



## Article

# On Coupled System of Langevin Fractional Problems with Different Orders of $\mu$ -Caputo Fractional Derivatives

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**Abstract:** In this paper, we study coupled nonlinear Langevin fractional problems with different orders of  $\mu$ -Caputo fractional derivatives on arbitrary domains with nonlocal integral boundary conditions. In order to ensure the existence and uniqueness of the solutions to the problem at hand, the tools of the fixed-point theory are applied. An overview of the main results of this study is presented through examples.

**Keywords:** Langevin problems; coupled system; integral boundary conditions; fixed-point theorems; existence; uniqueness;  $\mu$ -Caputo fractional derivatives

**MSC:** 34A08; 34G20; 26A33



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## 1. Introduction

Fractional differential equations (FDEs) have gained a lot of attention in recent years due to their numerous applications in engineering, physics, biology, chemistry, and other fields (see, for instance, [1–6], and the references therein for more information on the boundary value issues of FDEs and inclusions subject to diverse boundary conditions). Differential inclusion and differential equations are thought to be particularly helpful when studying dynamical systems and stochastic processes (see [7–10] for some recent related results).

The Langevin equation successfully captures Brownian motion when the random fluctuation force is assumed to be white noise. Otherwise, the extended Langevin equation represents the particle motion (see, e.g., [11,12]). In fractal media, Langevin's equation has become widely used to represent dynamical operations (see [13–18] for more recent interesting results). In [19], the authors utilized the fractional Langevin equation to recreate Brownian motion. By applying both fluctuation–dissipation theorems and fractional calculus techniques, they derived analytical expressions for the correlation functions. The fractional Langevin equation has drawn the attention of numerous researchers due to its wide-ranging applications in various fields such as physics, chemistry, biology, aerodynamics, economics, control theory, biophysics, signal and image processing, fitting of experimental data, blood flow phenomena, and others. Moreover, it has been studied under various conditions. Moreover, it has been studied under various conditions (see e.g., [20–28]). Due to its numerous applications, the coupled system of differential equations with fractional order is regarded as a crucial and worthwhile topic of study (see

e.g., [29,30]). It is important to note that the majority of the research on the coupled systems of FDEs focuses on fixed domains. The mixed-order coupled system offers a more comprehensive approach and is a valuable addition to the existing literature.

The ability to describe a variety of physical and technical systems, including viscoelastic materials, diffusion and transport processes, and electromagnetic phenomena, necessitates the study of FDEs on arbitrary domains. The domain on which the system is defined is frequently irregular or has a complicated border rather than a straightforward geometric shape. In contrast to conventional integer-order differential equations, we are able to model these systems more precisely and successfully by using fractional differential operators on arbitrary domains. Additionally, the study of FDEs on arbitrary domains results in the creation of fresh analytical and numerical approaches for resolving these equations, which have applications in materials science, biology, finance, and control theory.

A new class of coupled FDEs of different orders with nonlocal multi-point boundary conditions was studied in [31]. Since then, numerous studies have focused on these types of systems of equations, including [32–34]. In the latter, the authors focused on the study of a coupled system of FDEs of Caputo type with different derivatives orders, which inspired us to study coupled systems of Langevin fractional problems with different orders of  $\mu$ -Caputo fractional derivatives with nonlocal integral boundary conditions of the form:

$${}^c D^{\gamma_i, \mu} ({}^c D^{\sigma_i, \mu} + \alpha_i) \varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b], i = 1, 2. \quad (1)$$

subject to the specific boundary conditions

$$\begin{aligned} \varphi_i(a) &= 0, & I^{\vartheta_i, \mu} \varphi_i(b) &= 0, \\ {}^c D^{\sigma_1, \mu} \varphi_1(a) &= \kappa \int_a^\xi \varphi_2(s) ds. \end{aligned} \quad (2)$$

where for  $i = 1, 2$ ,  ${}^c D^{\gamma_i, \mu}$  and  ${}^c D^{\sigma_i, \mu}$  are  $\mu$ -Caputo fractional derivatives,  $0 < \sigma_i, \gamma_2 < 1$ ,  $1 < \gamma_1 \leq 2$ ,  $\alpha_i, \kappa \in \mathbb{R}$ .

As indicated in the above system, we present this study with a  $\mu$ -Caputo fractional derivative operator (FDO), which is a generalization of the Riemann–Liouville FDO. Below, we highlight some of its advantages, which have been discussed in various research papers and articles in the field of fractional calculus and its applications (see e.g., [1,2,35–37]):

- **Flexibility:** The  $\mu$ -Caputo FDO is more flexible than the Riemann–Liouville FDO because it allows for the use of different kernels (functions that define the fractional derivative), depending on the application.
- **Smoothing property:** The  $\mu$ -Caputo FDO has a smoothing property that can be used to eliminate noise from a signal or image. This property makes it useful in image processing, signal processing, and other applications where noise reduction is important.
- **Nonlocality:** The  $\mu$ -Caputo FDO is nonlocal, meaning that the value of the derivative at a point depends on the values of the function at all other points. This property allows for the detection of long-range correlations in data, which can be useful for studying complex systems.
- **Fractional order:** The  $\mu$ -Caputo FDO allows for the use of non-integer orders, which can be used to model phenomena that do not conform to integer-order models. This property makes it efficient for various applications such as physics, engineering, and other fields where non-integer orders are needed to accurately model systems.
- **Numerical methods:** The  $\mu$ -Caputo FDO can be efficiently computed using numerical methods, which makes it useful for computer simulations and other applications where analytical solutions are not available.

The structure of the rest of this paper is as follows. Section 2 outlines the fundamental principles of fractional calculus and defines the key terms and symbols. In Section 3, we present the main finding for fractional differential derivatives. Section 4 discusses the use of the Leray–Schauder alternative and Krasnoselskii’s theorem to establish the existence

of a solution. In contrast, in Section 5, we prove that there is a unique solution based on Banach's contraction mapping principle. Section 6 includes examples that illustrate the key points of our study and the last section represents the conclusions.

## 2. Preliminaries

In this section, we introduce a number of fundamental concepts and relevant lemmas in fractional calculus. Let  $\mathfrak{J} = [a, b]$ . We define  $\mathfrak{C} = \mathcal{C}(\mathfrak{J}, \mathbb{R})$  as the Banach space of all continuous functions  $g : \mathfrak{J} \rightarrow \mathbb{R}$  with the norm

$$\|g\| = \sup\{|g(t)| : t \in \mathfrak{J}\},$$

and we represent the Banach space of Lebesgue-integrable functions  $g : \mathfrak{J} \rightarrow \mathbb{R}$  by  $\mathbf{L}^1(\mathfrak{J}, \mathbb{R})$  with the norm

$$\|g\|_{\mathbf{L}^1} = \int_{\mathfrak{J}} |g(t)| dt.$$

Assume that  $g : \mathfrak{J} \rightarrow \mathbb{R}$  is integrable and that  $\mu \in \mathcal{C}^m(\mathfrak{J}, \mathbb{R})$  is increasing such that  $\mu'(t) \neq 0$  for every  $t$  included in  $\mathfrak{J}$ .

**Definition 1** ([38]). The  $q$ - $\mu$ -Riemann–Liouville integral of a function  $g$  is defined as

$$I_{a+}^{q;\mu} g(t) = \frac{1}{\Gamma(q)} \int_a^t \mu'(\sigma) (\mu(t) - \mu(\sigma))^{q-1} g(\sigma) d\sigma, \quad q > 0.$$

**Definition 2** ([38]). The  $q$ - $\mu$ -Riemann–Liouville fractional derivative of a function  $g$  is

$$D_{a+}^{q;\mu} g(t) = \left( \frac{1}{\mu'(t)} \frac{d}{dt} \right)^m I_{a+}^{(m-q);\mu} g(t),$$

where  $m = [q] + 1$ .

**Definition 3** ([38]). For a function  $g \in \mathcal{AC}^m(\mathfrak{J}, \mathbb{R})$ , the  $\mu$ -fractional derivative of order  $q$  in a Caputo sense is given as follows

$${}^C D_{a+}^{q;\mu} g(t) = I_{a+}^{(m-q);\mu} g^{[m]}(t),$$

where  $g^{[m]}(t) = \left( \frac{1}{\mu'(t)} \frac{d}{dt} \right)^m g(t)$  and  $m = [q] + 1, m \in \mathbb{N}$ .

**Lemma 4** ([38]). Let  $q_1, q_2 > 0$ . Then:

1.  $I_{a+}^{q_1;\mu} (\mu(\sigma) - \mu(a))^{q_2-1}(t) = \frac{\Gamma(q_2)}{\Gamma(q_1+q_2)} (\mu(t) - \mu(a))^{q_1+q_2-1}$
2.  ${}^C D_{a+}^{q_1;\mu} (\mu(\sigma) - \mu(a))^{q_2-1}(t) = \frac{\Gamma(q_2)}{\Gamma(q_2-q_1)} (\mu(t) - \mu(a))^{q_2-q_1-1}$

**Lemma 5** ([38]). If  $g \in \mathcal{AC}^m(\mathfrak{J}, \mathbb{R})$  and  $q \in (m-1, m)$ ,

$$I_{a+}^{q;\mu} {}^C D_{a+}^{q;\mu} g(t) = g(t) - \sum_{k=0}^{m-1} \frac{g^{[m]}(a^+)}{k!} (\mu(t) - \mu(a))^k.$$

**Lemma 6.** Let  $h_1, h_2 \in C([a, b], \mathbb{R})$ . Then, for  $\alpha_i, \kappa \in \mathbb{R}; i = 1, 2$  the linear-type system

$$\begin{cases} {}^C D^{\gamma_i;\mu} ({}^C D^{\sigma_i;\mu} \varphi_i(t) + \alpha_i) \varphi_i(t) = h_i(t), & t \in [a, b], 0 < \sigma_i, \gamma_2 < 1, 1 < \gamma_1 \leq 2, \\ \varphi_i(a) = 0, & I^{\vartheta_i;\mu} \varphi_i(b) = 0, \\ {}^C D^{\sigma_1;\mu} \varphi_1(a) = \kappa \int_a^\xi \varphi_2(s) ds, \end{cases}$$

has a unique solution, which is

$$\begin{aligned} \varphi_1(t) = & I^{\sigma_1+\gamma_1,\mu} h_1(t) - \alpha_1 I^{\sigma_1,\mu} \varphi_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \left[ I^{\vartheta_1+\sigma_1+\gamma_1,\mu} h_1(b) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu} \varphi_1(b) \right] \\ & + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t) \int_a^\xi \varphi_2(s) ds, \end{aligned} \quad (3)$$

and

$$\varphi_2(t) = I^{\sigma_2+\gamma_2,\mu} h_2(t) - \alpha_2 I^{\sigma_2,\mu} \varphi_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} \left[ I^{\vartheta_2+\sigma_2+\gamma_2,\mu} h_2(b) - \alpha_2 I^{\vartheta_2+\sigma_2,\mu} \varphi_2(b) \right].$$

where

$$\begin{aligned} \Delta_t &= \mu(t) - \mu(a), \\ \Lambda_1 &= \frac{\Gamma(\vartheta_1 + \sigma_1 + 2)}{\Delta_b^{\vartheta_1 + \sigma_1 + 1}}, \Lambda_2 = \frac{\Gamma(\vartheta_2 + \sigma_2 + 1)}{\Delta_b^{\vartheta_2 + \sigma_2}}, \\ \varsigma_1(t) &= \frac{(\sigma_1 + 1)\Delta_b - (\vartheta_1 + \sigma_1 + 1)\Delta_t}{(\sigma_1 + 1)\Delta_b}. \end{aligned}$$

**Proof.** From Lemma [2], we have

$$\varphi_i = I^{\sigma_i+\gamma_i,\mu} h_i(t) - \alpha_i I^{\sigma_i,\mu} \varphi_i(t) + c_{i0} + \sum_{k=1}^n c_{ik} \frac{\Delta_t^{\sigma_i+k-1}}{\Gamma(\sigma_i+k)}, \quad i = 1, 2, n = [\gamma_i] + 1. \quad (4)$$

From  $\varphi_i(a) = 0$  and  ${}^c D^{\sigma_1,\mu} \varphi_1(0) = \kappa \int_0^\xi \varphi_2(s) ds$ , we find that  $c_{i0} = 0$ ,  $c_{11} = \kappa \int_0^\xi \varphi_2(s) ds$ , and the last two conditions enable us to directly obtain

$$\begin{aligned} c_{12} &= -\Lambda_1 \left[ I^{\vartheta_1+\sigma_1+\gamma_1,\mu} h_1(b) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu} \varphi_1(b) \right] - \frac{\kappa(\vartheta_1 + \sigma_1 + 1)}{\Delta_b} \int_a^\xi \varphi_2(s) ds, \\ c_{21} &= -\Lambda_2 \left[ I^{\vartheta_2+\sigma_2+\gamma_2,\mu} h_2(b) - \alpha_2 I^{\vartheta_2+\sigma_2,\mu} \varphi_2(b) \right]. \end{aligned}$$

By substituting in Equation (4), we obtain the desired result.  $\square$

### 3. Main Results

Let  $\mathfrak{C} = \mathcal{C}([a, b], \mathbb{R})$  denote the Banach space of all continuous functions from  $[a, b]$  to  $\mathbb{R}$ . Let us introduce the space  $\mathfrak{X} = \{u(t) | u(t) \in C([a, b])\}$  endowed with the norm  $\|u(t)\| = \sup \{|u(t)|, t \in [a, b]\}$ . Obviously,  $(\mathfrak{X}, \|\cdot\|)$  is a Banach space. The product space  $(\mathfrak{X} \times \mathfrak{X}, \|(u, v)\|)$  is a Banach space with the norm  $\|(u, v)\| = \|u\| + \|v\|$ .

According to Lemma 6, an operator  $\mathfrak{K} : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$  is defined as follows:

$$\mathfrak{K} = \begin{pmatrix} \check{\mathfrak{O}}(\varphi_1, \varphi_2)(t) \\ \wp(\varphi_1, \varphi_2)(t) \end{pmatrix}, \quad (5)$$

where

$$\begin{aligned} \check{\mathfrak{O}}(\varphi_1, \varphi_2)(t) = & I^{\sigma_1+\gamma_1,\mu} h_1(t, \varphi_1(t), \varphi_2(t)) - \alpha_1 I^{\sigma_1,\mu} \varphi_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \\ & \left[ I^{\vartheta_1+\sigma_1+\gamma_1,\mu} h_1(b, \varphi_1(b), \varphi_2(b)) - \alpha_1 I^{\vartheta_1+\sigma_1,\mu} \varphi_1(b) \right] + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t) \int_a^\xi \varphi_2(s) ds, \end{aligned} \quad (6)$$

and

$$\wp(\varphi_1, \varphi_2)(t) = I^{\sigma_2+\gamma_2, \mu} h_2(t, \varphi_1(t), \varphi_2(t)) - \alpha_2 I^{\sigma_2, \mu} \varphi_2(t) - \Lambda_2 \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} \\ \left[ I^{\vartheta_2+\sigma_2+\gamma_2, \mu} h_2(b, \varphi_1(b), \varphi_2(b)) - \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b) \right]. \quad (7)$$

We simplify the notations using the following constants:

$$\Phi_1 = \left[ \frac{\Delta_b^{\sigma_1+\gamma_1}}{\Gamma(\sigma_1+\gamma_1+1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1+\gamma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)} \right] \quad (8)$$

$$\Phi_2 = \left[ \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)} \right] \quad (9)$$

$$\Phi_3 = \left[ \kappa |\zeta_1| \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} (\xi - a) \right] \quad (10)$$

$$\Psi_1 = \left[ \frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)} \right] \quad (11)$$

and

$$\Psi_2 = \left[ \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)} \right] \quad (12)$$

#### 4. Existence Results

Fixed-point theorems have recently played a vital role in proving many interesting results (see, e.g., [39–41]).

**Lemma 7** ([42]). *Let  $\mathcal{W}$  be a closed convex and nonempty subset of a Banach space  $E$ . Let  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  be two operators such that:*

1.  $\mathfrak{F}_1 \mathcal{X} + \mathfrak{F}_2 \mathcal{Y} \in \mathcal{W}, \mathcal{X}, \mathcal{Y} \in \mathcal{W}$ ,
2.  $\mathfrak{F}_1$  is compact and continuous on  $\mathcal{W}$ ,
3.  $\mathfrak{F}_2$  is a contraction mapping on  $\mathcal{W}$ .

*Then, there exists  $\mathcal{Z} \in \mathcal{W}$  such that  $\mathcal{Z} = \mathfrak{F}_1 \mathcal{Z} + \mathfrak{F}_2 \mathcal{Z}$ .*

**Theorem 8.** *Suppose that the following conditions are satisfied:*

- (C<sub>1</sub>)  $|h_1(t, \varphi_1, \varphi_2)| \leq \mathfrak{w}_1(t)$
- (C<sub>2</sub>)  $|h_2(t, \varphi_1, \varphi_2)| \leq \mathfrak{w}_2(t)$

*If*

$$\mathfrak{m} = \max\{|\alpha_1| \Phi_2, \Phi_3 + |\alpha_2| \Psi_2\} \leq 1 \quad (13)$$

*where  $\Phi_2, \Phi_3$ , and  $\Psi_2$  are defined by (9), (10), and (12). Then, Problems (1) and (2) have at least one solution for  $[a, b]$ .*

**Proof.** To prove our results, we set  $\sup_{t \in [a, b]} |\mathfrak{w}_1(t)| = \|\mathfrak{w}_1\|$ ,  $\sup_{t \in [a, b]} |\mathfrak{w}_2(t)| = \|\mathfrak{w}_2\|$  and chose

$$\mathfrak{r} \geq \frac{\|\mathfrak{w}_1\| \Phi_1 + \|\mathfrak{w}_2\| \Psi_1}{1 - \mathfrak{m}} \quad (14)$$

where  $\Phi_1$  and  $\Psi_1$  are defined by (8) and (11). Let  $\mathfrak{B}_\tau = \{(\varphi_1, \varphi_2) \in \mathfrak{X} \times \mathfrak{X} : \|(\varphi_1, \varphi_2)\| \leq \tau\}$ . Now, we represent the four operators as follows:

$$\begin{aligned}\bar{\partial}_1(\varphi_1, \varphi_2)(t) &= I^{\sigma_1+\gamma_1, \mu} \hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1, \mu} \hat{h}_1(b), \\ \bar{\partial}_2(\varphi_1, \varphi_2)(t) &= -\alpha_1 I^{\sigma_1, \mu} \varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \alpha_1 I^{\vartheta_1+\sigma_1, \mu} \varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t) \int_a^\xi \varphi_2(s) ds, \\ \wp_1(\varphi_1, \varphi_2)(t) &= I^{\sigma_2+\gamma_2, \mu} \hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2, \mu} \hat{h}_2(b), \\ \wp_2(\varphi_2)(t) &= -\alpha_2 I^{\sigma_2, \mu} \varphi_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b).\end{aligned}$$

where  $\hat{h}_i(\tau) = \hat{h}_i(\tau, \varphi_1(\tau), \varphi_2(\tau))$ ,  $i = 1, 2$ , and

$$\mathfrak{K}_1 = \begin{pmatrix} \bar{\partial}_1(\varphi_1, \varphi_2)(t) \\ \wp_1(\varphi_1, \varphi_2)(t) \end{pmatrix}, \quad \mathfrak{K}_2 = \begin{pmatrix} \bar{\partial}_2(\varphi_1, \varphi_2)(t) \\ \wp_2(\varphi_2)(t) \end{pmatrix}. \quad (15)$$

Note that  $\bar{\partial} = \bar{\partial}_1 + \bar{\partial}_2$ ,  $\wp = \wp_1 + \wp_2$  and  $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$ :

$$\begin{aligned}|\bar{\partial}_1(\varphi_1, \varphi_2) + \bar{\partial}_2(\varphi_1, \varphi_2)| &= \left| I^{\sigma_1+\gamma_1, \mu} \hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1, \mu} \hat{h}_1(b) \right. \\ &\quad \left. - \alpha_1 I^{\sigma_1, \mu} \varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \alpha_1 I^{\vartheta_1+\sigma_1, \mu} \varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t) \int_a^\xi \varphi_2(s) ds \right| \\ &\leq \|\mathfrak{w}_1\| \left[ \frac{\Delta_b^{\sigma_1+\gamma_1}}{\Gamma(\sigma_1+\gamma_1+1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1+\gamma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)} \right] + \|\varphi_1\| |\alpha_1| \left[ \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} \right. \\ &\quad \left. + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)} \right] + \|\varphi_2\| \left[ \kappa |\varsigma_1| \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} (\xi - a) \right] \\ &\leq \|\mathfrak{w}_1\| \Phi_1 + \|\varphi_1\| |\alpha_1| \Phi_2 + \|\varphi_2\| \Phi_3.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}|\wp_1(\varphi_1, \varphi_2) + \wp_2(\varphi_2)| &= \left| I^{\sigma_2+\gamma_2, \mu} \hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2, \mu} \hat{h}_2(b) \right. \\ &\quad \left. - \alpha_2 I^{\sigma_2, \mu} \varphi_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b) \right| \\ &\leq \|\mathfrak{w}_2\| \left[ \frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)} \right] \\ &\quad + \|\varphi_2\| |\alpha_2| \left[ \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)} \right] \\ &\leq \|\mathfrak{w}_2\| \Psi_1 + \|\varphi_2\| |\alpha_2| \Psi_2.\end{aligned}$$

which implies that  $\|\mathfrak{K}_1 + \mathfrak{K}_2\| \leq \tau$ . This shows that  $\mathfrak{K}_1 + \mathfrak{K}_2 \in \mathfrak{B}_\tau$ . For  $(\varphi_1, \varphi_2), (\varphi_1^*, \varphi_2^*) \in \mathfrak{X} \times \mathfrak{X}$  and  $t \in [a, b]$ , we have

$$\|\bar{\partial}_2(\varphi_1, \varphi_2) - \bar{\partial}_2(\varphi_1^*, \varphi_2^*)\| \leq |\alpha_1| \Phi_2 \|\varphi_1 - \varphi_1^*\| + \Phi_3 \|\varphi_2 - \varphi_2^*\|,$$

and

$$\|\wp_2(\varphi_2) - \wp_2(\varphi_2^*)\| \leq |\alpha_2| \Psi_2 \|\varphi_2 - \varphi_2^*\|.$$

Thus,

$$\|\mathfrak{K}_2(\varphi_1, \varphi_2) - \mathfrak{K}_2(\varphi_1^*, \varphi_2^*)\| \leq \mathfrak{m}\|\varphi_1 - \varphi_1^*\| + \mathfrak{m}\|\varphi_2 - \varphi_2^*\| = \mathfrak{m}\|(\varphi_1 - \varphi_1^*, \varphi_2 - \varphi_2^*)\|,$$

which implies that  $\mathfrak{K}_2$  is a contraction mapping by (13). The continuity of  $h_i$ ,  $i = 1, 2$  implies that the operator  $\mathfrak{K}_1$  is continuous. In addition,  $\mathfrak{K}_1$  is uniformly bounded on  $\mathfrak{B}_\tau$  as

$$\|\tilde{\mathfrak{O}}_1(\varphi_1, \varphi_2)\| \leq \|\mathfrak{w}_1\|\Phi_1, \quad \text{and} \quad \|\wp_1(\varphi_1, \varphi_2)\| \leq \|\mathfrak{w}_2\|\Psi_1.$$

Thus,

$$\|\mathfrak{K}_1(\varphi_1, \varphi_2)\| \leq \|\mathfrak{w}_1\|\Phi_1 + \|\mathfrak{w}_2\|\Psi_1.$$

Next, we prove the compactness of the operator  $\mathfrak{K}_1$ . Let  $t_1, t_2 \in [a, b]$  with  $t_1 < t_2$ . Then, we obtain

$$\begin{aligned} & |\tilde{\mathfrak{O}}_1(\varphi_1, \varphi_2)(t_2) - \tilde{\mathfrak{O}}_1(\varphi_1, \varphi_2)(t_1)| \\ & \leq \left| I^{\sigma_1+\gamma_1, \mu} \hat{h}_1(t_2) - \Lambda_1 \frac{\Delta_{t_2}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1, \mu} \hat{h}_1(b) - I^{\sigma_1+\gamma_1, \mu} \hat{h}_1(t_1) + \Lambda_1 \frac{\Delta_{t_1}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} I^{\vartheta_1+\sigma_1+\gamma_1, \mu} \hat{h}_1(b) \right| \\ & \leq \|\mathfrak{w}_1\| \left[ \frac{1}{\Gamma(\sigma_1+\gamma_1+1)} (\Delta_{t_2}^{\sigma_1+\gamma_1} - \Delta_{t_1}^{\sigma_1+\gamma_1}) + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+\sigma_1+\gamma_1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+\gamma_1+1)} (\Delta_{t_2}^{\sigma_1+1} - \Delta_{t_1}^{\sigma_1+1}) \right], \end{aligned}$$

and

$$\begin{aligned} & |\wp_1(\varphi_1, \varphi_2)(t_2) - \wp_1(\varphi_1, \varphi_2)(t_1)| \\ & \leq \left| I^{\sigma_2+\gamma_2, \mu} \hat{h}_2(t_2) - \Lambda_2 \frac{\Delta_{t_2}^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2, \mu} \hat{h}_2(b) - I^{\sigma_2+\gamma_2, \mu} \hat{h}_2(t_1) + \Lambda_2 \frac{\Delta_{t_1}^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2, \mu} \hat{h}_2(b) \right| \\ & \leq \|\mathfrak{w}_2\| \left[ \frac{1}{\Gamma(\sigma_2+\gamma_2+1)} (\Delta_{t_2}^{\sigma_2+\gamma_2} - \Delta_{t_1}^{\sigma_2+\gamma_2}) + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)} (\Delta_{t_2}^{\sigma_2} - \Delta_{t_1}^{\sigma_2}) \right]. \end{aligned}$$

As  $t_1 \rightarrow t_2$ , we have  $|\mathfrak{K}_1 - \mathfrak{K}_2| \rightarrow 0$ . Hence,  $\mathfrak{K}_1$  is equicontinuous. By the Arzelà–Ascoli theorem,  $\mathfrak{K}_1$  is compact.  $\square$

**Theorem 9.** Let  $h_1, h_2 : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions. Suppose that (13) holds. Additionally, we assume that:

( $\mathcal{H}_1$ ) there exist a non-negative function  $\mathfrak{z}_1(t)$ ,  $\mathfrak{z}_1(t) \in C([a, b], \mathbb{R})$  and nondecreasing functions  $\psi_1, \psi_2$

$$|h_1(t, (\varphi_1, \varphi_2))| \leq [\|\mathfrak{z}_1\|\psi_1(\tau) + \|\mathfrak{z}_2\|\psi_2(\tau)]$$

( $\mathcal{H}_2$ ) there exist a non-negative function  $\mathfrak{s}_1(t)$ ,  $\mathfrak{s}_2(t) \in C([a, b], \mathbb{R})$  and nondecreasing functions  $\chi_1, \chi_2$

$$|h_2(t, (\varphi_1, \varphi_2))| \leq [\|\mathfrak{s}_1\|\chi_1(\tau) + \|\mathfrak{s}_2\|\chi_2(\tau)]$$

Then, the problem in (1) has at least one solution for  $[a, b]$ .

**Proof.** Observe that the continuity of the operator  $\mathfrak{K} : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$  follows that of the functions  $h_1$  and  $h_2$ . Next, let  $\Omega_\tau \subset \mathfrak{X} \times \mathfrak{X}$  be bounded so we need to prove some steps.

The set  $\mathfrak{K}(\Omega_\tau)$  is bounded. We first show that  $\mathfrak{K}_1$  is bounded. For any  $(\varphi_1, \varphi_2) \in \Omega_\tau$ , we have

$$\|\tilde{\mathfrak{O}}_1(\varphi_1, \varphi_2)\| \leq [\|\mathfrak{z}_1\|\psi_1(\tau) + \|\mathfrak{z}_2\|\psi_2(\tau)]\Phi_1,$$

and

$$|\wp_1(\varphi_1, \varphi_2)| \leq [\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})]\Psi_1,$$

This proves that  $\mathfrak{K}_1$  is uniformly bounded. Similarly, we have

$$\|\mathfrak{K}_2(\varphi_1, \varphi_2)\| \leq \mathfrak{m}\mathfrak{r}.$$

In this step, to show that  $\mathfrak{K}$  is equicontinuous, we only have to prove that  $\mathfrak{K}_2$  is equicontinuous (in the previous theorem we proved that  $\mathfrak{K}_1$  was equicontinuous). Let  $t_2, t_1 \in [a, b]$  with  $t_2 < t_1$ . Then, we have

$$\begin{aligned} & |\tilde{\mathfrak{O}}_2(\varphi_1, \varphi_2)(t_2) - \tilde{\mathfrak{O}}_2(\varphi_1, \varphi_2)(t_1)| \\ &= \left| -\alpha_1 I^{\sigma_1, \mu} \varphi_1(t_2) + \Lambda_1 \frac{\Delta_{t_2}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \alpha_1 I^{\vartheta_1+\sigma_1, \mu} \varphi_1(b) + \kappa \frac{\Delta_{t_2}^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t_2) \int_a^{\xi} \varphi_2(s) ds \right. \\ &\quad \left. - [-\alpha_1 I^{\sigma_1, \mu} \varphi_1(t_1) + \Lambda_1 \frac{\Delta_{t_1}^{\sigma_1+1}}{\Gamma(\sigma_1+2)} \alpha_1 I^{\vartheta_1+\sigma_1, \mu} \varphi_1(b) + \kappa \frac{\Delta_{t_1}^{\sigma_1}}{\Gamma(\sigma_1+1)} \varsigma_1(t_1) \int_a^{\xi} \varphi_2(s) ds] \right| \\ &\leq |\alpha_1| |\Lambda_1| \frac{\Delta_b^{\sigma_1+1} (\Delta_{t_2}^{\vartheta_1+\sigma_1} - \Delta_{t_1}^{\vartheta_1+\sigma_1})}{\Gamma(\sigma_1+2) \Gamma(\vartheta_1+\sigma_1+1)} + [|\alpha_1| + \kappa |\varsigma_1(t_2) - \varsigma_1(t_1)| (\xi - a)] \frac{(\Delta_{t_2}^{\sigma_1} - \Delta_{t_1}^{\sigma_1})}{\Gamma(\sigma_1+1)}. \end{aligned}$$

In a similar manner, we can obtain

$$\begin{aligned} & |\wp_2(\varphi_2)(t_2) - \wp_2(\varphi_2)(t_1)| \\ &= | -\alpha_2 I^{\sigma_2, \mu} \varphi_2(t_2) + \Lambda_2 \frac{\Delta_{t_2}^{\sigma_2}}{\Gamma(\sigma_2+1)} \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b) \\ &\quad - [-\alpha_2 I^{\sigma_2, \mu} \varphi_2(t_1) + \Lambda_2 \frac{\Delta_{t_1}^{\sigma_2}}{\Gamma(\sigma_2+1)} \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b)] | \\ &\leq \left[ \frac{|\alpha_2| (\Delta_{t_2}^{\sigma_2} - \Delta_{t_1}^{\sigma_2})}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\sigma_2} (\Delta_{t_2}^{\vartheta_2+\sigma_2} - \Delta_{t_1}^{\vartheta_2+\sigma_2})}{\Gamma(\sigma_2+1) \Gamma(\vartheta_2+\sigma_2+1)} \right] \end{aligned}$$

In the last step, it is verified that the set  $\Pi = \{(\varphi_1, \varphi_2) \in \mathfrak{X} \times \mathfrak{X} : (\varphi_1, \varphi_2) = \delta \mathfrak{K}(\varphi_1, \varphi_2)\}$  is bounded. Let  $(\varphi_1, \varphi_2) \in \Pi$  with  $(\varphi_1, \varphi_2) = \delta \mathfrak{K}$  so we have

$$\varphi_1(t) = \delta [\tilde{\mathfrak{O}}_1(\varphi_1, \varphi_2)(t) + \tilde{\mathfrak{O}}_2(\varphi_1, \varphi_2)(t)],$$

and

$$\varphi_2(t) = \delta [\wp_1(\varphi_1, \varphi_2)(t) + \wp_2(\varphi_2)(t)].$$

Then,

$$\|\varphi_1\| \leq [\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})]\Phi_1 + \|\varphi_1\| |\alpha_1| \Phi_2 + \|\varphi_2\| \Phi_3,$$

and

$$\|\varphi_2\| \leq [\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})]\Psi_1 + \|\varphi_2\| |\alpha_2| \Psi_2,$$

As a consequence, this implies that

$$\|(\varphi_1, \varphi_2)\| \leq \frac{[\|\mathfrak{z}_1\|\psi_1(\mathfrak{r}) + \|\mathfrak{z}_2\|\psi_2(\mathfrak{r})]\Phi_1 + [\|\mathfrak{s}_1\|\chi_1(\mathfrak{r}) + \|\mathfrak{s}_2\|\chi_2(\mathfrak{r})]\Psi_1}{1 - \mathfrak{m}}$$



By using this result, it can be established that set  $\Pi$  is bounded as a result of the Leray–Schauder alternative [43]. As a result, at least one solution exists for Systems (1) and (2).  $\square$

### 5. Uniqueness Result

**Theorem 10.** Suppose that: the functions  $h_1, h_2 : [a, b] \times \mathbb{R} \times \mathbb{R}$  are continuous functions and  $\ell_1, \ell_2$  are positive constants such that for all  $t \in [a, b]$  and  $\varphi_i^* \in \mathbb{R}$ , we have:

$$\begin{aligned} \|h_1(t, \varphi_1, \varphi_2) - h_1(t, \varphi_1^*, \varphi_2^*)\| &\leq \ell_1(\|\varphi_1 - \varphi_1^*\| + \|\varphi_2 - \varphi_2^*\|), \\ \|h_2(t, \varphi_1, \varphi_2) - h_2(t, \varphi_1^*, \varphi_2^*)\| &\leq \ell_2(\|\varphi_1 - \varphi_1^*\| + \|\varphi_2 - \varphi_2^*\|), \end{aligned}$$

If

$$\mathfrak{m} + \ell_1 \Phi_1 + \ell_2 \Psi_1 < 1, \quad (16)$$

Systems (1) and (2) have unique solutions for  $[a, b]$ .

**Proof.** Consider the two assumptions  $\sup_{t \in [a, b]} h_1(t, 0, 0) = \mathcal{N}_1$  and  $\sup_{t \in [a, b]} h_2(t, 0, 0) = \mathcal{N}_2$ . Choose a number  $\mathfrak{r}$  that satisfies the condition below.

Here, we prove that  $\mathfrak{K}\mathcal{B}_{\mathfrak{r}} \subset \mathcal{B}_{\mathfrak{r}}$ , where  $\mathcal{B}_{\mathfrak{r}} = \{(u, v) : \| (u, v) \| \leq \mathfrak{r}\}$  and  $\mathfrak{K}$  is defined by (5). Based on assumption (L), for  $(\varphi_1, \varphi_2) \in \mathcal{B}_{\mathfrak{r}}$ , we have:

$$\begin{aligned} |h_1(t, \varphi_1, \varphi_2)| &\leq |h_1(t, \varphi_1, \varphi_2) - h_1(t, 0, 0)| + |h_1(t, 0, 0)| \\ &\leq \ell_1(|\varphi_1| + |\varphi_2|) + \mathcal{N}_1 \\ &\leq \ell_1(\|\varphi_1\| + \|\varphi_2\|) + \mathcal{N}_1 \leq \ell_1 \mathfrak{r} + \mathcal{N}_1, \end{aligned}$$

and

$$\begin{aligned} |h_2(t, \varphi_1, \varphi_2)| &\leq |h_2(t, \varphi_1, \varphi_2) - h_2(t, 0, 0)| + |h_2(t, 0, 0)| \\ &\leq \ell_2(|\varphi_1| + |\varphi_2|) + \mathcal{N}_2 \\ &\leq \ell_2(\|\varphi_1\| + \|\varphi_2\|) + \mathcal{N}_2 \leq \ell_2 \mathfrak{r} + \mathcal{N}_2. \end{aligned}$$

As a result, we obtain

$$\begin{aligned} |\tilde{\mathfrak{d}}_1(\varphi_1, \varphi_2) + \tilde{\mathfrak{d}}_2(\varphi_1, \varphi_2)| &= \left| I^{\sigma_1 + \gamma_1, \mu} \hat{h}_1(t) - \Lambda_1 \frac{\Delta_t^{\sigma_1 + 1}}{\Gamma(\sigma_1 + 2)} I^{\vartheta_1 + \sigma_1 + \gamma_1, \mu} \hat{h}_1(b) \right. \\ &\quad \left. - \alpha_1 I^{\sigma_1, \mu} \varphi_1(t) + \Lambda_1 \frac{\Delta_t^{\sigma_1 + 1}}{\Gamma(\sigma_1 + 2)} \alpha_1 I^{\vartheta_1 + \sigma_1, \mu} \varphi_1(b) + \kappa \frac{\Delta_t^{\sigma_1}}{\Gamma(\sigma_1 + 1)} \varsigma_1(t) \int_a^{\xi} \varphi_2(s) ds \right| \\ &\leq (\ell_1 \mathfrak{r} + \mathcal{N}_1) \left[ \frac{\Delta_b^{\sigma_1 + \gamma_1}}{\Gamma(\sigma_1 + \gamma_1 + 1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1 + 2\sigma_1 + \gamma_1}}{\Gamma(\sigma_1 + 2) \Gamma(\vartheta_1 + \sigma_1 + \gamma_1 + 1)} \right] + \|\varphi_1\| |\alpha_1| \left[ \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1 + 1)} \right. \\ &\quad \left. + |\Lambda_1| \frac{\Delta_b^{\vartheta_1 + 2\sigma_1}}{\Gamma(\sigma_1 + 2) \Gamma(\vartheta_1 + \sigma_1 + 1)} \right] + \|\varphi_2\| \left[ \kappa |\varsigma_1| \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1 + 1)} (\xi - a) \right] \\ &\leq (\ell_1 \mathfrak{r} + \mathcal{N}_1) \Phi_1 + \|\varphi_1\| |\alpha_1| \Phi_2 + \|\varphi_2\| \Phi_3. \end{aligned}$$

and

$$\begin{aligned}
 |\wp_1(\varphi_1, \varphi_2) + \wp_2(\varphi_2)| &= \left| I^{\sigma_2+\gamma_2, \mu} \hat{h}_2(t) - \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} I^{\vartheta_2+\sigma_2+\gamma_2, \mu} \hat{h}_2(b) \right. \\
 &\quad \left. - \alpha_2 I^{\sigma_2, \mu} \varphi_2(t) + \Lambda_2 \frac{\Delta_t^{\sigma_2}}{\Gamma(\sigma_2+1)} \alpha_2 I^{\vartheta_2+\sigma_2, \mu} \varphi_2(b) \right| \\
 &\leq (\ell_2 \mathfrak{r} + \mathcal{N}_2) \left[ \frac{\Delta_b^{\sigma_2+\gamma_2}}{\Gamma(\sigma_2+\gamma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2+\gamma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+\gamma_2+1)} \right] \\
 &\quad + \|\varphi_2\| |\alpha_2| \left[ \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2+1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2+2\sigma_2}}{\Gamma(\sigma_2+1)\Gamma(\vartheta_2+\sigma_2+1)} \right] \\
 &\leq (\ell_2 \mathfrak{r} + \mathcal{N}_2) \Psi_1 + \|\varphi_2\| |\alpha_2| \Psi_2.
 \end{aligned}$$

so

$$\|\mathfrak{K}(\varphi_1, \varphi_2)\| \leq [\ell_1 \Phi_1 + \ell_2 \Psi_1] \mathfrak{r} + [\mathcal{N}_1 \Phi_1 + \mathcal{N}_2 \Psi_1] + \mathfrak{r} \mathfrak{m},$$

and from (16), we obtain  $\|\mathfrak{K}(\varphi_1, \varphi_2)\| \leq \mathfrak{r}$ .

Next, for  $(\varphi_1, \varphi_2), (\varphi_1^*, \varphi_2^*)$ , as we have already established that  $\mathfrak{K}_2$  is a contraction mapping, it is similarly easy to find:

$$\|\mathfrak{K}(\varphi_1, \varphi_2) - \mathfrak{K}(\varphi_1^*, \varphi_2^*)\| \leq [\mathfrak{m} + \ell_1 \Phi_1 + \ell_2 \Psi_1] [\|\varphi_1 - \varphi_1^*\| + \|\varphi_2 - \varphi_2^*\|].$$

Since  $\mathfrak{m} + \ell_1 \Phi_1 + \ell_2 \Psi_1 < 1$ , this indicates that  $\mathfrak{K}$  is a contraction. Accordingly, Problems (1) and (2) have unique solutions based on Banach's contraction mapping principle. The proof is completed.  $\square$

## 6. Examples

**Example 1.** The following fractional Langevin equation system can be considered:

$$\begin{cases} {}^c D^{\frac{8}{7}, t} ({}^c D^{\frac{1}{2}, t} + \frac{1}{10}) \varphi_1(t) = \frac{t \sin 2t \tan^{-1} \varphi_1(t)}{(t+1)(7|\varphi_1(t)|+2)} + \frac{\cos t \sin \varphi_2(t)}{(5t^2+1)(2|\varphi_2(t)|+1)}, & t \in [1, 2], \\ {}^c D^{\frac{1}{3}, t} ({}^c D^{\frac{7}{10}, t} + \frac{1}{5}) \varphi_2(t) = \frac{2t^2(2\varphi_1(t))}{(3t+1)(5|\varphi_1(t)|+1)} + \frac{(2\varphi_2(t)+3)}{(3|\varphi_2(t)|+\frac{9}{2})}, & t \in [1, 2], \\ \varphi_1(1) = 0, & I^{\frac{3}{2}, t} \varphi_1(2) = 0, {}^c D^{\frac{1}{2}, t} \varphi_1(1) = \frac{1}{4} \int_1^{\frac{6}{5}} \varphi_2(s) ds, \\ \varphi_2(1) = 0, & I^{\frac{9}{8}, t} \varphi_2(2) = 0. \end{cases} \quad (17)$$

Here,  $\gamma_1 = \frac{8}{7}, \gamma_2 = \frac{1}{3}, \sigma_1 = \frac{1}{2}, \sigma_2 = \frac{7}{10}, \vartheta_1 = \frac{3}{2}, \vartheta_2 = \frac{9}{8}, \alpha_1 = \frac{1}{10}, \alpha_2 = \frac{1}{5}, \zeta_1 = \frac{6}{5}, \kappa = \frac{1}{4}$ , and

$$h_1(t, \varphi_1, \varphi_2) = \frac{t \sin 2t \tan^{-1} \varphi_1(t)}{(t+1)(7|\varphi_1(t)|+2)} + \frac{\cos t \sin \varphi_2(t)}{(5t^2+1)(2|\varphi_2(t)|+1)},$$

$$h_2(t, \varphi_1, \varphi_2) = \frac{2t^2(2\varphi_1(t))}{(3t+1)(5|\varphi_1(t)|+1)} + \frac{(2\varphi_2(t)+3)}{(3|\varphi_2(t)|+\frac{9}{2})}.$$

Since  $h_1(t, \varphi_1, \varphi_2) \leq \frac{t \sin 2t}{(7t+7)} + \frac{\cos t}{(10t^2+2)}, h_2(t, \varphi_1, \varphi_2) \leq \frac{4t^2}{(15t+5)} + \frac{2}{3}$ . The Maple program can be used to determine the following values:

$$\Phi_2 = \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} + |\Lambda_1| \frac{\Delta_b^{\vartheta_1+2\sigma_1+1}}{\Gamma(\sigma_1+2)\Gamma(\vartheta_1+\sigma_1+1)}$$

$$\approx 3.385137501,$$

$$\Phi_3 = \kappa |\zeta_1| \frac{\Delta_b^{\sigma_1}}{\Gamma(\sigma_1+1)} (\zeta - a)$$

$$\approx 0.1692568750,$$

and

$$\Psi_2 = \frac{\Delta_b^{\sigma_2}}{\Gamma(\sigma_2 + 1)} + |\Lambda_2| \frac{\Delta_b^{\vartheta_2 + 2\sigma_2}}{\Gamma(\sigma_2 + 1)\Gamma(\vartheta_2 + \sigma_2 + 1)}$$

$$\approx 2.201094810.$$

Thus,  $m \approx 0.6094758370 < 1$ . For  $[1, 2]$ , System (17) must have at least one solution according to Theorem 8.

**Example 2.** The following fractional Langevin equation system can be considered.

$$\begin{cases} {}^c D^{\frac{5}{3}, \rho^t} ({}^c D^{\frac{2}{7}, \rho^t} + \frac{5}{9}) \varphi_1(t) = \frac{5}{(3t^2+9)} \left( \frac{\varphi_1^2(t)}{7(|\varphi_1(t)|+1)} + \frac{3}{8} \right) + \frac{1}{(3+t)} \left( \frac{\varphi_2}{3(|\varphi_2(t)|+1)} \right), & t \in [0, \ln 2], \\ {}^c D^{\frac{2}{9}, \rho^t} ({}^c D^{\frac{3}{4}, \rho^t} + \frac{1}{6}) \varphi_2(t) = \frac{\sin t}{(3+t)} \left( \frac{3\varphi_1}{|\varphi_1(t)|+7} \right) + \frac{1}{(5+t^2)} \left( \frac{\varphi_2}{2|\varphi_2(t)|+\frac{5}{3}} \right), & t \in [0, \ln 2], \\ \varphi_1(0) = 0, & I^{\frac{3}{5}, \rho^t} \varphi_1(\ln 2) = 0, {}^c D^{\frac{2}{3}, \rho^t} \varphi_1(0) = \frac{1}{9} \int_0^{\frac{5}{8}} \varphi_2(s) ds, \\ \varphi_2(0) = 0, & I^{\frac{7}{8}, \rho^t} \varphi_2(\ln 2) = 0. \end{cases} \quad (18)$$

Here,  $\gamma_1 = \frac{5}{3}, \gamma_2 = \frac{2}{9}, \sigma_1 = \frac{2}{7}, \sigma_2 = \frac{3}{4}, \vartheta_1 = \frac{3}{5}, \vartheta_2 = \frac{7}{8}, \alpha_1 = \frac{5}{9}, \alpha_2 = \frac{1}{6}, \xi_1 = \frac{5}{8}, \kappa = \frac{1}{9}$ , and

$$h_1(t, \varphi_1, \varphi_2) = \frac{5}{(3t^2+9)} \left( \frac{\varphi_1^2(t)}{7(|\varphi_1(t)|+1)} + \frac{3}{8} \right) + \frac{1}{(3+t)} \left( \frac{\varphi_2}{3(|\varphi_2(t)|+1)} \right),$$

$$h_2(t, \varphi_1, \varphi_2) = \frac{\sin t}{(3+t)} \left( \frac{3\varphi_1}{|\varphi_1(t)|+7} \right) + \frac{1}{(5+t^2)} \left( \frac{\varphi_2}{2|\varphi_2(t)|+\frac{5}{3}} \right).$$

Since  $h_1(t, \varphi_1, \varphi_2) \leq \frac{1}{9} (\frac{5}{7} |\varphi_1| + |\varphi_2| + \frac{3}{8})$ ,  $h_2(t, \varphi_1, \varphi_2) \leq \frac{1}{7} |\varphi_1| + \frac{3}{25} |\varphi_2|$ . The Maple program can be used to determine that  $m = 0.5190783766 < 1$ .

Thus, System (18) must have at least one solution according to Theorem 9.

**Example 3.** The following fractional Langevin equation system can be considered.

$$\begin{cases} {}^c D^{\frac{11}{7}, t^2} ({}^c D^{\frac{2}{3}, t^2} + \frac{2}{11}) \varphi_1(t) = \frac{1}{10} \frac{\varphi_1^2 + |\varphi_1|}{(|\varphi_1|+3)} \sin^2 t + \frac{5}{9} (1 + t \cos t \varphi_2), & t \in [0, 1], \\ {}^c D^{\frac{2}{3}, t^2} ({}^c D^{\frac{1}{2}, t^2} + \frac{3}{10}) \varphi_2(t) = \frac{2}{5} \left( \frac{|\varphi_1|}{(|\varphi_1|+5)} \cos^2 t + 1 \right) + \frac{3}{7} (1 + \frac{t}{1+t} \sin t \varphi_2), & t \in [0, 1], \\ \varphi_1(0) = 0, & I^{\frac{5}{4}, t} \varphi_1(1) = 0, {}^c D^{\frac{1}{2}, t} \varphi_1(1) = \frac{1}{7} \int_1^{\frac{5}{2}} \varphi_2(s) ds, \\ \varphi_2(0) = 0, & I^{\frac{3}{2}, t} \varphi_2(1) = 0. \end{cases} \quad (19)$$

Here,  $\gamma_1 = \frac{11}{7}, \gamma_2 = \frac{2}{3}, \sigma_1 = \frac{2}{3}, \sigma_2 = \frac{1}{2}, \vartheta_1 = \frac{5}{4}, \vartheta_2 = \frac{3}{2}, \alpha_1 = \frac{2}{11}, \alpha_2 = \frac{3}{10}, \xi_1 = \frac{2}{5}, \kappa = \frac{1}{7}$ , and

$$h_1(t, \varphi_1, \varphi_2) = \frac{1}{10} \frac{\varphi_1^2 + |\varphi_1|}{(|\varphi_1|+3)} \sin^2 t + \frac{5}{9} (1 + t \cos t \varphi_2),$$

$$h_2(t, \varphi_1, \varphi_2) = \frac{2}{5} \left( \frac{|\varphi_1|}{(|\varphi_1|+5)} \cos^2 t + 1 \right) + \frac{3}{7} (1 + \frac{t}{1+t} \sin t \varphi_2).$$

Since  $|h_1(t, \varphi_1, \varphi_2) - h_1(t, \varphi_1^*, \varphi_2^*)| \leq \frac{5}{9} (|\varphi_1 - \varphi_1^*| + |\varphi_2 - \varphi_2^*|)$ ,  $|h_2(t, \varphi_1, \varphi_2) - h_2(t, \varphi_1^*, \varphi_2^*)| \leq \frac{3}{7} (|\varphi_1 - \varphi_1^*| + |\varphi_2 - \varphi_2^*|)$ . The Maple program can be used to obtain:

$$m + \ell_1 \Phi_1 + \ell_2 \Psi_1 < 0.8496699665 < 1,$$

which means (based on Theorem 10) that the given system has only one solution for  $[0, 1]$ .

## 7. Conclusions

In this work, we investigate coupled nonlinear Langevin fractional problems with different orders of  $\mu$ -Caputo fractional derivatives on arbitrary domains with nonlocal integral boundary conditions. We address the original problem by transforming it into an equivalent fixed-point problem and applying the standard tools of modern functional

analysis to determine its existence and uniqueness. Our results are not only new in this setting but also provided some special cases that we obtained by fixing certain parameters or giving a function-specific definition to the appropriate interval, for example:

(1)  $\kappa = 0$

$${}^c D^{\gamma_i, \mu} ({}^c D^{\sigma_i, \mu} + \alpha_i) \varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b], i = 1, 2.$$

Subjected to the specific boundary conditions

$$\begin{aligned} \varphi_i(a) &= 0, & I^{\vartheta_i, \mu} \varphi_i(b) &= 0, \\ {}^c D^{\sigma_1, \mu} \varphi_1(a) &= \kappa \int_a^{\xi} \varphi_2(s) ds. \end{aligned}$$

(2)  $\mu(t) = t$

$${}^c D^{\gamma_i} ({}^c D^{\sigma_i} + \alpha_i) \varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b], i = 1, 2.$$

Subjected to the specific boundary conditions

$$\begin{aligned} \varphi_i(a) &= 0, & I^{\vartheta_i, \mu} \varphi_i(b) &= 0, \\ {}^c D^{\sigma_1, \mu} \varphi_1(a) &= \kappa \int_a^{\xi} \varphi_2(s) ds. \end{aligned}$$

(3)  $\mu(t) = t^\rho$

$${}^c D^{\gamma_i, \rho} ({}^c D^{\sigma_i, \rho} + \alpha_i) \varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b], i = 1, 2.$$

Subjected to the specific boundary conditions

$$\begin{aligned} \varphi_i(a) &= 0, & I^{\vartheta_i, \mu} \varphi_i(b) &= 0, \\ {}^c D^{\sigma_1, \mu} \varphi_1(a) &= \kappa \int_a^{\xi} \varphi_2(s) ds. \end{aligned}$$

(4)  $\mu(t) = \log t$

$${}^c D^{\gamma_i, \mathcal{H}} ({}^c D^{\sigma_i, \mathcal{H}} + \alpha_i) \varphi_i(t) = \Xi_i(t, \varphi_1(t), \varphi_2(t)), \quad t \in [a, b], a > 0, i = 1, 2.$$

Subjected to the specific boundary conditions

$$\begin{aligned} \varphi_i(a) &= 0, & I^{\vartheta_i, \mu} \varphi_i(b) &= 0, \\ {}^c D^{\sigma_1, \mu} \varphi_1(a) &= \kappa \int_a^{\xi} \varphi_2(s) ds. \end{aligned}$$

In future work, we could investigate our results based on other FDs such as the Abu-Shady–Kaabar FD, Katugampola derivative, or conformable derivative.

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## References

- Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
- Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006.
- Salem, A.; Al-dosari, A. Existence results of solution for fractional Sturm-Liouville inclusion involving composition with multi-maps. *J. Taibah Univ. Sci.* **2020**, *14*, 721–733. [\[CrossRef\]](#)
- Salem, A.; Al-dosari, A. Positive Solvability for Conjugate Fractional Differential Inclusion of  $(k, n - k)$  Type without Continuity and Compactness. *Axioms* **2021**, *10*, 170. [\[CrossRef\]](#)
- Wang, H. Existence of solutions for fractional anti-periodic BVP. *Results Math.* **2015**, *68*, 227–245. [\[CrossRef\]](#)
- Baghani, H. An analytical improvement of a study of nonlinear Langevin equation involving two fractional orders in different intervals. *J. Fixed Point Theory Appl.* **2019**, *21*, 95. [\[CrossRef\]](#)
- Ahmad, B.; Ntouyas, S.K.; Zhou, Y.; Alsaedi, A. A study of fractional differential equations and inclusions with nonlocal Erdlyi-Kober type integral boundary conditions. *Bull. Iran. Math. Soc.* **2018**, *44*, 1315–1328. [\[CrossRef\]](#)
- Ahmad, B.; Alghanmi, M.; Ntouyas, S.K.; Alsaedi, A. A study of fractional differential equations and inclusions involving generalized Caputo-type derivative equipped with generalized fractional integral boundary conditions. *AIMS Math.* **2019**, *4*, 12–28. [\[CrossRef\]](#)
- Salem, A.; Al-dosari, A. A Countable System of Fractional Inclusions with Periodic, Almost, and Antiperiodic Boundary Conditions. *Complexity* **2021**, *2021*, 6653106. [\[CrossRef\]](#)
- Salem, A.; Almaghamisi, L. Existence Solution for Coupled System of Langevin Fractional Differential Equations of Caputo Type with Riemann Stieltjes Integral Boundary Conditions. *Symmetry* **2021**, *13*, 2123. [\[CrossRef\]](#)
- Zwanzig, R. *Nonequilibrium Statistical Mechanics*; Oxford University Press: New York, NY, USA, 2001.
- Sun, H.-G.; Chen, Y.-Q.; Chen, W. Random order fractional differential equation models. *Signal Process.* **2011**, *91*, 525–530. [\[CrossRef\]](#)
- Zhao, K. Existence and UH-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions. *Filomat* **2023**, *37*, 1053–1063.
- Awadalla, M.; Subramanian, M.; Abuasbeh, K. Existence and Ulam–Hyers Stability Results for a System of Coupled Generalized Liouville–Caputo Fractional Langevin Equations with Multipoint Boundary Conditions. *Symmetry* **2023**, *15*, 198. [\[CrossRef\]](#)
- Salem, A.; Almaghamisi, L.; Alzahrani, F. An Infinite System of Fractional Order with p-Laplacian Operator in a Tempered Sequence Space via Measure of Noncompactness Technique. *Fractal Fract.* **2021**, *5*, 182. [\[CrossRef\]](#)
- Salem, A.; Alshehri, H.M.; Almaghamisi, L. Measure of noncompactness for an infinite system of fractional Langevin equation in a sequence space. *Adv. Differ. Equ.* **2021**, *2021*, 132. [\[CrossRef\]](#)
- Salem, A.; Alnegga, M. Measure of Noncompactness for Hybrid Langevin Fractional Differential Equations. *Axioms* **2020**, *9*, 59. [\[CrossRef\]](#)
- Salem, A. Existence results of solutions for anti-periodic fractional Langevin equation. *J. Appl. Anal. Comput.* **2020**, *10*, 2557–2574. [\[CrossRef\]](#)
- Mainradi, F.; Pironi, P. The fractional Langevin equation: Brownian motion revisited. *Extracta Math.* **1996**, *10*, 140–154.
- Baghani, H.; Nieto, J.J. On fractional Langevin equation involving two orders in different intervals. *Nonlinear Anal. Model. Control* **2019**, *24*, 884–897. [\[CrossRef\]](#)
- Baghani, H. Existence and uniqueness of solutions to fractional Langevin equations involving two fractional orders. *J. Fixed Point Theory Appl.* **2018**, *20*, 63. [\[CrossRef\]](#)
- Fazli, H.; Sun, H.-G.; Nieto, J.J. Fractional Langevin Equation Involving Two Fractional Orders: Existence and Uniqueness Revisited. *Mathematics* **2020**, *8*, 743. [\[CrossRef\]](#)
- Hilal, K.; Ibnelaziz, L.; Guida, K.; Melliani, S. Fractional Langevin Equations with Nonseparated Integral Boundary Conditions. *Adv. Math. Phys.* **2020**, *2020*, 3173764. [\[CrossRef\]](#)
- Salem, A.; Alghamdi, B. Multi-Strip and Multi-Point Boundary Conditions for Fractional Langevin Equation. *Fractal Fract.* **2020**, *4*, 18. [\[CrossRef\]](#)
- Salem, A.; Mshary, N. On the Existence and Uniqueness of Solution to Fractional-Order Langevin Equation. *Adv. Math. Phys.* **2020**, *2020*, 8890575. [\[CrossRef\]](#)
- Salem, A.; Alzahrani, F.; Almaghamisi, L. Fractional Langevin Equation With Nonlocal Integral Boundary Conditions. *Mathematics* **2019**, *7*, 402. [\[CrossRef\]](#)
- Salem, A.; Alghamdi, B. Multi-Point and Anti-Periodic Conditions for Generalized Langevin Equation with Two Fractional Orders. *Fractal Fract.* **2019**, *3*, 51. [\[CrossRef\]](#)
- Salem, A.; Mshary, N.; El-Shahed, M.; Alzahrani, F. Compact and Noncompact Solutions to Generalized Sturm-Liouville and Langevin Equation with Caputo-Hadamard Fractional Derivative. *Math. Probl. Eng.* **2021**, *2021*, 9995969. [\[CrossRef\]](#)

29. Chen, Y.; An, H.-L. Numerical solutions of coupled Burgers equations with time- and space-fractional derivatives. *Appl. Math. Comput.* **2008**, *200*, 87–95. [[CrossRef](#)]
30. Salem, A.; Alzahrani, F.; Alnegga, M. Coupled system of non-linear fractional Langevin equations with multi-point and nonlocal integral boundary conditions. *Math. Probl. Eng.* **2020**, *2020*, 7345658. [[CrossRef](#)]
31. Ahmad, B.; Ntouyas, S.K.; Alsaedi, A. On fully coupled nonlocal multi-point boundary value problems of nonlinear mixed-order fractional differential equations on an arbitrary domain. *Filomat* **2018**, *32*, 4503–4511. [[CrossRef](#)]
32. Ahmad, B.; Hamdan, S.; Alsaedi, A.; Ntouyas, S.K. On a nonlinear mixed-order coupled fractional differential system with new integral boundary conditions. *AIMS Math.* **2021**, *6*, 5801–5816. [[CrossRef](#)]
33. Ahmad, B.; Hamdan, S.; Alsaedi, A.; Ntouyas, S.K. Existence results for coupled nonlinear fractional differential equations of different orders with nonlocal coupled boundary conditions. *J. Inequalities Appl.* **2021**, *15*, 95.
34. Alruwaily, Y.; Aljoudi, S.; Almaghamisi, L.; Makhoulf, A.B.; Alghamdi, N. Existence and Uniqueness Results for Different Orders Coupled System of Fractional Integro-Differential Equations with Anti-Periodic Nonlocal Integral Boundary Conditions. *Symmetry* **2023**, *15*, 182. [[CrossRef](#)]
35. Nigmatullin, R.R. The realization of the generalized transfer equation in a medium with fractal geometry. *Phys. Status Solidi B* **1986**, *133*, 425–430. [[CrossRef](#)]
36. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [[CrossRef](#)]
37. Awadalla, M.; Noupoue, Y.Y.Y.; Asbeh, K.A.; Ghiloufi, N. Modeling Drug Concentration Level in Blood Using Fractional Differential Equation Based on Psi-Caputo Derivative. *J. Math.* **2022**, *2022*, 9006361. [[CrossRef](#)]
38. Jarad, F.; Abdeljawad, T. Generalized fractional derivatives and Laplace transform. *Discrete Contin. Dyn. Syst. S* **2019**, *13*, 709–722. [[CrossRef](#)]
39. El-hady, E.S. Ben Makhoulf, A.; Boulaaras, S.; Mchiri, L. Ulam-Hyers-Rassias stability of nonlinear differential equations with riemann-liouville fractional derivative. *J. Funct. Spaces* **2022**, *2022*, 7827579.
40. Makhoulf, A.B.; El-Hady, E.S.; Arfaoui, H.; Boulaaras, S.; Mchiri, L. Stability of some generalized fractional differential equations in the sense of Ulam–Hyers–Rassias. *Bound. Value Probl.* **2023**, *2023*, 8. [[CrossRef](#)] [[PubMed](#)]
41. Alruwaily, Y.; Almaghamisi, L.; Karthikeyan, K.; El-hady, E.S. Existence and uniqueness for a coupled system of fractional equations involving Riemann-Liouville and Caputo derivatives with coupled Riemann-Stieltjes integro-multipoint boundary conditions. *AIMS Math.* **2023**, *8*, 10067–10094. [[CrossRef](#)]
42. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk.* **1955**, *10*, 123–127.
43. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2005.

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