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Brief Report

Synchronization in Finite Time of Fractional-Order Complex-Valued Delayed Gene Regulatory Networks

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Abstract: The synchronization in finite time of fractional-order complex-valued gene networks with time delays is studied in this paper. Several sufficient conditions of the synchronization in finite time for the relevant network models are explored based on feedback controllers and adaptive controllers. Then, the setting time of the response is estimated by the theory of fractional calculus. Finally, to validate the theoretical results, a numerical example is presented using the proposed two controllers, showing that the setting time based on the adaptive controller is shorter than the that based on the feedback controller.

Keywords: fractional-ordergene regulatory networks; time delay; synchronization in finite time; feedback control; adaptive control

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1. Introduction

Gene regulatory networks control gene expression and describe the relationship between deoxyribonucleic acid, ribonucleic acid, and small molecules in organisms. The abundance of the gene product leads to the aggregation of the molecular types of action between them, playing an essential role in the cycle, differentiation, metabolic processes, and signal transduction of cells, which are controlled by gene networks [1]. There are numerous gene regulatory network models, including linear models [2,3], Bayesian networks [4,5], neural networks [6], differential equations [7], and models including stochastic components at the molecular level [8,9]. In recent decades, the extensive application of gene regulatory networks has been explored in various areas, such as biotechnological practices [10–12], integrated networks [13,14], and mechanical systems [15,16].

As is well known, fruitful results based on the gene network of integer-order differential equations have been reported [17–19]. With the development of the theory of fractional-order calculus and fractional-order differential equations [20–23], various applications of gene regulatory networks that employ fractional-order calculus, such as the fields of medical science, control, and biotechnology, have shown distinct advantages due to the merits of memory and heredity properties, see [24–26] and the references therein. In [24], the results indicated that the most significant benefit of using gene regulatory networks with fractional orders for their memorability and hereditary properties is the enhancement in the dexterity and accuracy of models. The authors found that by combining fractional derivatives, the basic computing power of gene regulatory networks can be enhanced and the processing of various signals can be efficiently processed [25]. In [26], the authors extended the general form of the Lyapunov–Krasovskii function to a new fractional form and derived the stability criteria for gene regulatory network systems with time delays and fractional-order dynamics.

Generally speaking, the time delays, which are usually caused by oscillation, instability, and other poor performance, unavoidably exist in most dynamical networks [27–31], including gene networks [32], neural networks [33–36], and evolutionary dynamics. As for the gene network models, there always exists certain time delays in the expression of most genes due to the fact that these genes and their regulatory interactions are usually not implemented immediately. It should be noted that, recently, the research interest has been transferring from traditional real-valued gene regulatory networks to complex-valued models. This is because complex-valued gene networks with complex numbers for various parameters [37] are more practical when compared with real-valued networks. Nowadays, complex-valued gene regulatory networks with fractional order are profoundly studies and many excellent results have been found, such as projective multi-synchronization [38] and global synchronization [39]. However, most of the methods in these works are based on separating the complex-valued system into two real-valued systems, which leads to a more tedious proof process and increases the complexity of the algorithm. Hence, it is important to study fractional-order complex-valued gene regulatory networks directly in the complex field [40–42].

Synchronization plays a prominent role in the dynamical behaviors of gene regulatory networks, which has increasingly attracted experts' interests [43–45]. Two or more systems communicating among gene molecules to realize a synergistic behavior and to adjust their dynamic characteristics is called synchronization of gene regulatory network. Due to the extensive application of synchronization in information processing, it has developed rapidly [46–48]. In [47], by using Lyapunov stability, the synchronization and asymptotic stability of a sort of fractional-order gene regulatory network were studied. Furthermore, in [48], through the designed adaptive controller, the gene regulatory network synchronization problem of fractional order was analyzed.

Inspired by the above description, this paper aims to analyze the problem of synchronization in finite time of fractional-order complex-valued gene regulatory networks. Analyzing complex-valued gene regulatory networks is more challenging than analyzing real-valued models. The chief contributions are summarized as follows:

- To analyze the synchronization in finite time for the addressed models, two different controllers are designed to achieve a flexible control of synchronization.
- The presented complex-valued gene regulatory networks are implemented as an entirety form without any decomposition.
- A novel complex-valued sign function is employed to design the adaptive controller to achieve a more efficient control strategy for the problem of synchronization in finite time.

The article is structured as follows: Section 2 presents the Riemann–Liouville integral and Caputo fractional derivative, introduces appropriate symbolic functions of the complex number field, and cites some complex-number-related lemmas. Section 3 introduces models of gene regulatory networks. In Sections 4 and 5, two controllers are presented and two theorems regarding synchronization in finite time of complex-valued gene regulatory networks are given. Section 6 provides a numerical example to demonstrate the feasibility and effectiveness of the theoretical results. Finally, the article concludes with a summary.

Notations: $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the collection of n-dimensional real-valued and complex-valued vectors, respectively. $\imath$ denotes the imaginary unit. For any $z \in \mathbb{C}$, Re(z) represents the real part of z and Im(z) represents the imaginary part. $\bar{z}$ is the conjugate of z; $|z|_1 = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$, and $|z|_2 = \sqrt{z\bar{z}}$. For any $z \in \mathbb{C}^n$, $\|z\|_1 = \sum_{k=1}^{n} |z_i|_1$, $\|z\|_2 = \sqrt{\sum_{k=1}^{n} |z_i|_2^2}$.

2. Preliminaries

Definition 1 ([49])**.** *For a function $\varphi(t) : [0, +\infty] \to \mathbb{C}$, the Riemann–Liouville fractional integral is described as*

$$ {}_{0}^{RL}D_t^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}\varphi(\varsigma)d\varsigma, $$

where $0 < \alpha < 1$ and $\Gamma(\alpha) = \int_0^{+\infty} \frac{t^{\alpha-1}}{e^{\mu}} d\mu$.

Definition 2 ([49]). *For a function $\varphi(t) : [0, +\infty] \to \mathbb{C}$, the Caputo fractional derivative is defined by*

$$ {}_0^C D_t^\alpha \varphi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\varsigma)^{-\alpha} \varphi'(\varsigma) d\varsigma. $$

where $\alpha \in (0, 1)$.

Lemma 1 ([50]). *For any $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, and for any positive number $\eta \in \mathbb{R}$, it holds that*

$$ \alpha\overline{\beta} + \overline{\alpha}\beta \leq \eta\alpha\overline{\alpha} + \frac{1}{\eta}\beta\overline{\beta}. $$

Definition 3 ([41]). *$\forall z(t) \in \mathbb{C}$, the symbolic functions in complex fields is defined as*

$$ [z(t)] = \text{sign}(\text{Re}(z(t))) + \text{isign}(\text{Im}(z(t))). $$

Lemma 2 ([41]). *$\forall z(t) \in \mathbb{C}^n$, the following three formulas are true.*

$$ (i)\ [z(t)] \times \overline{z(t)} + \overline{[z(t)]} \times z(t) = 2|z(t)|_1 \geq 2|z(t)|_2, $$
$$ (ii)\ 2{}_0^C D_t^\alpha |z(t)|_1 \leq \left(\overline{[z(t)]} \times {}_0^C D_t^\alpha z(t) + [z(t)] \times {}_0^C D_t^\alpha \overline{z(t)} \right), \text{ where } 0 < \alpha < 1, $$
$$ (iii)\ [z(t)] \times \overline{[z(t)]} = |\text{sign}(\text{Re}(z(t)))| + |\text{sign}(\text{Im}(z(t)))| = |[z(t)]|_1. $$

Lemma 3 ([41]). *For any $\alpha \in \mathbb{C}$, there is*

$$ \alpha + \overline{\alpha} = 2Re(\alpha) \leq 2|\alpha|_2 \leq 2|\alpha|_1. $$

Lemma 4 ([51]). *Let analytic function $\varphi(t) \in \mathbb{C}$ be continuous, and for any $\vartheta \in \mathbb{C}$, it holds*

$$ {}_0^C D_t^\alpha \left((\varphi(t) - \vartheta) \times (\overline{\varphi(t) - \vartheta}) \right) \leq (\overline{\varphi(t) - \vartheta}) \times {}_0^C D_t^\alpha \varphi(t) + (\varphi(t) - \vartheta) \times {}_0^C D_t^\alpha \overline{\varphi(t)},\ 0 < \alpha < 1. $$

3. System Description

Considering that gene regulatory networks are formed by the interplay of genes, the gene model proposed in this paper is designed as a two-dimensional model, which is written as the following fractional-order differential equations:

$$ \begin{cases} {}_0^C D_t^\alpha \varphi_i(t) = -a_i \varphi_i(t) + \sum\limits_{j=1}^{n} \omega_{ij} \zeta_j(\phi_j(t-\tau_1)) + B_i, \\ {}_0^C D_t^\alpha \phi_i(t) = -c_i \phi_i(t) + d_i \varphi_i(t-\tau_2), \quad i \in \mathbb{N}^+ \end{cases} \tag{1} $$

for $\alpha \in (0, 1)$, $t \geqslant 0$, where $\varphi_i(t)$, $\phi_i(t)$ represent the density of the ith messenger RNA as well as protein at t. The degradation rate of messenger RNA and protein are denoted by a_i and c_i, respectively. $d_i > 0$ describes the translation rate. τ_1 and τ_2 correspond to the delay in different gene transcription processes. ω_{ij} represents the coupling relations of the complex-valued system, which is depicted as follows:

$$ \omega_{ij} = \begin{cases} b_{ij}, & \textit{the activating factor of gene } i \textit{ is transcription factor } j; \\ 0, & \textit{there is no connection between } j \textit{ and } i; \\ -b_{ij}, & \textit{the antirepressor of gene } i \textit{ is transcription factor } j. \end{cases} \tag{2} $$

$B_i = \sum_{j \in I_i} b_{ij}$, where b_{ij} are the transcriptional rates of factor j to i which are bounded constants and have no units of measure and I_i is a repressor of gene i that is the aggregate of all values of j. $\zeta(\phi_j(t))$ is referred to as the feedback regulation of transcription by proteins, which is described as follows:

$$\zeta(\phi_j(t)) = \frac{(\phi_j(t)/\beta_j)^{\Theta_j}}{(\phi_j(t)/\beta_j)^{\Theta_j} + 1}, \tag{3}$$

where Θ_j means the Hill coefficient and $\beta_j > 0$ describes the constant. Obviously, $\zeta_j(\phi_j(t))$ is a monotonically increasing function. In addition, the initial conditions are $\varphi_i(s) = \delta_i^{(1)}(s)$, $s \in [-\tau_1, 0]$, $\phi_i(s) = \gamma_i^{(1)}(s)$, and $s \in [-\tau_2, 0]$.

The corresponding response system of model (1) is depicted as

$$\begin{cases} {}_0^C D_t^\alpha \kappa_i(t) = -a_i \kappa_i(t) + \sum_{j=1}^{n} \omega_{ij} \zeta_j(\psi_j(t-\tau_1)) + B_i + u_i(t), \\ {}_0^C D_t^\alpha \psi_i(t) = -c_i \psi_i(t) + d_i \kappa(t-\tau_2) + \widetilde{u}_i(t), \qquad i \in \mathbb{N}^+, \end{cases} \tag{4}$$

where the messenger RNA and protein concentrations of the system (4) are denoted by $\kappa_i(t), \psi_i(t)$, and $t \geqslant 0$. The system (4) and the system (1) have identical coefficients. $u_i(t)$ and $\widetilde{u}_i(t)$ stand for the designed controller which will be given later, and the initial values of the model (4) are given by $\kappa_i(s) = \delta_i^{(2)}(s)$, $s \in [-\tau_1, 0]$, $\psi_i(s) = \gamma_i^{(2)}(s)$, and $s \in [-\tau_2, 0]$.

Let $\varrho_{\varphi i}(t) = \kappa_i(t) - \varphi_i(t)$, and $\varrho_{\phi i}(t) = \psi_i(t) - \phi_i(t)$, then the following error systems are obtained

$$\begin{cases} {}_0^C D_t^\alpha \varrho_{\varphi i}(t) = -a_i \varrho_{\varphi i}(t) + \sum_{j=1}^{n} \omega_{ij} \zeta_j(\varrho_{\phi j}(t-\tau_1)) + u_i(t), \\ {}_0^C D_t^\alpha \varrho_{\phi i}(t) = -c_i \varrho_{\phi i}(t) + d_i \varrho_{\varphi i}(t-\tau_2) + \widetilde{u}_i(t), \quad t \geqslant 0, \end{cases} \tag{5}$$

with initial conditions

$$\begin{cases} \varrho_{\varphi i}(s) = \delta_i(s), s \in [-\tau_1, 0], \\ \varrho_{\phi i}(s) = \gamma_i(s), s \in [-\tau_2, 0], \end{cases} \tag{6}$$

where $\zeta_j(\varrho_{\phi j}(t-\tau_1)) = \zeta_j(\psi_j(t-\tau_1)) - \zeta_j(\phi_j(t-\tau_1))$, $i \in \mathbb{N}^+$.

Definition 4 ([52]). *If there exists a setting time $T \geq 0$ such that $\lim_{t\to T}(||\varrho_\varphi(t)||_i + ||\varrho_\phi(t)||_i) = 0$ and $||\varrho_\varphi(t)||_i + ||\varrho_\phi(t)||_i \equiv 0$, $\forall t \geq T, i = 1, 2$, then the driving system (1) is synchronized in finite time with the response system (4).*

In this paper, it is necessary to make the following assumptions.

Assumption 1. *For any $u, v \in \mathbb{C}$, there exists a real number $l_j > 0$ such that*

$$|\zeta_j(u) - \zeta_j(v)|_i \leqslant l_j |u - v|_i, \quad i, j \in \mathbb{N}^+. \tag{7}$$

Assumption 2. *In model (5), the initial values satisfy*

$$\begin{aligned} ||\delta(s)||_1 &= \sup_{-\tau_1 \leq s \leq 0} \sum_{i=1}^{n} |\delta_i(s)|_2 \leq \sum_{i=1}^{n} |\delta_i(0)|_2, \ s \in [-\tau_1, 0]), \\ ||\gamma(s)||_1 &= \sup_{-\tau_2 \leq s \leq 0} \sum_{i=1}^{n} |\gamma_i(s)|_2 \leq \sum_{i=1}^{n} |\gamma_i(0)|_2, \ s \in [-\tau_2, 0]). \end{aligned} \tag{8}$$

4. Synchronization in Finite Time with a Feedback Controller

In this section, in order to establish the criteria of synchronization in finite time, the following complex-valued feedback controller is designed:

$$\begin{cases} u_i(t) = -k_i \varrho_{\varphi i}(t) - \dfrac{\eta}{\varrho_{\varphi i}(t)} \dfrac{|\varrho_{\varphi i}(t)|_2}{||\varrho_{\varphi}(t)||_2}, \\ \widetilde{u}_i(t) = -\widetilde{k}_i \varrho_{\phi i}(t) - \dfrac{\widetilde{\eta}}{\varrho_{\phi i}(t)} \dfrac{|\varrho_{\phi i}(t)|_2}{||\varrho_{\phi}(t)||_2}, \end{cases} \tag{9}$$

where q_i and g_i are adaptive coupling strengths and $k_i, \widetilde{k}_i, \eta$, and $\widetilde{\eta}$ are arbitrary complex numbers, $i = 1, 2, ..., n$.

Assumption 3. *The parameters of the model (5) and controller (9) satisfy the following conditions*

$$a_i + \overline{a_i} + k_i + \overline{k_i} - \sum_{j=1}^{n} |\omega_{ij}|^2 \geq 0, \quad c_i + \overline{c_i} + \widetilde{k}_i + \overline{\widetilde{k}_i} - |d_i|^2 \geq 0.$$

Theorem 1. *Under Assumptions 1–3, the drive system (1) and the response system (4) can achieve synchronization in finite time under the action of the adaptive controller (9). Meanwhile, the time of synchronization is estimated as*

$$T_1 = \left[\frac{\Gamma(\alpha+1)(\sum_{i=1}^{n} |\varrho_{\varphi i}(0)|_2^2 + \sum_{i=1}^{n} |\varrho_{\phi i}(0)|_2^2)}{2 \times \mathrm{Re}(\eta + \widetilde{\eta})} \right]^{\frac{1}{\alpha}}.$$

Proof of Theorem 1. Consider the following Lyapunov function,

$$V(t) = \sum_{i=1}^{n} |\varrho_{\varphi i}(t)|_2^2 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_2^2. \tag{10}$$

By calculating the derivative of $V(t)$ along the error system (5), we can obtain from Lemma 4 that

$$\begin{aligned}
&{}_0^C D_t^{\alpha} V(t) \\
&= \sum_{i=1}^{n} {}_0^C D_t^{\alpha} |\varrho_{\varphi i}(t)|_2^2 + \sum_{i=1}^{n} {}_0^C D_t^{\alpha} |\varrho_{\phi i}(t)|_2^2 \\
&= \sum_{i=1}^{n} \varrho_{\varphi i}(t) [{}_0^C D_t^{\alpha} \overline{\varrho_{\varphi i}(t)} + \overline{\varrho_{\varphi i}(t)} {}_0^C D_t^{\alpha} \varrho_{\varphi i}(t)] + \sum_{i=1}^{n} \varrho_{\phi i}(t) [{}_0^C D_t^{\alpha} \overline{\varrho_{\phi i}(t)} + \overline{\varrho_{\phi i}(t)} {}_0^C D_t^{\alpha} \varrho_{\phi i}(t)] \\
&= \sum_{i=1}^{n} \varrho_{\varphi i}(t) \left[-\overline{a_i} \overline{\varrho_{\varphi i}(t)} - \overline{k_i} \overline{\varrho_{\varphi i}(t)} + \sum_{j=1}^{n} \overline{\omega_{ij}} \overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))} - \frac{\eta}{\varrho_{\varphi i}(t)} \frac{|\varrho_{\varphi i}(t)|_2}{||\varrho_{\varphi}(t)||_2} \right] \\
&\quad + \sum_{i=1}^{n} \overline{\varrho_{\varphi i}(t)} \left[-a_i \varrho_{\varphi i}(t) - k_i \varrho_{\varphi i}(t) + \sum_{j=1}^{n} \omega_{ij} \zeta_j(\varrho_{\phi j}(t-\tau_1)) - \frac{\overline{\eta}}{\overline{\varrho_{\varphi i}(t)}} \frac{|\varrho_{\varphi i}(t)|_2}{||\varrho_{\varphi}(t)||_2} \right] \\
&\quad + \sum_{i=1}^{n} \varrho_{\phi i}(t) \left[-\overline{c_i} \overline{\varrho_{\phi i}(t)} - \overline{\widetilde{k}_i} \overline{\varrho_{\phi i}(t)} + \overline{d_i} \overline{\varrho_{\varphi i}(t-\tau_2)} - \frac{\widetilde{\eta}}{\varrho_{\phi i}(t)} \frac{|\varrho_{\phi i}(t)|_2}{||\varrho_{\phi}(t)||_2} \right] \\
&\quad + \sum_{i=1}^{n} \overline{\varrho_{\phi i}(t)} \left[-c_i \varrho_{\phi i}(t) - \widetilde{k}_i \varrho_{\phi i}(t) + d_i \varrho_{\varphi i}(t-\tau_2) - \frac{\overline{\widetilde{\eta}}}{\overline{\varrho_{\phi i}(t)}} \frac{|\varrho_{\phi i}(t)|_2}{||\varrho_{\phi}(t)||_2} \right] \\
&= \sum_{i=1}^{n} \left[-(a_i + \overline{a_i} + k_i + \overline{k_i}) |\varrho_{\varphi i}(t)|_2^2 + \sum_{j=1}^{n} \Big(\overline{\omega_{ij}} \varrho_{\varphi i}(t) \overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))} \right. \\
&\quad \left. + \omega_{ij} \overline{\varrho_{\varphi i}(t)} \zeta_j(\varrho_{\phi j}(t-\tau_1)) \Big) \right] + \sum_{i=1}^{n} \Big[-(c_i + \overline{c_i} + \widetilde{k}_i + \overline{\widetilde{k}_i}) |\varrho_{\phi i}(t)|_2^2 \\
&\quad + \overline{d_i} \overline{\varrho_{\varphi i}(t-\tau_2)} \varrho_{\phi i}(t) + d_i \varrho_{\varphi i}(t-\tau_2) \overline{\varrho_{\phi i}(t)} \Big] - 2 \times \mathrm{Re}(\eta + \widetilde{\eta}).
\end{aligned} \tag{11}$$

According to Assumption 1 and Lemma 1, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\overline{\omega_{ij}}\varrho_{\varphi i}(t)\overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))}+\omega_{ij}\overline{\varrho_{\varphi i}(t)}\zeta_j(\varrho_{\phi j}(t-\tau_1))\Big)\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\omega_{ij}\overline{\omega_{ij}}\varrho_{\varphi i}(t)\overline{\varrho_{\varphi i}(t)}+\zeta_j(\varrho_{\phi j}(t-\tau_1))\overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))}\Big)\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|\omega_{ij}|_2^2|\varrho_{\varphi i}|_2^2+l_j^2|\varrho_{\phi i}(t-\tau_1)|_2^2\Big).
\end{aligned}\tag{12}$$

Similarly,

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(\overline{d_i\varrho_{\varphi i}(t-\tau_2)}\varrho_{\phi i}(t)+d_i\varrho_{\varphi i}(t-\tau_2)\overline{\varrho_{\phi i}(t)}\Big)\\
&\leq\sum_{i=1}^{n}\Big(d_i\overline{d_i}\varrho_{\phi i}(t)\overline{\varrho_{\phi i}(t)}+\varrho_{\varphi i}(t-\tau_2)\overline{\varrho_{\varphi i}(t-\tau_2)}\Big)\\
&\leq\sum_{i=1}^{n}|d_i|_2^2|\varrho_{\phi i}(t)|_2^2+|\varrho_{\varphi i}(t-\tau_2)|_2^2.
\end{aligned}\tag{13}$$

Based on Assumption 2, applying the Razumikhin condition (see pages 55–56 in ref. [53]) yields

$$\begin{aligned}
&\sum_{i=1}^{n}|\varrho_{\phi i}(\varepsilon)|\leq\sum_{i=1}^{n}|\varrho_{\phi i}(t)|,\ t-\tau_1\leq\varepsilon\leq t,\\
&\sum_{i=1}^{n}|\varrho_{\varphi i}(\varepsilon)|\leq\sum_{i=1}^{n}|\varrho_{\varphi i}(t)|,\ t-\tau_2\leq\varepsilon\leq t.
\end{aligned}\tag{14}$$

Then, from (12)–(14), we can obtain

$$\begin{aligned}
{}_0^CD_t^\alpha V(t)\leq&-\sum_{i=1}^{n}\Big[(a_i+\overline{a_i}+k_i+\overline{k_i}-\sum_{j=1}^{n}|\omega_{ij}|_2^2-1)|\varrho_{\varphi i}(t)|_2^2\\
&+(c_i+\overline{c_i}+\widetilde{k}_i+\overline{\widetilde{k}_i}-l_j^2-|d_i|_2^2)|\varrho_{\phi i}(t)|_2^2\Big]\\
&-2\times\mathrm{Re}(\eta+\widetilde{\eta}).
\end{aligned}\tag{15}$$

Under Assumption 3, one has

$$ {}_0^CD_t^\alpha V(t)\leq-2\times\mathrm{Re}(\eta+\widetilde{\eta}),$$

therefore, let $F(t)$ be a non-negative function which satisfies the following conditions:

$$ {}_0^CD_t^\alpha V(t)+F(t)=-2\times\mathrm{Re}(\eta+\widetilde{\eta}).\tag{16}$$

Taking the α-order integral from 0 to t of Equation (16), it can be converted to

$$\begin{aligned}
V(t)=&V(0)-{}_0^{RL}D_t^\alpha F(t)+{}_0^{RL}D_t^\alpha\Big(-2\times\big(\mathrm{Re}(\eta)+\mathrm{Re}(\widetilde{\eta})\big)\Big)\\
=&V(0)-\frac{1}{\Gamma(\alpha)}\times\int_0^t\Big(F(s)+2\times\big(\mathrm{Re}(\eta)+\mathrm{Re}(\widetilde{\eta})\big)\Big)(t-s)^{\alpha-1}ds.
\end{aligned}\tag{17}$$

Obviously, $\Gamma(\alpha)>0$; therefore, ${}_0^{RL}D_t^\alpha F(t)\geq0$ and the following can be obtained:

$$V(t)\leq V(0)-\frac{2\times\mathrm{Re}(\eta+\widetilde{\eta})}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds=V(0)-\frac{(2\times\mathrm{Re}(\eta+\widetilde{\eta}))t^\alpha}{\Gamma(\alpha+1)}.\tag{18}$$

$(t-s)^{\alpha-1}F(s)$ $\big(s \in [0,t)\big)$ is a non-negative function. Hence,

$$\begin{aligned} V(t) &\leq V(0) - \frac{M_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &= V(0) - \frac{M_1 t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{19}$$

Letting $\Phi_1(t) = V(0) - \dfrac{\big(2\times \mathrm{Re}(\eta+\widetilde{\eta})\big)t^\alpha}{\Gamma(\alpha+1)}$ and $\Phi(t) = 0$ yields

$$\begin{aligned} T_1 &= \left[\frac{V(0)\Gamma(\alpha+1)}{2\times \mathrm{Re}(\eta+\widetilde{\eta})}\right]^{\frac{1}{\alpha}} \\ &= \left[\frac{(\sum_{i=1}^n |\varrho_{\varphi i}(0)|_2^2 + \sum_{i=1}^n |\varrho_{\phi i}(0)|_2^2)\Gamma(\alpha+1)}{2\times \mathrm{Re}(\eta+\widetilde{\eta})}\right]^{\frac{1}{\alpha}}. \end{aligned} \tag{20}$$

When $t \geq T_1$, we obtain $V(t) \leq \Phi_1(t) \leq 0$, $V(t)$ as a non-negative function. In addition,

$$\begin{aligned} &\sum_{i=1}^n |\varrho_{\varphi i}(t)|_2 + \sum_{i=1}^n |\varrho_{\phi i}(t)|_2 \\ \leq&\sqrt{(|\varrho_{\varphi 1}|^2 + \cdots + |\varrho_{\varphi n}|^2 + |\varrho_{\phi 1}|^2 + \cdots + |\varrho_{\phi n}|^2)(1^2 + \cdots + 1^2 + 1^2 + \cdots + 1^2)} \\ =&\sqrt{(\sum_{i=1}^n |\varrho_{\varphi i}(t)|^2 + \sum_{i=1}^n |\varrho_{\phi i}(t)|^2)(2n)} \\ =&\sqrt{2n}\sqrt{V(t)}, \end{aligned} \tag{21}$$

which leads to

$$\sum_{i=1}^n |\varrho_{\varphi i}(t)| + \sum_{i=1}^n |\varrho_{\phi i}(t)| \equiv 0,\ \forall t \geq T_1. \tag{22}$$

According to Definition 3, the drive system (1) and the response system (4) can achieve synchronization in finite time under the state feedback controller (9). □

5. Synchronization in Finite Time with Adaptive Controller

Based on a novel complex-valued sign function, a complex-valued adaptive controller is designed by

$$\begin{cases} u_i(t) = -(q_i(t) + \theta_i)[\varrho_{\varphi i}(t)], \\ \widetilde{u_i}(t) = -(g_i(t) + \widetilde{\theta}_i)[\varrho_{\phi i}(t)], \\ {}_0^C D_t^\alpha q_i(t) = \delta_i |[\varrho_{\varphi i}(t)]|_1, \\ {}_0^C D_t^\alpha g_i(t) = \widetilde{\delta}_i |[\varrho_{\phi i}(t)]|_1, \end{cases} \tag{23}$$

where $q_i(t)$ and $g_i(t)$ are complex-valued functions and $\theta_i, \widetilde{\theta}_i, \delta_i, \widetilde{\delta}_i \in \mathbb{C}$, and $i \in \mathbb{N}^+$.

Assumption 4. *For the parameters of model (5) and controller (23), the following inequalities hold*

$$\mathrm{Re}(a_i) - |\mathrm{Im}(a_i)| - |d_i|_1 \geq 0,$$

$$\mathrm{Re}(c_i) - |\mathrm{Im}(c_i)|\sum_{j=1}^n l_j^2 |\omega_{ij}|_1 \geq 0,$$

$$\mathrm{Re}(q_i) \geq 0, \quad \mathrm{Re}(g_i) \geq 0.$$

Theorem 2. *Under the Assumptions 1 and 2 and condition (4), the drive system (1) and the response system (4) can achieve synchronization in finite time under the action of the adaptive controller (23). Moreover, the time of synchronization is estimated as*

$$T_2 = \left[\frac{\Gamma(\alpha+1)\Big(\sum_{i=1}^{n} |\varrho_{\varphi i}(0)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(0)|_1 - \sum_{i=1}^{n} \frac{|q_i(0)-q_i|^2}{\delta_i} - \sum_{i=1}^{n} \frac{|g_i(0)-g_i|^2}{\widetilde{\delta}_i} \Big)}{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}_i})} \right]^{\frac{1}{\alpha}}.$$

Proof. A Lyapunov function is constructed as

$$\begin{aligned} V(t) = &\sum_{i=1}^{n} |\varrho_{\varphi i}(t)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_1 + \sum_{i=1}^{n} \frac{1}{2\delta_i}(q_i(t)-q_i)(\overline{q_i(t)-q_i}) \\ &+ \sum_{i=1}^{n} \frac{1}{2\widetilde{\delta}_i}(g_i(t)-g_i)(\overline{g_i(t)-g_i}). \end{aligned}$$

By calculating the derivative of $V(t)$ along the error system (5) with adaptive controller (23), we can obtain from Lemmas 2 and 4 that

$$\begin{aligned} &{}_0^C D_t^\alpha V(t) \\ =&\sum_{i=1}^{n} {}_0^C D_t^\alpha |\varrho_{\varphi i}(t)|_1 + \sum_{i=1}^{n} {}_0^C D_t^\alpha |\varrho_{\phi i}(t)|_1 + \sum_{i=1}^{n} {}_0^C D_t^\alpha \left(\frac{1}{2\delta_i}(q_i(t)-q_i)(\overline{q_i(t)-q_i}) \right) \\ &+ \sum_{i=1}^{n} {}_0^C D_t^\alpha \left(\frac{1}{2\widetilde{\delta}_i}(g_i(t)-g_i)(\overline{g_i(t)-g_i}) \right) \\ \leq&\frac{1}{2}\sum_{i=1}^{n} \Big(\overline{[\varrho_{\varphi i}(t)]}{}_0^C D_t^\alpha \varrho_{\varphi i}(t) + [\varrho_{\varphi i}(t)]{}_0^C D_t^\alpha \overline{\varrho_{\varphi i}(t)} + \overline{[\varrho_{\phi i}(t)]}{}_0^C D_t^\alpha \varrho_{\phi i}(t) + [\varrho_{\phi i}(t)]{}_0^C D_t^\alpha \overline{\varrho_{\phi i}(t)} \Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\delta_i} \times \Big((q_i(t)-q_i){}_0^C D_t^\alpha \overline{q_i(t)} + (\overline{q_i(t)-q_i}){}_0^C D_t^\alpha q_i(t) \Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\widetilde{\delta}_i} \times \Big((g_i(t)-g_i){}_0^C D_t^\alpha \overline{g_i(t)} + (\overline{g_i(t)-g_i}){}_0^C D_t^\alpha g_i(t) \Big) \\ =&\frac{1}{2}\sum_{i=1}^{n} \overline{[\varrho_{\varphi i}(t)]} \left(-a_i \varrho_{\varphi i}(t) + \sum_{j=1}^{n} \omega_{ij}\zeta_j(\varrho_{\phi j}(t-\tau_1)) - q_i(t)[\varrho_{\varphi i}(t)] - \theta_i[\varrho_{\varphi i}(t)] \right) \\ &+ \frac{1}{2}\sum_{i=1}^{n} [\varrho_{\varphi i}(t)] \left(\overline{-a_i}\,\overline{\varrho_{\varphi i}(t)} + \sum_{j=1}^{n} \overline{\omega_{ij}}\,\overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))} - \overline{q_i(t)}\,\overline{[\varrho_{\varphi i}(t)]} - \overline{\theta_i}\,\overline{[\varrho_{\varphi i}(t)]} \right) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \overline{[\varrho_{\phi i}(t)]} \Big(-c_i \varrho_{\phi i}(t) + d_i \varrho_{\varphi i}(t-\tau_2) - g_i(t)[\varrho_{\phi i}(t)] - \widetilde{\theta}_i[\varrho_{\phi i}(t)] \Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n} [\varrho_{\phi i}(t)] \Big(\overline{-c_i}\,\overline{\varrho_{\phi i}(t)} + \overline{d_i}\,\overline{\varrho_{\varphi i}(t-\tau_2)} - \overline{g_i(t)}\,\overline{[\varrho_{\phi i}(t)]} - \overline{\widetilde{\theta}_i}\,\overline{[\varrho_{\phi i}(t)]} \Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \left(\frac{1}{\delta_i}(q_i(t)-q_i)\delta_i \big|[\varrho_{\varphi i}(t)]\big|_1 + \frac{1}{\delta_i}(\overline{q_i(t)-q_i})\delta_i \big|[\varrho_{\varphi i}(t)]\big|_1 \right) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \left(\frac{1}{\widetilde{\delta}_i}(g_i(t)-g_i)\widetilde{\delta}_i \big|[\varrho_{\phi i}(t)]\big|_1 + \frac{1}{\widetilde{\delta}_i}(\overline{g_i(t)-g_i})\widetilde{\delta}_i \big|[\varrho_{\phi i}(t)]\big|_1 \right) \\ =&-\frac{1}{2}\sum_{i=1}^{n} \Big(a_i \overline{[\varrho_{\varphi i}(t)]}\varrho_{\varphi i}(t) + \overline{a_i}[\varrho_{\varphi i}(t)]\overline{\varrho_{\varphi i}(t)} \Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \Big(\omega_{ij}\overline{[\varrho_{\varphi i}(t)]}\zeta_j(\varrho_{\phi j}(t-\tau_1)) + \overline{\omega_{ij}}[\varrho_{\varphi i}(t)]\overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))} \Big) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}(q_i(t)+\overline{q_i(t)}+\theta_i+\overline{\theta}_i)[\overline{\varrho_{\varphi i}(t)}][\varrho_{\varphi i}(t)]\\
&-\frac{1}{2}\sum_{i=1}^{n}\Big(c_i[\overline{\varrho_{\phi i}(t)}]\varrho_{\phi i}(t)+\overline{c_i}[\varrho_{\phi i}(t)]\overline{\varrho_{\phi i}(t)}\Big)\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(d_i[\overline{\varrho_{\phi i}(t)}]\varrho_{\varphi i}(t-\tau_2)+\overline{d_i}[\varrho_{\phi i}(t)]\overline{\varrho_{\varphi i}(t-\tau_2)}\Big)\\
&-\frac{1}{2}\sum_{i=1}^{n}(g_i(t)+\overline{g_i(t)}+\widetilde{\theta}_i+\overline{\widetilde{\theta}}_i)[\overline{\varrho_{\phi i}(t)}][\varrho_{\phi i}(t)]\\
&+\frac{1}{2}\sum_{i=1}^{n}\left((q_i(t)-q_i)\big|[\overline{\varrho_{\varphi i}(t)}]\big|_1+(\overline{q_i(t)-q_i})\big|[\varrho_{\varphi i}(t)]\big|_1\right)\\
&+\frac{1}{2}\sum_{i=1}^{n}\left((g_i(t)-g_i)\big|[\overline{\varrho_{\phi i}(t)}]\big|_1+(\overline{g_i(t)-g_i})\big|[\varrho_{\phi i}(t)]\big|_1\right),
\end{aligned}
\tag{24}
$$

which, by Lemma 3, gives

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}\left(a_i[\overline{\varrho_{\varphi i}(t)}]\varrho_{\varphi i}(t)+\overline{a_i}[\varrho_{\varphi i}(t)]\overline{\varrho_{\varphi i}(t)}\right)\\
&=\sum_{i=1}^{n}(-\mathrm{Re}(a_i)\big|\varrho_{\varphi i}(t)\big|_1-\mathrm{Im}(a_i)\times\\
&\quad(\mathrm{sign}(\mathrm{Im}(\varrho_{\varphi i}(t)))\times\mathrm{Re}(\varrho_{\varphi i}(t))-\mathrm{sign}(\mathrm{Re}(\varrho_{\varphi i}(t))\times\mathrm{Im}(\varrho_{\varphi i}(t)))\\
&\leq\sum_{i=1}^{n}\Big(|\mathrm{Im}(a_i)|-\mathrm{Re}(a_i)\Big)\big|\varrho_{\varphi i}(t)\big|_1.
\end{aligned}
\tag{25}
$$

Similarly,

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}\left(c_i[\overline{\varrho_{\phi i}(t)}]\varrho_{\phi i}(t)+\overline{c_i}[\varrho_{\phi i}(t)]\overline{\varrho_{\phi i}(t)}\right)\\
&\leq\sum_{i=1}^{n}\Big(|\mathrm{Im}(c_i)|-\mathrm{Re}(c_i)\Big)|\varrho_{\phi i}(t)|_1.
\end{aligned}
\tag{26}
$$

From Lemma 3 and Assumption 1, it holds that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\omega_{ij}[\overline{\varrho_{\varphi i}(t)}]\zeta_j(\varrho_{\phi j}(t-\tau_1))+\overline{\omega_{ij}}[\varrho_{\varphi i}(t)]\overline{\zeta_j(\varrho_{\phi j}(t-\tau_1))}\right)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}[\mathrm{Re}(\zeta_j(\varrho_{\phi j}(t-\tau_1)))(\mathrm{sign}(\mathrm{Re}(\varrho_{\varphi i}(t)))\mathrm{Re}(\omega_{ij})+\mathrm{sign}(\mathrm{Im}(\varrho_{\varphi i}(t)))\mathrm{Im}(\omega_{ij}))]\\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}[\mathrm{Im}(\zeta_j(\varrho_{\phi j}(t-\tau_1)))(\mathrm{sign}(\mathrm{Im}(\varrho_{\varphi i}(t)))\mathrm{Re}(\omega_{ij})-\mathrm{sign}(\mathrm{Re}(\varrho_{\varphi i}(t)))\mathrm{Im}(\omega_{ij}))]\\
&\leq\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|\mathrm{Re}(\zeta_j(\varrho_{\phi j}(t-\tau_1)))||\omega_{ij}|_1+|\mathrm{Im}(\zeta_j(\varrho_{\phi j}(t-\tau_1)))||\omega_{ij}|_1\Big)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}|\omega_{ij}|_1 l_j^2|\varrho_{\phi j}(t-\tau_1)|_1,
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\Big[d_i[\overline{\varrho_{\phi i}(t)}]\varrho_{\varphi i}(t-\tau_2)+\overline{d_i}[\varrho_{\phi i}(t)]\overline{\varrho_{\varphi i}(t-\tau_2)}\Big]\\
&\leq\sum_{i=1}^{n}|d_i|_1|\varrho_{\varphi i}(t-\tau_2)|_1.
\end{aligned}
\tag{28}
$$

By applying Lemma 2, one has

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}(\theta_i + q_i(t) + \overline{\theta_i} + \overline{q_i(t)})[\varrho_{\varphi i}(t)][\overline{\varrho_{\varphi i}(t)}] \\
&= -\sum_{i=1}^{n}\mathrm{Re}(q_i(t))\big|[\varrho_{\varphi i}(t)]\big|_1 - \frac{1}{2}\sum_{i=1}^{n}(\theta_i + \overline{\theta_i})\big|[\varrho_{\varphi i}(t)]\big|_1 \\
&\leq -\sum_{i=1}^{n}\left(\mathrm{Re}(q_i(t))\big|[\varrho_{\varphi i}(t)]\big|_1 + \frac{\theta_i + \overline{\theta_i}}{2}\right),
\end{aligned} \tag{29}$$

and

$$-\tfrac{1}{2}\textstyle\sum_{i=1}^{n}(\widetilde{\theta}_i + g_i(t) + \overline{\widetilde{\theta}_i} + \overline{g_i(t)})[\varrho_{\phi i}(t)][\overline{\varrho_{\phi i}(t)}] \leq -\sum_{i=1}^{n}\left(\mathrm{Re}(g_i(t))\big|[\varrho_{\phi i}(t)]\big|_1 + \dfrac{\widetilde{\theta}_i + \overline{\widetilde{\theta}_i}}{2}\right). \tag{30}$$

Based on the definition of a conjugate complex number, the following equation is obtained:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\Big((q_i(t) - q_i)\big|[\varrho_{\varphi i}(t)]\big|_1 + (\overline{q_i(t) - q_i})\big|[\varrho_{\varphi i}(t)]\big|_1\Big) \\
&+ \frac{1}{2}\sum_{i=1}^{n}\Big((g_i(t) - g_i)\big|[\varrho_{\phi i}(t)]\big|_1 + (\overline{g_i(t) - g_i})\big|[\varrho_{\phi i}(t)]\big|_1\Big) \\
&= \sum_{i=1}^{n}\Big(\mathrm{Re}(g_i(t) - g_i)\big|[\varrho_{\phi i}(t)]\big|_1 + \mathrm{Re}(g_i(t) - g_i)\big|[\varrho_{\phi i}(t)]\big|_1\Big).
\end{aligned} \tag{31}$$

Based on Assumption 2, applying the Razumikhin condition (see pages 55–56 in ref. [53]) yields

$$\begin{aligned}
&\sum_{i=1}^{n}|\varrho_{\phi i}(\varepsilon)| \leq \sum_{i=1}^{n}|\varrho_{\phi i}(t)|,\ t - \tau_1 \leq \varepsilon \leq t, \\
&\sum_{i=1}^{n}|\varrho_{\varphi i}(\varepsilon)| \leq \sum_{i=1}^{n}|\varrho_{\varphi i}(t)|,\ t - \tau_2 \leq \varepsilon \leq t.
\end{aligned}$$

Then, it follows from (24)–(32) that

$$\begin{aligned}
{}_{0}^{C}D_t^{\alpha}V(t) \leq &-\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\mathrm{Re}(a_i) - |\mathrm{Im}(a_i)| - |d_i|_1\Big)\big|\varrho_{\varphi i}(t)\big|_1 \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\mathrm{Re}(c_i) - |\mathrm{Im}(c_i)| - \sum_{j=1}^{n}l_j^2|\omega_{ij}|_1\Big)\big|\varrho_{\phi i}(t)\big|_1 \\
&- \sum_{i=1}^{n}\mathrm{Re}(q_i)\big|[\varrho_{\varphi i}(t)]\big|_1 - \sum_{i=1}^{n}\mathrm{Re}(g_i)\big|[\varrho_{\phi i}(t)]\big|_1 \\
&- \frac{1}{2}\sum_{i=1}^{n}(\theta_i + \overline{\theta_i} + \widetilde{\theta}_i + \overline{\widetilde{\theta}_i}).
\end{aligned} \tag{32}$$

Furthermore, by condition (4), we can deduce that

$${}_{0}^{C}D_t^{\alpha}V(t) \leq -2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}_i}). \tag{33}$$

Now, let $G(t)$ be a non-negative function such that

$${}_{0}^{C}D_t^{\alpha}V(t) + G(t) = -2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}_i}). \tag{34}$$

Taking the α-order integral from 0 to t on both sides of Equation (34) yields

$$\begin{aligned} V(t) =& V(0) - {}_0^{RL}D_t^{\alpha} G(t) + {}_0^{RL}D_t^{\alpha}(-2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i)) \\ =& V(0) - \frac{1}{\Gamma(\alpha)} \times \int_0^t (G(s) + 2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i))(t-s)^{\alpha-1} ds. \end{aligned} \tag{35}$$

Since $(t-s)^{\alpha-1}G(s)$, $s \in [0, t)$ is a non-negative function, it holds that

$$\begin{aligned} V(t) \leq& V(0) - \frac{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ =& V(0) - \frac{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i) t^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{36}$$

Evidently, the right part of the inequality (36) is a strictly decreasing function. Therefore, let $\Phi_2(t) = V(0) - \frac{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i) t^{\alpha}}{\Gamma(\alpha+1)}$. Then, $\Phi_2(t) = 0$ when and only when

$$\begin{aligned} T_2 =& \left[\frac{V(0)\Gamma(\alpha+1)}{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i)} \right]^{\frac{1}{\alpha}} \\ =& \left[\frac{\Gamma(\alpha+1)\left(\sum_{i=1}^{n} |\varrho_{\varphi i}(0)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_1 - \sum_{i=1}^{n} \frac{|q_i(0)-q_i|^2}{\delta_i} - \sum_{i=1}^{n} \frac{|g_i(0)-g_i|^2}{\widetilde{\delta}_i} \right)}{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}}_i)} \right]^{\frac{1}{\alpha}}. \end{aligned} \tag{37}$$

Hence, $V(t) \equiv 0, \forall t \geq T_2$, i.e.,

$$\begin{aligned} &\sum_{i=1}^{n} |\varrho_{\varphi i}(t)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_1 + \sum_{i=1}^{n} \frac{1}{\delta_i}(q_i(t) - q_i) \times (\overline{q_i(t) - q_i}) \\ &+ \sum_{i=1}^{n} \frac{1}{\widetilde{\delta}_i}(g_i(t) - g_i) \times (\overline{g_i(t) - g_i}) \equiv 0, \ \ \forall t \geq T_2. \end{aligned}$$

Thus,

$$\forall t \geq T_2, \quad \sum_{i=1}^{n} |\varrho_{\varphi i}(t)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_1 \equiv 0,$$

which implies that the drive system (1) and the response system (4) can achieve synchronization in finite time. □

Particularly, in adaptive controller (23), if $\delta_i = \widetilde{\delta}_i = 0$, then $q_i(t)$ and $g_i(t)$ are constants and the controller (23) becomes

$$\begin{cases} u_i(t) = -(q_i + \theta_i)[\varrho_{\varphi i}(t)], \\ \widetilde{u}_i(t) = -(g_i + \widetilde{\theta}_i)[\varrho_{\phi i}(t)], \end{cases} \tag{38}$$

where $i \in \mathbb{N}^+$ and $q_i, g_i, \theta_i, \widetilde{\theta}_i \in \mathbb{C}$.

Corollary 1. *If the following inequalities hold:*

$$\mathrm{Re}(a_i) - |\mathrm{Im}(a_i)| - |d_i|_1 \geq 0,$$

$$\mathrm{Re}(c_i) - |\mathrm{Im}(c_i)| + \sum_{j=1}^{n} l_j^2| + \omega_{ij}|_1 \geq 0,$$

then system (1) is synchronized with system (4) in finite time with the controller (38), and the time of synchronization is estimated as

$$T = \left[\frac{\Gamma(\alpha+1) \times V(0)}{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}_i})} \right]^{\frac{1}{\alpha}} = \left[\frac{(\sum\limits_{i=1}^{n} |\varrho_{\varphi i}(0)|_1 + \sum\limits_{i=1}^{n} |\varrho_{\phi i}(t)|_1)\Gamma(\alpha+1)}{2n \times (\theta + \widetilde{\theta} + \overline{\theta_i} + \overline{\widetilde{\theta}_i})} \right]^{\frac{1}{\alpha}}. \tag{39}$$

Proof. We choose the Lyapunov function:

$$V(t) = \sum_{i=1}^{n} |\varrho_{\varphi i}(t)|_1 + \sum_{i=1}^{n} |\varrho_{\phi i}(t)|_1 - \sum_{i=1}^{n} \frac{1}{2\delta_i} q_i(t)\overline{q_i(t)} - \sum_{i=1}^{n} \frac{1}{2\widetilde{\delta}_i} g_i(t)\overline{g_i(t)}.$$

The proof is similar to Theorems (1) and (2), and thus we omit it here. □

6. Numerical Examples

In this section, with the selection of values that match Assumptions 3 and 4, a numerical example is given to prove the effectiveness of proposed schemes.

Example 1. *Considering that the model is formed by the interaction of various small molecules, the following gene regulatory network with two dimensions is built as the drive system:*

$$\begin{cases} {}_0^C D_t^\alpha \varphi_i(t) = -a_i \varphi_i(t) + \sum\limits_{j=1}^{n} \omega_{ij} \zeta_j(\phi_j(t-\tau_1)) + B_i, \\ {}_0^C D_t^\alpha \phi_i(t) = -c_i \phi_i(t) + d_i \varphi_i(t-\tau_2), \qquad t \geqslant 0, \end{cases} \tag{40}$$

where $i = 1,2$, $\alpha = 0.92$, $\tau_1 = \tau_2 = 0.5$ $\zeta_j(\phi_j(t-\tau)) = \dfrac{\phi_j^2(t-\tau)}{\phi_j^2(t-\tau)+1}$, $j = 1,2$, *and* $B_1 = B_2 = 0$. *Let the initial values of the model (40) be* $\varphi_1(t) = e^{0.3t - i0.9t}$, $\varphi_2(t) = e^{1.2t+i0.9t}$, $\phi_1(t) = e^{0.8t-i2t}$, $\phi_2(t) = e^{1t+i3t}$, $t \in [-0.5, 0]$, *and*

$$A = \begin{pmatrix} 1.2+4i & 0 \\ 0 & 1.4+4i \end{pmatrix}, W = (\omega_{ij})_{2\times 2} = \begin{pmatrix} 2.3-0.3i & 3.5+0.6i \\ 2.8-0.5i & 1.5+0.4i \end{pmatrix},$$

$$C = \begin{pmatrix} 2+5i & 0 \\ 0 & 3+5i \end{pmatrix}, D = \begin{pmatrix} 3+1i & 0 \\ 0 & 2-1i \end{pmatrix}.$$

The response system is

$$\begin{cases} {}_0^C D_t^\alpha \kappa_i(t) = -a_i \kappa_i(t) + \sum\limits_{j=1}^{n} \omega_{ij} \zeta_j(\psi_j(t-\tau_1)) + B_i + u_i(t), \\ {}_0^C D_t^\alpha \psi_i(t) = -c_i \psi_i(t) + d_i \kappa_i(t-\tau_2) + \widetilde{u}_i(t), \qquad t \geqslant 0, \end{cases} \tag{41}$$

where the initial values are $\kappa_1(t) = e^{0.4-1i}$, $\kappa_2(t) = e^{1.3+2i}$, $\psi_1(t) = e^{0.7-2i}$, *and* $\psi_2(t) = e^{1.5+3i}$, $t \in [-0.5, 0]$.

First, as shown in Figure 1, it is clear that systems (40) and (41) cannot be synchronized without controllers.

Now, we will verify the synchronization between (40) and (41) by our proposed control schemes (9) and (23).

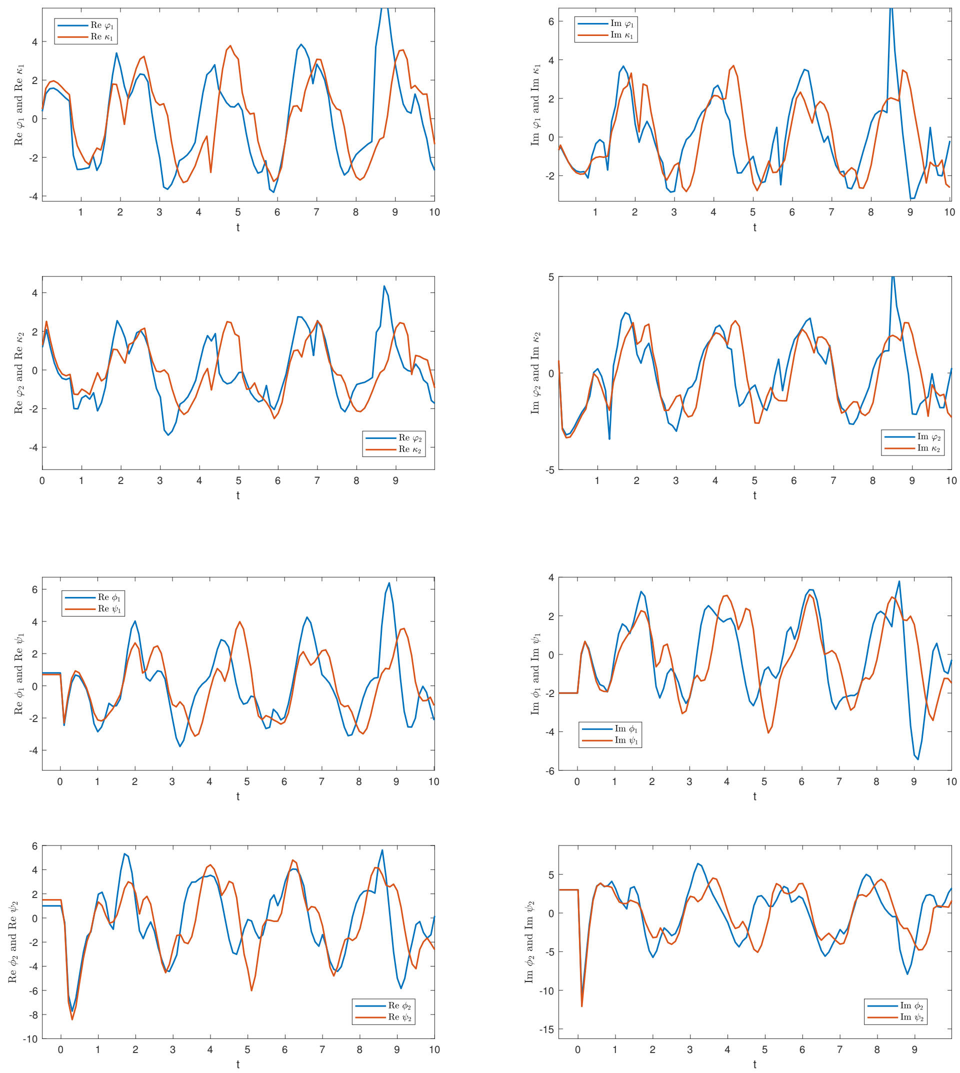

Figure 1. State trajectories of (40) and (41) without controller.

6.1. Synchronization by State Feedback Controller (9)

In the controller (9), let $k_1 = k_2 = 10 - 5i$, $\widetilde{k}_1 = \widetilde{k}_2 = 4 - i$, and $\eta = \widetilde{\eta} = 0.01 + 0.5i$. Then, we have

$$a_1 + \overline{a_1} + k_1 + \overline{k_1} - \sum_{j=1}^{n} |\omega_{1j}|_2^2 = 7.93 \geq 0, \quad c_1 + \overline{c_1} + \widetilde{k_1} + \overline{\widetilde{k_1}} - l_1^2 - |d_1|_2^2 = 2 \geq 0,$$

$$a_2 + \overline{a_2} + k_2 + \overline{k_2} - \sum_{j=1}^{n} |\omega_{2j}|_2^2 = 6.78 \geq 0, \quad c_1 + \overline{c_2} + \widetilde{k_2} + \overline{\widetilde{k_2}} - l_2^2 - |d_2|_2^2 = 7 \geq 0.$$

Thus, all assumptions and conditions in Theorem 1 hold, and it is easy to calculate that the setting time is estimated as $T = 58.3433$. Thereupon, according to Theorem 1, (40) and (41) can achieve synchronization in finite time by controller (9). Figure 2 presents the state trajectories of drive system (40) and response system (41), and Figure 3 depicts the state trajectory of the error system between (40) and (41).

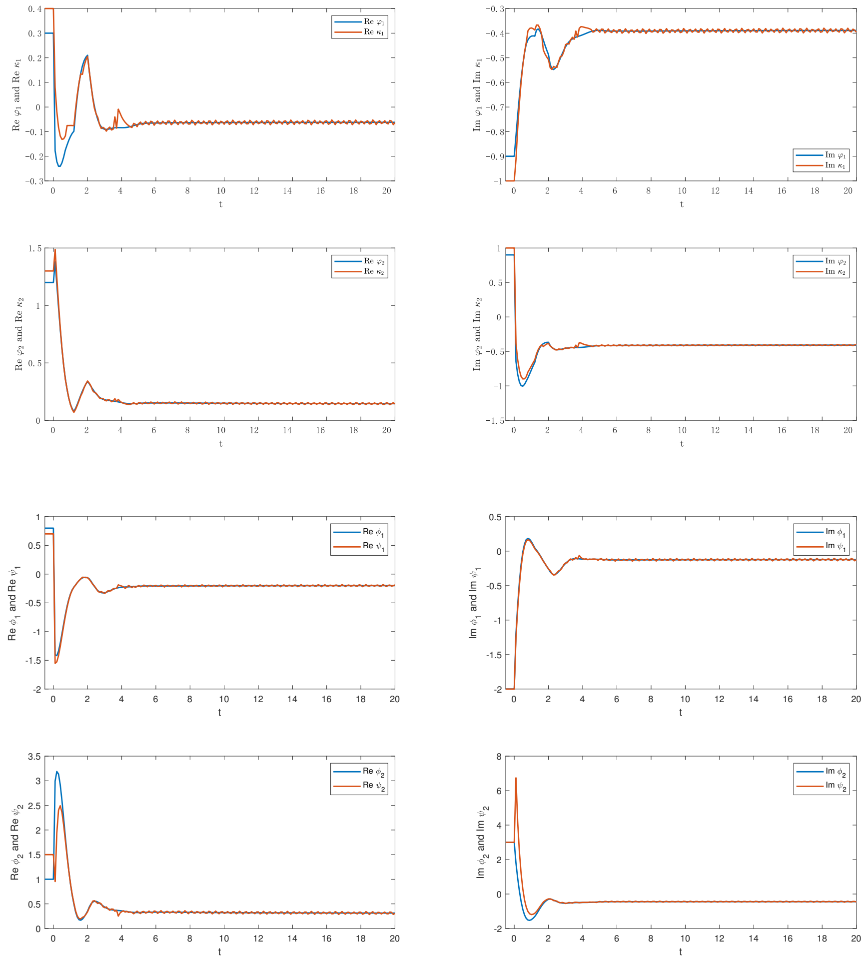

Figure 2. State trajectories of (40) and (41) under controller (9).

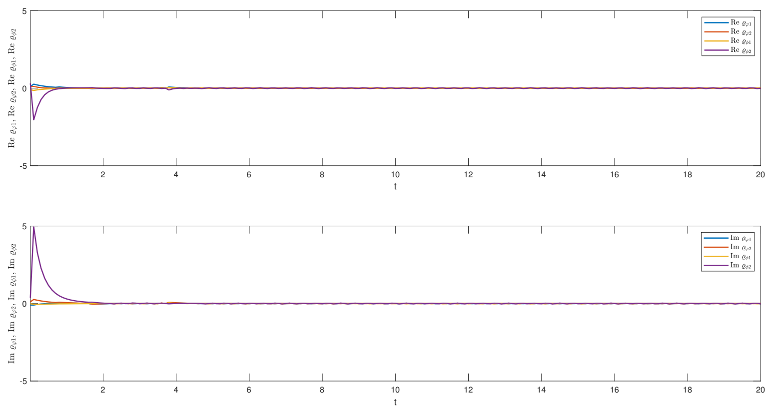

Figure 3. State trajectories of error system under controller (9).

6.2. Synchronization by Adaptive Controller (23)

In the controller (23), let $\theta_1 = \theta_2 = 0.03 + 0.8i$, $\widetilde{\theta_1} = \widetilde{\theta_2} = 0.02 - 0.7i$, $\delta_1 = \delta_2 = 0.4 + i$, $\widetilde{\delta_1} = \widetilde{\delta_2} = 7 + 3i$, $q_1 = q_2 = 9 + 3i$, and $g_1 = g_2 = 2 + 4i$. Similarly, according to the conditions of Theorem 2, in the controller (23), the synchronization of the drive-response system in finite time is shown in Figure 4, with an estimated time of $T = 6.7739$, and the trajectory of the synchronization error system is shown in Figure 5.

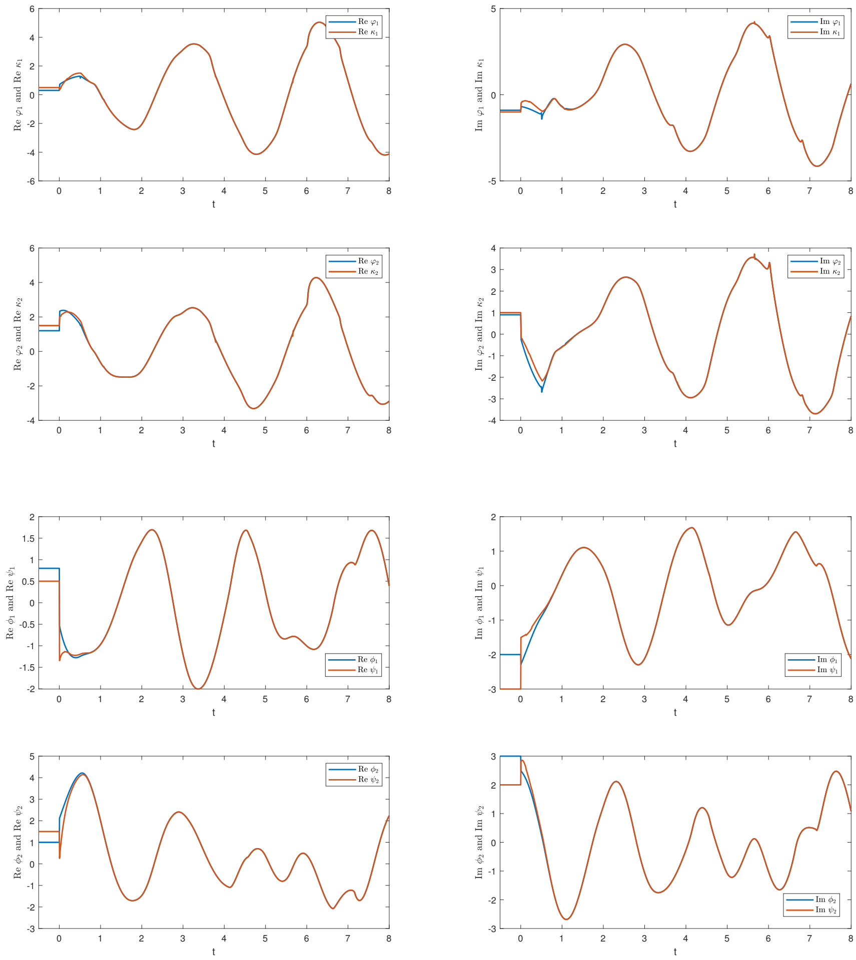

Figure 4. State trajectories of (40) and (41) under controller (23).

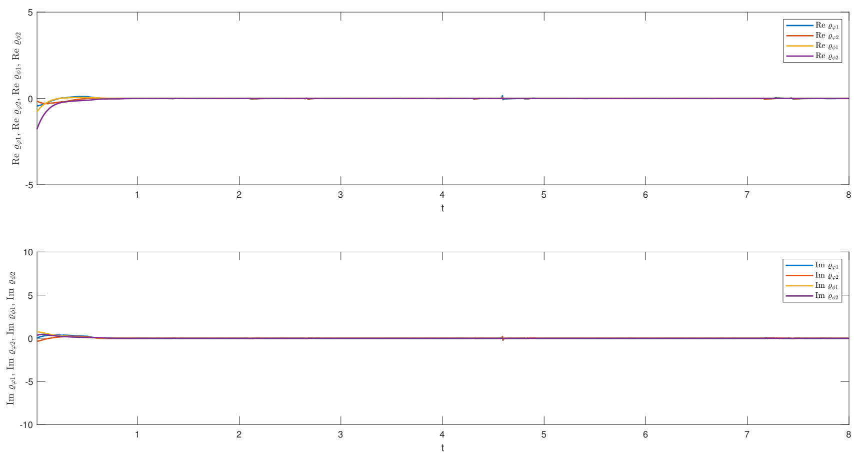

Figure 5. The synchronization trajectories of error system under controller (23).

7. Conclusions

This paper has investigated the problem of synchronization in finite time for fractional-order complex-valued gene regulatory networks with delays. Two different controllers have been designed to address this problem and a complex-valued sign function has been employed to solve this problem directly in a complex field instead of separating the system into two real-valued systems. Then, several appropriate criteria ensuring synchronization in finite time based on the designed controllers by using suitable Lyapunov functions and the stability theory of fractional systems have been presented. Additionally, the effectiveness of these theoretical results has been illustrated through a numerical example, and the simulation results have demonstrated that the instant of complete synchronization for the considered models based on the proposed adaptive controller is shorter than that based on the feedback controller.

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