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Analysis of Cauchy Problems and Diffusion Equations Associated with the Hilfer–Prabhakar Fractional Derivative via Kharrat–Toma Transform

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Abstract: In this paper, the Kharrat–Toma transforms of the Prabhakar integral, a Hilfer–Prabhakar (HP) fractional derivative, and the regularized version of the HP fractional derivative are derived. Moreover, we also compute the solution of some Cauchy problems and diffusion equations modeled with the HP fractional derivative via Kharrat–Toma transform. The solutions of Cauchy problems and the diffusion equations modeled with the HP fractional derivative are computed in the form of the generalized Mittag–Leffler function.

Keywords: Hilfer–Prabhakar fractional derivative; Cauchy problems; diffusion equations; Kharrat–Toma transform; Mittag–Leffler function

MSC: 26A33; 42B10; 33E12; 34A08; 44A35

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1. Introduction

Integral transforms are very relevant in solving differential and integral equations with initial and boundary value conditions. Recently, Kharrat and Toma [1] suggested a new integral transform called Kharrat–Toma (KT) transform with applications to initial value problems. In the field of fractional calculus, various notable works have been reported. Hilfer [2], Podlubny [3], and Kilbas [4] have provided efficient literature on the theory of fractional differential equations (FDEs). Hilfer [5] generalized the Caputo and Riemann–Liouville (RL) fractional derivative operators (FDOs). The Prabhakar integral and derivative were introduced by replacing the kernel of the RL integral operator with the three-parameter Mittag–Leffler function [6]. The generalization of the Prabhakar integral appears as an important tool in solving the problems involving the Hilfer–Prabhakar (HP) fractional derivative utilizing the integral transform technique. The HP fractional derivative (HPFD) and its regularized Caputo version were introduced in [6]. Panchal et al. [7] computed the Sumudu integral transform of HP fractional derivatives and demonstrated its applications to Cauchy problems. Moreover, Singh et al. [8] provided the solution for a free-electron laser (FEL) equation modeled with the HP fractional order derivative using the Elzaki transform. Furthermore, Singh et al. [9] adopted a new method to investigate the Cattaneo–Hristov diffusion model and fractional diffusion equations with the HP derivative.

This paper derives the KT transform of the Prabhakar integral (PI) and the HP fractional derivative and its regularized version, and the derived formulae were used further to explore the solutions of some non-homogeneous Cauchy-type FDEs [6] modeled with the HP fractional derivative. Moreover, the diffusion equations in 1D and 2D spaces [10–12] modeled with the HP fractional derivative were also solved by an integral method consisting of Fourier sine transform (FST) [13] and KT transform [1]. The remaining paper is organized as follows: Section 2 presents the definition and properties of the KT transform and HP fractional derivatives. Section 3 derives the KT transform of the Prabhakar integral and HP fractional derivatives. In Section 4, the derived results from the previous section are utilized in solving Cauchy problems involving the HP derivative and its regularized version. Section 5 presents the application of the integral method consisting of FST and KT transform in solving diffusion equations with the HP fractional derivative. Finally, Section 6 presents the concluding remarks.

2. Preliminaries: Kharrat–Toma Transform and HP Fractional Derivative Operator

In the present paper, we follow these definitions, theorems, and symbols:

Definition 1 ([1]). *A function $\Im(\xi)$ is called of exponential order on every finite interval in $[0,\infty)$ if $\exists$ a positive number K such that $|\Im(\xi)| \leq Ke^{\alpha\xi}$, $K > 0$, $\alpha > 0$, $\forall\xi \geq 0$.*

Definition 2 ([1]). *Let $\Im(\xi)$ be a real valued function such that $\Im(\xi) > 0$ for $\xi \geq 0$ and $\Im(\xi) = 0$ for $\xi < 0$. If $\Im(\xi)$ is piece-wise continuous and of exponential order, the KT transform (KTT) of the function $\Im(\xi)$ denoted by $B[\Im(\xi)]$ is expressed by:*

$$B[\Im(\xi)] = G(s) = s^3 \int_0^\infty e^{-\frac{\xi}{s^2}} \Im(\xi)\, d\xi;\ s > 0, \tag{1}$$

providing the integral on the right exists. Here, s denotes the transform variable. Here, $\Im(\xi)$ is called the inverse KT or the inverse of $G(s)$ and is expressed by $\Im(\xi) = B^{-1}[G(s)]$.

Theorem 1 ([1]). *(Sufficient criterion for the existence of the KT transform)*

The KT transform $B[\Im(\xi)]$ exists if it has exponential order and $\int_0^b |\Im(\xi)| d\xi$ exists for any $b > 0$.

Theorem 2 ([1]). *(Linearity property)*

If a and b are any constants and $\Im_1(\xi)$ and $\Im_2(\xi)$ are functions, then:

$$B\{a\Im_1(\xi) + b\Im_2(\xi)\} = aB[\Im_1(\xi)] + bB[\Im_2(\xi)]. \tag{2}$$

Theorem 3 ([1]). *(KT transform of the mth-order derivative)*

If $\Im^m(\xi)$ is the mth-order derivative of the function $\Im(\xi)$ in respect of ξ, then its KT transform is given by:

$$B\left[\Im^{(m)}(\xi)\right] = G_m(s) = \frac{1}{s^{2m}} G(s) - \sum_{k=0}^{m-1} s^{-2m+2k+5} \Im^{(k)}(0),\ m \geq 1. \tag{3}$$

For $m = 1,\ 2,\ 3$, we have:

$$B\left[\Im'(\xi)\right] = G_1(s) = \frac{1}{s^2} G(s) - s^3 \Im(0), \tag{4}$$

$$B[\Im''(\xi)] = G_2(s) = \frac{1}{s^4} G(s) - s\Im(0) - s^3 \Im'(0). \tag{5}$$

Similarly, the KT transform of the first-order partial derivative of a function $\Im(y,\xi)$ is written as:

$$B\left[\frac{\partial\Im}{\partial\xi}(y,\xi)\right] = G(y,s) = \frac{1}{s^2} G(y,s) - s^3 \Im(y,\ 0). \tag{6}$$

Theorem 4 ([1]). *(Convolution theorem for KT transform)*

If $\overline{\Im}_1(s)$ & $\overline{\Im}_2(s)$, respectively, are the KT transforms of $\Im_1(\xi)$ & $\Im_2(\xi)$, then:

$$B[\Im_1 * \Im_2] = \frac{1}{s^3}\overline{\Im}_1(s)\overline{\Im}_2(s), \tag{7}$$

where $\Im_1 * \Im_2$ is the convolution of two functions $\Im_1(\xi)$ and $\Im_2(\xi)$.

Some important formulae for the KT transform of special functions described in refs. [1] *are constituted here: $B[1] = s^5$, $B[\xi^m] = s^{2m+5}.m\,!$, $B[\sin(kx)] = \frac{ks^7}{1+k^2s^4}$, $B[\cos(kx)] = \frac{s^5}{1+k^2s^4}$, $B[\xi^m] = s^{2m+5}.m\,!$ for $m = 1,\ 2\ldots$.*

Definition 3 ([1]). *(Inverse of the KT transform)*

The inverse of the KT transform is defined by:

$$B^{-1}[G(s)](\xi) = \Im(\xi) = B^{-1}\left[s^3\int_0^\infty \Im(\xi)e^{-\frac{\xi}{s^2}}d\xi\right],\ \xi > 0, \tag{8}$$

where $G(s)$ denotes the KT transform of the function $\Im(\xi)$.

Definition 4 ([14]). *Let $\Im(\xi)$ be a real valued function taken on $]-\infty,\infty[$ and be section-wise continuous in each partial interval of finite length and completely integrable in $(-\infty,\infty)$; then, $F[\Im(\xi)](p) = \int_{-\infty}^{\infty} e^{-ip\xi}\Im(\xi)d\xi$ is called Fourier transform (FT) of $\Im(\xi)$ and is denoted by $F[\Im(\xi)](p) = \hat{\Im}(p)$. The function $\Im(\xi)$ called the inverse FT of $\hat{\Im}(p)$ expressed by $\Im(\xi) = F^{-1}\left[\hat{\Im}(p)\right](\xi)$ and is formulated as:*

$$\Im(\xi) = F^{-1}\left[\hat{\Im}(p)\right](\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\xi}\hat{\Im}(p)dp. \tag{9}$$

Definition 5 ([13]). *The Fourier sine transform of a function $\Im(\xi)$ defined in Definition 4 is given by:*

$$F_{\sin e}[\Im(\xi)](p) = \hat{\Im}_{fst}(p) = \frac{2}{\pi}\int_0^\infty \Im(\xi)\sin p\xi\, d\xi.$$

The function $\Im(\xi)$ is called the inverse Fourier transform of $\hat{\Im}_{fst}(p)$ and is formulated as:

$$\Im(\xi) = F_{\sin e}{}^{-1}\left[\hat{\Im}_{fst}(p)\right](\xi) = \frac{2}{\pi}\int_0^\infty \hat{\Im}_{fst}(p)\sin p\xi\, dp. \tag{10}$$

Definition 6 ([2]). *(Hilfer derivative)*

Let $\sigma \in (0,\ 1)$, $\lambda \in [0,\ 1]$, $\Im \in L^1(\alpha,\beta)$, $-\infty < \alpha < \xi < \beta < \infty$, $\left(\Im * I_{a+}^{(1-\sigma)(1-\lambda)}\right)(\xi) \in AC^1[\alpha,\beta]$.

Then, the Hilfer fractional derivative (HFD) of $\Im(\xi)$ of order σ and type λ is given by:

$$\left(D_{a+}^{\sigma,\lambda}\Im\right)(\xi) = \left(I_{a+}^{\lambda(1-\sigma)}\frac{d}{d\xi}I_{a+}^{(1-\sigma)(1-\lambda)}\Im\right)(\xi). \tag{11}$$

The HFD operator is considered as an interpolator between the RL and Caputo derivative operators.

Definition 7 ([6]). *(Regularized version of HFD)*

Let $\sigma \in (0, 1)$, $\lambda \in [0, 1]$, and $\Im \in AC^1[0, \beta]$, $0 < \xi < \beta < \infty$. Then, the regularized version of HFD of $\Im(\xi)$ of order σ and type λ is defined as:

$$\left(D_{0+}^{\sigma,\lambda}\Im\right)(\xi) - \frac{\xi^{-\sigma}\Im(0^+)}{\Gamma(1-\sigma)} = {}^{C}D_{0+}^{\sigma}\Im(\xi), \tag{12}$$

where ${}^{C}D_{0+}^{\sigma}\Im(\xi)$ denotes the Caputo fractional derivative (CFD) operator.

Definition 8 ([15]). *The generalized three parameters Mittag–Leffler function (MLF) given by Prabhakar is formulated as $E_{\eta,\sigma}^{\Theta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)} \frac{z^k}{k\,!}$ for $\eta, \sigma, \Theta \in C$ and $\mathrm{Re}(\eta) > 0$, where C specifies the set of complex numbers.*

Definition 9. *Garra et al. [6] introduced the generalization form of the Hilfer derivative by substituting the RL integral in the formula of the Hilfer derivative with a more general integral operator with kernel $e_{\eta,\sigma,\theta}^{\Theta}(\xi) = \xi^{\sigma-1}E_{\eta,\sigma}^{\Theta}(\theta\xi^{\eta})$, where $\xi \in R$, $\eta, \sigma, \theta, \Theta \in C$, $\mathrm{Re}(\sigma)$, $\mathrm{Re}(\eta) > 0$, and $E_{\eta,\sigma}^{\Theta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)} \frac{z^k}{k\,!}$ is the generalized MLF investigated for the first time in [15].*

Definition 10 ([15,16]). *(Prabhakar integral)*
Let $\Im \in L^1(0, \beta)$, $0 < \xi < \beta < \infty$. Then, the Prabhakar integral is written as:

$$E_{\eta,\sigma,\theta,0^+}^{\Theta}\Im(\xi) = \int_0^{\xi} (\xi - y)^{\sigma-1} E_{\eta,\sigma}^{\Theta}\left[\theta(\xi - y)^{\eta}\right]\Im(y)dy = \left(\Im * e_{\eta,\sigma,\theta}^{\Theta}\right)(\xi), \tag{13}$$

where $*$ denotes the convolution operation; $\eta, \sigma, \theta, \Theta \in C$ with $\mathrm{Re}(\eta)$, $\mathrm{Re}(\sigma) > 0$ and

$$e_{\eta,\sigma,\theta}^{\Theta}(\xi) = \xi^{\sigma-1}E_{\eta,\sigma}^{\Theta}(\theta\xi^{\eta}). \tag{14}$$

Definition 11 ([6]). *(HP fractional derivative)*
Let $\sigma \in (0, 1)$, $\lambda \in [0, 1]$, $\Im \in L^1(0, \beta)$, $0 < \xi < \beta < \infty$ and let $\left(\Im * e_{\eta,(1-\lambda)(1-\sigma),\theta}^{-\Theta(1-\lambda)}\right)(\xi) \in AC^1[0, \beta]$. The HP fractional derivative (HPFD) of $\Im(\xi)$ of order σ illustrated as $D_{\eta,\theta,0^+}^{\Theta,\sigma,\lambda}\Im(\xi)$ is defined by:

$$D_{\eta,\theta,0^+}^{\Theta,\sigma,\lambda}\Im(\xi) = \left(E_{\eta,\lambda(1-\sigma),\theta,0^+}^{-\Theta\lambda} \frac{d}{d\xi}\left(E_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}^{-\Theta(1-\lambda)}\Im\right)\right)(\xi), \tag{15}$$

where $\Theta, \theta \in R$, $\eta > 0$ and $E_{\eta,0,\theta,0^+}^{0}\Im = \Im$.
Here, it is observed that the mathematical expression (15) reduces to the Hilfer derivative for $\Theta = 0$.

Definition 12 ([6]). *(Regularized version of the HP fractional derivative)*
Let $\Im \in AC^1[0, \beta]$, $0 < \xi < \beta < \infty$ and let $\sigma \in (0, 1)$, $\lambda \in [0, 1]$, $\Theta, \theta \in R$, $\eta > 0$. The regularized version of the HPFD of $\Im(\xi)$ of order σ is written as ${}^{C}D_{\eta,\theta,0^+}^{\Theta,\sigma}\Im(\xi)$ and defined by:

$${}^{C}D_{\eta,\theta,0^+}^{\Theta,\sigma}\Im(\xi) = \left(E_{\eta,\lambda(1-\sigma),\theta,0^+}^{-\Theta\lambda} E_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}^{-\Theta(1-\lambda)} \frac{d}{d\xi}\Im\right)(\xi). \tag{16}$$

In addition:

$$\left(E_{\eta,\sigma,\theta,0^+}^{\Theta} E_{\eta,\lambda,\theta,0^+}^{\Lambda}\Im\right)(\xi) = \left(E_{\eta,\sigma+\lambda,\theta,0^+}^{\Theta+\Lambda}\Im\right)(\xi). \tag{17}$$

3. Kharrat–Toma Transform of the Prabhakar Integral and HP Fractional Derivatives

This section derives the formula for the KT transform of the Prabhakar integral and HPFD and its regularized Caputo version.

Lemma 1. *The KT transform of the kernel function* $e^{\Theta}_{\eta,\sigma,\theta}(\xi)$ *is given by:*

$$B\left[e^{\Theta}_{\eta,\sigma,\theta}(\xi)\right] = s^{2\sigma+3}\left[1-\theta\ s^{2\eta}\right]^{-\Theta}$$

for $\sigma \in (0,\ 1)$, $\Theta, \theta \in R$, $\eta > 0$ *and consequently, the KT transform of the Prabhakar integral is obtained as:*

$$B\left[E^{\Theta}_{\eta,\sigma,\theta,0^+}\Im(\xi)\right] = s^{2\sigma}B(\Im)\left[1-\theta\ s^{2\eta}\right]^{-\Theta}.$$

Proof. From Definition 8 of the three-parameter generalized Mittag–Leffler function, we have:

$$E^{\Theta}_{\eta,\sigma}(\theta\xi^{\eta}) = \sum_{k=0}^{\infty}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)}\frac{(\theta\xi^{\eta})^k}{k\,!}. \tag{18}$$

From Equations (18) and (14), we have:

$$e^{\Theta}_{\eta,\sigma,\theta}(\xi) = \sum_{k=0}^{\infty}\xi^{\sigma-1}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)}\frac{(\theta\xi^{\eta})^k}{k\,!}. \tag{19}$$

Taking the KT transform of Equation (19), we obtain:

$$\begin{aligned} B\left[e^{\Theta}_{\eta,\sigma,\theta}(\xi)\right] = {} & B\left[\sum_{k=0}^{\infty}\xi^{\sigma-1}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)}\frac{(\theta\xi^{\eta})^k}{k\,!}\right] \\ & = \sum_{k=0}^{\infty}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)}\frac{\theta^k}{k\,!}B\left(\xi^{\eta k+\sigma-1}\right) \\ & = \sum_{k=0}^{\infty}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)\Gamma(\eta k+\sigma)}\frac{\theta^k}{k\,!}s^{2(\eta k+\sigma-1)+5}(\eta k+\sigma-1)\,! \\ & = s^{2\sigma+3}\sum_{k=0}^{\infty}\frac{\Gamma(\Theta+k)}{\Gamma(\Theta)}\frac{\left(\theta\ s^{2\eta}\right)^k}{k\,!}. \end{aligned} \tag{20}$$

After simplification, we obtain:

$$B\left[e^{\Theta}_{\eta,\sigma,\theta}(\xi)\right] = s^{2\sigma+3}\left[1-\theta\ s^{2\eta}\right]^{-\Theta}$$

□

Now, by using the formula of the Prabhakar integral and the convolution transform of KT, the KT transform of the Prabhakar integral is obtained as:

$$B\left[E^{\Theta}_{\eta,\sigma,\theta,0^+}\Im(\xi)\right] = s^{2\sigma}B(\Im)\left[1-\theta\ s^{2\eta}\right]^{-\Theta}. \tag{21}$$

The above obtained formulae will be used frequently in forthcoming lemmas and theorems.

Lemma 2. *The KT transform of the HP fractional derivative*$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)$*provided in Equation (15) is given by:*

$$\begin{gathered} B\left(E^{-\Theta\lambda}_{\eta,\lambda(1-\sigma),\theta,0^+}\frac{d}{d\xi}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)\right)(s) \\ = s^{-2\sigma}\left[1-\theta\ s^{2\eta}\right]^{\Theta}B[\Im]\ (s) - s^{2\lambda(1-\sigma)+3}\left[1-\theta\ s^{2\eta}\right]^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}. \end{gathered} \tag{22}$$

Proof. The KT transform of the HP fractional derivative is given as:

$$B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)\right)\ (s) = B\left(E^{-\Theta\lambda}_{\eta,\lambda(1-\sigma),\theta,0^+}\frac{d}{d\xi}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)\right)(s). \tag{23}$$

Utilizing the formula of the Prabhakar integral, Equation (23) is reduced as:

$$B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)\right)(s)=B\left(\frac{d}{d\xi}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right)*e^{-\Theta\lambda}_{\eta,\lambda(1-\sigma),\theta}(\xi)\right)(s). \tag{24}$$

Through the convolution theorem for KT transform and using Lemma 1, Equation (24) is expressed as:

$$\begin{aligned}
B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)\right)(s) &= \tfrac{1}{s^3}B\left(e^{-\Theta\lambda}_{\eta,\lambda(1-\sigma),\theta}(\xi)\right)B\left(\tfrac{d}{d\xi}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right)\right)(s)\\
&= \tfrac{1}{s^3}s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}B\left(\tfrac{d}{d\xi}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right)\right)(s)\\
&= s^{2\lambda(1-\sigma)}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left[\tfrac{1}{s^2}B\left[E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right](s)-s^3\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right)_{\xi=0^+}\right]\\
&= s^{2\lambda(1-\sigma)}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left[\tfrac{1}{s^2}B\left[\Im*e^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta}(\xi)\right](s)-s^3\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}\right]\\
&= s^{2\lambda(1-\sigma)}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left[\tfrac{1}{s^5}B(\Im)B\left(e^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta}(\xi)\right)-s^3\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}\right]\\
&= s^{2\lambda(1-\sigma)}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left[\tfrac{1}{s^5}B(\Im)s^{2(1-\lambda)(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta(1-\lambda)}-s^3\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}\right].
\end{aligned}$$

After simplification, we obtain:

$$B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)\right)=s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}B[\Im](s)-s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}$$

□

Lemma 3. *The KT transform of the regularized version of the HP fractional derivative of order σ is computed as:*

$$B\left({}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im\right)(s)=\left[1-\theta\, s^{2\eta}\right]^{\Theta}\left[s^{-2\sigma}B[\Im(\xi)](s)-s^{5-2\sigma}\Im\left(0^+\right)\right]. \tag{25}$$

Proof. The regularized version of the HPFD of order σ is given by:

$${}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im(\xi)=\left(E^{-\Theta\lambda}_{\eta,\lambda(1-\sigma),\theta,0^+}E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\frac{d}{d\xi}\Im\right)(\xi). \tag{26}$$

Using Equation (17), Equation (26) is simplified as:

$${}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im(\xi)=\left(E^{-\Theta}_{\eta,1-\sigma,\theta,0^+}\frac{d}{d\xi}\Im\right)(\xi). \tag{27}$$

Using the formula of the Prabhakar integral and exerting the KT transform on Equation (27), we obtain:

$$B\left({}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im\right)(s,u)=B\left(\frac{d}{d\xi}\Im*e^{-\Theta}_{\eta,1-\sigma,\theta}(\xi)\right)(s). \tag{28}$$

In view of the convolution theorem of the KT transform, Equation (28) is transformed as:

$$B\left({}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im\right)(s)=\frac{1}{s^3}B\left(\frac{d}{d\xi}\Im\right)(s)B\left(e^{-\Theta}_{\eta,1-\sigma,\theta}(\xi)\right)(s). \tag{29}$$

With the help of Equation (4) and Lemma 1, Equation (29) reduces to:

$$
\begin{aligned}
B\left({}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Im\right)(s) &= \tfrac{1}{s^3}s^{2(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta}\left[\tfrac{1}{s^2}B[\Im(\xi)]\,(s)-s^3\Im(0^+)\right] \\
&= \left[1-\theta\, s^{2\eta}\right]^{\Theta}\left[s^{-2\sigma}B[\Im(\xi)]\,(s)-s^{5-2\sigma}\Im(0^+)\right].
\end{aligned}
$$

This completes the proof. □

4. Cauchy Problems via the HP Fractional Derivative and KT Transform

This section presents the solution of some Cauchy problems modeled with the HP fractional derivative with the help of KT and FT methods.

Theorem 5. *The solution of the Cauchy problem [6] modeled with the HP derivative:*

$$
\begin{cases}
D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)=\vartheta\, E^{\delta}_{\eta,\sigma,\theta,0^+}\Im(\xi)+\wp(\xi)\,, \\
\left[E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im(\xi)\right]_{\xi=0^+}=M,\ M\geq 0,
\end{cases}
\tag{30}
$$

where $\xi\in(0,\infty)$, $\wp(\xi)\in L^1[0,\infty)$, $\sigma\in(0,\,1)$, $\lambda\in[0,\,1]$, $\vartheta,\theta\in R$, $\xi,\eta>0$, *and* $\Theta,\delta\geq 0$ *is given by:*

$$
\Im(\xi)=M\sum_{m=0}^{\infty}\vartheta^m\xi^{\lambda(1-\sigma)+\sigma(1+2m)-1}E^{\Theta(1-\lambda)+(\delta+\Theta)m}_{\eta,\lambda(1-\sigma)+\sigma(1+2m)}(\theta\xi^{\eta})+\sum_{m=0}^{\infty}\vartheta^m E^{\Theta(m+1)+\delta m}_{\eta,(1+2m)\sigma,\theta,0^+}\wp(\xi).
\tag{31}
$$

Proof. Exerting the KT transform on both sides of Equation (30), we have:

$$
B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(\xi)\right)=\vartheta\, B\left(E^{\delta}_{\eta,\sigma,\theta,0^+}\Im(\xi)\right)\,+B[\wp(\xi)].
\tag{32}
$$

Now, using Lemma 2, Equation (13), the convolution theorem of the KT transform and Lemma 1, Equation (32) reduces to:

$$
\begin{aligned}
&s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}B[\Im]\,(s)-s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+} \\
&\quad=\vartheta\, B\left[\left(\Im*e^{\delta}_{\eta,\sigma,\theta}\right)(\xi)\right](s)+B[\wp(\xi)](s) \\
&\quad=\vartheta\,\tfrac{1}{s^3}B(\Im)(s)B\left(e^{\delta}_{\eta,\sigma,\theta}(\xi)\right)(s)+B[\wp(\xi)](s) \\
&\quad=\vartheta\, B(\Im)(s)s^{2\sigma}\left[1-\theta\, s^{2\eta}\right]^{-\delta}+B[\wp(\xi)](s).
\end{aligned}
\tag{33}
$$

After rearrangement of the terms, we have:

$$
\begin{aligned}
B[\Im](s)&=\frac{s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}+B[\wp(\xi)](s)}{s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}-\vartheta\, s^{2\sigma}\left[1-\theta\, s^{2\eta}\right]^{-\delta}} \\
&=\frac{s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}+B[\wp(\xi)](s)}{s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}\left\{\,1-\vartheta\, s^{4\sigma}\left[1-\theta\, s^{2\eta}\right]^{-(\delta+\Theta)}\right\}} \\
&=\frac{s^{2[\lambda(1-\sigma)+\sigma]+3}\left[1-\theta\, s^{2\eta}\right]^{-\Theta(1-\lambda)}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Im\right)_{\xi=0^+}+s^{2\sigma}\left[1-\theta\, s^{2\eta}\right]^{-\Theta}B[\wp(\xi)](s)}{\left\{\,1-\vartheta\, s^{4\sigma}\left[1-\theta\, s^{2\eta}\right]^{-(\delta+\Theta)}\right\}} \\
&=\left(s^{2[\lambda(1-\sigma)+\sigma]+3}\left[1-\theta\, s^{2\eta}\right]^{-\Theta(1-\lambda)}M+s^{2\sigma}\left[1-\theta\, s^{2\eta}\right]^{-\Theta}B[\wp(\xi)](s)\right)\times\sum_{m=0}^{\infty}s^{4\sigma m}\vartheta^m\left[1-\theta\, s^{2\eta}\right]^{-m(\delta+\Theta)} \\
&=M\sum_{m=0}^{\infty}\vartheta^m s^{2[\lambda(1-\sigma)+\sigma(1+2m)]+3}\left[1-\theta\, s^{2\eta}\right]^{-[m(\delta+\Theta)+\Theta(1-\lambda)]} \\
&\quad+\sum_{m=0}^{\infty}\vartheta^m s^{2\sigma(1+2m)}\left[1-\theta\, s^{2\eta}\right]^{-[\Theta(m+1)+\delta m]}B[\wp](s).
\end{aligned}
\tag{34}
$$

Applying the inverse KT transform operator B^{-1} on both sides of Equation (34) and applying Lemma 1, we have:

$$\begin{aligned}\Im(\xi) = M\sum_{m=0}^{\infty}\vartheta^m e^{m(\delta+\Theta)+\Theta(1-\lambda)}_{\eta,\lambda(1-\sigma)+(1+2m)\sigma,\theta}(\xi)\\ +\sum_{m=0}^{\infty}\vartheta^m B^{-1}\left(\tfrac{1}{s^3}B[\wp]\ (s)s^{2\sigma(1+2m)+3}\left[1-\theta\ s^{2\eta}\right]^{-[\Theta(m+1)+\delta m]}\right).\end{aligned} \tag{35}$$

On account of the convolution theorem of the KT transform and Lemma 1 in Equation (36), we obtain:

$$\begin{aligned}\Im(\xi) &= M\sum_{m=0}^{\infty}\vartheta^m e^{m(\delta+\Theta)+\Theta(1-\lambda)}_{\eta,\lambda(1-\sigma)+(1+2m)\sigma,\theta}(\xi)+\sum_{m=0}^{\infty}\vartheta^m\times\wp * B^{-1}\left(s^{2\sigma(1+2m)+3}\left[1-\theta\ s^{2\eta}\right]^{-[\Theta(m+1)+\delta m]}\right)\\ &= M\sum_{m=0}^{\infty}\vartheta^m e^{m(\delta+\Theta)+\Theta(1-\lambda)}_{\eta,\lambda(1-\sigma)+(1+2m)\sigma,\theta}(\xi)+\sum_{m=0}^{\infty}\vartheta^m\left(\wp * e^{\Theta(m+1)+\delta m}_{\eta,(1+2m)\sigma,\theta}\right)(\xi).\end{aligned} \tag{36}$$

Using Equations (13) and (14) in Equation (36), we obtain:

$$\Im(\xi) = M\sum_{m=0}^{\infty}\vartheta^m\xi^{\lambda(1-\sigma)+(1+2m)\sigma-1}E^{m(\delta+\Theta)+\Theta(1-\lambda)}_{\eta,\lambda(1-\sigma)+(1+2m)\sigma}(\theta\xi^\eta)+\sum_{m=0}^{\infty}\vartheta^m E^{\Theta(m+1)+\delta m}_{\eta,(1+2m)\sigma,\theta,0^+}\wp(\xi).$$

□

Remark 1. *For $\vartheta=-ip\beta$, $\wp(\xi)=0$, $\Theta=\lambda=0$, $\eta=\delta=1$, $\theta=i\alpha$, $p,\alpha\in R$, $\xi\in(0,\ 1]$, the above Cauchy problem transforms to the following FEL equation [17]*

$$\begin{cases}\frac{d\Im}{d\xi} = -ip\beta\int_0^\xi(\xi-\tau)e^{i\alpha(\xi-\tau)}\Im(\tau)d\tau,\\ \Im(0)=1.\end{cases} \tag{37}$$

Theorem 6. *The solution of the Cauchy problem [7] modeled with the HP derivative:*

$$\begin{cases}{}^{C}D^{\Theta,\sigma}_{\eta,-\theta,0^+}\Pi(\zeta,\xi) = -\vartheta(1-\zeta)\Pi(\zeta,\xi),\ |\zeta|\le 1\\ \Pi(\zeta,0^+)=1,\end{cases} \tag{38}$$

with $\xi>0$, $\vartheta>0$, $\Theta\ge 0$, $\eta\in(0,\ 1]$, $\sigma\in(0,\ 1]$, $\vartheta,\theta\in R$, $\zeta,\eta>0$, and $\Theta,\delta\ge 0$ is given by

$$\Pi(\zeta,\xi) = \sum_{m=0}^{\infty}(-\vartheta)^m(1-\zeta)^m\xi^{m\sigma}E^{m\Theta}_{\eta,m\sigma+1}(-\theta\xi^\eta). \tag{39}$$

Proof. Taking the KT transform of Equation (38) with respect to ξ and using initial condition $\Pi(\zeta,0)=1$ and Lemma 3, we obtain:

$$\begin{aligned}&B\left({}^{C}D^{\Theta,\sigma}_{\eta,-\theta,0^+}\Pi(\zeta,\xi)\right) = -\vartheta(1-\zeta)B(\Pi(\zeta,\xi))(\zeta,s)\\ &s^{-2\sigma}\left[1+\theta\ s^{2\eta}\right]^{\Theta}B[\Pi(\zeta,\xi)]\ (\zeta,s)-s^{5-2\sigma}\left[1+\theta\ s^{2\eta}\right]^{\Theta}\Pi(\zeta,0^+) = -\vartheta(1-\zeta)B[\Pi(\zeta,\xi)].\end{aligned}$$

After rearranging the terms, we obtain:

$$\begin{aligned}B[\Pi(\zeta,\xi)]\ (\zeta,s) &= \frac{s^{5-2\sigma}\left[1+\theta\ s^{2\eta}\right]^{\Theta}}{s^{-2\sigma}\left[1+\theta\ s^{2\eta}\right]^{\Theta}+\vartheta(1-\zeta)}\\ &= \frac{s^5}{\left[1+\frac{\vartheta(1-\zeta)}{s^{-2\sigma}\left[1+\theta\ s^{2\eta}\right]^{\Theta}}\right]}\\ B[\Pi(\zeta,\xi)]\ (\zeta,s) &= \sum_{m=0}^{\infty}(-\vartheta)^m(1-\zeta)^m s^{2m\sigma+5}\left[1+\theta\ s^{2\eta}\right]^{-m\Theta}.\end{aligned} \tag{40}$$

Applying the inverse KT transform operator B^{-1} on Equation (40) and utilizing Lemma 1 and Equation (14), we have:

$$\begin{aligned}\Pi(\zeta,\xi) &= \sum_{m=0}^{\infty}(-\vartheta)^m(1-\zeta)^m B^{-1}\left(s^{2(m\sigma+1)+3}\left[1+\theta\, s^{2\eta}\right]^{-m\Theta}\right)\\ &= \sum_{m=0}^{\infty}(-\vartheta)^m(1-\zeta)^m e^{m\Theta}_{\eta,m\sigma+1,-\theta}(\zeta,\xi)\\ &= \sum_{m=0}^{\infty}(-\vartheta)^m(1-\zeta)^m \xi^{m\sigma}E^{m\Theta}_{\eta,m\sigma+1}(-\theta\xi^{\eta}).\end{aligned}$$

□

Theorem 7. *The solution of the Cauchy problem [6] modeled with the HP derivative:*

$$\begin{cases} D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Pi(\zeta,\xi) = T\frac{\partial^2}{\partial\zeta^2}\Pi(\zeta,\xi),\\ \left[E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\Pi(\zeta,\xi)\right]_{\xi=0^+} = \psi(\zeta),\\ \lim\limits_{\zeta\to\pm\infty}\Pi(\zeta,\xi)=0,\end{cases} \tag{41}$$

with $\sigma\in(0,\,1)$, $\lambda\in[0,\,1]$, $\zeta,\theta\in R$, $\xi,T,\eta>0$, and $\Theta\geq 0$ is given by

$$\Pi(\zeta,\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip\zeta}\hat{\psi}(p)dp\sum_{m=0}^{\infty}\left(-Tp^2\right)^m\xi^{\lambda(1-\sigma)+\sigma(m+1)-1}E^{\Theta[m+(1-\lambda)]}_{\eta,\lambda(1-\sigma)+\sigma(m+1)}(\theta\xi^{\eta}). \tag{42}$$

Proof. Let $\overline{\Pi}(\zeta,s)$ and $\hat{\Pi}(p,\xi)$ denote the KT and Fourier transforms of $\Pi(\zeta,\xi)$, respectively. Furthermore, let $\hat{\overline{\Pi}}(p,s)$ and $\hat{\psi}(p)$ denote the Fourier-KT transform and Fourier transform of $\Pi(\zeta,\xi)$ and $\psi(\zeta)$, respectively.

Employing Fourier-KT transform of Equation (41) and using Lemma 2, we have:

$$s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}\hat{\overline{\Pi}}(p,s)-s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\hat{\psi}(p)=-Tp^2\hat{\overline{\Pi}}(p,s). \tag{43}$$

On simplification, we obtain:

$$\begin{aligned}\hat{\overline{\Pi}}(p,s) &= \frac{s^{2\lambda(1-\sigma)+3}\left[1-\theta\, s^{2\eta}\right]^{\Theta\lambda}\hat{\psi}(p)}{s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}+Tp^2}\\ &= \frac{s^{2\lambda(1-\sigma)+2\sigma+3}\left[1-\theta\, s^{2\eta}\right]^{-(1-\lambda)\Theta}\hat{\psi}(p)}{1+\frac{Tp^2}{s^{-2\sigma}\left[1-\theta\, s^{2\eta}\right]^{\Theta}}}\\ &= \sum_{m=0}^{\infty}\left(-Tp^2\right)^m\hat{\psi}(p)s^{2[\lambda(1-\sigma)+(m+1)\sigma]+3}\left[1-\theta\, s^{2\eta}\right]^{-\Theta[(1-\lambda)+m]}.\end{aligned} \tag{44}$$

Inverting Fourier transform in Equation (44), we obtain:

$$\overline{\Pi}(\zeta,s)=\sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Tp^2\right)^m e^{ip\zeta}\hat{\psi}(p)\,dp\; s^{2[\lambda(1-\sigma)+(m+1)\sigma]+3}\left[1-\theta\, s^{2\eta}\right]^{-\Theta[(1-\lambda)+m]}. \tag{45}$$

Applying the inverse KT transform on Equation (45) and using Lemma 1 and Equation (14), we obtain:

$$\begin{aligned}\Pi(\zeta,\xi) &= \sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Tp^2\right)^m e^{ip\zeta}\hat{\psi}(p)dp\; e^{\Theta[(1-\lambda)+m]}_{\eta,\lambda(1-\sigma)+(m+1)\sigma,\theta}(\zeta,\xi)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip\zeta}\hat{\psi}(p)dp\sum_{m=0}^{\infty}(-T)^m p^{2m}\xi^{\lambda(1-\sigma)+(m+1)\sigma-1}E^{\Theta[(1-\lambda)+m]}_{\eta,\lambda(1-\sigma)+(m+1)\sigma}(\theta\xi^{\eta}).\end{aligned}$$

□

Theorem 8. *The solution of the Cauchy problem [6] modeled with the regularized version of the HP derivative:*

$$\begin{cases} {}^{C}D^{\Theta,\sigma}_{\eta,\theta,0^+}\Pi(\zeta,\xi) = T\frac{\partial^2}{\partial\zeta^2}\Pi(\zeta,\xi),\ \xi > 0,\ \zeta \in R, \\ [\Pi(\zeta,0^+)]_{\xi=0^+} = \psi(\zeta), \\ \lim\limits_{\zeta\to\pm\infty}\Pi(\zeta,\xi) = 0, \end{cases} \tag{46}$$

with $\sigma \in (0,\ 1)$, $\lambda \in [0,\ 1]$, $\zeta,\theta \in R$, $T,\eta > 0$, *and* $\Theta \geq 0$ *is given by*:

$$\Pi(\zeta,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\zeta}\hat{\psi}(p)dp\sum_{m=0}^{\infty}\left(-Tp^2\right)^m \xi^{m\sigma}E^{m\Theta}_{\eta,m\sigma+1}(\theta\xi^\eta). \tag{47}$$

Proof. Let $\overline{\Pi}(\zeta,s)$ and $\hat{\Pi}(p,\xi)$ denote the KT and Fourier transforms of $\Pi(\zeta,\xi)$, respectively. Furthermore, let $\hat{\overline{\Pi}}(p,s)$ and $\hat{\psi}(p)$ denote the Fourier-KT transform and Fourier transform of $\Pi(\zeta,\xi)$ and $\psi(\zeta)$, respectively.

Taking the Fourier-KT transform of Equation (46) and applying Lemma 3, we have:

$$\left[1-\theta\, s^{2\eta}\right]^{\Theta} s^{-2\sigma}\hat{\overline{\Pi}}(p,s) - \left[1-\theta\, s^{2\eta}\right]^{\Theta} s^{5-2\sigma}\hat{\psi}(p) = -Tp^2\hat{\overline{\Pi}}(p,s).$$

After simplification, we obtain:

$$\begin{aligned} \hat{\overline{\Pi}}(p,s) &= \frac{\left[1-\theta\, s^{2\eta}\right]^{\Theta} s^{5-2\sigma}\hat{\psi}(p)}{\left[1-\theta\, s^{2\eta}\right]^{\Theta} s^{-2\sigma}+Tp^2} \\ &= \frac{s^5\hat{\psi}(p)}{1+\frac{Tp^2}{\left[1-\theta\, s^{2\eta}\right]^{\Theta} s^{-2\sigma}}} \\ &= \hat{\psi}(p)\sum_{m=0}^{\infty}\left(-Tp^2\right)^m s^{2\sigma m+5}\left[1-\theta\, s^{2\eta}\right]^{-m\Theta} \end{aligned} \tag{48}$$

Inverting the Fourier transform in Equation (48), we obtain:

$$\overline{\Pi}(\zeta,s) = \sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Tp^2\right)^m e^{ip\zeta}\hat{\psi}(p)dp\, s^{2(m\sigma+1)+3}\left[1-\theta\, s^{2\eta}\right]^{-m\Theta}. \tag{49}$$

Applying the inverse KT transform on Equation (45) and using Lemma 1 and Equation (14), we obtain:

$$\begin{aligned} \Pi(\zeta,\xi) &= \sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Tp^2\right)^m e^{ip\zeta}\hat{\psi}(p)dp\, e^{m\Theta}_{\eta,m\sigma+1,\theta}(\xi) \\ &= \sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Tp^2\right)^m e^{ip\zeta}\hat{\psi}(p)dp\, \xi^{m\sigma}E^{m\Theta}_{\eta,m\sigma+1}(\theta\xi^\eta) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\zeta}\hat{\psi}(p)dp\sum_{m=0}^{\infty}\left(-Tp^2\right)^m \xi^{m\sigma}E^{m\Theta}_{\eta,m\sigma+1}(\theta\xi^\eta). \end{aligned}$$

□

The results of Theorems 5–8 are exactly the same as the solutions reported by Panchal et al. [7].

5. Analytical Solution of Diffusion Equations with the HPFD in 1D and 2D Spaces

In this section, the fractional diffusion equations in 1D and 2D spaces are investigated.

The diffusion equation discussed in the works of Hristov [11] is reported as:

$$\frac{\partial \Im(u,\xi)}{\partial \xi} = k^2 \frac{\partial^2 \Im(u,\xi)}{\partial u^2},$$

where $k^2 = \frac{\mathrm{H}}{\Phi C_q}$, H is the thermal conductivity, Φ is the specific heat, C_q is the density of material, and $\Im$ is the temperature distribution of the material.

The 1D fractional diffusion equation discussed in refs. [10,12] modeled with the HP derivative is given as:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+} \Im(u,\xi) = k^2 \frac{\partial^2 \Im(u,\xi)}{\partial u^2},$$

where $\Theta, \theta \in R$, $\eta > 0$, k^2 denotes the diffusion coefficient, $D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}$ specifies the HP fractional derivative operator defined in Definition 11, and $E^{0}_{\eta,0,\theta,0^+}\Im = \Im$.

Here, Equation (51) is equipped with the following Dirichlet boundary conditions:

- $\Im(u,0) = 0$, when $u > 0$;
- $\Im(0,\xi) = 1$, when $\xi > 0$.

In the 2D space, the fractional diffusion equation [12] with the HP derivative is given by:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+} \Im(u,v,\xi) = k^2 \left\{ \frac{\partial^2 \Im(u,v,\xi)}{\partial u^2} + \frac{\partial^2 \Im(u,v,\xi)}{\partial v^2} \right\}$$

with Dirichlet boundary conditions:

- $\Im(u,v,0) = 0$, when $u > 0, v > 0$;
- $\Im(0,v,\xi) = \Im(u,0,\xi) = 1$, when $\xi > 0$.

The Dirichlet boundary conditions are very helpful in the context of an integral method. The form of analytical solutions is governed by the boundary conditions. This section presents the integral method consisting of the KT transform and Fourier sine transform to explore the solutions for diffusion equations with the HP fractional derivative.

Model I: Diffusion equation with the HPFD in 1D space

The HP fractional diffusion equation in 1D space is given as:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+} \Im(u,\xi) = k^2 \frac{\partial^2 \Im(u,\xi)}{\partial u^2} \tag{50}$$

subjected to Dirichlet boundary conditions:

$$\begin{cases} \Im(u,0) = 0, \ u > 0 \\ \Im(0,\xi) = 1, \ \xi > 0 \\ \lim\limits_{u \to \pm\infty} \Im(u,\xi) = 0\,. \end{cases} \tag{51}$$

Now, employing the Fourier sine transform on Equation (50), multiplying by $\frac{2}{\pi}\sin\mu u$, and integrating it within the range of 0 to ∞ with respect to u, we obtain:

$$\begin{aligned} D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+} \hat{\Im}_{fst}(\mu,\ \xi) &= k^2 \left\{ \tfrac{2}{\pi}\mu \Im(0,\xi) - \mu^2 \hat{\Im}_{fst}(\mu,\xi) \right\} \\ &= \tfrac{2k^2\mu}{\pi} - k^2\mu^2 \hat{\Im}_{fst}(\mu,\xi), \end{aligned} \tag{52}$$

where $\hat{\Im}_{fst}(\mu,\xi)$ denotes the Fourier sine transform of $\Im(u,\xi)$.

After rearrangement of the terms in Equation (52), we obtain the following FDE:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+} \hat{\Im}_{fst}(\mu,\ \xi) + k^2\mu^2 \hat{\Im}_{fst}(\mu,\xi) = \frac{2k^2\mu}{\pi}. \tag{53}$$

Applying the KT transform on both sides of Equation (53), we obtain:

$$B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)+k^2\mu^2\bar{\hat{\Im}}_{fst}(\mu,s)=\frac{2k^2\mu}{\pi}s^5. \tag{54}$$

Using Lemma 2, Equation (54) transforms into the following:

$$\begin{gathered}
s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}\bar{\hat{\Im}}_{fst}(\mu,s)\\
-s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}+k^2\mu^2\bar{\hat{\Im}}_{fst}(\mu,s)=\tfrac{2k^2\mu}{\pi}s^5\\
\bar{\hat{\Im}}_{fst}(\mu,s)\left[s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\mu^2\right]\\
=s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}+\tfrac{2k^2\mu}{\pi}s^5\\
\bar{\hat{\Im}}_{fst}(\mu,s)=\frac{s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}}{s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\mu^2}+\frac{2k^2\mu\ s^5}{\pi\left[s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\mu^2\right]}\\
\bar{\hat{\Im}}_{fst}(\mu,s)=s^{2[\lambda(1-\sigma)+\sigma]+3}\left(1-\theta\, s^{2\eta}\right)^{-(1-\lambda)\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}\left[1+\frac{k^2\mu^2}{s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}}\right]^{-1}\\
+\tfrac{2k^2\mu}{\pi}s^{2\sigma+5}\left(1-\theta\, s^{2\eta}\right)^{-\Theta}\left[1+\frac{k^2\mu^2}{s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}}\right]^{-1}\\
\bar{\hat{\Im}}_{fst}(\mu,s)=s^{2[\lambda(1-\sigma)+\sigma]+3}\left(1-\theta\, s^{2\eta}\right)^{-(1-\lambda)\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}\\
\times\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}s^{2\sigma m}\left(1-\theta\, s^{2\eta}\right)^{-m\Theta}\\
+\tfrac{2k^2\mu}{\pi}s^{2\sigma+5}\left(1-\theta\, s^{2\eta}\right)^{-\Theta}\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}s^{2\sigma m}\left(1-\theta\, s^{2\eta}\right)^{-m\Theta}\\
\bar{\hat{\Im}}_{fst}(\mu,s)=\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}s^{2[\lambda(1-\sigma)+\sigma(m+1)]+3}\left(1-\theta\, s^{2\eta}\right)^{-[(m+1)-\lambda]\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}\\
+\tfrac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\mu^{2m+1}s^{2[\sigma(m+1)+1]+3}\left(1-\theta\, s^{2\eta}\right)^{-(m+1)\Theta},
\end{gathered} \tag{55}$$

where $\bar{\hat{\Im}}_{fst}(\mu,s)$ denotes the KT transform of $\hat{\Im}_{fst}(\mu,\ \xi)$. Now, applying the inverse of the KT transform on both sides of Equation (55) and using Lemma 1 and Equation (14), we obtain:

$$\begin{gathered}
\hat{\Im}_{fst}(\mu,\ \xi)=\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}e^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1),\theta}(\xi)\\
+\tfrac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\mu^{2m+1}e^{(m+1)\Theta}_{\eta,\sigma(m+1)+1,\theta}(\xi)\\
\hat{\Im}_{fst}(\mu,\ \xi)=\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}\xi^{\lambda(1-\sigma)+\sigma(m+1)-1}E^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1)}(\theta\xi^{\eta})\\
+\tfrac{2}{\pi}\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\mu^{2m+1}\xi^{\sigma(m+1)}E^{(m+1)\Theta}_{\eta,\sigma(m+1)+1}(\theta\xi^{\eta}).
\end{gathered} \tag{56}$$

Applying the inverse of the Fourier sine transform on both sides of Equation (56), we obtain:

$$\begin{gathered}
\Im(u,\xi)=\tfrac{2}{\pi}\int_0^{\infty}\sin\mu u\\
\times\left(\sum_{m=0}^{\infty}(-1)^m k^{2m}\mu^{2m}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\ \xi)\right)_{\xi=0^+}\xi^{\lambda(1-\sigma)+\sigma(m+1)-1}E^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1)}(\theta\xi^{\eta})\right)d\mu\\
+\tfrac{4}{\pi^2}\int_0^{\infty}\sin\mu u\left(\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\mu^{2m+1}\xi^{\sigma(m+1)}E^{(m+1)\Theta}_{\eta,\sigma(m+1)+1}(\theta\xi^{\eta})\right)d\mu.
\end{gathered} \tag{57}$$

This is the complete analytical solution of the HP fractional diffusion Equation (50).

Model II: Diffusion equation with the HP fractional derivative in 2D space

The HP fractional diffusion equation in 2D space is illustrated as:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\Im(u,v,\xi) = k^2\left\{\frac{\partial^2\Im(u,v,\xi)}{\partial u^2} + \frac{\partial^2\Im(u,v,\xi)}{\partial v^2}\right\} \tag{58}$$

subjected to Dirichlet boundary conditions:

$$\begin{cases} \Im(u,v,\ 0) = 0,\ u > 0, v > 0 \\ \Im(0,v,\xi) = \Im(u,0,\xi) = 1,\ \xi > 0 \\ \lim\limits_{u,v\to\pm\infty} \Im(u,v,\xi) = 0\,. \end{cases} \tag{59}$$

Now, employing the Fourier sine transform on Equation (58), multiplying by $\frac{2}{\pi}\sin\mu u\ \sin\vartheta v$, and integrating it within the range of 0 to ∞ in respect of u and v, we obtain:

$$\begin{aligned} D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi) &= k^2\left\{\tfrac{2(\mu^2+\vartheta^2)}{\pi\mu\vartheta}\Im(0,0,\xi) - (\mu^2+\vartheta^2)\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right\} \\ &= \tfrac{2k^2(\mu^2+\vartheta^2)}{\pi\mu\vartheta} - k^2(\mu^2+\vartheta^2)\hat{\Im}_{fst}(\mu,\vartheta,\xi), \end{aligned} \tag{60}$$

where $\hat{\Im}_{fst}(\mu,\vartheta,\xi)$ denotes the Fourier sine transform of $\Im(u,v,\xi)$.

After rearrangement of the terms in Equation (60), we obtain the following FDE:

$$D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi) + k^2\left(\mu^2+\vartheta^2\right)\hat{\Im}_{fst}(\mu,\vartheta,\xi) = \frac{2k^2(\mu^2+\vartheta^2)}{\pi\mu\vartheta}. \tag{61}$$

Applying the KT transform on both sides of Equation (53), we obtain:

$$B\left(D^{\Theta,\sigma,\lambda}_{\eta,\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right) + k^2\left(\mu^2+\vartheta^2\right)\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s) = \frac{2k^2(\mu^2+\vartheta^2)}{\pi\mu\vartheta}s^5. \tag{62}$$

Using Lemma 2, Equation (62) transforms into:

$$
\begin{aligned}
&s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)-s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}\\
&+k^2\left(\mu^2+\vartheta^2\right)\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)=\frac{2k^2\left(\mu^2+\vartheta^2\right)}{\pi\mu\vartheta}s^5\\
&\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)\left[s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\left(\mu^2+\vartheta^2\right)\right]\\
&=s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}+\frac{2k^2\left(\mu^2+\vartheta^2\right)}{\pi\mu\vartheta}s^5\\
&\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)=\frac{s^{2\lambda(1-\sigma)+3}\left(1-\theta\, s^{2\eta}\right)^{\Theta\lambda}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}}{\left[s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\left(\mu^2+\vartheta^2\right)\right]}\\
&+\frac{2k^2\left(\mu^2+\vartheta^2\right)s^5}{\pi\mu\vartheta\left[s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}+k^2\left(\mu^2+\vartheta^2\right)\right]}\\
&\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)=s^{2[\lambda(1-\sigma)+\sigma]+3}\left(1-\theta\, s^{2\eta}\right)^{-(1-\lambda)\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}\left[1+\frac{k^2\left(\mu^2+\vartheta^2\right)}{s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}}\right]^{-1}\\
&+\frac{2k^2\left(\mu^2+\vartheta^2\right)}{\pi\mu\vartheta}s^{2\sigma+5}\left(1-\theta\, s^{2\eta}\right)^{-\Theta}\left[1+\frac{k^2\left(\mu^2+\vartheta^2\right)}{s^{-2\sigma}\left(1-\theta\, s^{2\eta}\right)^{\Theta}}\right]^{-1}\\
&\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)=s^{2[\lambda(1-\sigma)+\sigma]+3}\left(1-\theta\, s^{2\eta}\right)^{-(1-\lambda)\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}\\
&\times\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m s^{2\sigma m}\left(1-\theta\, s^{2\eta}\right)^{-m\Theta}\\
&+\frac{2k^2\left(\mu^2+\vartheta^2\right)}{\pi\mu\vartheta}s^{2\sigma+5}\left(1-\theta\, s^{2\eta}\right)^{-\Theta}\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m s^{2\sigma m}\left(1-\theta\, s^{2\eta}\right)^{-m\Theta}\\
&\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)\\
&=\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m s^{2[\lambda(1-\sigma)+\sigma(m+1)]+3}\left(1-\theta\, s^{2\eta}\right)^{-[(m+1)-\lambda]\Theta}\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}\\
&+\frac{2}{\pi}\frac{k^2\left(\mu^2+\vartheta^2\right)}{\mu\vartheta}\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m s^{2[\sigma(m+1)+1]+3}\left(1-\theta\, s^{2\eta}\right)^{-(m+1)\Theta},
\end{aligned}
\tag{63}
$$

where $\overline{\hat{\Im}}_{fst}(\mu,\vartheta,s)$ denotes the KT transform of $\hat{\Im}_{fst}(\mu,\vartheta,\xi)$. Now, applying the inverse of the KT transform on both sides of Equation (63) and using Lemma 1 and Equation (14), we obtain:

$$
\begin{aligned}
&\hat{\Im}_{fst}(\mu,\vartheta,\xi)=\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}e^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1),\theta}(\xi)\\
&+\frac{2}{\pi}\frac{\left(\mu^2+\vartheta^2\right)}{\mu\vartheta}\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\left(\mu^2+\vartheta^2\right)^m e^{(m+1)\Theta}_{\eta,\sigma(m+1)+1,\theta}(\xi)\\
&\hat{\Im}_{fst}(\mu,\vartheta,\xi)\\
&=\sum_{m=0}^{\infty}(-1)^m k^{2m}\left(\mu^2+\vartheta^2\right)^m\left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+}\hat{\Im}_{fst}(\mu,\vartheta,\xi)\right)_{\xi=0^+}\xi^{\lambda(1-\sigma)+\sigma(m+1)-1}E^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1)}(\theta\xi^{\eta})\\
&+\frac{2}{\pi}\frac{\left(\mu^2+\vartheta^2\right)}{\mu\vartheta}\sum_{m=0}^{\infty}(-1)^m k^{2(m+1)}\left(\mu^2+\vartheta^2\right)^m\xi^{\sigma(m+1)}E^{(m+1)\Theta}_{\eta,\sigma(m+1)+1}(\theta\xi^{\eta}).
\end{aligned}
\tag{64}
$$

Applying the inverse of the Fourier sine transform on both sides of Equation (64), we obtain:

$$\begin{aligned}\Im(u,\xi) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \sin\mu u \sin\vartheta v \left(E^{-\Theta(1-\lambda)}_{\eta,(1-\lambda)(1-\sigma),\theta,0^+} \hat{\Im}_{fst}(\mu,\vartheta,\xi) \right)_{\xi=0^+} \\ \times \left(\sum_{m=0}^{\infty} (-1)^m k^{2m} \left(\mu^2+\vartheta^2\right)^m \xi^{\lambda(1-\sigma)+\sigma(m+1)-1} E^{[(m+1)-\lambda]\Theta}_{\eta,\lambda(1-\sigma)+\sigma(m+1)}(\theta\xi^\eta) \right) d\mu d\vartheta \\ + \frac{4}{\pi^2}\int_0^\infty \frac{\sin\mu u}{\mu} \int_0^\infty \frac{\sin\vartheta v}{\vartheta} \left(\sum_{m=0}^{\infty} (-1)^m k^{2(m+1)} \left(\mu^2+\vartheta^2\right)^{m+1} \xi^{\sigma(m+1)} E^{(m+1)\Theta}_{\eta,\sigma(m+1)+1}(\theta\xi^\eta) \right) d\mu d\vartheta.\end{aligned} \tag{65}$$

This is the desired analytical solution of the HP fractional diffusion Equation (58).

6. Conclusions

In this work, the Kharrat–Toma transforms of the Prabhakar integral and HPFD and its regularized version are derived. Furthermore, the solutions of some fractional order Cauchy problems and diffusion equations arising in mathematical physics with the HPFD and its regularized Caputo version are also computed utilizing the derived results of the KT transform of the HP derivatives. The solutions of the HP fractional order Cauchy problems and diffusion equations were obtained in the form of generalized Mittag–Leffler function. The present work illustrates that the computation of solutions of fractional order Cauchy problems and diffusion equations with KT transform is very much easier than other integral transforms.

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