i^\alpha \int_a^b g(x) (I_T^{1-\alpha} \psi)(0, x) \, dx + \delta \int_0^T (t+1)^\tau \partial_x \psi(t, b) \, dt \\ &= -i^\alpha \int_{Q_T} u \partial_t (I_T^{1-\alpha} \psi) \, dx \, dt + \int_{Q_T} u \left(\partial_{xx} \psi + \frac{\lambda}{(x-a)^2} \psi \right) dx \, dt \end{aligned} \tag{10}$$

for all $T > 0$ and $\psi \in \Psi_T$.

The integral formulation (10) can be easily obtained by multiplying (1) by $\psi \in \Psi_T$, integrating over Q_T and using (9) and Lemma 1.

For $\lambda \leq \frac{1}{4}$, let

$$\kappa_\lambda = \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}. \tag{11}$$

We introduce the function H defined in $\mathbb{I}$ by

$$H(x)=\begin{cases}(b-a)^{1-2\kappa_\lambda}(x-a)^{\kappa_\lambda}\left(1-\left(\dfrac{x-a}{b-a}\right)^{1-2\kappa_\lambda}\right) & \text{if} \quad \lambda<\frac{1}{4},\\ \sqrt{x-a}\left(1-\dfrac{\ln(x-a)}{\ln(b-a)}\right) & \text{if} \quad \lambda=\frac{1}{4},\ b-a>1,\\ \dfrac{\sqrt{x-a}\ln(x-a)}{\ln(b-a)}\left(1-\dfrac{\ln(b-a)}{\ln(x-a)}\right) & \text{if} \quad \lambda=\frac{1}{4},\ b-a<1,\\ -\sqrt{x-a}\ln(x-a) & \text{if} \quad \lambda=\frac{1}{4},\ b-a=1.\end{cases} \tag{12}$$

Remark 1. *By taking equal the real parts and then the imaginary parts in (10), it can be easily seen that (10) is equivalent to the system of equations*

$$\begin{aligned}&\mu_1\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx\\ &+\delta_1\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt\\ &=-\int_{Q_T}\left(\cos\left(\frac{\alpha\pi}{2}\right)u_1-\sin\left(\frac{\alpha\pi}{2}\right)u_2\right)\partial_t(I_T^{1-\alpha}\psi)\,dx\,dt\\ &\quad+\int_{Q_T}u_1\left(\partial_{xx}\psi+\frac{\lambda}{(x-a)^2}\psi\right)dx\,dt\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\mu_2\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+\int_a^b\left(\sin\left(\frac{\alpha\pi}{2}\right)g_1(x)+\cos\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx\\ &+\delta_2\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt\\ &=-\int_{Q_T}\left(\sin\left(\frac{\alpha\pi}{2}\right)u_1+\cos\left(\frac{\alpha\pi}{2}\right)u_2\right)\partial_t(I_T^{1-\alpha}\psi)\,dx\,dt\\ &\quad+\int_{Q_T}u_2\left(\partial_{xx}\psi+\frac{\lambda}{(x-a)^2}\psi\right)dx\,dt.\end{aligned} \tag{14}$$

Let us denote by WSL the set of weak solutions to (1)–(3).
We first discuss the case $\delta=0$.

Theorem 1. *Let $0<\alpha<1$, $\lambda\le\frac{1}{4}$, $\mu\in\mathbb{C}\backslash\{0\}$, $\delta=0$ and $g\in L^1_{\text{loc}}(\mathbb{I},\mathbb{C})$. Assume that $gH\in L^1(\mathbb{I},\mathbb{C})$ and*

$$\mu_1\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)H(x)\,dx>0;\ or \tag{15}$$

$$\mu_2\int_a^b\left(\sin\left(\frac{\alpha\pi}{2}\right)g_1(x)+\cos\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)H(x)\,dx>0. \tag{16}$$

If

$$\rho>2,\ 1<p<1+\frac{\rho-2}{1-\kappa_\lambda}, \tag{17}$$

then WSL $=\emptyset$.

For $\delta\in\mathbb{C}\backslash\{0\}$, we have the following result.

Theorem 2. *Let $0 < \alpha < 1$, $\lambda \leq \frac{1}{4}$, $\mu, \delta \in \mathbb{C}\backslash\{0\}$, $\tau \in \mathbb{R}$ and $g \in L^1_{\text{loc}}(\mathbb{I}, \mathbb{C})$. Assume that*

$$\mu_1\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1 - \sin\left(\frac{\alpha\pi}{2}\right)g_2\right) \geq 0,\ \delta_1\mu_1 < 0;\ or \tag{18}$$

$$\mu_2\left(\sin\left(\frac{\alpha\pi}{2}\right)g_1 + \cos\left(\frac{\alpha\pi}{2}\right)g_2\right) \geq 0,\ \delta_2\mu_2 < 0. \tag{19}$$

Moreover, suppose that one of the following statements holds:

(I) $\tau > 0$ *and* $p > 1$*;*
(II) $-\alpha \leq \tau \leq 0$ *and* (17) *holds;*
(III) $\tau < -\alpha$ *and*

$$\rho > 2,\ 1 < p < 1 + \min\left\{\frac{\rho - 2}{1 - \kappa_\lambda}, \frac{-\alpha}{\tau + \alpha}\right\}. \tag{20}$$

Then WSL $= \emptyset$.

We shall prove Theorems 1 and 2 using nonlinear capacity estimates that are well adapted to the operators ∂_t^α and $\partial_{xx} + \frac{\lambda}{(x-a)^2}I$, the singularity $x = a$, and the boundary condition (3).

We provide below two examples to illustrate our obtained results. In the first example, we consider a homogeneous Dirichlet boundary condition ($\delta = 0$).

Example 1. *Let us consider the time-fractional Schrödinger equation*

$$\begin{cases} i^{\frac{1}{2}}\partial_t^{\frac{1}{2}}u + \partial_{xx}u + \dfrac{1}{4x^2}u = (-1+i)x^{-3}|u|^p, & t > 0,\ 0 < x \leq 1, \\ u(0,x) = \sqrt{x} + i, \quad 0 < x \leq 1, \\ u(t,1) = 0, \quad t > 0. \end{cases} \tag{21}$$

Problem (21) *is a special case of* (1)*,* (2)*,* (3) *with*

$$a = 0,\ b = 1,\ \alpha = \frac{1}{2},\ \lambda = \frac{1}{4},\ \rho = 3,\ \delta = 0$$

and

$$\mu = \mu_1 + i\mu_2 = -1 + i,\ g(x) = g_1(x) + ig_2(x) = \sqrt{x} + i.$$

Let us check that all the assumptions of Theorem 1 are satisfied. Clearly, $g \in L^1([0,1], \mathbb{C})$*. On the other hand, by* (12)*, one has*

$$H(x) = -\sqrt{x}\ln x, \quad 0 < x \leq 1.$$

Then

$$g(x)H(x) = -(\sqrt{x} + i)\sqrt{x}\ln x = -x\ln x - i\sqrt{x}\ln x, \quad 0 < x \leq 1$$

and

$$\int_0^1 |g(x)H(x)|\,dx = -\int_0^1 \sqrt{x(x+1)}\ln x\,dx < \infty,$$

which shows that $gH \in L^1([0,1], \mathbb{C})$*. Furthermore, we have*

$$\mu_1 \int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)H(x)\,dx = \frac{\sqrt{2}}{2}\int_0^1 (\sqrt{x} - 1)\sqrt{x}\ln x\,dx > 0.$$

Then, all the assumptions of Theorem 1 are satisfied. On the other hand, since $\lambda = \frac{1}{4}$*, we obtain by* (11) *that* $\kappa_\lambda = \frac{1}{2}$*. Consequently, for all*

$$1 < p < 1 + \frac{\rho - 2}{1 - \kappa_\lambda} = 3,$$

(21) *admits no weak solution.*

In the second example, we consider an inhomogeneous Dirichlet boundary condition ($\delta \neq 0$).

Example 2. *Let us consider the time-fractional Schrödinger equation*

$$\begin{cases} i^{\frac{1}{2}}\partial_t^{\frac{1}{2}} u + \partial_{xx} u + \dfrac{1}{5x^2} u = (-1+i) x^{-\frac{5}{2}} |u|^p, & t > 0,\ 0 < x \leq 1, \\ u(0,x) = \sqrt{x} + i, & 0 < x \leq 1, \\ u(t,1) = (1+2i)(t+1)^{-3}, & t > 0. \end{cases} \tag{22}$$

Problem (22) *is a special case of* (1)–(3) *with*

$$a = 0,\ b = 1,\ \alpha = \frac{1}{2},\ \lambda = \frac{1}{5},\ \tau = -3,\ \rho = \frac{5}{2},\ \delta = \delta_1 + i\delta_2 = 1 + 2i$$

and

$$\mu = \mu_1 + i\mu_2 = -1 + i,\ g(x) = g_1(x) + ig_2(x) = \sqrt{x} + i.$$

On the other hand, for all $0 < x \leq 1$, *one has*

$$\mu_1 \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x) \right) = \frac{\sqrt{2}}{2} (1 - \sqrt{x}) \geq 0.$$

Moreover, we have

$$\delta_1 \mu_1 = -1 < 0,\ \tau = -3 < -\frac{1}{2} = -\alpha,\ \rho = \frac{5}{2} > 2,\ \kappa_\lambda = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$$

and

$$\frac{\rho - 2}{1 - \kappa_\lambda} = \frac{5}{5 + \sqrt{5}},\ \frac{-\alpha}{\tau + \alpha} = \frac{1}{7}.$$

Thus, from Theorem 2 (III), we deduce that for all

$$1 < p < 1 + \min\left\{ \frac{\rho - 2}{1 - \kappa_\lambda}, \frac{-\alpha}{\tau + \alpha} \right\} = \frac{8}{7},$$

(22) *admits no weak solution.*

4. Preliminaries

Let $0 < \alpha < 1$, $\lambda \leq \frac{1}{4}$, $\mu \in \mathbb{C} \backslash \{0\}$, $\delta \in \mathbb{C}$, $\rho \geq 0$, $\tau \in \mathbb{R}$, $p > 1$ and $g \in L^1_{\text{loc}}(\mathbb{I}, \mathbb{C})$. We denote by L_λ the differential operator

$$L_\lambda \phi = \phi_{xx} + \frac{\lambda}{(x-a)^2} \phi.$$

For $T > 0$ and $\psi \in \Psi_T$, let

$$K_1(\psi) = \int_{\text{supp}\left(\partial_t (I_T^{1-\alpha} \psi)\right)} (x-a)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |\partial_t (I_T^{1-\alpha} \psi)|^{\frac{p}{p-1}} \, dx \, dt, \tag{23}$$

$$K_2(\psi) = \int_{\text{supp}(L_\lambda \psi)} (x-a)^{\frac{\rho}{p-1}} \psi^{\frac{-1}{p-1}} |L_\lambda \psi|^{\frac{p}{p-1}} \, dx \, dt. \tag{24}$$

We have the following a priori estimates.

Lemma 2. *Let* $u \in L^p_{\text{loc}}([0, \infty) \times \mathbb{I}, \mathbb{C})$ *be a weak solution to* (1)–(3).

(i) *If* $\mu_1 \neq 0$, *then*

$$\frac{1}{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx+\frac{\delta_1}{\mu_1}\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt \\ \leq C\sum_{j=1}^2 K_j(\psi) \tag{25}$$

for every $T>0$ and $\psi\in\Psi_T$, provided that $K_j(\psi)<\infty$, $j=1,2$.

(ii) *If $\mu_2\neq 0$, then*

$$\frac{1}{\mu_2}\int_a^b\left(\sin\left(\frac{\alpha\pi}{2}\right)g_1(x)+\cos\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx+\frac{\delta_2}{\mu_2}\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt \\ \leq C\sum_{j=1}^2 K_j(\psi) \tag{26}$$

for every $T>0$ and $\psi\in\Psi_T$, provided that $K_j(\psi)<\infty$, $j=1,2$.

Proof. Let $u\in L^p_{\text{loc}}([0,\infty)\times\mathbb{I},\mathbb{C})$ be a weak solution to (1), (2), (3). Let $\mu_1\neq 0$, $T>0$ and $\psi\in\Psi_T$ with $K_j(\psi)<\infty$, $j=1,2$. Then, by (13), one has

$$\begin{aligned}&\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+\frac{1}{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx\\&+\frac{\delta_1}{\mu_1}\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt\\&=-\frac{1}{\mu_1}\int_{Q_T}\left(\cos\left(\frac{\alpha\pi}{2}\right)u_1-\sin\left(\frac{\alpha\pi}{2}\right)u_2\right)\partial_t(I_T^{1-\alpha}\psi)\,dx\,dt+\frac{1}{\mu_1}\int_{Q_T}u_1L_\lambda\psi\,dx\,dt,\end{aligned}$$

which yields

$$\begin{aligned}&\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+\frac{1}{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx\\&+\frac{\delta_1}{\mu_1}\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt\\&\leq\frac{2}{|\mu_1|}\int_{Q_T}|u||\partial_t(I_T^{1-\alpha}\psi)|\,dx\,dt+\frac{1}{|\mu_1|}\int_{Q_T}|u||L_\lambda\psi|\,dx\,dt.\end{aligned}\tag{27}$$

On the other hand, by means of Young's inequality, for all $\varepsilon>0$, we obtain

$$\begin{aligned}\int_{Q_T}|u||\partial_t(I_T^{1-\alpha}\psi)|\,dx\,dt&=\int_{Q_T}\left[(x-a)^{\frac{-\rho}{p}}|u|\psi^{\frac{1}{p}}\right]\left[(x-a)^{\frac{\rho}{p}}\psi^{\frac{-1}{p}}|\partial_t(I_T^{1-\alpha}\psi)|\right]dx\,dt\\&\leq\varepsilon\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+CK_1(\psi).\end{aligned}\tag{28}$$

Similarly, one has

$$\int_{Q_T}|u||L_\lambda\psi|\,dx\,dt\leq\varepsilon\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt+CK_2(\psi).\tag{29}$$

Therefore, in view of (27)–(29), we obtain

$$\begin{aligned}&\left(1-\frac{3}{|\mu_1|}\varepsilon\right)\int_{Q_T}(x-a)^{-\rho}|u|^p\psi\,dx\,dt\\&+\frac{1}{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx+\frac{\delta_1}{\mu_1}\int_0^T(t+1)^\tau\partial_x\psi(t,b)\,dt\\&\leq C\sum_{j=1}^2 K_j(\psi)\end{aligned}$$

Hence, taking $\varepsilon=\frac{|\mu_1|}{3}$ in the above inequality, we obtain (25). Similarly, if $\mu_2\neq 0$, using (14) and proceeding as above, we obtain (26). □

Let H be the function defined in $\mathbb{I}$ by (12). It can be easily seen that

$$H\in C^2(\mathbb{I}),\ H\geq 0,\ L_\lambda H=0\text{ in }\mathbb{I},\ H(b)=0. \tag{30}$$

Let ξ be a function verifying

$$\xi\in C^\infty(\mathbb{R}),\ 0\leq\xi\leq 1,\ \xi=0\text{ in }\left[0,\frac{1}{2}\right],\ \xi=1\text{ in }[1,\infty). \tag{31}$$

For $R,\ell\gg 1$ (sufficient large), let

$$\xi_R(x)=H(x)\xi^\ell(R(x-a)),\quad x\in\mathbb{I},$$

that is,

$$\xi_R(x)=\begin{cases}0 & \text{if}\quad a<x\leq a+\frac{1}{2R},\\ H(x)\xi^\ell(R(x-a)) & \text{if}\quad a+\frac{1}{2R}<x<a+\frac{1}{R},\\ H(x) & \text{if}\quad a+\frac{1}{R}\leq x\leq b.\end{cases} \tag{32}$$

For $T>0$, let

$$\iota_T(t)=T^{-\ell}(T-t)^\ell,\quad t\in[0,T]. \tag{33}$$

We consider functions of the form

$$\psi(t,x)=\iota_T(t)\xi_R(x),\quad (t,x)\in Q_T. \tag{34}$$

Lemma 3. *Let $T>0$. For $R,\ell\gg 1$, we have $\psi\in\Psi_T$, where ψ is defined by (34).*

Proof. One can check easily that from (30)–(33), the function ψ satisfies the conditions (i) and (ii) of Definition 1. □

Lemma 4. *For $T,R,\ell\gg 1$, we have*

$$K_1(\psi)\leq CT^{\frac{(1-\alpha)p-1}{p-1}}\ln R. \tag{35}$$

Proof. Thanks to (23) and (34), we get

$$K_1(\psi)=\left(\int_0^T\iota_T^{\frac{-1}{p-1}}|(I_T^{1-\alpha}\iota_T)'|^{\frac{p}{p-1}}\,dt\right)\left(\int_{a+\frac{1}{2R}}^b(x-a)^{\frac{\rho}{p-1}}\xi_R(x)\,dx\right). \tag{36}$$

On the other hand, for all $0 < t < T$ and $\sigma > 0$, one has

$$\begin{aligned}(I_T^\sigma \iota_T)(t) &= \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} \iota_T(s)\, ds \\ &= \frac{T^{-\ell}}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} (T-s)^\ell\, ds \\ &= \frac{T^{-\ell}}{\Gamma(\sigma)} \int_t^T ((T-t)-(T-s))^{\sigma-1} (T-s)^\ell\, ds \\ &= \frac{T^{-\ell}(T-t)^{\sigma-1}}{\Gamma(\sigma)} \int_t^T \left(1-\frac{T-s}{T-t}\right)^{\sigma-1} (T-s)^\ell\, ds.\end{aligned}$$

Then, by the change of variable $\vartheta = \frac{T-s}{T-t}$, we get

$$\begin{aligned}(I_T^\sigma \iota_T)(t) &= \frac{T^{-\ell}(T-t)^{\sigma+\ell}}{\Gamma(\sigma)} \int_0^1 (1-\vartheta)^{\sigma-1} \vartheta^{(\ell+1)-1}\, d\vartheta \\ &= \frac{T^{-\ell}(T-t)^{\sigma+\ell}}{\Gamma(\sigma)} B(\sigma, \ell+1),\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. Next, by means of the property (see e.g., [25])

$$B(\sigma, \ell+1) = \frac{\Gamma(\sigma)\Gamma(\ell+1)}{\Gamma(\sigma+\ell+1)},$$

we obtain

$$(I_T^\sigma \iota_T)(t) = \frac{\Gamma(\ell+1)}{\Gamma(\sigma+\ell+1)} T^{-\ell}(T-t)^{\sigma+\ell}.$$

In particular, for $\sigma = 1-\alpha$, we have

$$(I_T^{1-\alpha} \iota_T)(t) = \frac{\Gamma(\ell+1)}{\Gamma(2-\alpha+\ell)} T^{-\ell}(T-t)^{1-\alpha+\ell}, \tag{37}$$

which yields

$$\iota_T^{\frac{-1}{p-1}}(t) |(I_T^{1-\alpha} \iota_T)'(t)|^{\frac{p}{p-1}} = C T^{-\ell} (T-t)^{\ell - \frac{\alpha p}{p-1}}.$$

Hence,

$$\begin{aligned}\int_0^T \iota_T^{\frac{-1}{p-1}} |(I_T^{1-\alpha} \iota_T)'|^{\frac{p}{p-1}}\, dt &= C T^{-\ell} \int_0^T (T-t)^{\ell-\frac{\alpha p}{p-1}}\, dt \\ &= C T^{\frac{(1-\alpha)p-1}{p-1}}.\end{aligned} \tag{38}$$

Moreover, by (31) and (32), we obtain

$$\begin{aligned}\int_{a+\frac{1}{2R}}^b (x-a)^{\frac{\rho}{p-1}} \xi_R(x)\, dx &= \int_{a+\frac{1}{2R}}^b (x-a)^{\frac{\rho}{p-1}} H(x) \xi^\ell(R(x-a))\, dx \\ &\le \int_{a+\frac{1}{2R}}^b (x-a)^{\frac{\rho}{p-1}} H(x)\, dx.\end{aligned}$$

On the other hand, by (12), we have

$$H(x) \le C \ln R, \quad a + \frac{1}{2R} < x < b.$$

Hence, there holds

$$\begin{aligned}\int_{a+\frac{1}{2R}}^{b}(x-a)^{\frac{\rho}{p-1}}\xi_R(x)\,dx &\leq C\ln R\int_{a+\frac{1}{2R}}^{b}(x-a)^{\frac{\rho}{p-1}}\,dx\\ &\leq C\ln R.\end{aligned} \tag{39}$$

Finally, (35) follows from (36), (38) and (39). □

Lemma 5. *For $T,R,\ell \gg 1$, we have*

$$K_2(\psi) \leq CTR^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}}(\ln R)^{\frac{p}{p-1}}. \tag{40}$$

Proof. Thanks to (24) and (34), we get

$$K_2(\psi) = \left(\int_0^T \iota_T\,dt\right)\left(\int_{a+\frac{1}{2R}}^{b}(x-a)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}|L_\lambda\xi_R|^{\frac{p}{p-1}}\,dx\right). \tag{41}$$

On the other hand, by (33), one has

$$\begin{aligned}\int_0^T \iota_T(t)\,dt &= T^{-\ell}\int_0^T (T-t)^\ell\,dt\\ &= CT.\end{aligned} \tag{42}$$

By (32), for all $a+\frac{1}{2R} < x < b$, we have

$$\begin{aligned}L_\lambda\xi_R(x) &= L_\lambda\left[H(x)\xi^\ell(R(x-a))\right]\\
&= \left[H(x)\xi^\ell(R(x-a))\right]'' + \frac{\lambda}{(x-a)^2}H(x)\xi^\ell(R(x-a))\\
&= H''(x)\xi^\ell(R(x-a)) + H(x)\left[\xi^\ell(R(x-a))\right]'' + 2H'(x)\left[\xi^\ell(R(x-a))\right]'\\
&\quad + \frac{\lambda}{(x-a)^2}H(x)\xi^\ell(R(x-a))\\
&= \xi^\ell(R(x-a))\left[H''(x) + \frac{\lambda}{(x-a)^2}H(x)\right] + H(x)\left[\xi^\ell(R(x-a))\right]''\\
&\quad + 2H'(x)\left[\xi^\ell(R(x-a))\right]'\\
&= \xi^\ell(R(x-a))L_\lambda H(x) + H(x)\left[\xi^\ell(R(x-a))\right]'' + 2H'(x)\left[\xi^\ell(R(x-a))\right]'.\end{aligned}$$

Since $L_\lambda H = 0$ by (30), there holds

$$L_\lambda\xi_R(x) = H(x)\left[\xi^\ell(R(x-a))\right]'' + 2H'(x)\left[\xi^\ell(R(x-a))\right]'. \tag{43}$$

Then, by (31), we deduce that

$$\int_{a+\frac{1}{2R}}^{b}(x-a)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}(x)|L_\lambda\xi_R(x)|^{\frac{p}{p-1}}\,dx = \int_{a+\frac{1}{2R}}^{a+\frac{1}{R}}(x-a)^{\frac{\rho}{p-1}}\xi_R^{\frac{-1}{p-1}}(x)|L_\lambda\xi_R(x)|^{\frac{p}{p-1}}\,dx. \tag{44}$$

Moreover, for all $a+\frac{1}{2R} < x < a+\frac{1}{R}$, one has

$$\left|\left[\xi^\ell(R(x-a))\right]''\right| \leq CR^2\xi^{\ell-2}(R(x-a)),\quad \left|\left[\xi^\ell(R(x-a))\right]'\right| \leq CR\xi^{\ell-2}(R(x-a)). \tag{45}$$

On the other hand, by (12), it can be easily seen that for all $a+\frac{1}{2R}<x<a+\frac{1}{R}$,

$$C_1 R^{-\kappa_\lambda} \le H(x) \le C_2 R^{-\kappa_\lambda}\ln R,\ \ H'(x) \le C R^{1-\kappa_\lambda}\ln R. \tag{46}$$

Consequently, if follows from (43), (45) and (46) that

$$|L_\lambda \xi_R(x)| \le C R^{2-\kappa_\lambda}\ln R\, \xi^{\ell-2}(R(x-a)), \quad a+\frac{1}{2R}<x<a+\frac{1}{R}. \tag{47}$$

Therefore, using (31), (32), (44), (46) and (47), we obtain

$$\begin{aligned}
&\int_{a+\frac{1}{2R}}^{b} (x-a)^{\frac{\rho}{p-1}} \xi_R^{\frac{-1}{p-1}}(x) |L_\lambda \xi_R(x)|^{\frac{p}{p-1}}\,dx \\
&\le C R^{\frac{(2-\kappa_\lambda)p+\kappa_\lambda}{p-1}} (\ln R)^{\frac{p}{p-1}} \int_{a+\frac{1}{2R}}^{a+\frac{1}{R}} (x-a)^{\frac{\rho}{p-1}} \xi^{\ell-\frac{2p}{p-1}}(R(x-a))\,dx \\
&\le C R^{\frac{(2-\kappa_\lambda)p+\kappa_\lambda}{p-1}} (\ln R)^{\frac{p}{p-1}} \int_{a+\frac{1}{2R}}^{a+\frac{1}{R}} (x-a)^{\frac{\rho}{p-1}}\,dx \\
&\le C R^{\frac{(2-\kappa_\lambda)p+\kappa_\lambda}{p-1}} R^{-\left(\frac{\rho}{p-1}+1\right)} (\ln R)^{\frac{p}{p-1}},
\end{aligned}$$

that is,

$$\int_{a+\frac{1}{2R}}^{b} (x-a)^{\frac{\rho}{p-1}} \xi_R^{\frac{-1}{p-1}}(x) |L_\lambda \xi_R(x)|^{\frac{p}{p-1}}\,dx \le C R^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}} (\ln R)^{\frac{p}{p-1}}. \tag{48}$$

Thus, in view of (41) , (42) and (48), we obtain (40). □

5. Proofs of the Main Results

In this section, we provide the proofs of Theorems 1 and 2.

Proof of Theorem 1. Suppose that $u \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{I},\mathbb{C})$ is a weak solution to (1)–(3).

We first consider the case (15). In this case, one has $\mu_1 \neq 0$. Hence, by (25) (with $\delta_1 = 0$) and Lemma 3, for $T, R, \ell \gg 1$, there holds

$$\frac{1}{\mu_1}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x)\right) (I_T^{1-\alpha}\psi)(0,x)\,dx \le C\sum_{j=1}^{2} K_j(\psi), \tag{49}$$

where ψ is the function defined by (34). On the other hand, by (34) and (37), one has

$$(I_T^{1-\alpha}\psi)(0,x) = C T^{1-\alpha}\xi_R(x), \quad x \in \mathbb{I},$$

which yields

$$\begin{aligned}
&\frac{1}{\mu_1}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x)\right) (I_T^{1-\alpha}\psi)(0,x)\,dx \\
&= \frac{C}{\mu_1} T^{1-\alpha}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x)\right)\xi_R(x)\,dx.
\end{aligned}$$

Since $gH \in L^1(\mathbb{I},\mathbb{C})$, by (32) and the dominated convergence theorem, we deduce that

$$\begin{aligned}
&\lim_{R\to\infty}\frac{1}{\mu_1}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x)\right)\xi_R(x)\,dx \\
&= \frac{1}{\mu_1}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right) g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right) g_2(x)\right) H(x)\,dx.
\end{aligned}$$

Hence, in view of (15), we deduce that (for $R \gg 1$)

$$\frac{1}{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x)-\sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx \geq CT^{1-\alpha}. \tag{50}$$

Then, (35), (40), (49) and (50) yield

$$1 \leq C\left(T^{\frac{-\alpha}{p-1}}\ln R + T^{\alpha}R^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}}(\ln R)^{\frac{p}{p-1}}\right). \tag{51}$$

Consider now a parameter θ satisfying

$$0<\theta<\frac{\rho-1-\kappa_\lambda-(1-\kappa_\lambda)p}{\alpha(p-1)}. \tag{52}$$

Notice that due to (17), the set of θ satisfying (52) is nonempty. Taking $T=R^\theta$, (51) reduces to

$$1 \leq C\left(R^{\frac{-\alpha\theta}{p-1}}\ln R + R^{\zeta}(\ln R)^{\frac{p}{p-1}}\right), \tag{53}$$

where

$$\zeta=\alpha\theta+\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}.$$

Observe that due to the choice (52) of the parameter θ, one has $\zeta<0$. Hence, passing to the limit as $R\to\infty$ in (53), we obtain a contradiction. Consequently, (1)–(3) admits no weak solution.

We next consider the case (16). As previously, we suppose that $u \in L^p_{\text{loc}}([0,\infty)\times\mathbb{I},\mathbb{C})$ is a weak solution to (1)–(3). Let $v=-iu$, that is,

$$v=v_1+iv_2,\ v_1=u_2,\ v_2=-u_1.$$

Then $v \in L^p_{\text{loc}}([0,\infty)\times\mathbb{I},\mathbb{C})$ is a weak solution to

$$i^\alpha\partial_t^\alpha v+\partial_{xx}v+\frac{\lambda}{(x-a)^2}v=\widetilde{\mu}(x-a)^{-\rho}|v|^p \quad \text{in } (0,\infty)\times\mathbb{I}$$

under the initial condition

$$v(0,x)=\widetilde{g}(x) \quad \text{in } \mathbb{I}$$

and the boundary condition

$$v(t,b)=0 \quad \text{in } (0,\infty),$$

where

$$\widetilde{\mu}=-i\mu=\widetilde{\mu_1}+i\widetilde{\mu_2},\ \widetilde{\mu_1}=\mu_2,\ \widetilde{\mu_2}=-\mu_1$$

and

$$\widetilde{g}=-ig=\widetilde{g_1}+i\widetilde{g_2},\ \widetilde{g_1}=g_2,\ \widetilde{g_2}=-g_1.$$

Observe that (16) is equivalent to

$$\widetilde{\mu_1}\int_a^b\left(\cos\left(\frac{\alpha\pi}{2}\right)\widetilde{g_1}(x)-\sin\left(\frac{\alpha\pi}{2}\right)\widetilde{g_2}(x)\right)H(x)\,dx>0.$$

Therefore, the nonexistence result follows from the case (15) already treated above. □

Proof of Theorem 2. Suppose that $u \in L^p_{\text{loc}}([0,\infty)\times\mathbb{I},\mathbb{C})$ is a weak solution to (1)–(3).

We first consider the case (18). In this case, one has $\mu_1 \neq 0$. Hence, by (25) and Lemma 3, for $T, R, \ell \gg 1$, there holds

$$\begin{aligned}
&\frac{1}{\mu_1}\int_a^b \left(\cos\left(\frac{\alpha\pi}{2}\right)g_1(x) - \sin\left(\frac{\alpha\pi}{2}\right)g_2(x)\right)(I_T^{1-\alpha}\psi)(0,x)\,dx \\
&+\frac{\delta_1}{\mu_1}\int_0^T (t+1)^\tau \partial_x\psi(t,b)\,dt \leq C\sum_{j=1}^2 K_j(\psi),
\end{aligned}$$

where ψ is the function defined by (34). Then, due to (18), we deduce that

$$\frac{\delta_1}{\mu_1}\int_0^T (t+1)^\tau \partial_x\psi(t,b)\,dt \leq C\sum_{j=1}^2 K_j(\psi). \tag{54}$$

On the other hand, by (32) and (34), we obtain

$$\begin{aligned}
\int_0^T (t+1)^\tau \partial_x\psi(t,b)\,dt &= H'(b)\int_0^T (t+1)^\tau \iota_T(t)\,dt \\
&= H'(b)T^{-\ell}\int_0^T (t+1)^\tau (T-t)^\ell\,dt.
\end{aligned}$$

Notice that by (12), one has $H'(b) < 0$. Then, there holds (for $T \gg 1$)

$$\begin{aligned}
\int_0^T (t+1)^\tau \partial_x\psi(t,b)\,dt &\leq H'(b)T^{-\ell}\int_{\frac{T}{2}}^T (t+1)^\tau (T-t)^\ell\,dt \\
&\leq H'(b)T^{\tau+1}.
\end{aligned}$$

Since $\delta_1\mu_1 < 0$ by (18), we obtain

$$\frac{\delta_1}{\mu_1}\int_0^T (t+1)^\tau \partial_x\psi(t,b)\,dt \geq CT^{\tau+1}. \tag{55}$$

Therefore, (35), (40), (54) and (55) yield

$$T^{\tau+1} \leq C\left(T^{\frac{(1-\alpha)p-1}{p-1}}\ln R + TR^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}}(\ln R)^{\frac{p}{p-1}}\right),$$

that is,

$$1 \leq C\left(T^{\frac{\tau-(\alpha+\tau)p}{p-1}}\ln R + T^{-\tau}R^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}}(\ln R)^{\frac{p}{p-1}}\right). \tag{56}$$

(I) The case $\tau > 1$. We have

$$\frac{\tau-(\alpha+\tau)p}{p-1} < 0 \text{ (since } p > 1), \; -\tau < 0.$$

Hence, fixing R and passing to the limit as $T \to \infty$ in (56), we obtain a contradiction.
(II) The case $-\alpha \leq \tau \leq 0$. If $\tau = 0$, then (56) reduces to

$$1 \leq C\left(T^{\frac{-\alpha p}{p-1}}\ln R + R^{\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1}}(\ln R)^{\frac{p}{p-1}}\right). \tag{57}$$

Observe that due to (17), one has

$$\frac{(1-\kappa_\lambda)p+\kappa_\lambda-\rho+1}{p-1} < 0.$$

Hence, taking $T = R$ and passing to the limit as $R \to \infty$ in (57), we obtain a contradiction. If $-\alpha \leq \tau < 0$, we take a parameter θ satisfying

$$0 < \theta < \frac{(1-\kappa_\lambda)p + \kappa_\lambda - \rho + 1}{\tau(p-1)}. \tag{58}$$

We point tout that due to (17) and $\tau < 0$, the set of θ satisfying (58) is nonempty. Taking $T = R^\theta$, (56) reduces to

$$1 \leq C\left(R^{\zeta_1} \ln R + R^{\zeta_2}\right), \tag{59}$$

where

$$\zeta_1 = \frac{\theta[\tau - (\alpha+\tau)p]}{p-1},\ \zeta_2 = -\tau\theta + \frac{(1-\kappa_\lambda)p + \kappa_\lambda - \rho + 1}{p-1}.$$

Notice that in this case, one has $\zeta_1 < 0$. Moreover, due to the choice (58) of the parameter θ, we have $\zeta_2 < 0$. Hence, passing to the limit as $R \to \infty$ in (59), we obtain a contradiction.

(III) The case $\tau < -\alpha$. Let θ verifies (58). We take $T = R^\theta$, and then (56) is reduced to (59). On the other hand, due to (20), one has $\zeta_1 < 0$ and $\zeta_2 < 0$. Hence, passing to the limit as $R \to \infty$ in (59), we obtain a contradiction.

We now assume that (19) holds and $u \in L^p_{\text{loc}}([0,\infty) \times \mathbb{I}, \mathbb{C})$ is a weak solution to (1)–(3). Then, taking $v = -iu$ and proceeding as in the proof of Theorem 1, we deduce the nonexistence result by the case (18) already treated above. $\square$

6. Conclusions

Problem (1)–(3) is studied in this paper. The time-fractional derivative is considered in the Caputo sense. Using nonlinear capacity and integral estimates, we obtained sufficient conditions, so that WSL $= \emptyset$. We investigated separately the homogeneous Dirichlet boundary condition (i.e., $\delta = 0$) and the inhomogeneous Dirichlet boundary condition (i.e., $\delta \in \mathbb{C}\backslash\{0\}$). It would be interested to study other types of boundary conditions, such as Neumann boundary condition and Robin boundary condition. For such problems, a judicious choice of the function ψ (given by (34) in the Dirichlet case) is needed for each type of boundary condition.

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