fractal and fractional

Article

Existence of Sobolev-Type Hilfer Fractional Neutral Stochastic Evolution Hemivariational Inequalities and Optimal Controls

Sivajiganesan Sivasankar [1], Ramalingam Udhayakumar [1,*], Venkatesan Muthukumaran [2], Saradha Madhrubootham [3], Ghada AlNemer [4,*] and Ahmed M. Elshenhab [5,6]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632014, Tamil Nadu, India
[2] Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, Tamilnadu, India
[3] Department of Mathematics, School of Applied Sciences, REVA University, Bangalore 560064, Karnataka, India
[4] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[5] School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
[6] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: udhayakumar.r@vit.ac.in (R.U.); gnnemer@pnu.edu.sa (G.A.)

Abstract: This article concentrates on a control system with a nonlocal condition that is driven by neutral stochastic evolution hemivariational inequalities (HVIs) of Sobolev-type Hilfer fractional (HF). In order to illustrate the necessary requirements for the existence of mild solutions to the required control system, we first use the characteristics of the modified Clarke sub-differential and a fixed point approach for multivalued functions. Then, we show that there are optimal state-control sets that are driven by Sobolev-type HF neutral stochastic evolution HVIs utilizing constrained Lagrange optimal systems. The optimal control (OC) results are created without taking the uniqueness of the control system solutions into account. Finally, the main finding is shown by an example.

Keywords: Hilfer fractional derivative (HFD); stochastic evolution Equation (SEE); Sobolev-type system; neutral system; optimal controls (OC); hemivariational inequalities; nonlocal condition

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1. Introduction

The hemivariational inequality (HVI) was invented by Panagiotopoulos in 1981 and is a weak framework for a range of engine issues using non-smooth and non-convex energy functionals [1,2]. The control issues with HVIs have recently drawn a lot of attention from investigators. In 2000, Migórski and Ochal [3] investigated the OC problems of the parabolic HVIs using the direct approach of the calculus of variations and the Galerkin technique. The existence of the OC sets for a hyperbolic quasi-linear HVI was proven in 2007 by researchers [4] employing the Faedo–Galerkin strategy and the iterative concept of the logic of alteration. A fixed point principle for multivalued maps was recently used by investigators [5] to investigate the approximate controllability of HVIs. Very recently, Muthukumar et al. [6] dealt with the optimal control issue of second-order SE-HVIs with Poisson jumps, applying the fixed point approach of multivalued maps and Balder's theorem. Harrat et al. [7] explored the OC and solvability of impulsive HF delay evolution inclusions with Clarke sub-differential utilizing semigroup theory, fractional calculus, the fixed point theorem, and multivalued evaluation, and the researchers proved the existence of an optimal control pair for the Lagrange problem under an appropriate set of sufficient conditions. Researchers [8] developed Hilfer fractional evolution hemivariational inequalities with nonlocal initial conditions and optimal controls by using the properties of generalized Clarke subdifferential and a fixed point theorem for condensing multivalued maps.

On the other hand, over the previous few decades, fractional calculus has become increasingly important in mathematics. For some physical problems, fractional-order differential equations are more appropriate than integer-order differential equations. Fractional differential Equations (FDEs) have been a hot research topic due to their application for control systems, chemical engineering, mechanical, natural sciences, dynamics, physical science, viscoelasticity materials, electrical circuits, neural networks, and so on [9–15]. We draw attention to the fact that over the past three decades, FDEs have developed significantly (see, for instance [16–21]), and are an effective tool for describing certain materials and processes [10,11].

Stochastic differential Equations (SDEs) are useful tools for characterizing diverse phenomena and procedures with stochastic disturbances in many branches of science and engineering. See the textbook [22] for details on the broad idea of SDEs. It should be highlighted as well that stochastic discomfort or noise may be found in both natural and artificial systems. SDEs are crucial for simulating real-world phenomena where an element of unpredictability is required. As a result of their numerous uses in the biological, physical, and pharmaceutical sciences (see [23–25]), SDEs have gained a lot of interest. The unpredictable events studied in microbiology, including entropy, neurobiology, and industrial engineering, are the inspiration for SEEs in infinite-dimensional spaces. Numerous writers have discussed in great detail the existence of mild solutions for different forms of SEEs and their OC in Hilbert spaces (see [26–28]).

The HFD, which also contains the R–L and Caputo fractional derivatives as special instances, was introduced by Hilfer [10]. It is used in conceptual simulations of electromagnetics in glass-forming components. Gu and Trujillo examined in [29] the possibility of mild solutions to an evolution equation including HFD. The solvability and OC of impulsive HF delay evolution systems with Clarke sub-differential were examined by Harrat et al. in [7]. Some intriguing results are presented for HF evolution equations under nonlocal situations. For instance, Yang and Wang examined in [30] the approximation of controllability of an HF differential system with nonlocal circumstances. Yang and Wang investigated the possibility of mild solutions to an HF differential equation with nonlocal circumstances in [31].

The mathematical structure of many physical processes, including the movement of fluids through cracked rocks and entropy, frequently reveals the differential system with Sobolev-type. The readers may refer to [32–34]. Neutral systems have gained more attention recently due to their extensive use in several pragmatic mathematics domains. Diverse neutral systems, including material's thermal expansion, stretchability, surface waves, and a number of biological advances, profit from neutral systems immediately or later. For further details on the neutral system and its use, readers might refer to the [21,24,35].

Numerous scientific fields, including ecology, quantum field theory, geography, and pharmacology, where a consequence or delay must be considered, utilize integral-differential equations. In practice, a model having hereditary properties is always denoted by an integro-differential system. The authors Ahmed et al. [36] produced the HF stochastic integro-differential equations by employing the ideas of fractional calculus, semigroups, and Sadovskii's fixed point principle. The existence and controllability of a fractional integro-differential system of order $1 < r < 2$ with infinite delay demonstrated using the mathematical concepts connected to the fractional derivatives and the MNC were recently studied by Mohan Raja et al. [37]. Researchers [24,35,38] have investigated the HF integro-differential systems with almost sectorial operators by utilizing the fixed point technique. Using the modified Clarke sub-differential and a fixed point principle for multivalued maps, Sivasankar et al. [39] recently created the OC issues for HF neutral stochastic evolution HVIs. Inspired by the above articles, the researchers developed the existence of Sobolev-type Hilfer fractional neutral stochastic evolution hemivariational inequalities and optimal controls.

It appears that there are still a lot of intriguing concepts and open-ended questions in the making, despite the fact that major advances have been made in the solvability and control problems of HVIs and fractional SEEs. It is generally recognized that the concept of controllability is important for the research and development of control systems. Moreover, it is crucial to study HVIs with fractional derivatives since they are connected to relevant fields such as selective memory entropy, thermoviscoelasticity, and water loss in heat transfer. To our knowledge, however, there has not been any research on OC problems for Sobolev-type HF neutral stochastic evolution HVI published in the literature as of yet.

This study proposes to close this gap by drawing inspiration from the above results. This article's goal is to demonstrate the existence of solutions and the optimal control problems for Sobolev-type HF neutral stochastic evolution HVIs of the following form:

$$\begin{cases} \langle D_{0^+}^{\lambda,\zeta}[\mathcal{J}\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))] + \widetilde{A}[\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))] + \mathscr{B}\mathfrak{u}(\mathfrak{t}) + \eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\frac{dW(\mathfrak{t})}{d\mathfrak{t}}, \omega\rangle_{\mathfrak{X}} \\ \qquad\qquad +\mathcal{G}^0(\mathfrak{t},\mathfrak{z}(\mathfrak{t});\omega) \geq 0, \quad \mathfrak{t} \in \mathscr{V} = [0,\tilde{c}],\ \forall\ \omega \in \mathfrak{X}, \\ \mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)}[\mathfrak{z}(\mathfrak{t})]_{\mathfrak{t}=0} + \hbar(\mathfrak{z}) = \mathfrak{z}_0, \end{cases} \tag{1}$$

where $D_{0^+}^{\lambda,\zeta}$ denotes the HFD, $\mathfrak{t} \in \mathscr{V}' = [0,\tilde{c}]$, $\lambda \in [0,1]$, $\zeta \in (0,1)$. The state variables $\mathfrak{z}(\cdot)$ takes the values in $\mathfrak{X}$ is the Hilbert space with $\|\cdot\|_{\mathfrak{X}}$ and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$. The almost sectorial operator of the strongly continuous semigroup operator $\{\mathcal{T}(\mathfrak{t}),\ \mathfrak{t} \geq 0\}$ in $\mathfrak{X}$ is $\widetilde{A} : D(\widetilde{A}) \subset \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{J} : D(\mathcal{J}) \subset \mathfrak{X} \to \mathfrak{X}$ is the linear operator on $\mathfrak{X}$. A control function is defined as $\mathfrak{u}$, and the collection of all admissible controls is classified as $\mathbb{U}$, which is a Hilbert space. Let the bounded linear operator $\mathscr{B} : \mathbb{U} \to \mathfrak{X}$, the function $\eth : \mathscr{V} \times \mathfrak{X} \to 2^{\mathfrak{X}}\backslash\{\emptyset\}$ be a bounded, non-empty, closed convex multivalued map, and $\psi : \mathscr{V} \times \mathfrak{X} \to \mathfrak{X}$ be the appropriate function. Assume that K is the another separable Hilbert space and $(\mho, \mathfrak{F}, P)$ is a complete probability space. Assume that the nuclear correlation operator $Q \geq 0$ and the finite trace of the Wiener process $\{W(\mathfrak{t}) : \mathfrak{t} > 0\}$ are characteristics of the K-Wiener process. The same expressions, $\|\cdot\|$, are used to represent the norm of $L(\mathrm{K},\mathfrak{X})$, where $L(\mathrm{K},\mathfrak{X})$ signifies the region of all bounded operators from $\mathrm{K} \to \mathfrak{X}$. If $\mathrm{K} = \mathfrak{X}$, then $L(\mathrm{K},\mathfrak{X}) = L(\mathfrak{X})$. The representation $\mathcal{G}^0(\mathfrak{t},\cdot;\cdot)$ represents the Clarke sub-differential of a universally Lipschitz function $\mathcal{G}(\mathfrak{t},\cdot) : \mathfrak{X} \to \Re$. Consider that E represents the predicted value of a random parameter or the Lebesgue integral with reference to the probability function P, and let $\mathscr{A}_{ad}$ be the collection of such admissible state control sets $(\mathfrak{z},\mathfrak{u})$. The cost functional on the set $\mathscr{A}_{ad}$ is supplied by

$$\mathscr{K}(\mathfrak{z},\mathfrak{u}) = E\int_0^{\tilde{c}} \mathscr{L}(\mathfrak{t},\mathfrak{z}^{\mathfrak{u}}(\mathfrak{t}),\mathfrak{u}(\mathfrak{t}))d\mathfrak{t}. \tag{2}$$

The following are the main contributions of our article:

- We construct and apply a set of sufficient conditions that demonstrate the existence and optimal control outcomes of Sobolev-type Hilfer fractional neutral stochastic evolution hemivariational inequalities under simple and fundamental system operator assumptions;
- In this study, we prove that the existence findings of Sobolev-type Hilfer fractional neutral stochastic evolution hemivariational inequalities satisfy certain requirements;
- We further expand the finding to derive the Lagrange problem for optimal controls findings for Sobolev-type Hilfer fractional neutral stochastic evolution hemivariational inequalities;
- These optimal control results are created without taking the originality of the control system's solutions into account;
- In particular, the optimal control problem is derived from the Lagrange problem and solved by the fixed point method.

This paper's outline is divided into five portions. In Section 2, we provide a list of required conditions. We show that system (1) has a mild solution in Section 3, assuming a few fair premises. We discuss the optimal control method governed by (1) under acceptable requirements in Section 4. A concrete example is given to illustrate our main conclusions in Section 5.1.

2. Preliminaries

The fundamental concepts, symbols, and lemmas of fractional derivatives that are required to explore the key outcomes are provided in this part, along with the necessary preliminary information.

Assume $\mathfrak{X}$ is a separable Hilbert space and that $\|\cdot\|_{\mathfrak{X}}$ represents its norm. Consider that $(\mho, \mathfrak{F}, P)$ represent the complete probability field with the regular identification $\{\mathfrak{F}_{\mathfrak{t}},\ \mathfrak{t} > 0\}$. $L^2(\mathfrak{F}, \mathfrak{X}) = L^2(\mho, \mathfrak{F}, P, \mathfrak{X})$ indicates the Hilbert space of all highly $\mathfrak{F}$-measurable square integrable $\mathfrak{X}$-valued random parameter fulfilling $E\|\mathfrak{z}\|^2_{\mathfrak{X}} < \infty$. Suppose $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$ is the Banach space of all continuous maps from $\mathscr{V} \to L^2(\mathfrak{F}, \mathfrak{X})$ with $\|\mathfrak{z}\|_{L^2} = \left[\sup_{\mathfrak{t}\in[0,\tilde{c}]} E\|\mathfrak{z}(\mathfrak{t})\|^2_{\mathfrak{X}}\right]^{\frac{1}{2}} < \infty$. $L^2_{\mathfrak{F}}(\mathscr{V}, \mathfrak{X})$ will represent the Hilbert space of all stochastic processes $\mathfrak{F}_{\mathfrak{t}}$-adapted measurable, described on $\mathscr{V}$ using values in $\mathfrak{X}$ and $\|\mathfrak{z}\|_{L^2_{\mathfrak{F}}(\mathscr{V},\mathfrak{X})} = \left[\int_0^{\tilde{c}} E\|\mathfrak{z}(\mathfrak{t})\|^2_{\mathfrak{X}} d\mathfrak{t}\right]^{\frac{1}{2}} < \infty$. The space $L^2_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})$ will stand for the Hilbert space of all stochastic processes $\mathfrak{F}_{\mathfrak{t}}$-adapted measurable characterized on $\mathscr{V}$ assuming values in $\mathbb{U}$ and $\|\mathfrak{u}\|_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})} = \left[\int_0^{\tilde{c}} E\|\mathfrak{u}(\mathfrak{t})\|^2_{\mathbb{U}} d\mathfrak{t}\right]^{\frac{1}{2}} < \infty$.

We suppose $\exists$ an entire orthonormal system $\{e_n\}$ in K, a bounded series of nonnegative real integers $\{\beta_n\} \ni Qe_n = \beta_n e_n,\ n = 1, 2, \cdots$, and a series $\{\mu_n\}$ of independent Wiener processes $\ni$

$$\langle W(\mathfrak{t}), \varepsilon\rangle = \sum_{n=1}^{\infty} \sqrt{\beta_n}\langle e_n, \varepsilon\rangle \mu_n(\mathfrak{t}), \quad \varepsilon \in \mathrm{K},\ \mathfrak{t} > 0.$$

Let $\sigma \in L(\mathrm{K}, \mathfrak{X})$ and specified by

$$\|\sigma\|^2_Q = Tr(\sigma Q \sigma^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\beta_n}\sigma e_n\right\|^2.$$

Suppose $\|\sigma\|_Q < \infty$, then σ is called a Q-Hilbert Schmidt operator. Let $L_Q(\mathrm{K}, \mathfrak{X})$ serve as the specification of the space of all Q-Hilbert Schmidt operators $\sigma : \mathrm{K} \to \mathfrak{X}$. The satisfaction of the equation $L_Q(\mathrm{K}, \mathfrak{X})$ of $L(\mathrm{K}, \mathfrak{X})$ with respect to the geometry resulting from the expression $\|\cdot\|_Q$, where $\|\sigma\|^2_Q = \langle\sigma, \sigma\rangle$ is a Hilbert space with the above norm geometry.

The operators $\widetilde{A} : D(\widetilde{A}) \subset \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{J} : D(\mathcal{J}) \subset \mathfrak{X} \to \mathfrak{X}$ fulfill the accompanying assumptions:

(A_1) $\widetilde{A}$ and $\mathcal{J}$ are closed linear operators;

(A_2) $D(\mathcal{J}) \subset D(\widetilde{A})$ and $\mathcal{J}$ is bijective;

(A_3) $\mathcal{J}^{-1} : \mathfrak{X} \to D(\mathcal{J})$ is continuous. Here, (A_1) and (A_2), together with the closed graph theorem, imply the boundedness of the linear operator $\widetilde{A}\mathcal{J}^{-1} : \mathfrak{X} \to \mathfrak{X}$. We designate $\|\mathcal{J}\| \le M_{\mathcal{J}}$ and $\|\mathcal{J}^{-1}\| \le M'_{\mathcal{J}}$.

The following presents the fundamentals of fractional calculus. (see [11,14]) for more information.

Definition 1 (see [11,14])**.** *The fractional integral of order $\lambda > 0$ with the lower bound zero is specified as for the function g*

$$I^{\lambda}_{\mathfrak{t}} g(\mathfrak{t}) = \frac{1}{\Gamma(\lambda)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \varsigma)^{\lambda-1} g(\varsigma) d\varsigma,\ \mathfrak{t} > 0,$$

if the RHS is given pointwise on the range $[0, +\infty)$*, where* Γ *is the gamma function. We simply assume that the gamma functions employed in this work are real without loss of generality.*

Definition 2 (see [11,14])**.** *The R–L derivative of order* $\lambda > 0$ *with lower limit zero may be expressed as for the function* $g : [0, +\infty) \to \mathbb{R}$

$$ {}^{L}D_{\mathfrak{t}}^{\lambda} g(\mathfrak{t}) = \frac{1}{\Gamma(m-\lambda)} \frac{d^m}{d\mathfrak{t}^m} \int_0^{\mathfrak{t}} \frac{g(\varsigma)}{(\mathfrak{t}-\varsigma)^{\lambda-m+1}} d\varsigma, \ m-1 < \lambda < m, \ m \in \mathbb{N}. $$

Definition 3 (see [11,14])**.** *The Caputo fractional derivative of order* $\lambda > 0$ *may be expressed by*

$$ D_{0^+}^{\lambda} g(\mathfrak{t}) = {}^{L} D_{\mathfrak{t}}^{\lambda} \left(g(\mathfrak{t}) - \sum_{n=1}^{m-1} \frac{\mathfrak{t}^n}{n!} g^n(0) \right), \ m = 1 < \lambda < m, \ m \in \mathbb{N}, $$

where the derivative of the function g *is absolutely continuous up to order* $m-1$*.*

Definition 4 (see [10])**.** *The HFD of order* $0 \le \lambda \le 1$, $0 < \zeta < 1$ *for the function* $\mathfrak{z}$ *is specified as*

$$ D_{0^+}^{\lambda,\zeta} g(\mathfrak{t}) = [\mathcal{I}_{0^+}^{\lambda(1-\zeta)} D(\mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)} g)](\mathfrak{t}). $$

Remark 1 (see [11,14])**.**

(a) *If* $g(\mathfrak{t}) \in C^m[0, \infty)$*, then*

$$ D_{\mathfrak{t}}^{\lambda} g(\mathfrak{t}) = \frac{1}{\Gamma(m-\lambda)} \int_0^{\mathfrak{t}} \frac{g^m(\varsigma)}{(\mathfrak{t}-\varsigma)^{\lambda-m+1}} d\varsigma = I^{(m-\lambda)} g^m(\mathfrak{t}), \ \mathfrak{t} > 0, \ m = 1 < \lambda < m. $$

(b) *The Caputo derivative of a constant is equal to zero;*

(c) *Suppose* g *is an arbitrary function with entries in E; then, the formulas in definitions 1 and 2 are interpreted in Bochner's sense.*

In the following, we discuss a number of essential properties of a multivalued map; for more information, please refer to the books of [40,41].

In a Banach space Y with $\|\cdot\|$, Y^* stands for Y's dual, and $\langle \cdot, \cdot \rangle$ represents the pairing of Y and Y^*. We shall employ the accompanying representations for our contentment:

$$ \mathcal{P}_{g(c)}(Y) = \{\mho \subseteq Y : \mho \text{ is non-empty, closed(convex)}\}, $$
$$ \mathcal{P}_{(w)k(c)}(Y) = \{\mho \subseteq Y : \mho \text{ is non-empty, (weakly) compact (convex)}\}. $$

In the next section, we will discuss how to describe the extended Clarke gradient for a globally Lipschitzian functional $\mathcal{G} : Y \to \mathbb{R}$. We indicate by $\mathcal{G}_0(\mathfrak{z}; \omega)$ the Clarke geometric gradient of $\mathcal{G}$ at $\mathfrak{z}$ in the direction ω, i.e.,

$$ \mathcal{G}^0(\mathfrak{z}, \omega) = \lim_{\mathfrak{z}' \to \mathfrak{z}} \sup_{\beta \to 0^+} \frac{\mathcal{G}(\mathfrak{z}' + \beta\omega) - \mathcal{G}(\mathfrak{z}')}{\beta}. $$

Recall also that the generalized Clarke sub-differential of $\mathcal{G}$ at $\mathfrak{z}$, represented by $\partial\mathcal{G}$, is a subset of Y^* produced by

$$ \partial\mathcal{G}(\mathfrak{z}) = \{\mathfrak{z}^* \in Y^* : \mathcal{G}^0(\mathfrak{z}, \omega) \ge \langle \mathfrak{z}^*, \omega \rangle, \ \forall \, \omega \in Y\}. $$

Our main conclusions depend heavily on the preceding essential characteristics of the generalized directional derivative and the generalized gradient.

Proposition 1 (see [42])**.** *Suppose* $g : \mho \to \mathbb{R}$ *is a globally Lipschitz function on an open set* $\mho$ *of* $\mathfrak{X}$*, then*

(i) $\forall\, z \in \mathfrak{X}$, *one has* $g^0(\mathfrak{z}, \omega) = \max\{\langle \mathfrak{z}^*, \omega\rangle : \forall\, \mathfrak{z} \in \partial g(\mathfrak{z})\}$;

(ii) $\forall\, \mathfrak{z} \in \mho$, *the derivative* $\partial g(x)$ *is a convex, non-empty, weak***-compact subset of* $\mathfrak{X}^*$ *and* $\|\mathfrak{z}^*\|_{\mathfrak{X}^*} \leq \mathbb{K}\, \forall\, \mathfrak{z} \in g(\mathfrak{z})$ *(where* $\mathbb{K}$ *is the Lipschitz constant of* g *near* $\mathfrak{z}$*);*

(iii) *the graph of the generalized derivative* ∂g *is closed in* $\mho \times \mathfrak{X}^*_{v^*}$ *topology, i.e., suppose* $\{y_n\} \subset \mho$ *and* $\{y_n^*\} \subset \mathfrak{X}$ *are series* $\ni \mathfrak{z}_n \in \partial g(\mathfrak{z}_n)$ *and* $\mathfrak{z}_n \to \mathfrak{z}$ *in* $\mathfrak{X}$, $\mathfrak{z}_n \to$ weakly* *in* $\mathfrak{X}$, *then* $\mathfrak{z} \in \partial g(\mathfrak{z})$ *(where* $\mathfrak{X}^*_{v^*}$ *represent the Banach space* $\mathfrak{X}$ *related with the* v^**-topology);*

(iv) *the multivalued function* $\mho \ni \mathfrak{z} \to \partial g(\mathfrak{z}) \subseteq \mathfrak{X} : \mho \to \mathfrak{X}^*_{v^*}$ *is u.s.c.*

Lemma 1 (see [43]). *Let* $G : [0, \tilde{c}] \times \mho \to L_0^2$ *be a strongly measurable mapping* $\ni \int_0^{\tilde{c}} E\|G(\mathfrak{t})\|^p_{L_0^2} d\mathfrak{t} < \infty$. *Then*

$$E\left\|\int_0^{\tilde{c}} G(\varsigma)dW(\varsigma)\right\|^p \leq L_G \int_0^{\tilde{c}} E\|G(\varsigma)\|^p_{L_0^2} d\varsigma,$$

$\forall\, 0 \leq \mathfrak{t} \leq \tilde{c}$ *and* $p \geq 2$, *where* L_G *is the constant involving* p *and* $\tilde{c}$.

Theorem 1 (see [44]). *Let* Y *be a locally convex Banach space and* $\mathcal{G} : Y \to 2^Y$ *be a compact convex valued, u.s.c. multivalued map* $\ni\ \exists$ *of a closed neighborhood* V *of 0 for which* $\mathcal{G}(V)$ *is a relatively compact set. Assume the set*

$$\mho = \{\mathfrak{z} \in Y : \mho\mathfrak{z} \in \mathcal{G}(\mathfrak{z}) \text{ for some } \lambda > 1\}$$

is bounded; then, $\mathcal{G}$ *has a fixed point.*

3. Existence

In this section, we investigate the existence of mild solutions for system (1). To achieve this goal, we shall study the following system:

$$\begin{cases} D_{0^+}^{\lambda,\zeta}[\mathcal{J}\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] \in \widetilde{A}[\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] + \mathscr{B}(\mathfrak{t})\mathfrak{u}(\mathfrak{t}) + \eth(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))\frac{dW(\mathfrak{t})}{d\mathfrak{t}} \\ \qquad\qquad + \partial\mathcal{G}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})), \quad \mathfrak{t} \in \mathscr{V}', \\ \mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)}[\mathfrak{z}(\mathfrak{t})]_{\mathfrak{t}=0} + \hbar(\mathfrak{z}) = \mathfrak{z}_0, \end{cases} \tag{3}$$

where $\partial\mathcal{G}$ represents the generalized Clarke sub-differential of a locally Lipschitz functional $\mathcal{G}(\mathfrak{t}, \cdot) : \mathfrak{X} \to \mathbb{R}$. $\mathscr{B} : \mathbb{U} \to \mathfrak{X}$ is a bounded linear operator, the control function $\mathfrak{u}(\cdot)$ is a stochastic process provided in $L^2_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})$ of admissible control functions, and the collection $\mathbb{U}$ is a Hilbert space. $\eth : \mathscr{V} \times \mathfrak{X} \times \mathfrak{X} \to L_0^2$ is a suitable function, and $\mathfrak{z}_0$ is a quantifiable $\mathfrak{X}$-valued random variable independent of W.

The fact that every solution to system (3) also resolves system (1) is obvious. Assume that in reality, $\mathfrak{z}(\mathfrak{t}) \in C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$ is true. If system (1) has a solution, then a function $g(\mathfrak{t})$ exists in $\partial\mathcal{G}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))$, also known as $\mathfrak{t} \in \mathscr{V}$, is a number that fulfills the following equation:

$$\begin{cases} D_{0^+}^{\lambda,\zeta}[\mathcal{J}\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] \in \widetilde{A}[\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] + \mathscr{B}(\mathfrak{t})\mathfrak{u}(\mathfrak{t}) + \eth(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))\frac{dW(\mathfrak{t})}{d\mathfrak{t}} \\ \qquad\qquad + \partial\mathcal{G}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})), \quad \mathfrak{t} \in \mathscr{V}', \\ \mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)}[\mathfrak{z}(\mathfrak{t})]_{\mathfrak{t}=0} + \hbar(\mathfrak{z}) = \mathfrak{z}_0. \end{cases}$$

In view of the above equation, we obtain

$$\begin{cases} \langle D_{0^+}^{\lambda,\zeta}[\mathcal{J}\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] + \widetilde{A}[\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))] + \mathscr{B}(\mathfrak{t})\mathfrak{u}(\mathfrak{t}) + \eth(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))\frac{dW(\mathfrak{t})}{d\mathfrak{t}}, \omega\rangle_{\mathfrak{X}} \\ \qquad\qquad + \langle \mathcal{G}(\mathfrak{t}), \omega\rangle_{\mathfrak{X}} = 0, \quad \text{a.e. } \mathfrak{t} \in \mathscr{V}',\ \forall\, \omega \in \mathfrak{X}, \\ \mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)}[\mathfrak{z}(\mathfrak{t})]_{\mathfrak{t}=0} + \hbar(\mathfrak{z}) = \mathfrak{z}_0. \end{cases}$$

Since $g(\mathfrak{t}) \in \partial\mathcal{G}(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))$ and $\langle \mathcal{G}(\mathfrak{t}), \omega\rangle_{\mathfrak{X}} \le \mathcal{G}^0(\mathfrak{t},\mathfrak{z}(\mathfrak{t});\omega)$, we obtain

$$\begin{cases} \langle D_{0^+}^{\lambda,\zeta}[\mathcal{J}\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))] + \widetilde{A}[\mathfrak{z}(\mathfrak{t}) - \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))] + \mathscr{B}(\mathfrak{t})\mathfrak{u}(\mathfrak{t}) + \eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\frac{dW(\mathfrak{t})}{d\mathfrak{t}}, \omega\rangle_{\mathfrak{X}} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \mathcal{G}^0(\mathfrak{t},\mathfrak{z}(\mathfrak{t});\omega) \ge 0, \quad \mathfrak{t} \in \mathscr{V}', \ \forall\, \omega \in \mathfrak{X}, \\ \mathcal{I}_{0^+}^{(1-\lambda)(1-\zeta)}[\mathfrak{z}(\mathfrak{t})]_{\mathfrak{t}=0} + \hbar(\mathfrak{z}) = \mathfrak{z}_0. \end{cases}$$

It is shown that we can investigate system (1) by the corresponding evolution inclusion system (3).

Now, using the Wright function, we specify the mild solution to system (3):

$$\mathfrak{M}_\zeta(\vartheta) = \sum_{n=1}^{\infty} \frac{(-\vartheta)^{n-1}}{(n-1)\Gamma(1-\mu n)}, \quad 0 < \mu < 1, \ \vartheta \in \mathbb{C},$$

that fulfills the equality

$$\int_0^\infty \theta^\iota \mathfrak{M}_\zeta(\theta)d\theta = \frac{\Gamma(1+\iota)}{\Gamma(1+\zeta\iota)}, \quad \theta \ge 0.$$

Lemma 2 (see [21])**.** *The operators $\mathcal{M}_{\lambda,\zeta}$, $\mathcal{N}_\zeta$ and $\mathbb{Q}_\lambda$ admit the accompanying requirements:*

(*a*) *For every fixed $\mathfrak{t} > 0$, $\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})$, $\mathcal{N}_\zeta(\mathfrak{t})$ and $\mathbb{Q}_\lambda(\mathfrak{t})$ are bounded linear operators $\ni$, $\forall\, \mathfrak{z} \in \mathfrak{X}$,*

$$\|\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})\mathfrak{z}\| \le \kappa_1 \mathfrak{t}^{-1+\zeta-\lambda\zeta+\lambda\vartheta}\|\mathfrak{z}\|, \quad \|\mathcal{N}_\zeta(\mathfrak{t})\mathfrak{z}\| \le \kappa_2 \mathfrak{t}^{\lambda(\vartheta-1)}\|\mathfrak{z}\| \text{ and}$$
$$\|\mathbb{Q}_\zeta(\mathfrak{t})\mathfrak{z}\| \le \kappa_2 \mathfrak{t}^{\lambda\vartheta-1}\|\mathfrak{z}\|,$$

where $\kappa_1 = \frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}$, $\quad \kappa_2 = \frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}$

(*b*) *$\{\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}),\ \mathfrak{t} > 0\}$, $\{\mathcal{N}_\lambda(\mathfrak{t}),\ \mathfrak{t} > 0\}$ and $\{\mathbb{Q}_\lambda(\mathfrak{t}),\ \mathfrak{t} > 0\}$ are strongly continuous;*

(*c*) *It $T(\mathfrak{t})$ is compact, then $\forall\ \mathfrak{t} > 0$, $\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})$, $\mathcal{N}_\lambda(\mathfrak{t})$, and $\mathbb{Q}_\lambda(\mathfrak{t})$ are also compact operators.*

Lemma 3 (see [43])**.** *If $\{\mathcal{T}(\mathfrak{t})\}_{\mathfrak{t}>0}$ is a compact C_0-semigroup for $\mathfrak{t} > 0$, then for $\mathfrak{t} > 0$, it is uniformly continuous.*

Remark 2 (see [45])**.** *A semigroup $\mathcal{T}(\mathfrak{t})$ is uniformly continuous if $\lim_{|\varsigma-\mathfrak{t}|\to 0} \|\mathcal{T}(\mathfrak{t}) - \mathcal{T}(\varsigma)\| = 0$. Lemma 3, together with Lemma 4, proves that $\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})$ and $\mathbb{Q}_\lambda(\mathfrak{t})$ are uniformly continuous provided $\mathcal{T}(\mathfrak{t})$ is compact for $\mathfrak{t} > 0$.*

Proposition 2. *Consider $\zeta \in (0,1)$, $q \in (0,1]$ and $\forall\, \mathfrak{z} \in D(\widetilde{A})$, then $\exists$ a $\kappa_q > 0$ $\ni$*

$$\|\widetilde{A}^q \mathbb{Q}_\zeta(\mathfrak{t})\mathfrak{z}\| \le \frac{\zeta\kappa_q\Gamma(2-q)}{\mathfrak{t}^{\zeta q}\Gamma(1+\zeta(1-q))}\|\mathfrak{z}\|, \ 0 < \mathfrak{t} < \tilde{c}.$$

Definition 5. *For each $\mathfrak{u} \in L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})$, an $\mathfrak{F}_\mathfrak{t}$-adapted stochastic process $\mathfrak{z} \in C(\mathscr{V}, L^2(\mathfrak{F},\mathfrak{X}))$ is said to be a mild solution of the control system (3) if $\mathfrak{z}(0) = \mathfrak{z}_0 \in \mathfrak{X}$ and $\exists$ a $g \in L^2_{\mathfrak{F}}(\mathscr{V},\mathfrak{X})$ $\ni$ $g(\mathfrak{t}) \in \partial\mathcal{G}(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))$ a.e. $\mathfrak{t} \in \mathscr{V}$ and*

$$\begin{aligned} \mathfrak{z}(\mathfrak{t}) &= \mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}(0) - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \mathcal{J}^{-1}\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\ &\quad + \int_0^\mathfrak{t} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma \\ &\quad + \int_0^\mathfrak{t} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma), \end{aligned}$$

where

$$\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}) = \mathcal{I}^{\zeta(1-\lambda)}\vartheta\mathcal{N}_\lambda(\mathfrak{t}),\ \mathcal{N}_\lambda(\mathfrak{t}) = \mathfrak{t}^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}) \text{ and } \mathbb{Q}_\lambda(\mathfrak{t}) = \int_0^\infty \zeta\xi\mathfrak{M}_\zeta(\xi)\mathcal{N}(\mathfrak{t}^\zeta\xi)d\xi.$$

In this paper, the following hypotheses are used:

(H_0) For $\mathfrak{t} > 0$, the operator $\mathcal{M}(\mathfrak{t})$ is compact;

(H_1) The function $\mathfrak{t} \to \mathcal{M}(\mathfrak{t})$ is continuous in $\mathscr{B}(\mathfrak{X})$ $\forall$ $\mathfrak{t} > 0$, and $\exists$ a constant $M > 1 \ni \|\mathcal{M}(\mathfrak{t})\| \leq M$;

(H_2) The function $\mathcal{G} : \mathscr{V} \times \mathscr{X} \to \mathbb{R}$ meets the accompanying criteria:

(a) $\forall$ $\mathfrak{z} \in \mathfrak{X}$, $\mathcal{G}(\cdot, \mathfrak{z})$ is measurable;

(b) for a.e. $\mathfrak{t} \in \mathscr{V}$, $\mathcal{G}(\mathfrak{t}; \cdot)$ is globally Lipschitz continuous;

(c) $\exists$ a function $b \in L^{\frac{1}{\delta}}(\mathscr{V}, \mathbb{R}^+)$, $\delta \in (0, 2\zeta - 1)$ and a constant $c \geq 0 \ni$

$$E\|\partial\mathcal{G}(\mathfrak{t}, \mathfrak{z})\|^2 = \sup\{E\|g(\mathfrak{t})\|^2 : g(\mathfrak{t}) \in \partial\mathcal{G}(\mathfrak{t}, \mathfrak{z})\} \leq b(\mathfrak{t}) + c\|\mathfrak{z}\|^2$$

for a.e. $\mathfrak{t} \in \mathscr{V}$ and $\forall$ $\mathfrak{z} \in \mathfrak{X}$.

(H_3) $\eth : \mathscr{V} \times \mathfrak{X} \to L_0^2$ is continuous in the second parameter for a.e. $\mathfrak{t} \in \mathscr{V}$ and $\exists$ a function $d \in L^{\frac{1}{\delta}}(\mathscr{V}, \mathbb{R}^+)$, $\delta \in (0, 2\zeta - 1)$ and a constant $e \geq 0 \ni$

$$E\|\eth(\mathfrak{t}, \mathfrak{z}(\mathfrak{t}))\|^2 \leq d(\mathfrak{t}) + e\|\mathfrak{z}\|^2.$$

(H_4) $\exists$ a constant $L_\hbar \ni \forall$ $\mathfrak{z}_1; \mathfrak{z}_2 \in C$,

$$E\|\hbar(\mathfrak{z}_1) - \hbar(\mathfrak{z}_2)\|^2 \leq L_\hbar\|\mathfrak{z}_1 - \mathfrak{z}_2\|^2.$$

(H_5) $\psi : \mathscr{V} \times \mathfrak{X} \to \mathfrak{X}$ is a continuous function and $\exists$ constants $q \in (0, 1)$ and $M_\psi > 0 \ni \psi$ is $\mathfrak{X}_q$-valued and satisfies the following conditions:

$$E\|\psi(\mathfrak{t}, \mathfrak{z})\|^2 \leq M_\psi^2(1 + \mathfrak{t}^{2(1-\zeta+\lambda\zeta\lambda\vartheta)}\|\mathfrak{z}\|_{\mathfrak{X}}^2), \ \mathfrak{z} \in \mathfrak{X}, \ \mathfrak{t} \in \mathscr{V}.$$

Define the admissible set as follows:

$$\mathscr{U}_{ad} = \{\mathfrak{u}(\cdot) \in L_{\mathfrak{F}}^p(\mathscr{V}, \mathbb{U}); \mathfrak{u}(\mathfrak{t}) \in \mathbb{U} \text{ a.e. } \mathfrak{t} \in \mathscr{V}\}.$$

In addition, $\mathscr{U}_{ad}$ is a bounded, convex, and closed subset of $L^p(\mathscr{V}, \mathbb{U})$ with $1 < p < \infty$. This is shown by Proposition 2.1.7 and Lemma 2.3.2 of [41], $\mathscr{U}_{ad} \neq \emptyset$. Clearly, $\mathscr{B}\mathfrak{u} \in L^p(\mathscr{V}, \mathfrak{X})$ $\forall$ $\mathfrak{u} \in \mathscr{U}_{ad}$. Next, define an operator $\Upsilon : L_{\mathfrak{F}}^2(\mathscr{V}, \mathfrak{X})$ by

$$\Upsilon(\mathfrak{z}) = \{W \in L_{\mathfrak{F}}^2(\mathscr{V}, \mathfrak{X}) : W(\mathfrak{t}) \in \partial\mathcal{G}(\mathfrak{t}, \mathfrak{z}(\mathfrak{t})) \text{ a.e. } \mathfrak{t} \in \mathscr{V} \text{ for } \mathfrak{z} \in L_{\mathfrak{F}}^2(\mathscr{V}, \mathfrak{X})\}.$$

In order to obtain the essential findings that we need, we also need the following lemmas:

Lemma 4 (see [26]). *Assume that* (H_2) *holds, then* $\forall$ $\mathfrak{z} \in L^2(\mathscr{V}, \mathfrak{X})$, *the set* $\Upsilon(\mathfrak{z})$ *has non-empty, convex, and weakly compact values.*

Lemma 5 (see [26]). *Suppose* (H_2) *holds,* Υ *fulfills the following: if* $\mathfrak{z}_n \to \mathfrak{z} \in L^2(\mathscr{V}, \mathfrak{X})$, $z_k \to z$ *weakly in* $L^2(\mathscr{V}, \mathfrak{X})$ *and* $z_k \in \Upsilon(\mathfrak{z}_k)$, *then* $z \in \Upsilon(\mathfrak{z})$.

Lemma 6 (see [6]). *Suppose* (H_2) *holds and the operator* Υ *fulfills the following: if* $\mathfrak{z}_n \to \mathfrak{z}$ *in* $L_{\mathfrak{F}}^2(\mathscr{V}, \mathfrak{X})$, $W_n \to W$ *weakly in* $L_{\mathfrak{F}}^2(\mathscr{V}, \mathfrak{X})$ *and* $W_n \in \Upsilon(\mathfrak{z}_n)$, *then we obtain* $W \in \Upsilon(\mathfrak{z})$.

Theorem 2. *Suppose* (H_0)–(H_5) *holds, then the HF stochastic system (1) has a mild solution on* $\mathscr{V}$ *provided by*

$$S = 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M_{\mathcal{J}}'^2\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}(\tilde{c}c + L_G e) < 1.$$

Proof. Let us assume that the operator of the multivalued map $\Xi : C \to 2^C$ is as follows: for any $\mathfrak{z} \in C \subset L^2(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$, by corresponding Lemma (4),

$$\Xi(\mathfrak{z}) = \begin{cases} z \in C : z(\mathfrak{t}) = \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\Big[\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}(0) - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \mathcal{J}^{-1}\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\ \qquad + \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma \\ \qquad + \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Big],\ \mathfrak{t} \in (0,\tilde{c}]. \end{cases}$$

It is now obvious that the major goal of our concentrated effort was to locate a stationary point of Ξ. In this section, we will show that Ξ fulfills each and every requirement of Theorem 1. To make our findings more usable, we divided them into the six steps listed below.

Step 1: We will now prove that $\forall\ \mathfrak{z} \in C$, $\Xi(\mathfrak{z})$ has convex, non-empty, and weakly compact values.

We can easily show that $\Xi(\mathfrak{z})$ has non-empty and weakly compact values by using Lemma 4. Because $\Upsilon(\mathfrak{z})$ has convex values, we can now draw a result for every $\alpha \in [0,1]$ by first providing $\chi_1, \chi_2 \in \Upsilon(\mathfrak{z})$ and then $\alpha\chi_1 + (1-\alpha)\chi_2 \in \Upsilon(\mathfrak{z})$. The function $\Xi(\mathfrak{z})$ is convex.

Step 2: For every $q > 0$, $\mathfrak{D}_q = \{\mathfrak{z} \in C : \|\mathfrak{z}\|^2 \le q\}$, and $\Xi \subseteq C$. The closed, convex, and bounded set of C is undoubtedly $\mathfrak{D}_q$.

In fact, it suffices to show that $\exists$ a positive constant $r^* \ni \|\vartheta\| \le r^*$, $\forall\ \Upsilon \in \Xi(\mathfrak{z})$, and $\mathfrak{z} \in \mathfrak{D}_q$. Assume $\vartheta \in \Xi(\mathfrak{z})$, then $\exists$ a function $g \in \Upsilon(\mathfrak{z}) \ni$

$$\begin{aligned} \varkappa(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\Big[\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \mathcal{J}^{-1}\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\ &+ \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g(\varsigma)d\varsigma + \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\ &+ \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Big], \quad \mathfrak{t} \in \mathscr{V}'. \end{aligned}$$

From (H_0) to (H_5), Lemma 1, and the Hölder inequality, we get

$$\begin{aligned} E\|\varkappa\|^2 \le& 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\Big[E\|\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))]\|^2 + E\|\mathcal{J}^{-1}\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\|^2 \\ &+ \int_0^{\mathfrak{t}} E\|\mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g(\varsigma)d\varsigma\|^2 \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{2(\lambda-1)}E\|\mathcal{J}^{-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)\|^2 d\varsigma \\ &+ L_G\int_0^{\mathfrak{t}} E\|\mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))d\varsigma\|^2_{L_0^2}\Big] \\ \le& 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\Big[M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}\right)^2\mathfrak{t}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)}(2M^2_{\mathcal{J}}E[\|\mathfrak{z}_0\|^2 \\ &+ \|\hbar(0)\|^2] + L_\hbar r + 2M^2_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)) \\ &+ M'^2_{\mathcal{J}}M^2_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) + M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\left(\frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}\int_0^{\mathfrak{t}} E\|\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\|^2\right. \\ &\left.+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{2(\lambda-1)}[E\|g(\varsigma)\|^2 + L_G E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2}]d\varsigma\right)\Big] \\ \le& 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\Big[M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}\right)^2\mathfrak{t}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)}(2M^2_{\mathcal{J}}E[\|\mathfrak{z}_0\|^2 \\ &+ \|\hbar(0)\|^2] + L_\hbar r + 2M^2_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)) \\ &+ M'^2_{\mathcal{J}}M^2_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) + M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Big\{\left(\frac{1-\delta}{2\lambda-1-\delta}\right)^{\delta-1}\tilde{c}^{2\lambda-1-\delta}[\|b\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)} \\ &+ L_G\|d\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)}] + \frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}[(c+L_G e)q + \|\mathscr{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})}]\Big\}\Big] := r^*. \end{aligned}$$

Thus, $\Xi(\mathfrak{D}_q)$ is bounded in $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$.

Step 3: $\{\Xi(\mathfrak{z}) : \mathfrak{z} \in \mathfrak{D}_q\}$ is equicontinuous.

Firstly, for all $\mathfrak{z} \in \mathfrak{D}_q$,

$$\begin{aligned}\varkappa(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\bigg\{\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) + \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g(\varsigma)d\varsigma \\ &+ \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma + \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\bigg\}.\end{aligned}$$

Next, given small enough $\epsilon > 0$ and $0 < \mathfrak{t}_1 < \mathfrak{t}_2 \leq \tilde{c}$, we get

$$\begin{aligned}
&E\|\varkappa(\mathfrak{t}_2) - \varkappa(\mathfrak{t}_1)\|^2 \\
&\leq E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}_2)[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t}_2,\mathfrak{z}(\mathfrak{t}_2)) \\
&\quad + \int_0^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma + \int_0^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\
&\quad + \int_0^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\bigg] \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}_1)[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1)) \\
&\quad + \int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)g(\varsigma)d\varsigma + \int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\
&\quad + \int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\bigg]\bigg\|^2 \\
&\leq 5E\bigg\|[\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}_2) - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t}_1)](\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0)))\bigg\|^2 \\
&\quad + 5E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\psi(\mathfrak{t}_2,\mathfrak{z}(\mathfrak{t}_2)) - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\psi(\mathfrak{t}_1,\mathfrak{z}(\mathfrak{t}_1))\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma) \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma) \\
&\quad - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\bigg\|^2 \\
&\quad + 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\bigg\|^2 \\
&= \sum_{i=1}^{11} J_i.
\end{aligned}$$

By the strong continuity of $\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})(\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0, \mathfrak{z}(0)))$, we obtain

$$J_1 \to 0 \text{ as } \mathfrak{t}_2 \to \mathfrak{t}_1.$$

The equicontinuity of ψ assures that

$$J_2 \to 0 \text{ as } \mathfrak{t}_2 \to \mathfrak{t}_1.$$

By assumptions (H_0)–(H_4) and the same method applied in Lemma 3.1 of [23], we get

$$\begin{aligned}
J_3 \leq\, & 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma \\
& - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
\leq\, & 15E\bigg\|\int_0^{\mathfrak{t}_1}\Big(\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_2-\varsigma)^{\lambda-1} - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\Big)\mathcal{J}^{-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
\leq\, & 15\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\bigg[\int_0^{\mathfrak{t}_1}\Big(\mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_2-\varsigma)^{2(\lambda-1)} \\
& - \mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_1-\varsigma)^{2(\lambda-1)}\Big)M_{\mathcal{J}}'^2(\mathfrak{t}_2-\varsigma)^{2(\lambda\vartheta-1)}E\|g(\varsigma)\|^2 d\varsigma\bigg].
\end{aligned}$$

We obtain $J_3 \to 0$ as $\mathfrak{t}_2 \to \mathfrak{t}_1$. Additionally,

$$\begin{aligned}
J_4 \leq\, & 15E\bigg\|\mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}\Big((\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma) - (\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)g(\varsigma)\Big)d\varsigma\bigg\|^2 \\
\leq\, & 15\mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M_{\mathcal{J}}'^2\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{2(\lambda-1)}\|\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)-\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\|^2 E\|g(\varsigma)\|^2 d\varsigma.
\end{aligned}$$

By Theorem 3 and strong continuity of $\mathbb{Q}_\lambda(\mathfrak{t})$, $J_4 \to 0$ as $\mathfrak{t}_2 \to \mathfrak{t}_1$.

$$\begin{aligned}
J_5 \leq\, & 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)g(\varsigma)d\varsigma\bigg\|^2 \\
\leq\, & 15\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2 M_{\mathcal{J}}'^2\mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_2-\mathfrak{t}_1)^{2(\lambda-1)}E\|g(\varsigma)\|^2.
\end{aligned}$$

Furthermore, in a similar way, we can get

$$\begin{aligned}
J_6 \leq\, & 15E\bigg\|\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma \\
& - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}_1}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\bigg\|^2 \\
\leq\, & 15E\bigg\|\int_0^{\mathfrak{t}_1}\Big(\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_2-\varsigma)^{\lambda-1} - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\Big)\mathcal{J}^{-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\bigg\|^2 \\
\leq\, & 15\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\bigg[\int_0^{\mathfrak{t}_1}\Big(\mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_2-\varsigma)^{2(\lambda-1)} \\
& - \mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_1-\varsigma)^{2(\lambda-1)}\Big)M_{\mathcal{J}}'^2(\mathfrak{t}_2-\varsigma)^{2(\lambda\vartheta-1)}\|\mathscr{B}\|^2 E\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathcal{V},\mathbb{U})}d\varsigma\bigg].
\end{aligned}$$

$$J_7 \leq 15E\left\| \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_0^{\mathfrak{t}_1} \Big((\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma) - (\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)\Big) d\varsigma \right\|^2$$
$$\leq 15\mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} M_{\mathcal{J}}'^2 \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varsigma)^{2(\lambda-1)} \|\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)-\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\|^2 \|\mathscr{B}\|^2 E\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})} d\varsigma$$
$$\leq 15\mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} M_{\mathcal{J}}'^2 \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varsigma)^{2(\lambda-1)} \|\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)-\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\|^2 \|\mathscr{B}\|^2 E\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})} d\varsigma.$$

$$J_8 \leq 15E\left\| \mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma) d\varsigma \right\|^2$$
$$\leq 15\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2 M_{\mathcal{J}}'^2 \mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} (\mathfrak{t}_2-\mathfrak{t}_1)^{2(\lambda-1)} \|\mathscr{B}\|^2 E\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})}.$$

$$J_9 \leq 15E\left\| \mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma)) dW(\varsigma) - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma)) dW(\varsigma) \right\|^2$$
$$\leq 15E\left\| \int_0^{\mathfrak{t}_1} \Big(\mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_2-\varsigma)^{\lambda-1} - \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta}(\mathfrak{t}_1-\varsigma)^{\lambda-1}\Big) \mathcal{J}^{-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma)) dW(\varsigma) \right\|^2$$
$$\leq 15L_G\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2 \Bigg[\int_0^{\mathfrak{t}_1} \Big(\mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_2-\varsigma)^{2(\lambda-1)} - \mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}(\mathfrak{t}_1-\varsigma)^{2(\lambda-1)}\Big) M_{\mathcal{J}}'^2 (\mathfrak{t}_2-\varsigma)^{2(\lambda\vartheta-1)} E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2} d\varsigma \Bigg].$$

$$J_{10} \leq 15E\left\| \mathfrak{t}_1^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_0^{\mathfrak{t}_1} \Big((\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma)) - (\mathfrak{t}_1-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))\Big) dW(\varsigma) \right\|^2$$
$$\leq 15L_G M_{\mathcal{J}}'^2 \mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varsigma)^{2(\lambda-1)} \|\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)-\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\|^2 E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2} d\varsigma$$
$$\leq 15L_G M_{\mathcal{J}}'^2 \mathfrak{t}_1^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} \int_0^{\mathfrak{t}_1} (\mathfrak{t}_1-\varsigma)^{2(\lambda-1)} \|\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)-\mathbb{Q}_\lambda(\mathfrak{t}_1-\varsigma)\|^2 E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2} d\varsigma.$$

$$J_{11} \leq 15E\left\| \mathfrak{t}_2^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}_2-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma)) dW(\varsigma) \right\|^2$$
$$\leq 15L_G\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2 M_{\mathcal{J}}'^2 \mathfrak{t}_2^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} (\mathfrak{t}_2-\mathfrak{t}_1)^{2(\lambda-1)} E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2}.$$

Hence, using LDCT, we deduce the RHS of the aforementioned constraints $\to 0$ as $\mathfrak{t}_2 - \mathfrak{t}_1 \to 0$. We conclude that $\Xi(\mathfrak{z})(\mathfrak{t})$ is continuous from the right in $(0, \tilde{c}]$; a similar argument shows that it is also continuous from the left in $(0, \tilde{c}]$.

Similar to the previous case, we can show that $E\|\vartheta(\mathfrak{t}_2) - \mathfrak{z}_0\|^2 \to 0$ operates independently of $\mathfrak{z} \in \mathfrak{D}_q$ for $\mathfrak{t}_2 \to 0$ and $\mathfrak{t}_1 = 0$ and $0 < \mathfrak{t}_2 \leq \tilde{c}$.

The aforementioned justifications lead us to believe that $\{\Xi(\mathfrak{z}) : \mathfrak{z} \in \mathfrak{D}_q\}$ is an equicontinuous set of functions in $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$.

Step 4: Ξ is completely continuous.

Consider the case when $\mathfrak{t} \in \mathscr{V}$ is fixed. We demonstrate that the collection $\psi(\mathfrak{t}) = \{\vartheta(\mathfrak{t}) : \vartheta \in \Xi(\mathfrak{D}_q)\}$ is relatively compact in $\mathfrak{X}$. There is no doubt that $\psi(0) = \{\mathfrak{z}_0\}$ is compact.

The only variable that has to be considered is $\mathfrak{t} > 0$. Assume $0 < \mathfrak{t} \leq \tilde{c}$ to be fixed and $\forall\, \mathfrak{z} \in \mathfrak{D}_q,\ \vartheta \in \Xi(\mathfrak{z}), \exists$ a $g \in \Upsilon(\mathfrak{z}) \ni$

$$\begin{aligned}\vartheta(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\\ &+ \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma\\ &+ \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Bigg], \quad \mathfrak{t} \in \mathscr{V}.\end{aligned}$$

For each $\epsilon \in (0, \mathfrak{t}),\ \mathfrak{t} \in (0, \tilde{c}]$ and any $\mathfrak{z} \in \mathfrak{D}_q$, we specify

$$\begin{aligned}\vartheta^\epsilon(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\\ &+ \int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma\\ &+ \int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Bigg].\end{aligned}$$

According to the boundedness of $\int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma$, $\int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma),(H\mathfrak{z})(\varsigma))dW(\varsigma)$, and the compactness of $\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})$, $\mathbb{Q}_\lambda$, we obtain that the set $\psi_\epsilon(\mathfrak{t})\{\vartheta^\epsilon(\mathfrak{t}) : \vartheta \in \Xi(\mathfrak{D}_q)\}$ is relatively compact in $\mathfrak{X}\ \forall\ \epsilon \in (0, \mathfrak{t})$. Moreover, $\forall$ $\vartheta \in \Xi(\mathfrak{D}_q)$, we obtain

$$\begin{aligned}E\|\vartheta(\mathfrak{t}) - \vartheta^\epsilon(\mathfrak{t})\|^2 \leq& E\Bigg\|\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\\ &+ \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma\\ &+ \int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Bigg]\\ &- \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Bigg[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\\ &+ \int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma\\ &+ \int_0^{\mathfrak{t}-\epsilon}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\Bigg]\Bigg\|^2\\ \leq& 2\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Bigg[\frac{\epsilon^{2\lambda-1}}{2\lambda-1}\int_{\mathfrak{t}-\epsilon}^{\mathfrak{t}} E\|g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)\|^2 d\varsigma\\ &+ L_G\int_{\mathfrak{t}-\epsilon}^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{2(\lambda-1)}E\|\eth(\varsigma,\mathfrak{z}(\varsigma))\|^2_{L_0^2}d\varsigma\Bigg]\\ \leq& 2\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Bigg[\frac{2\epsilon^{2\lambda-1}}{2\lambda-1}(\epsilon^{1-\delta}\|b\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)} + cq\epsilon + \|\mathscr{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})}\epsilon)\\ &+ L_G\left(\frac{1-\delta}{2\lambda-\delta}\right)^{\delta-1}\epsilon^{2\lambda-1-\delta}\|d\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)} + L_G\frac{\epsilon^{2\lambda-1}}{2\lambda-1}eq\Bigg],\end{aligned}$$

$\Longrightarrow$ the set $\psi_\epsilon(\mathfrak{t})\{\vartheta^\epsilon(\mathfrak{t}) : \vartheta \in \Xi(\mathfrak{D}_q)\}$ is totally bounded, i.e., relatively compact in $\mathfrak{X}$. The Ascoli–Arzela theorem allows us to prove Ξ is completely continuous.

Step 5: Ξ has a closed graph.

Assume $\mathfrak{z}_n \to \mathfrak{z}_*$ in $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$, $\vartheta_n \in \Xi(\mathfrak{z}_n)$ and $\vartheta_n \to \vartheta_*$ in $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$. We'll demonstrate $\vartheta_* \in \Xi(\mathfrak{z}_*)$. Accordingly, $\vartheta_n \in \Xi(\mathfrak{z}_n)$ means that $\exists$ a $g_n \in \Upsilon(\mathfrak{z}_n) \ni$

$$\begin{aligned}\vartheta_n(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}_n(\mathfrak{t})) \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g_n(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}_n(\varsigma))dW(\varsigma)\Big]. \end{aligned} \tag{4}$$

We can demonstrate that $\{(g_n, f(\cdot, \mathfrak{z}_n))\}_{n\geq 1} \subseteq L^2_{\mathfrak{F}}(\mathscr{V}, \mathfrak{X}) \times L^2_0$ is bounded by using $(H_3)(iii)$ and (H_4). In light of this, we may assume and go on to a related idea, if necessary, that

$$(g_n, f(\cdot, \mathfrak{z}_n)) \to (g_*, f(\cdot, \mathfrak{z}_*)) \text{ weakly in } L^2_{\mathfrak{F}}(\mathscr{V}, \mathfrak{X}) \times L^2_0. \tag{5}$$

From the compactness of $\mathbb{Q}_\lambda$, (H_4), 4 and 5, we get

$$\begin{aligned}\vartheta_n(\mathfrak{t}) \to& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \psi(\mathfrak{t},\mathfrak{z}_*(\mathfrak{t})) \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g_*(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}_*(\varsigma))dW(\varsigma)\Big]. \end{aligned} \tag{6}$$

It should be noticed that g_n and $\vartheta_n \to \vartheta_*$ in $C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X}))$ and $g_n \in \Upsilon(\mathfrak{z}_n)$, respectively. In accordance with Lemma 6 and (6), we obtain $g_* \in \Upsilon(\mathfrak{z}_*)$. Because of this, we have shown that $\Longrightarrow$ Ξ has a closed graph for $\vartheta_* \in \Xi(\mathfrak{z}_*)$. It follows from Proposition 3.3.12(2) of [42] that Ξ is u.s.c.

Step 6: A priori estimate.

From Steps 1 to 5, we know that Ξ is a compact convex value and u.s.c., and $\Xi(\mathfrak{D}_q)$ is a relatively compact set. According to Theorem 1, we remain to prove the set

$$\Pi = \{\mathfrak{z} \in C(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})) : \lambda\mathfrak{z} \in \Xi(\mathfrak{z}),\ \lambda > 1\}$$

is bounded to obtain a fixed point of Ξ.

Let $\mathfrak{z} \in \Pi$ and assume $\exists$ a $f \in \Upsilon(\mathfrak{z}) \ni$

$$\begin{aligned}\vartheta(\mathfrak{t}) =& \lambda^{-1}\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))] + \lambda^{-1}\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) \\ &+ \lambda^{-1}\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g(\varsigma) + \mathscr{B}\mathfrak{u}(\varsigma)]d\varsigma \\ &+ \lambda^{-1}\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma). \end{aligned}$$

By using hypotheses (H_0)–(H_5), Lemma 1 and the Hölder inequality, we get

$$
\begin{aligned}
E\|\mathfrak{z}(\mathfrak{t})\|^2 \leq\ & 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\mathcal{J}^{-1}\Big[E\|\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0,\mathfrak{z}(0))]\|^2 + E\|\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))\|^2 \\
& + E\|\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g(\varsigma)d\varsigma\|^2 \\
& + E\|\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}(\varsigma)d\varsigma\|^2 \\
& + E\|\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}(\varsigma))dW(\varsigma)\|^2\Big] \\
\leq\ & 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}\right)^2 M'^2_{\mathcal{J}}\mathfrak{t}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)}[M^2_{\mathcal{J}}E\|\mathfrak{z}_0\|^2 + E\|\hbar(0)\|^2 \\
& + L_\hbar r + M_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)] + 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}M_\psi(1+\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) \\
& + 5\mathfrak{t}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Bigg[\left(\frac{\mathfrak{t}^{2\lambda-1}}{2\lambda-1}\right)\int_0^{\mathfrak{t}}[b(\varsigma)+cE\|\mathfrak{z}(\varsigma)\|^2] \\
& + \left(\frac{\mathfrak{t}^{2\lambda-1}}{2\lambda-1}\right)\int_0^{\mathfrak{t}}E\|\mathscr{B}\mathfrak{u}(\mathfrak{z})\|^2 d\varsigma + L_G\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{2(\lambda-1)}[d(\varsigma)+eE\|\mathfrak{z}(\varsigma)\|^2]\Bigg] \\
\leq\ & 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}\right)^2 M'^2_{\mathcal{J}}\tilde{c}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)}[M'^2_{\mathcal{J}}E\|\mathfrak{z}_0\|^2 + E\|\hbar(0)\|^2 \\
& + L_\hbar r + M_\psi(1+\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)] + 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}M_\psi(1+\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) \\
& + 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Bigg[\frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}(\tilde{c}^{\lambda-1}\|b\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)} + (\tilde{c}c + L_G e)\|\mathfrak{z}\|^2 \\
& + \|\mathscr{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})}) + L_G\left(\frac{1-\delta}{2\lambda-1-\delta}\right)^{\delta-1}\tilde{c}^{2\lambda-1-\delta}\|d\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)}\Bigg] \\
\leq\ & \mathsf{l} + S\|\mathfrak{z}\|^2,
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\mathsf{l} =\ & 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)}\right)^2 M'^2_{\mathcal{J}}\tilde{c}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)}[M^2_{\mathcal{J}}E\|\mathfrak{z}_0\|^2 + E\|\hbar(0)\|^2 \\
& + L_\hbar r + M_\psi(1+\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)] + 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}M_\psi(1+\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) \\
& + 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}M'^2_{\mathcal{J}}\left(\frac{\kappa_0\Gamma(\vartheta)}{\Gamma(\lambda\vartheta)}\right)^2\Bigg[\frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}(\tilde{c}^{\lambda-1}\|b\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)} + \frac{\tilde{c}^{2\lambda-1}}{2\lambda-1}\|\mathscr{B}\|^2\|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V},\mathbb{U})}) \\
& + L_G\left(\frac{1-\delta}{2\lambda-1-\delta}\right)^{\delta-1}\tilde{c}^{2\lambda-1-\delta}\|d\|_{L^{\frac{1}{\delta}}(\mathscr{V},\mathbb{R}^+)}\Bigg].
\end{aligned}
$$

As a result, from $S < 1$, the inequality (7)

$$\Longrightarrow \|\mathfrak{z}\|^2 = \sup_{\mathfrak{t}\in\mathscr{V}} E\|\mathfrak{z}(\mathfrak{t})\|^2 \leq \mathsf{l} + S\|\mathfrak{z}\|^2 \leq \frac{\mathsf{l}}{S-1} := p_0.$$

As a result, the collection Π is bounded. By Theorem 1, we obtain that Ξ has a fixed point. The proof is completed. □

4. Optimal Controls

This section examines the prior Lagrange problem (LP):
(P) Find a set $(\mathfrak{z}^0,\mathfrak{u}^0) \in C(\mathscr{V}, L^2(\mathfrak{F},\mathfrak{X})) \times \mathscr{U}_{ad} \ni$

$$\mathscr{K}(\mathfrak{z}^0,\mathfrak{u}^0) \leq \mathscr{K}(\mathfrak{z},\mathfrak{u}),\ \forall\ (\mathfrak{z},\mathfrak{u}) \in C(\mathscr{V}, L^2(\mathfrak{F},\mathfrak{X})) \times \mathscr{U}_{ad},$$

where

$$\mathscr{K}(\mathfrak{z},\mathfrak{u}) = E\int_0^{\tilde{c}} \mathscr{L}(\mathfrak{t},\mathfrak{z}^{\mathfrak{u}}(\mathfrak{t}),\mathfrak{u}(\mathfrak{t}))d\mathfrak{t}.$$

Here, $\mathfrak{z}^{\mathfrak{u}}$ stands for the mild solution of system (1) relating to the control $\mathfrak{u} \in \mathscr{U}_{ad}$. We use the preceding assertion (H_6) to explore the Lagrange problem (P):

(i) The functional $\mathscr{L} : \mathscr{V} \times \mathfrak{X} \times \mathbb{U} \to \mathbb{R} \bigcup \{\infty\}$ is Borel measurable;
(ii) For almost all $\mathfrak{t} \in \mathscr{V}$, $\mathscr{L}$ is sequentially l.s.c. on $\mathfrak{X} \times \mathbb{U}$;
(iii) For each $\mathfrak{z} \in \mathfrak{X}$ and almost all $\mathfrak{t} \in \mathscr{V}$, $\mathscr{L}(\mathfrak{t}, \mathfrak{z}, \cdot)$ is convex on $\mathbb{U}$;
(iv) $\exists$ constants $r_1 \geq 0$, $r_2 > 0$, ξ is positive and $\xi \in L^1(\mathscr{V}, \mathbb{R}) \ni$

$$\mathscr{L}(\mathfrak{t}, \mathfrak{z}, \mathfrak{u}) \geq \xi(\mathfrak{t}) + r_1 E\|\mathfrak{z}\|^2_{\mathfrak{X}} + r_2 E\|\mathfrak{u}\|^p_{\mathbb{U}}.$$

Theorem 3. *Suppose the assertions (H_0)–(H_6) are satisfied. The OC problem (P) allows at least one optimal set if $\mathscr{B}$ is a strongly continuous operator.*

Proof. The Lagrange problem (P) has a single optimum set if $\inf\{\mathscr{K}(\mathfrak{z}, \mathfrak{u}) | \mathfrak{u} \in \mathscr{U}_{ad}\} = +\infty$. Assume $\inf\{\mathscr{K}(\mathfrak{z}, \mathfrak{u}) : \mathfrak{u} \in \mathscr{U}_{ad}\} = v < +\infty$ without loss of generality. Then, $v > -\infty$ is implied by the condition $(H_6)(d)$. According to the concept of infimum, a minimizing sequence of possible couples $\{(\mathfrak{z}^n, \mathfrak{u}^n)\} \subset \mathscr{A}_{ad} \ni \mathscr{L}(\mathfrak{z}^n, \mathfrak{u}^n) \to v$ as $n \to +\infty$ exists. Therefore, $\{\mathfrak{u}^n\} \subseteq \mathscr{U}_{ad}$, $n = 1, 2, \cdots$, $\{\mathfrak{u}^n\}$ is bounded on $L^p_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})$, due to the reflexivity of $L^p_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})$, $\exists$ a subsequence of $\{\mathfrak{u}^n\}$, denoted again by $\{\mathfrak{u}^n\}$, and $\mathfrak{u}^* \in L^p_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})$, fulfilling

$$\mathfrak{u}^n \xrightarrow{\text{weakly}} \mathfrak{u}^* \in L^p_{\mathfrak{F}}(\mathscr{V}, \mathbb{U}).$$

According to Mazur's lemma, $\mathfrak{u}^* \in \mathscr{U}_{ad}$, since $\mathscr{U}_{ad}$ is convex and closed. Remember that the corresponding sequence of answers to the previous integral equation is indicated by the symbol $\{\mathfrak{z}^n\}$:

$$\begin{aligned} \mathfrak{z}^n(\mathfrak{t}) =& \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta} \mathcal{J}^{-1} \Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0 - \hbar(\mathfrak{z}) - \psi(0, \mathfrak{z}(0))] + \psi(\mathfrak{t}, \mathfrak{z}^n(\mathfrak{t})) \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t} - \varsigma)^{\lambda-1} \mathbb{Q}_\lambda(\mathfrak{t} - \varsigma)[g^n(\varsigma) + \mathscr{B}\mathfrak{u}^n(\varsigma)] d\varsigma \\ &+ \int_0^{\mathfrak{t}} (\mathfrak{t} - \varsigma)^{\lambda-1} \mathbb{Q}_\lambda(\mathfrak{t} - \varsigma) \eth(\varsigma, \mathfrak{z}^n(\varsigma)) dW(\varsigma) \Big], \end{aligned} \tag{8}$$

where $\eth(\varsigma, \mathfrak{z}^n(\varsigma), (H\mathfrak{z}^n)(\varsigma)) \in S_{\eth, \mathfrak{z}^n}$ and $g^n \in \Upsilon(\mathfrak{z}^n)$.

The next step is to show that $\{\mathfrak{z}^n\}$ is a relatively compact subset of $C\big(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})\big)$. First, similar to the proof in (7), we obtain

$$\begin{aligned} E\|\mathfrak{z}^n(\mathfrak{t})\|^2 \leq& 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} \left(\frac{\kappa_0 \Gamma(\vartheta)}{\Gamma(\zeta(1-\lambda)+\lambda\vartheta)} \right)^2 M'^2_{\mathcal{J}} \tilde{c}^{2(-1+\zeta-\lambda\zeta+\lambda\vartheta)} [M^2_{\mathcal{J}} E\|\mathfrak{z}_0\|^2 + E\|\hbar(0)\|^2 \\ &+ L_\hbar r + M_\psi(1 + \tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}_0\|^2)] + 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} M'^2_{\mathcal{J}} M_\psi(1 + \tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)}\|\mathfrak{z}\|^2) \\ &+ 5\tilde{c}^{2(1-\zeta+\lambda\zeta-\lambda\vartheta)} M'^2_{\mathcal{J}} \left(\frac{\kappa_0 \Gamma(\vartheta)}{\Gamma(\lambda\vartheta)} \right)^2 \Bigg[\frac{\tilde{c}^{2\lambda-1}}{2\lambda-1} \big(\tilde{c}^{\lambda-1} \|b\|_{L^{\frac{1}{\delta}}(\mathscr{V}, \mathbb{R}^+)} + \tilde{c}c\|\mathfrak{z}\|^2 + \|\mathscr{B}\|^2 \|\mathfrak{u}\|^2_{L^2_{\mathfrak{F}}(\mathscr{V}, \mathbb{U})} \big) \\ &+ L_G \left(\frac{1-\delta}{2\lambda-1-\delta} \right)^{\delta-1} \tilde{c}^{2\lambda-1-\delta} \|d\|_{L^{\frac{1}{\delta}}(\mathscr{V}, \mathbb{R}^+)} + L_G e \frac{\tilde{c}^{2\lambda-1}}{2\lambda-1} \|\mathfrak{z}\|^2 \Bigg]. \end{aligned} \tag{9}$$

We deduce that $\exists$ a constant $\mu > 0 \ni \|\mathfrak{z}^n\| \leq \mu, \Longrightarrow \{\mathfrak{z}^n\}$ is uniformly bounded due to the boundedness of $\{\mathfrak{u}_n\}$, (9), and Gronwall's inequality.

After that, we can demonstrate that $\{\mathfrak{z}^n(\mathfrak{t})\}$ is equicontinuous on $\mathscr{V}$, and that $\{\mathfrak{z}^n(\mathfrak{t})\}$ is relatively compact for any $\mathfrak{t} \in \mathscr{V}$ using an equivalent argument to Steps 3 and 4 in Theorem 2. There is a function $\mathfrak{z}^* \in C\big(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})\big)$ since the Ascoli–Arzela theorem is a relatively compact subset of $C\big(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})\big) \ni$

$$\mathfrak{z}^n \to \mathfrak{z}^* \text{ in } C\big(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})\big) \subset L^2\big(\mathscr{V}, L^2(\mathfrak{F}, \mathfrak{X})\big). \tag{10}$$

The dominated convergence theorem, together with the properties of boundedness and compactness of $\{\mathfrak{u}_n\}$ and $\mathcal{N}_\lambda(\mathfrak{t}-\varsigma)$,

$$\Longrightarrow \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathcal{N}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}^n(\varsigma)d\varsigma \to \int_0^{\mathfrak{t}} \mathcal{J}^{-1}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathcal{N}_\lambda(\mathfrak{t}-\varsigma)\mathscr{B}\mathfrak{u}^*(\varsigma)d\varsigma. \tag{11}$$

Due to the compactness of $\mathbb{Q}_\lambda$, $(H_2)(c)$, (H_3), (10), and Lemma 6, one gets a result similar to Step 5 in Theorem 2.

$$\begin{aligned}
&\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0-\hbar(\mathfrak{z})-\psi(0,\mathfrak{z}(0))]+\psi(\mathfrak{t},\mathfrak{z}^n(\mathfrak{t}))+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g^n(\varsigma)d\varsigma\\
&\qquad+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}^n(\varsigma))dW(\varsigma)\Big]\\
&\to \mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0-\hbar(\mathfrak{z})-\psi(0,\mathfrak{z}(0))]+\psi(\mathfrak{t},\mathfrak{z}^*(\mathfrak{t}))+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)g^*(\varsigma)d\varsigma\\
&\qquad+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}^*(\varsigma))dW(\varsigma)\Big],
\end{aligned} \tag{12}$$

where $\eth(\varsigma,\mathfrak{z}^*(\varsigma)) \in S_{\eth,\mathfrak{z}^*}$ and $g^* \in \Upsilon(\mathfrak{z}^*)$. Thus, it may be deduced from (11) and (12) that

$$\begin{aligned}
\mathfrak{z}^*(\mathfrak{t}) =&\mathfrak{t}^{1-\zeta+\lambda\zeta-\lambda\vartheta}\mathcal{J}^{-1}\Big[\mathcal{M}_{\lambda,\zeta}(\mathfrak{t})[\mathcal{J}\mathfrak{z}_0-\hbar(\mathfrak{z})-\psi(0,\mathfrak{z}(0))]+\psi(\mathfrak{t},\mathfrak{z}^*(\mathfrak{t}))\\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)[g^*(\varsigma)+\mathscr{B}\mathfrak{u}^*(\varsigma)]d\varsigma\\
&+\int_0^{\mathfrak{t}}(\mathfrak{t}-\varsigma)^{\lambda-1}\mathbb{Q}_\lambda(\mathfrak{t}-\varsigma)\eth(\varsigma,\mathfrak{z}^*(\varsigma))dW(\varsigma)\Big],
\end{aligned} \tag{13}$$

This proves that $\mathfrak{z}^*$ is a mild solution to problem (1) corresponding to the control $\mathfrak{u} \in \mathscr{U}_{ad}$.

As we can see from Theorem 2.1 of [46], all of Balder's assumptions are satisfied by (a)–(d). Consequently, Balder's Theorem illustrates that the functional

$$(\mathfrak{z},\mathfrak{u}) \mapsto E\int_0^{\tilde{c}} \mathscr{L}(\mathfrak{t},\mathfrak{z}^{\mathfrak{u}}(\mathfrak{t}),\mathfrak{u}(\mathfrak{t}))d\mathfrak{t}$$

is sequentially l.s.c. in the strong topology of $L^1_{\mathfrak{F}}(\mathscr{V},\mathfrak{X})$ and weak topology of $L^p_{\mathfrak{F}}(\mathscr{V},\mathfrak{X}) \subset L^1_{\mathfrak{F}}(\mathscr{V},\mathbb{U})$. Since $L^p_{\mathfrak{F}}(\mathscr{V},\mathbb{U}) \subset L^1_{\mathfrak{F}}(\mathscr{V},\mathbb{U})$, we conclude that $\mathscr{K}$ is weakly l.s.c. on $L^p_{\mathfrak{F}}(\mathscr{V},\mathbb{U})$. From the hypotheses $(H_5)(d)$, we know that $\mathscr{K} > -\infty$. Consequently, we determine that $\mathscr{K}$ achieves its infimum at $\mathfrak{u}^* \in \mathscr{U}_{ad}$, and so

$$v = \lim_{n\to\infty} E\int_0^{\tilde{c}} \mathscr{L}(\mathfrak{t},\mathfrak{z}^n(\mathfrak{t}),\mathfrak{u}^n(\mathfrak{t}))d\mathfrak{t} \geq E\int_0^{\tilde{c}} \mathscr{L}(\mathfrak{t},\mathfrak{z}^*(\mathfrak{t}),\mathfrak{u}^*(\mathfrak{t}))d\mathfrak{t} \geq v.$$

The result is now finished. □

5. Example

5.1. Example-1

We examine the following control system represented by hemivariational inequality as an implementation of our major findings:

$$\begin{cases}
D_{0^+}^{\lambda,\frac{4}{7}}[\mathfrak{z}(\mathfrak{t},\xi)-\mathfrak{z}_{\xi\xi}(\mathfrak{t},\xi)-\mathbb{k}(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))] \in \frac{\partial^2\mathfrak{z}}{\partial\xi^2}[(\mathfrak{t},\xi)-\mathbb{k}(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))]+\partial\mathcal{G}(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))+\mathscr{B}\mathfrak{u}(\mathfrak{t},\xi)\\
\qquad\qquad+\eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))\frac{dW(\mathfrak{t})}{d\mathfrak{t}},\ \mathfrak{t}\in(0,1],\ \xi\in[0,\pi],\\
\mathfrak{z}(\mathfrak{t},0)=\mathfrak{z}(\mathfrak{t},\pi)=0,\ \mathfrak{t}\in[0,1],\\
\mathcal{I}_{0^+}^{(\frac{3}{7})(1-\lambda)}(\mathfrak{z}(0,\xi))+\sum_{i=1}^m\int_0^\pi k(\mathfrak{t},\varsigma)\mathfrak{z}(\mathfrak{t}_i,\varsigma)d\varsigma=\mathfrak{z}_0(\xi),\ \xi\in[0,\pi],
\end{cases} \tag{14}$$

where $D_{0^+}^{\lambda,\frac{4}{7}}$ is the HFD of order $\frac{4}{7}$ and type $\lambda \in [0,1]$, $\mathcal{I}_{0^+}^{(\frac{3}{7})(1-\lambda)}$ is the R–L integral of order $(\frac{3}{7})(1-\lambda)$ and $k(\mathfrak{t},\varsigma) \in L^2([0,\pi]\times[0,\pi])$, m is a nonnegative number, and $0 < \mathfrak{t}_1 < \mathfrak{t}_2 < \cdots < \mathfrak{t}_m \leq 1$. Consider $\mathfrak{z}(\cdot)(\xi) = \mathfrak{z}(\cdot,\xi)$, $\mathscr{B}(\cdot)\mathfrak{u}(\cdot)(\xi) = \mathscr{B}\mathfrak{u}(\cdot,\xi)$, and

$$\mathfrak{F}(\mathfrak{z},\mathfrak{u}) = E\int_0^\pi\int_0^1 |\mathfrak{z}(\mathfrak{t},\xi)|^2 d\mathfrak{t}d\xi + \int_0^\pi\int_0^1 |\mathfrak{u}(\mathfrak{t},\xi)|^2 d\mathfrak{t}d\xi.$$

Let $W(\mathfrak{t})$ be the two-sided, one-dimensional Brownian motion in $\mathfrak{X}$ specified on filtered probability area $(\Lambda,\mathfrak{F},P)$, and $\partial\mathcal{G}$ represent the generalized gradient of a locally Lipschitz function $\mathcal{G}$. A simple example of $\mathcal{G}$ fulfilling the requirements (H_2) is $\mathcal{G}(\mathfrak{t},\mu) = \mathcal{G}(\mu) = \min\{g_1(\mu), g_2(\mu)\}$ where $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are convex quadratic functions (see [20,42]). The function $\psi(\mathfrak{t},\mathfrak{z}(\mathfrak{t})) = \mathbb{k}(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))$, satisfies the condition (H_5).

To write the above system (14) into the abstract form (1), set $\mathfrak{X} = Y := L^2[0,\pi]$. Specify, $\widetilde{A} : D(\widetilde{A}) \subset \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{J} : D(\mathcal{J}) \subset \mathfrak{X} \to \mathfrak{X}$ by $\widetilde{A}\mathfrak{z} = \mathfrak{z}_{\xi\xi}$ and $\mathcal{J}\mathfrak{z} = \mathfrak{z} - \mathfrak{z}_{\xi\xi}$, where domains $D(\widetilde{A})$ and $D(\mathcal{J})$ are provided by $\{\mathfrak{z} \in \mathfrak{X} : \mathfrak{z}, \mathfrak{z}_\xi$ are absolutely continuous, $\mathfrak{z}_{\xi\xi} \in \mathfrak{X},\ \mathfrak{z}(0) = \mathfrak{z}(\pi) = 0\}$. Then, $\widetilde{A}$ and $\mathcal{J}$ can be written as

$$\widetilde{A}\mathfrak{z} = \sum_{n=1}^{\infty} -n^2\langle \mathfrak{z}, e_n\rangle e_n,\ \mathfrak{z} \in D(\widetilde{A}),$$

$$\mathcal{J}\mathfrak{z} = \sum_{n=1}^{\infty} (1+n^2)\langle \mathfrak{z}, e_n\rangle e_n,\ \mathfrak{z} \in D(\widetilde{A}),$$

where $e_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny)$, $n = 1,2,\cdots$ is the orthonormal basis of eigenvectors. Moreover, for $\mathfrak{z} \in \mathfrak{X}$, we have

$$\mathcal{J}^{-1}\mathfrak{z} = \sum_{n=1}^{\infty} \frac{1}{1+n^2}\langle \mathfrak{z}, e_n\rangle e_n,$$

$$\widetilde{A}\mathcal{J}^{-1}\mathfrak{z} = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}\langle \mathfrak{z}, e_n\rangle e_n,$$

and $\mathbb{Q}_\lambda(\mathfrak{t}) = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2}{1+n^2}\right)\langle \mathfrak{z}, e_n\rangle e_n$. Here, $\widetilde{A}\mathcal{J}^{-1}$ generates the infinitesimal generator of a compact semigroup $T(\mathfrak{t})(\mathfrak{t} > 0)$ on $\mathfrak{X}$, presented as

$$T(\mathfrak{t})\mathfrak{z} = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}\mathfrak{t}}\langle \mathfrak{z}, e_n\rangle e_n,\ \mathfrak{z} \in \mathfrak{X},\ \|T(\mathfrak{t})\| \leq e^{-\mathfrak{t}},\ \forall\, \mathfrak{t} \geq 0.$$

Specify $\mathfrak{z}(\mathfrak{t})(\xi) = \mathfrak{z}(\mathfrak{t},\xi)$ and $\eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t}))(\xi) = \eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t},\xi))$ that fulfill the condition (H_3). Let the infinite-dimensional space $\mathbb{U}$ be defined by

$$\mathbb{U} = \{\mathfrak{u} : \mathfrak{u} = \sum_{n=2}^{\infty} \mathfrak{u}_n e_n \text{ with } \sum_{n=2}^{\infty} \mathfrak{u}_n^2 < \infty\}.$$

The norm in $\mathbb{U}$ is specified by $\|\mathfrak{u}\|_{\mathbb{U}} = (\sum_{n=2}^{\infty} \mathfrak{u}_n^2)^{\frac{1}{2}}$. Explain a bounded linear operator $\mathscr{B} : \mathbb{U} \to \mathfrak{X}$ as follows:

$$\mathscr{B}\mathfrak{u} = 2\mathfrak{u}_2 e_1 + \sum_{n=2}^{\infty} \mathfrak{u}_n e_n \text{ for } \mathfrak{u} = \sum_{n=2}^{\infty} \mathfrak{u}_n e_n \in \mathbb{U}.$$

Since all of the criteria of the Theorem 2 are met, system (14) may be written in the abstract form (1). The hemivariational stochastic control system (14) has a mild solution and at least one feasible optimal pair, according to Theorem 2.

5.2. Example-2

In this section, we give an example to illustrate our main results. Let $\mathfrak{X} = \mathbb{U} = L^2[0,\pi]$. Define $\widetilde{A} : D(A) \subset \mathfrak{X} \to \mathfrak{X}$ by $\widetilde{A}\mathfrak{z} = \mathfrak{z}_{yy}$, where domain $D(\widetilde{A})$ is given by

$$D(\widetilde{A}) = \{\mathfrak{z} \in \mathfrak{X} : \mathfrak{z},\ \mathfrak{z}_y \text{ are absolutely continuous}, \mathfrak{z}_{yy} \in \mathfrak{X}\mathfrak{z}(\mathfrak{t},0) = \mathfrak{z}(\mathfrak{t},\pi) = 0\}.$$

Then, $\widetilde{A}$ can be written as follows:

$$\widetilde{A}\mathfrak{z} = -\sum_{n=1}^{\infty} n^2 \langle \mathfrak{z}, e_n \rangle e_n,\ \mathfrak{z} \in D(\widetilde{A}),$$

where

$$e_n(y) = \sqrt{\frac{2}{\pi}} \sin ny,\ n = 1, 2, \cdots$$

is the orthonormal basis of eigenfunctions of $\widetilde{A}$. It is known that A is the infinitesimal generator of a compact semigroup $T(\mathfrak{t})(\mathfrak{t} > 0)$ on $\mathfrak{X}$ given by

$$T(\mathfrak{t})\mathfrak{z} = \sum_{n=1}^{\infty} e^{-n^2\mathfrak{t}} \langle \mathfrak{z}, e_n \rangle e_n,\ \mathfrak{z} \in \mathfrak{X},\ \|T(\mathfrak{t})\| \le e^{-\mathfrak{t}},\ \forall\, \mathfrak{t} \ge 0.$$

The admissible controls set $\mathscr{U}_{ad}$ is defined by $\mathscr{U}_{ad} = \{\mathfrak{u} \in \mathbb{U} | \|\mathfrak{u}\|_{L^2([0,1],\mathbb{U})} \le 1\}$. Consider the problem of finding the controls $\mathfrak{u}(\mathfrak{z}; y)$ to minimize the cost function

$$\mathfrak{F}(\mathfrak{z}, \mathfrak{u}) = E\int_0^{\pi} \left[\int_0^1 |\mathfrak{z}(\mathfrak{t}, y)|^2 d\mathfrak{t} dy + \int_0^{\pi}\int_0^1 |\mathfrak{u}(\mathfrak{t}, y)|^2 d\mathfrak{t} \right] dy,$$

subject to the following system:

$$\begin{cases} \Big\langle -D_{\mathfrak{t}}^{\lambda,\frac{3}{5}}[\mathfrak{z}(\mathfrak{t},y) - \kappa(\mathfrak{t},\mathfrak{z}(\mathfrak{t},y))] + [\mathfrak{z}_{yy}(\mathfrak{t},y) - \kappa(\mathfrak{t},\mathfrak{z}(\mathfrak{t},y))] + \int_0^1 q(y,\delta)\mathfrak{u}(\mathfrak{t},y)d\delta \\ +\eth(\mathfrak{t},\mathfrak{z}(\mathfrak{t},y))\frac{dW(\mathfrak{t})}{d\mathfrak{t}},\ \omega \Big\rangle + \mathcal{G}^0(\mathfrak{t},y,\mathfrak{z}(\mathfrak{t},y);\omega) \ge 0,\ \mathfrak{t} \in (0,1] = (0,\tilde{c}], \\ \mathfrak{z}(\mathfrak{t},0) = \mathfrak{z}(\mathfrak{t},\pi) = 0,\ \mathfrak{t} \in [0,1], \\ \mathcal{I}_{0^+}^{(\frac{2}{5})(1-\lambda)}(\mathfrak{z}(0,\xi)) + \beta(\mathfrak{z}(\mathfrak{t},y)) = \mathfrak{z}_0(\xi),\ \xi \in [0,\pi]. \end{cases} \tag{15}$$

where $\{W(\mathfrak{t})\}_{\mathfrak{t}\in\mathbb{R}}$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Lambda, \mathfrak{F}, P)$; $\eth : \mathscr{V} \times \mathfrak{X} \to L_0^2$ is a non-empty, bounded, closed, and convex multivalued map which satisfies the assumptions (H_3); $\psi : \mathscr{V} \times \mathfrak{X} \to \mathfrak{X}$ be the appropriate function; $q : [0,1] \times [0,1] \to \mathbb{R}$ is continuous.

Let a functional $\mathcal{G} : [0,1] \times \mathscr{X} \to \mathbb{R}$ be defined by

$$\mathcal{G}(\mathfrak{t}, \mathfrak{z}) = \int_0^1 g(\mathfrak{t}, y, \mathfrak{z}(y)) dy, \quad \mathfrak{t} \in (0,1),\ \mathfrak{z} \in \mathfrak{X},$$

where

$$g(\mathfrak{t}, y, z) = \int_0^z \gamma(\mathfrak{t}, y, \theta) d\theta, \quad (\mathfrak{t}, y) \in (0,1) \times (0,\pi), \quad z \in \mathbb{R}.$$

Suppose that $\gamma : (0,\pi) \times (0,1) \times \mathbb{R} \to \mathbb{R}$ is a functional such that

(i) for all $y \in (0,\pi)$, $z \in \mathbb{R}$, $\gamma(\cdot, y, z) : (0,1) \to \mathbb{R}$ is measurable;
(ii) for all $\mathfrak{t} \in (0,1)$, $z \in \mathbb{R}$, $\gamma(\mathfrak{t}, \cdot, z) : (0,\pi) \to \mathbb{R}$ is continuous;
(iii) for all $z \in \mathbb{R}$ there exists a constant $c_1 > 0$ satisfying

$$|\gamma(\cdot, \cdot, z)| \le c_1(1 + |z|);$$

(iv) for all $z \in \mathbb{R}$, $\gamma(\cdot, \cdot, z \pm 0)$ exists.

If γ satisfies (iii), then one has $\partial\mathcal{G}(z) \subset [\underline{\gamma}(z), \bar{\gamma}(z)]$ for $z \in \mathbb{R}$ (we omit $(\mathfrak{t}, y)$ here), where $\underline{\gamma}(z)$ and $\bar{\gamma}(z)$ stand for the essential infimum and essential supremum of η at a point z. If γ satisfies $(i) - (iv)$, then the function $g(\cdot,\cdot,\cdot)$ defined above has properties as follows:

(i) for all $y \in (0,\pi)$, $z \in \mathbb{R}$, $g(\cdot, y, z) : (0,1) \to \mathbb{R}$ is measurable and $g(\cdot,\cdot,0) \in L^2((0,1) \times (0,\pi))$;
(ii) for all $\mathfrak{t} \in (0,1)$, $z \in \mathbb{R}$, $g(\mathfrak{t},\cdot,z) : (0,\pi) \to \mathbb{R}$ is continuous;
(iii) for all $(\mathfrak{t}, y) \in (0,1) \times (0,\pi)$, $g(\mathfrak{t}, y, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz;
(iv) there exists a constant $c_2 > 0$ satisfying

$$|\zeta| \leq c_2(1 + |z|), \text{ for all } \zeta \in \partial g(\mathfrak{t}, y, \cdot),\ (\mathfrak{t}, y) \in (0,1) \times (0,\pi);$$

(v) there exists a constant $c_3 > 0$ satisfying

$$g^0(\mathfrak{t}, y, z, -z) \leq c_2(1 + |z|), \text{ for all } (\mathfrak{t}, y) \in (0,1) \times (0,\pi).$$

Moreover, the functional $\mathcal{G}(\cdot,\cdot)$ defined above satisfies conditions (H_2).

Define

$$\mathfrak{z}(\mathfrak{t}, y) = \mathfrak{z}(y)(\mathfrak{t}), \quad \mathscr{B}\mathfrak{u}(\mathfrak{t}) = \int_0^1 q(y,\delta)\mathfrak{u}(\mathfrak{t}, y)d\delta.$$

We also suppose that condition (H_6) holds. Thus, problem (15) can be written as the abstract form of problem (1) with the cost function

$$\mathfrak{F}(\mathfrak{z}, \mathfrak{u}) = E\int_0^\pi [\|\mathfrak{z}(\mathfrak{t})\|^2 + \|\mathfrak{u}(\mathfrak{t})\|^2]d\mathfrak{t}.$$

Summarizing the above, we know that the hypotheses (H_0)–(H_6) hold. Thus, all conditions of Theorems 2 and 3 are satisfied, and we conclude that problem (15) has a mild solution and at least one optimal pair.

6. Conclusions

This study explores the existence of mild solutions and OC for a class of Sobolev-type HF neutral stochastic evolution hemivariational inequalities. The properties of generalized Clarke's subdifferential and a fixed point approach for multivalued maps were first employed to provide appropriate standards for the existence of mild solutions to the focused control system. The existence of optimal state-control sets that are governed by Sobolev-type HF neutral stochastic evolution HVIs was then proven using limited Lagrange optimal systems. The OC results are gathered without considering how unique the control system's solutions are. The main result is eventually illustrated with an example. In the next study, we will talk about the optimal controls for HF stochastic Volterra–Fredholm integro-differential evolution HVIs with Poisson jumps, as well as the existence of mild solutions.

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Abbreviations

The following abbreviations are used in this manuscript:

HF	Hilfer fractional
HFD	Hilfer fractional derivative
FDEs	Fractional differential equations
HVI	Hemivariational inequality
HVIs	Hemivariational inequalities
OC	Optimal control
SDEs	Stochastic differential equations
SEEs	Stochastic evolution equations
SEHVIs	Stochastic evolution hemivariational inequalities
R–L	Riemann–Liouville
RHS	Right-hand side
LDCT	Lebesgue dominated convergence theorem

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