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# Fractional Scale Calculus: Hadamard vs. Liouville

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**Abstract:** A general fractional scale derivative is introduced and studied. Its relation with the Hadamard derivatives is established and reformulated. A new derivative similar to the Grünwald–Letnikov's is deduced. Tempered versions are also introduced. Scale-invariant systems are described and exemplified. For solving the corresponding differential equations, a new logarithmic Mittag-Leffler series is proposed.

**Keywords:** scale derivative; Liouville derivative; Hadamard derivative; tempered derivative; logarithmic series

**MSC:** 26A33

## 1. Introduction

Scale is a term we find often but that we do not know how to define. L. Cohen [1] considered it as a physical attribute similar to frequency; he proposed a scale operator for representing it and introduced a scale transform. However, L. Nottale [2] defined scale as the resolution with which measurements are performed and considered it as a relative state of the reference system. He paid attention mainly to the scale transformations he obtained with the operators based on the scale derivative obtained from fractal considerations [3,4]. A similar procedure was developed by J. Cresson [5,6]. For these approaches, the important factor is the scale invariance [7,8], which led to establishing bridges to other formulations such as the Lamperti transformations for stochastic processes [9,10] and the operator theory [11]. However, the importance of scale was fully recognized in the multiscale/multiresolution analysis introduced during the eighties of the last century in signal processing, mainly in image processing, obtained from wavelet transform [12–16]. In applications, the discrete version is normally used [17–19]. However, this approach goes further in the sense that it simultaneously explores the shift and scale characteristics of the wavelet used to define the transform such that it keeps the form but changes the position and scale. Contrary to this route, we are interested in the study of tools and systems that are scale invariant. It is not surprising that the Mellin transform has appeared in this context (see J. Bertrand et al. [20]) and, as consequence, in fractional calculus through the Hadamard derivative [21–24].

In this paper, we join Mellin transform and Hadamard's idea to introduce a general framework for the scale fractional derivative. We follow a path identical to the one proposed by Liouville for the shift-invariant fractional derivatives [25,26].

The first reference to the possibility of non-integer order derivatives was found in a letter from G. Leibniz to J. Bernoulli [27]. Although Euler (1730), Fourier (1822), and Abel (1823) touched on the problem, we must consider Liouville as the true father of fractional calculus (FC) through two papers he published in 1832 [25,26]. His approach was based on generalizing the formula for the derivative of the exponential. This brought many difficulties in imposing his vision, since, at that time, the inverse Laplace integral was unknown; so, Liouville could not find a simple way of expressing a function in terms of



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exponentials. The main definitions of fractional derivatives (FD) we find nowadays are based on the formulæ presented by Liouville, mainly the Riemann–Liouville [22,28,29], (Dzherbashian)–Caputo [28–30], and Grünwald–Letnikov [22,26,31] definitions. However, and based on these derivatives, other ones were proposed. Among the most important, we refer to Hadamard's [21,28] and Marchaud's [22].

It is important to remark that Liouville's approach was based on the fact that the order $\alpha$ fractional derivative, $D^\alpha$, of

$$f(t) = \sum_{n\geq 0} a_n e^{\gamma_n t},$$

with $a_n, \gamma_n \in \mathbb{R}$, would be given by

$$D^\alpha f(t) = \sum_{n\geq 0} a_n \gamma_n^\alpha e^{\gamma_n t},$$

Hadamard [21] arrived at FC through Riemann's formulation [32] and had in mind the search for a derivative, $\mathfrak{D}$, such that if the function at hand has a MacLaurin expansion,

$$g(t) = \sum_{n\geq 0} a_n t^n,$$

then

$$\mathfrak{D} g(t) = \sum_{n\geq 0} a_n n^\alpha t^n.$$

With a straightforward modification to the Riemann formulation followed by a logarithmic transformation, he could devise the first formula for what is called the Hadamard integral. If $\alpha > 0$, it reads

$$\mathfrak{D}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (-\ln(t))^{\alpha-1} f(t\eta) \frac{d\eta}{\eta} \tag{1}$$

It is interesting to remark that, in passing, he introduced the operator $t^\alpha D^\alpha f(t)$ that we will recover later; however, he did not formally present the derivative expression. This was achieved, for the first time, by Samko et al. [22].

The framework we will propose starts from two different types of linear systems, shift and dilation (scale)-invariant, such that we establish a parallel way into the Liouville and Hadamard derivatives, having their eigenfunctions as "seeds". The derivatives are defined from the corresponding eigenvalues by Laplace and Mellin inverse transforms. The shift-invariant systems led to the Grünwald–Letnikov (GL) and Liouville (L) derivatives [33]. Similar procedures here will lead to the corresponding Hadamard and new GL-like scale derivatives. The formulation clarifies the difference between left and right derivatives through their regions of convergence: right or left complex half-planes.

The paper is outlined as follows. In Section 2, we make a brief introduction to the linear systems. In Section 3, the derivatives based on the Liouville formulation and referred in [33–35] are then obtained sequentially; these result from the shift-invariant systems. Similar development is achieved from the scale-invariant systems in Section 4, where we obtain their eigenfunctions and corresponding eigenvalue that is the base for the definition of scale derivative. In passing, we established its relation with the classic and quantum derivatives and deduced a logarithmic series analog to the Taylor series. The properties of the scale derivative allowed us to reformulate the Hadamard derivatives that we generalized to obtain the tempered scale derivatives. Finally, we conclude the paper with a reflection on the results we obtained.

**Remark 1.** *The questions associated with the existence and uniqueness of the fractional derivatives are very important. They have been treated in many good texts: [22,28,30] for Liouville type derivatives, [22,31] for the GL, and [22,28,36–43] for the Hadamard derivatives. Therefore, we will not pay attention to the subject. We will assume always to work with functions having Laplace or Mellin transform.*

## 2. On the Linear Systems

In a simple way, we can define a system as a combination of components (mathematical, physical, biological, social, etc.) acting together to fulfill certain predefined goals. Interaction with systems is performed by means of functions called signals [44]. Mathematically, a system is defined as an application on the set of signals—that is, a transformation of one signal, $x(t)$, into another, $y(t)$. Let $T[.]$ be an operator that symbolically represents such a transformation,

$$y(t) = T[x(t)], \tag{2}$$

where $x(t)$ is the input or excitation and $y(t)$ is the output or response. A system is linear if it is formally represented by a linear operator, which means that it satisfies the properties of additivity and homogeneity (superposition principle) [44,45]

$$\begin{aligned} y(t) &= T[a_1 x_1(t) + a_2 x_2(t)] \\ &= a_1 T[x_1(t)] + a_2 T[x_2(t)] \\ &= a_1 y_1(t) + a_2 y_2(t) \end{aligned} \tag{3}$$

There are many classes of linear systems (LS). Among them, we consider the shift-invariant and dilation (scale)-invariant.

**Definition 1.** *We call a linear system shift-invariant (SI) if its input–output relation is given by the usual convolution:*

$$y(t) = x(t) * g(t) = \int_{-\infty}^{+\infty} x(t-\eta) g(\eta) d\eta, \tag{4}$$

*where $t \in \mathbb{R}$ and $g(t)$ is its impulse response (or Green function): the response to $x(t) = \delta(t)$.*

**Definition 2.** *We call a linear system scale-invariant or dilation-invariant (DI) if its input–output relation is given by the Mellin convolution*

$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta) \frac{d\eta}{\eta}, \tag{5}$$

*where $\tau \in \mathbb{R}^+$ and $g(\tau)$ is the impulse response: the response to $x(\tau) = \delta(\tau - 1)$.*

We demand that the impulse response, $g(t)$, be at least

- piecewise continuous,
- with bounded variation.

The specific applications impose constraints on the signals. However, we will assume they are square integrable: "signals type energy" [46]. For "enough good" functions, the derivative (suitably defined) of a convolution is equal to the convolution of one function with the derivative of the other,

$$D^\alpha x(t) * y(t) = x(t) * D^\alpha y(t).$$

**Remark 2.** *It is important to emphasize that, if $D$ represents a SI derivative, from a numerical point of view, $D^\alpha (x * y)$ is preferable relative to $D^\alpha x * y$ or $x * D^\alpha y$. For the Mellin convolution, the situation is similar.*

## 3. Shift-Invariant Systems: The Liouville Derivatives

Consider a linear shift-invariant system (LTIS) defined by (4). Then, the following apply [44]:

1. The exponentials are the eigenfunctions of LTIS

$$y(t) = G(s)e^{st}, \quad t \in \mathbb{R}, \tag{6}$$

with eigenvalues given by $G(s)$ given by

$$G(s) = \int_{-\infty}^{+\infty} g(t)e^{-st}dt$$

which is the transfer function and the (bilateral) Laplace transform (LT) of the impulse response of the system.

2. If the region of convergence (ROC) of $G(s)$ contains the imaginary axis, we can set $s = i\omega,\ \omega \in \mathbb{R}$, in such a way that the response of an LS to a sinusoid is also a sinusoid with the same frequency. In such a situation, the LT degenerates into the Fourier transform and we say that the system is stable.

**Definition 3.** *Let $\alpha \in \mathbb{R}$. The $\alpha$-order SI derivative, $D^\alpha$, is an operator such that*

$$D^\alpha e^{st} = s^\alpha e^{st}, \quad t \in \mathbb{R}, s \in \mathbb{C}, \tag{7}$$

*for $Re(s) > 0$ (causal case) or $Re(s) < 0$ (anti-causal case). We can extend the validity of this definition to $Re(s) = 0$ if $s \neq 0$.*

*This definition dates back to Liouville who introduced it in the first paper on fractional calculus [25].*

*The inverse LT theorem (Bromwich integral) [47] allows us to state that if $x(t)$ is a function with LT, $X(s)$, then $x(t)$ has a derivative in the Liouville sense and*

$$D^\alpha x(t) = \frac{1}{2\pi i}\int_\gamma s^\alpha X(s)e^{st}ds, \tag{8}$$

*where $\gamma$ is a vertical straight line located in the ROC of $X(s)$.*

**Remark 3.** *In the following, we shall be working with derivative orders that are always real numbers. The results remain valid if they are complex numbers, but the resulting derivatives are not hermitian operators, implying that they are not useful in applications [48].*

Liouville's definition highlights the importance of the transfer function, $G(s) = s^\alpha$. The way how we express it in the inverse time domain leads to several expressions for the derivative. To start, we can obtain direct generalizations of classic derivatives by noting that

$$s^\alpha = \lim_{h\to 0^+} \begin{cases} \left(\dfrac{(1-e^{-sh})}{h}\right)^\alpha & Re(s) > 0, \\[2ex] \left(\dfrac{(e^{sh}-1)}{h}\right)^\alpha & Re(s) < 0, \end{cases} \tag{9}$$

that lead to the forward and backward Grünwald–Letnikov (GL) derivatives, respectively [22,31]. However, the first proposal of such a derivative was also presented by Liouville [26]. Let $(a)_n,\ n = 1, 2, \cdots$ define the Pochhammer symbol for the rising factorial

$$(a)_0 = 1,\ (a)_n = \prod_{k=0}^{n-1}(a+k).$$

As

$$(1-e^{-sh})^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!}e^{-nsh}$$

it is a simple matter to obtain the GL derivatives using the translation property of the LT. For the causal (forward) case, we have

$$D_{+}^{\alpha} f(t) := \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{+\infty} \frac{(-\alpha)_n}{n!} f(t - nh). \tag{10}$$

The backward is obtained through the substitution $-h \to h$.

However, we may be more interested here in obtaining the convolutional versions of the SI derivatives. As is well known, the impulse response, $g(t)$, of the SI system with transfer function $s^{\alpha}$ is given by [47]

$$g(t) = \pm \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \varepsilon(\pm t) \tag{11}$$

where $\varepsilon(t)$ is the Heaviside unit step; the $+$ sign corresponds to the causal case ($Re(s) > 0$) and $-$ to the anti-causal ($Re(s) < 0$) [31]. Consequently, attending to (7) and to the convolution property of the LT, we conclude that

$$D_{\pm}^{\alpha} x(t) = x(t) * g(t) = \pm \frac{1}{\Gamma(-\alpha)} \int\limits_0^{\infty} \eta^{-\alpha-1} f(t \mp \eta)\, d\eta, \tag{12}$$

The formula corresponding to the $-$ sign was introduced first by Liouville [25] who noted that the derivative case ($\alpha > 0$) leads to a singular convolution. However, instead of a regularization, he devised a trick for solving the singularity problem that we can describe as

$$s^{\alpha} = s^N s^{\alpha-N} = s^{\alpha-N} s^N, \tag{13}$$

where $N > \alpha$. The most usual choice is the first integer greater than $\alpha$, $N - 1 < \alpha \leq N$, that we denote by $N = \lceil \alpha \rceil$. Basically, it consists of transferring the singular behavior to an integer order derivative. The first approach leads to the so-called (Riemann–)Liouville derivatives [22],

$$D_{+}^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^{\infty} f(t-\eta)\eta^{N-\alpha-1}\, d\eta = \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_{-\infty}^{t} f(\eta)(t-\eta)^{N-\alpha-1}\, d\eta \tag{14}$$

and

$$D_{-}^{\alpha} f(t) = \frac{(-1)^N}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^{\infty} f(t+\eta)\eta^{N-\alpha-1}\, d\eta = \frac{(-1)^N}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_{t}^{\infty} f(\eta)(t-\eta)^{N-\alpha-1}\, d\eta. \tag{15}$$

Liouville's second procedure leads to what can be called Liouville–Caputo derivatives [44,49] that read

$$D_{+}^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_{-\infty}^{t} f^{(N)}(\eta)(t-\eta)^{N-\alpha-1}\, d\eta \tag{16}$$

and

$$D_{-}^{\alpha} f(t) = \frac{(-1)^N}{\Gamma(N-\alpha)} \int_{t}^{\infty} f^{(N)}(\eta)(t-\eta)^{N-\alpha-1}\, d\eta. \tag{17}$$

**Remark 4.** *The usually called Riemann–Liouville and (Dzherbashian–)Caputo derivatives are particular cases of these, obtained for functions defined in intervals of* $\mathbb{R}$.

The problem raised by the singularity of the integral (12) can be solved in an alternative way through a regularization [31,44]. For the forward case, it reads

$$D_{+}^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int\limits_0^{\infty} \eta^{-\alpha-1} \left[ f(t-\eta) - \sum_{0}^{N-1} \frac{(-)^m f^{(m)}(t)}{m!} \eta^m \right] d\eta, \tag{18}$$

which we will call the regularized forward Liouville derivative. The backward case is similar

$$D^{\alpha}_{+}f(t) = \frac{1}{\Gamma(-\alpha)} \int\limits_{0}^{\infty} \eta^{-\alpha-1} \left[ f(t+\eta) - \sum_{0}^{N-1} \frac{f^{(m)}(t)}{m!} \eta^{m} \right] d\eta, \tag{19}$$

**Remark 5.** *The method introduced in (13) can be generalized. For example,*

$$s^{\alpha} = s^{N-k} s^{\alpha-N} s^{k}, \qquad \alpha \in \mathbb{R}^{+}, N \in \mathbb{Z}^{+}_{0}, \tag{20}$$

*with $k < N$ and $N > \alpha$. This procedure was used by Davidson, Essex, and Cavanati [35]. Similarly, Hilfer [50] proposed the decomposition*

$$s^{\alpha} = s^{\mu(1-\alpha)} s s^{(1-\mu)(1-\alpha)}, \tag{21}$$

*where $0 < \alpha, \mu < 1$. This approach can be generalized and can lead to the invention of many alternative derivatives [33].*

The derivatives that we described in this section are suitable for modeling SI linear systems. However, they are useful in dealing with nonlinear systems too [51].

## 4. On the Scale-Invariant Systems: Hadamard Derivatives

### *4.1. From the System to the Derivative*

In this section, we are going to reproduce the approach used in the SI case to the DI. We start from (5), which we repeat here

$$y(\tau) = x(\tau) \star g(\tau) = \int_0^{\infty} x\left(\frac{\tau}{\eta}\right) g(\eta) \frac{d\eta}{\eta}.$$

We must remark that the procedure in the following only makes sense if the working domain is $\mathbb{R}^{+}$. Therefore, we will assume it implicitly in the following.

Similarly to the SI case, the powers $x(\tau) = \tau^{v}$, $\tau \in \mathbb{R}^{+}, v \in \mathbb{C}$ are the eigenfunctions of the DI systems. In fact, if the input is $x(\tau) = \tau^{v}$, then the output $y(\tau) = x(\tau) \star g(\tau)$ is

$$y(\tau) = G(v)\tau^{v} \tag{22}$$

where $G(v)$ is the transfer function given by

$$G(v) = \int_0^{\infty} g(u) u^{-v-1} du, \tag{23}$$

which is the Mellin transform (MT) of the impulse response.

The MT in (23) has a parameter sign change $-v \rightarrow v$ relatively to the usual MT [20,52,53]. However, we will keep this definition, since it establishes a better parallelism with the LT, which concerns the region of convergence. The corresponding inverse Mellin transform is given by the integral

$$x(\tau) = \mathcal{M}^{-1}[X(v)] = \frac{1}{2\pi i} \int_{\gamma} X(v) \tau^{v} dv, \quad \tau \in \mathbb{R}^{+} \tag{24}$$

where $\gamma$ is a vertical straight line in the ROC of the transform.

**Remark 6.** *As in the SI case, if the ROC of the transfer function includes the imaginary axis, we will say that the system is stable. This corresponds to write*

$$\int_0^{\infty} |g(u)| u^{-1} du < \infty.$$

Consider the relation (22). We are going to use it as a starting point to obtain the Hadamard derivatives and two new ones, analog to the GL derivatives.

**Definition 4.** *Let $\alpha \in \mathbb{R}$. We define the $\alpha$-order scale derivative (SD) as the operator $\mathfrak{D}_s$ obeying the rule*

$$\mathfrak{D}_s^\alpha \tau^v = v^\alpha \tau^v, \quad \tau \in \mathbb{R}^+, v \in \mathbb{C}, \tag{25}$$

*for $Re(v) > 0$ (expansion case) or $Re(v) < 0$ (shrinkage case).*

If a function, $x(\tau)$, has MT, $X(v)$, then it has a fractional *scale-derivative* that is given by

$$\mathfrak{D}_s^\alpha x(\tau) = \frac{1}{2\pi i} \int_\gamma v^\alpha X(v) \tau^v dv, \quad t \in \mathbb{R}^+ \tag{26}$$

where $\gamma$ is a vertical straight line located in the ROC of $X(v)$. Similarly to the shift-invariant case, (25) shows that the scale derivative is an elemental system with transfer function $v^\alpha$. Again, the way we express it in the inverse scale domain leads to various expressions for the derivative.

**Theorem 1.** *Let $q > 1$. The following expressions,*

$$\mathfrak{D}_{s+}^\alpha x(\tau) = \lim_{q \to 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(\tau q^{-n}), \tag{27}$$

*and*

$$\mathfrak{D}_{s-}^\alpha x(\tau) = \lim_{q \to 1^+} (-1)^\alpha \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} x(\tau q^{n}), \tag{28}$$

*represent scale-derivatives that we can call stretching and shrinking GL-type derivatives, respectively.*

**Proof.** As it is not hard to deduce,

$$v^\alpha = \lim_{q \to 1^+} \begin{cases} \left[\dfrac{(1 - q^{-v})}{\ln q}\right]^\alpha & Re(v) > 0, \\ \left[\dfrac{(q^{v} - 1)}{\ln q}\right]^\alpha & Re(v) < 0, \end{cases} \tag{29}$$

Using the binomial theorem,

$$(1 - q^{\pm v})^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{\pm nv},$$

and the dilation property of the MT,

$$\mathcal{M}[x(a\tau)] = a^v X(v) \quad a \in \mathbb{R}^+, \tag{30}$$

we obtain (27) and (28). □

We can extend these results to $Re(v) = 0,\ v \neq 0$.

Particular cases are interesting. Let $\alpha = \pm 1$.

1. From (27), we obtain

$$\mathfrak{D}_{s+} x(\tau) = \lim_{q \to 1^+} \frac{x(\tau) - x(\tau q^{-1})}{\ln q} \tag{31}$$

and

$$\mathfrak{D}_{s+}^{-1} x(\tau) = \lim_{q \to 1^+} \ln q \sum_{n=0}^{\infty} x(\tau q^{-n}),$$

2. Ref. (28) gives

$$\mathfrak{D}_{s-}x(\tau) = \lim_{q\to 1^+} \frac{x(\tau q) - x(t)}{\ln q} \tag{32}$$

and

$$\mathfrak{D}_{s-}^{-1}x(\tau) = -\lim_{q\to 1^+} \ln q \sum_{n=0}^{\infty} x(\tau q^n),$$

**Example 1.**

1. *Power functions:* $\tau^a$
*We have*

$$\mathfrak{D}_{s+}\tau^a = \lim_{q\to 1^+} \frac{\tau^a - \tau^a q^{-a}}{\ln q} = \tau^a \lim_{q\to 1^+} \frac{1-\left(1 - a\ln(q) + a^2 a \ln^2(q)/2 - \cdots\right)}{\ln q} = a\tau^a$$

*and*

$$\mathfrak{D}_{s-}\tau^a = \lim_{q\to 1^+} \frac{\tau^a q^a - \tau^a}{\ln q} = \tau^a \lim_{q\to 1^+} \frac{q^a - 1}{\ln q} = a\tau^a$$

2. *Logarithm:* $\ln^a(\tau)$
*As above, we obtain*

$$\begin{aligned}\mathfrak{D}_{s+}\ln^a(\tau) &= \lim_{q\to 1^+} \frac{\ln^a(\tau) - (\ln(\tau) - \ln(q))^a}{\ln(q)} \\ &= \ln^a(\tau) \lim_{q\to 1^+} \frac{a\frac{\ln(q)}{\ln(\tau)} - \frac{a^2}{2}\left(\frac{\ln(q)}{\ln(\tau)}\right)^2 + \cdots}{\ln(q)} = a\ln^{a-1}(\tau)\end{aligned} \tag{33}$$

*and*

$$\begin{aligned}\mathfrak{D}_{s-}\ln^a(\tau) &= \lim_{q\to 1^+} \frac{(\ln(\tau) + \ln(q))^a - \ln^a(\tau)}{\ln(q)} \\ &= \ln^a(\tau) \lim_{q\to 1^+} \frac{a\frac{\ln(q)}{\ln(\tau)} + \frac{a^2}{2}\left(\frac{\ln(q)}{\ln(\tau)}\right)^2 + \cdots}{\ln(q)} = a\ln^{a-1}(\tau)\end{aligned} \tag{34}$$

### *4.2. Properties of the Scale Derivatives*

Definition 4 together with (26), which serve as base for the introduction of the scale derivatives, allow us to obtain their corresponding properties that are similar to those of the Liouville derivatives [31]. Let $\alpha,\ \beta \in \mathbb{R}$. The scale derivative enjoys the following properties:

- Linearity
It is obvious from (26).
- Additivity and Commutativity of the orders

$$\mathfrak{D}_s^{\alpha}\mathfrak{D}_s^{\beta}x(\tau) = \mathfrak{D}_s^{\alpha+\beta}x(\tau). \tag{35}$$

This comes from (25).
- Neutral and inverse elements
Let $\alpha = -\beta$. Then,

$$\mathfrak{D}_s^{\alpha}\mathfrak{D}_s^{-\alpha}x(\tau) = \mathfrak{D}_{s-}^{0}x(\tau) = x(\tau). \tag{36}$$

From (36), we conclude that there is always an inverse element—that is, for every $\alpha$ there is always the $-\alpha$ order that we call anti-derivative.
- The generalized Leibniz rule

This rule gives the FD of the product of two functions and assumes the format of other fractional derivatives [31]

$$\mathfrak{D}_s^\alpha[x(\tau)y(\tau)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} \mathfrak{D}_s^k x(\tau) \mathfrak{D}_s^{\alpha-k} y(\tau). \tag{37}$$

To prove this relation, we note first that

$$\mathcal{M}[x(\tau)y(\tau)] = X(v) \star Y(v).$$

Using the Bromwich inverse Mellin transform, we can write

$$\mathfrak{D}_s^\alpha[x(\tau)y(\tau)] = \frac{1}{2\pi i} \int_{\gamma_1} v^\alpha \int_{\gamma_2} X(u)Y(v-u)du\tau^v dv,$$

where $\gamma_1$ and $\gamma_2$ are vertical straight lines in the intersection of the region of convergence of both transforms. With a trick, with which we must be careful,

$$v^\alpha = (v-u+u)^\alpha = (v-u)^\alpha \left[1 + \frac{u}{v-u}\right]^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} k^k (v-u)^{\alpha-k},$$

and evident manipulations, we obtain

$$\mathfrak{D}_s^\alpha[x(\tau)y(\tau)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{1}{2\pi i} \int_{\gamma_1} \int_{\gamma_2} u^k X(u)(v-u)^{\alpha-k} Y(v-u)du\tau^v dv,$$

from where the property (37) results.

These properties refer to derivatives defined on $\mathbb{R}^+$. In mathematical texts, it is usual to restrict the definitions by including the domain of the functions that are assumed to have bounded support. In such situations, the properties are not exactly as those we presented [37].

*4.3. Relation with Classic and Quantum Derivatives*

Consider derivative (31). Assume that we apply the l'Hôpital rule for the indeterminacy. We obtain

$$\mathfrak{D}_{s+}x(\tau) = \lim_{q\to 1^+} \frac{x(\tau) - x(\tau q^{-1})}{\ln q} = \lim_{q\to 1^+} \frac{\tau x'(\tau q^{-1})}{1/q} = \tau x'(\tau),$$

which expresses the usual way of representing $\mathfrak{D}_{s+}x(\tau)$ [22]. Therefore, the scale derivative of order $n$, $\mathfrak{D}_{s+}^n$ is the result of applying $\tau x'(\tau)$ $n$ times. Such interpretation is not so obvious in the fractional case. Anyway, we can write $\mathfrak{D}_{s+}^\alpha = (\tau D)^\alpha$. As for $\mathfrak{D}_{s-}x(\tau)$, the conclusion is the same.

These results suggest we go into the usual derivative definition

$$x'(t) = \lim_{h\to 0^+} \frac{x(t) - x(t-h)}{h}$$

and make the substitution $t - h \to tq^{-1}$ to obtain

$$x'(t) = \lim_{q\to 1^+} \frac{x(t) - x(tq^{-1})}{(1-q^{-1})t}. \tag{38}$$

Similarly, with

$$x'(t) = \lim_{h\to 0^+} \frac{x(t+h) - x(t)}{h}$$

and the substitution $t + h \to tq$, we obtain

$$x'(t) = \lim_{q\to 1^+} \frac{x(qt) - x(t)}{(q-1)t}. \quad (39)$$

The Formulae (38) and (39) express the so-called "quantum derivatives", frequently written without the limit operation [54,55]. These results lead to other formulations for the scale derivative alternative to (31) and (32). We may write

$$\mathfrak{D}_{q+}x(\tau) = \lim_{q\to 1^+} \frac{x(\tau) - x(\tau q^{-1})}{(1-q^{-1})}. \quad (40)$$

It must be noted that $\mathfrak{D}_{q+}x(\tau) = \mathfrak{D}_{s+}x(\tau)$, since $\lim_{q\to 1^+} \ln q = \lim_{q\to 1^+} 1 - q^{-1} = 0$. Similarly, we define

$$\mathfrak{D}_{q-}x(\tau) = \lim_{q\to 1^+} \frac{x(q\tau) - x(\tau)}{(q-1)}. \quad (41)$$

The fractionalization of derivatives (38) and (39) was performed in [56]. For the first case, we obtained a Grünwald–Letnikov like fractional quantum derivative

$$D^{\alpha}_{q+}x(\tau) = \tau^{-\alpha} \lim_{q\to 1^+} \frac{\sum_{j=0}^{\infty} \left[\begin{array}{c} \alpha \\ j \end{array}\right]_{q^{-1}} (-1)^j q^{-j(j+1)/2} q^{j\alpha} x\left(q^{-j}\tau\right)}{(1-q^{-1})^{\alpha}} \quad (42)$$

where the q-binomial coefficients are given by

$$\left[\begin{array}{c} \alpha \\ i \end{array}\right]_q = \frac{[\alpha]_q!}{[j]_q![\alpha - i]_q!}$$

and

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}$$

Identically, we obtain

$$D^{\alpha}_{q-}x(\tau) = \tau^{-\alpha} \lim_{q\to 1^+} \frac{\sum_{j=0}^{\infty} \left[\begin{array}{c} \alpha \\ j \end{array}\right]_{q^{-1}} (-1)^j q^{-j(j-1)/2} x\left(q^{j}\tau\right)}{(q-1)^{\alpha}} \quad (43)$$

Formulas (42) and (43) represent derivatives that are essentially Liouville derivatives. From them, we obtain new scale derivatives through

$$\mathfrak{D}^{\alpha}_{q\pm}x(\tau) = \tau^{\alpha} D^{\alpha}_{q\pm}x(\tau).$$

However, these are different from those we presented above, (27) and (28). Its usefulness is in solving the Euler–Cauchy differential equations [57].

### *4.4. Scale Conversion: Logarithmic Series*

The previous results show that the logarithm here plays the role performed by the power in the Liouville derivatives. Therefore, we expect to obtain a series analog to Taylor/MacLaurin's.

On the other hand, we considered a DI system, but nothing was said about the scale modification. With the tools developed above, we can obtain a scale conversion.

**Theorem 2.** *Let $f(\tau)$ be an indefinitely scale differentiable bounded function and $a > 0$. Then,*

$$f(a\tau) = \sum_{n=0}^{\infty} \frac{\mathfrak{D}_s^n f(a)}{n!} \ln^n(\tau) \tag{44}$$

**Proof.** Computing the MT, (23), of $f(a\tau)$ relative to $a$ (the two variables play the same role), it is not hard to show that

$$\mathcal{M}[f(a\tau)](v) = \tau^v F(v). \tag{45}$$

Therefore, there is an operator with a transfer function $\tau^v$ that performs a scale conversion. We can write

$$\tau^v = \sum_{n=0}^{\infty} \frac{v^n}{n!} \ln^n(\tau),$$

which leads immediately to the result. We can obtain another justification by performing successive scale derivations using (33) followed by $\tau \to 1$. We must note that $\mathfrak{D}_s^n \ln^n(\tau) = n \ln^{n-1}(\tau)$. □

The convergence of the series can be stated by the traditional methods used in the study of power series.

Expression (44) shows how ideally we can obtain a scale conversion.

**Example 2.** *Let $N \in \mathbb{Z}^+$ and $f(\tau) = \tau^N$. According to Example 1,*

$$\mathfrak{D}_s^n f(a) = N^n f(a) = N^n a^N$$

*and*

$$f(a\tau) = a^N \sum_{n=0}^{\infty} \frac{N^n}{n!} \ln^n(\tau) = a^N e^{N\ln(\tau)} = a^N \tau^N$$

*as expected.*

**Corollary 1.** *From (44), we obtain*

$$f(a/\tau) = \sum_{n=0}^{\infty} (-1)^n \frac{\mathfrak{D}_{s+}^n f(a)}{n!} \ln^n(\tau) \tag{46}$$

The proof is obvious.

Relations (44) and (46) are valid for both $\mathfrak{D}_{s+}^n$ and $\mathfrak{D}_{s-}^n$ The logarithmic series will be useful in the next sub-section to regularize a fractional derivative.

### *4.5. Hadamard Derivatives*

In Section 4.1, we presented two GL-like scale derivatives. If we apply the MT to (27) and (28), and attend to (26), we are led to the generalization of a well-known property:

$$\mathcal{M}[\mathfrak{D}_{s\pm}^{\alpha} x(\tau)] = v^{\alpha} X(v), \quad \pm Re(v) > 0. \tag{47}$$

This result means that there should exist two, $g_{\pm}(t)$, MT inverses of $v^{\alpha}$, one for each ROC.

**Theorem 3.** *Let us assume that $\alpha < 0$. The MT inverse of $v^{\alpha}$ is given by*

$$\mathcal{M}^{-1}[v^{\alpha}](\tau) = \begin{cases} \dfrac{\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)}\varepsilon(\tau-1) & Re(v) > 0, \\[2ex] \dfrac{\ln^{-\alpha-1}(1/\tau)}{\Gamma(-\alpha)}\varepsilon(1-\tau) & Re(v) < 0. \end{cases} \tag{48}$$

**Proof.** We can write for any $\beta < 0$,

$$\frac{1}{v^\beta} = \int_0^\infty \frac{\eta^{\beta-1}}{\Gamma(\beta)} e^{-v\eta} d\eta = \int_1^\infty \frac{\ln^{\beta-1}(\tau)}{\Gamma(\beta)} \tau^{-v-1} d\tau,$$

from where the first expression results. The second is obtained with the substitution $\tau \to 1/\tau$. $\square$

These MT inverses allow us to obtain two new scale anti-derivatives.

**Corollary 2.** *Let $\alpha < 0$ and $\tau \in \mathbb{R}^+$. The relations stated in (47) and (48) used in the Mellin convolution lead to the scale anti-derivatives given by*

$$\mathfrak{D}^\alpha_{s+} x(\tau) = \frac{1}{\Gamma(-\alpha)} \int_1^\infty x(\tau/\eta) \ln^{-\alpha-1}(\eta) \frac{d\eta}{\eta} = \frac{1}{\Gamma(-\alpha)} \int_0^\tau x(u) \ln^{-\alpha-1}(\tau/u) \frac{du}{u} \tag{49}$$

*and*

$$\mathfrak{D}^\alpha_{s-} x(\tau) = \frac{1}{\Gamma(-\alpha)} \int_0^1 x(\tau/\eta) \ln^{-\alpha-1}(1/\eta) \frac{d\eta}{\eta} = \frac{1}{\Gamma(-\alpha)} \int_\tau^\infty x(u) \ln^{-\alpha-1}(u/\tau) \frac{du}{u} \tag{50}$$

These anti-derivatives are the Hadamard integrals ([22,28]).

**Theorem 4.** *As in the shift-invariant cases, these integrals become singular when the derivative order is positive. We can proceed as in (13) to obtain the following:*

1. *Hadamard right derivative [22,28]*

$$\mathfrak{D}^\alpha_{s+} x(\tau) = \frac{1}{\Gamma(N-\alpha)} \mathfrak{D}^N_{s+} \int_1^\infty x(\tau/\eta) \ln^{N-\alpha-1}(\eta) \frac{d\eta}{\eta} \tag{51}$$

2. *Hadamard left derivative [22,28]*

$$\mathfrak{D}^\alpha_{s-} x(\tau) = \frac{1}{\Gamma(N-\alpha)} \mathfrak{D}^N_{s-} \int_0^1 x(\tau/\eta) \ln^{N-\alpha-1}(1/\eta) \frac{d\eta}{\eta} \tag{52}$$

3. *Hadamard–Liouville right derivative*

$$\mathfrak{D}^\alpha_{s+} x(\tau) = \frac{1}{\Gamma(N-\alpha)} \int_1^\infty \left[\mathfrak{D}^N_{s+} x(\tau)\right] \ln^{N-\alpha-1}(\tau/u) \frac{du}{u} \tag{53}$$

4. *Hadamard–Liouville left derivative*

$$\mathfrak{D}^\alpha_{s-} x(\tau) = \frac{1}{\Gamma(N-\alpha)} \int_0^1 \left[\mathfrak{D}^N_{s-} x(\tau)\right] \ln^{N-\alpha-1}(u/\tau) \frac{du}{u} \tag{54}$$

To prove these results, it is enough to note that

$$v^\alpha = v^N v^{\alpha-N} = v^{\alpha-N} v^N,\ \alpha \le N \in \mathbb{Z}^+_0.$$

**Remark 7.** *The designation "Hadamard–Liouville" is more correct than "Hadamard–Caputo" or "Caputo–Hadamard" [58–60], since the idea of performing an integration after a derivation was proposed by Liouville [25]. On the other hand, M. Caputo did not publish anything about this derivative.*

Similarly to the procedure used in (18), we can regularize the integrals in (49) and (50) (see [61] for an alternative). To do it, we use the logarithmic series (44) and (46).

**Definition 5.** *We define regularized Hadamard derivatives by*

$$\mathfrak{D}^{\alpha}_{s+}x(\tau) = \frac{1}{\Gamma(-\alpha)}\int_0^{\tau}\left[x(u) - \sum_{n=0}^{N-1}(-1)^n\frac{\mathfrak{D}^N_{s+}x(\tau)}{n!}\ln^n(\tau/u)\right]\ln^{-\alpha-1}(\tau/u)\frac{du}{u} \tag{55}$$

*and*

$$\mathfrak{D}^{\alpha}_{s-}x(\tau) = \frac{1}{\Gamma(-\alpha)}\int_{\tau}^{\infty}\left[x(u) - \sum_{n=0}^{N-1}\frac{\mathfrak{D}^N_{s+}x(\tau)}{n!}\ln^n(u/\tau)\right]\ln^{-\alpha-1}(u/\tau)\frac{du}{u} \tag{56}$$

We can consider these two expressions valid for any real order, provided that we assume the summation to be null for $N \leq 0$.

*4.6. Tempered Scale-Invariant Derivatives*

The tempered scale-invariant derivatives were already studied in the context of a generalization of the Hadamard derivatives [36,37,62] but without referring directly to the concept. In fact, the SI tempered fractional derivatives are obtained from the SI Liouville derivatives, described previously (Section 3), through a translation in their transfer function. As shown there, the transfer function of the Liouville derivatives is given by $G(s) = s^{\alpha}$ with $\pm Re(s) > 0$, according to the causality. Let $\lambda \in \mathbb{R}^+_0$. The tempered Liouville derivatives have transfer functions defined by [63]

$$G(s) = (s+\lambda)^{\alpha} \quad \pm(Re(s)+\lambda) > 0. \tag{57}$$

The corresponding impulse functions are given by

$$\mathcal{L}^{-1}[(s+\lambda)^{\alpha}] = \pm e^{-\lambda t}\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(\pm t). \tag{58}$$

With (57) and (58), and proceeding as in Section 3, we obtain tempered versions of the Liouville derivatives. The causal derivatives (the anti-causal are readily obtained) read [63]

1. Forward Grünwald–Letnikov

$$D^{\alpha}_{\lambda,f}f(t) = \lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}e^{-n\lambda h}f(t-nh), \tag{59}$$

for $\alpha \in \mathbb{R}$.

2. Forward regularized derivative

$$D^{\alpha}_{\lambda,f}f(t) = \int_0^{\infty}\left[f(t-\tau) - \sum_{0}^{N-1}\frac{(-1)^m f^{(m)}(t)}{m!}\tau^m\right]e^{-\lambda\tau}\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau \tag{60}$$

where $\alpha \leq N$, again. If $N \leq 0$, the summation is null.

It can be shown that [63]

$$D^{\beta}_{\lambda,f}f(t) = e^{-\lambda t}D^{\beta}_{0,f}\left[e^{\lambda t}f(t)\right] \tag{61}$$

and

$$D^{\beta}_{\lambda,b}f(t) = e^{\lambda t}D^{\beta}_{0,b}\left[e^{-\lambda t}f(t)\right]. \tag{62}$$

The approach to introducing tempered scale derivatives is similar.

**Definition 6.** *Let $\mu \in \mathbb{R}^+_0$. The tempered SD have transfer functions defined by*

$$G(v) = (v+\mu)^{\alpha} \quad \pm(Re(v)+\mu) > 0. \tag{63}$$

**Theorem 5.** *Let $q > 1$. The following expressions*

$$\mathfrak{D}^{\alpha}_{\mu,s+}x(\tau) = \lim_{q\to 1^+} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{-n\mu} x(\tau q^{-n}), \tag{64}$$

*and*

$$\mathfrak{D}^{\alpha}_{\mu,s-}x(\tau) = \lim_{q\to 1^+} (-1)^{\alpha} \ln^{-\alpha}(q) \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} q^{n\mu} x(\tau q^{n}) \tag{65}$$

*represent the tempered scale-derivatives corresponding to the stretching (27) and shrinking (28) GL-type derivatives, respectively.*

The proof is readily found from

$$(v+\mu)^{\alpha} = \lim_{q\to 1^+} \begin{cases} \left[\dfrac{(1-q^{-v-\mu})}{\ln q}\right]^{\alpha} & Re(v) > -\mu, \\[2ex] \left[\dfrac{(q^{v+\mu}-1)}{\ln q}\right]^{\alpha} & Re(v) < -\mu, \end{cases}$$

To obtain the Hadamard tempered derivatives, we have to compute the corresponding impulse response functions, a simple task attending to the properties of the MT. We obtain

$$\mathcal{M}^{-1}[(v+\mu)^{\alpha}](\tau) = \begin{cases} \dfrac{\tau^{-\mu}\ln^{-\alpha-1}(\tau)}{\Gamma(-\alpha)}\varepsilon(\tau-1) & (Re(v)+\mu) > 0 \\[2ex] \dfrac{\tau^{\mu}\ln^{-\alpha-1}(1/\tau)}{\Gamma(-\alpha)}\varepsilon(1-\tau) & (Re(v)+\mu) < 0 \end{cases} \tag{66}$$

With this relation, we can obtain the tempered Hadamard derivatives. In particular, the tempered regularized Hadamard derivatives are given by

$$\mathfrak{D}^{\alpha}_{s+}x(\tau) = \frac{\tau^{-\mu}}{\Gamma(-\alpha)} \int_0^{\tau} u^{\mu} \left[x(u) - \sum_{n=0}^{N-1} (-1)^n \frac{\mathfrak{D}^{N}_{s+}x(\tau)}{n!} \ln^n(\tau/u)\right] \ln^{-\alpha-1}(\tau/u)\frac{du}{u} \tag{67}$$

and

$$\mathfrak{D}^{\alpha}_{s-}x(\tau) = \frac{\tau^{\mu}}{\Gamma(-\alpha)} \int_{\tau}^{\infty} u^{-\mu} \left[x(u) - \sum_{n=0}^{N-1} \frac{\mathfrak{D}^{N}_{s+}x(\tau)}{n!} \ln^n(u/\tau)\right] \ln^{-\alpha-1}(u/\tau)\frac{du}{u}. \tag{68}$$

From (67) and (68), we obtain

$$\mathfrak{D}^{\alpha}_{\mu,s+}x(\tau) = \tau^{-\mu}\mathfrak{D}^{\alpha}_{0,s+}x(\tau)[\tau^{\mu}f(t)] \tag{69}$$

and

$$\mathfrak{D}^{\alpha}_{\mu,s-}x(\tau) = \tau^{\mu}\mathfrak{D}^{\alpha}_{0,s-}x(\tau)\left[\tau^{-\mu}f(t)\right], \tag{70}$$

similar to (69) and (70).

## 5. Scale-Invariant Systems

**Definition 7.** *In agreement with the concepts introduced above, we define the dilation (scale)-invariant fractional autoregressive-moving average (DI-FARMA) system through*

$$\sum_{k=0}^{N_0} a_k \mathfrak{D}^{\alpha_k}_{s\pm} y(\tau) = \sum_{k=0}^{M_0} b_k \mathfrak{D}^{\beta_k}_{s\pm} x(\tau) \qquad \tau \in \mathbb{R}^+ \tag{71}$$

*where* $\mathfrak{D}_{s\pm}^{\alpha_k,(\beta_k)}$, $k = 0, 1, 2, \cdots$ *mean the fractional* $\alpha_k(\beta_k)$*-order scale-derivatives and* $N_0, M_0$ *are the system orders. The parameters* $a_k, b_k$, $k = 0, 1, \cdots$ *are considered real numbers. Without losing generality, we set* $a_{N_0} = 1$.

We could use tempered SD but we will not do so here. The results in [63] can be easily adapted. The name "autoregressive-moving average" was borrowed from a current nomenclature used for shift-invariant systems [64]. As the power $t^v$ is the eigenfunction of (71), we obtain easily the transfer function

$$G(v) = \frac{\sum\limits_{k=0}^{M_0} b_k v^{\beta_k}}{\sum\limits_{k=0}^{N_0} a_k v^{\alpha_k}}. \tag{72}$$

which is a bit difficult to manipulate [65]. In the following, we shall be considering the so-called "commensurate" systems described by differential equations with the format

$$\sum_{k=0}^{N_0} a_k \mathfrak{D}_{s\pm}^{k\alpha} y(\tau) = \sum_{k=0}^{M_0} b_k \mathfrak{D}_{s\pm}^{k\alpha} x(\tau) \qquad \tau \in \mathbb{R}^+ \tag{73}$$

The corresponding transfer function is

$$G(v) = \frac{\sum\limits_{k=0}^{M_0} b_k v^{k\alpha}}{\sum\limits_{k=0}^{N_0} a_k v^{k\alpha}}, \tag{74}$$

where we assume that $M_0 < N_0$ for simplicity, and all the roots, $p_k, k = 1, 2, \cdots$, of $\sum\limits_{k=0}^{N_0} a_k w^k$ are simple, which allows us to write

$$G(v) = \sum_{k=1}^{N} \frac{A_k}{v^\alpha - p_k}, \tag{75}$$

where the $A_k$ are the residues obtained by substituting $w$ for $v^\alpha$ in (72). The impulse response results from the inversion of a combination of partial fractions such as

$$F(v) = \frac{1}{v^\alpha - p}. \tag{76}$$

We will treat the case ($Re(v) > 0$). We write

$$\frac{1}{v^\alpha - p} = \sum_{n=0}^{\infty} p^n v^{-n\alpha-1}, \quad Re(v) > |p|.$$

The corresponding inverse MT is

$$f(\tau) = \sum_{n=0}^{\infty} p^n \frac{\ln^{n\alpha}(\tau)}{\Gamma(n\alpha+1)}, \quad \tau \geq 1, \tag{77}$$

which is analog to the $\alpha-$exponential function [28] and can be expressed as

$$f(\tau) = \ln^{\beta-1}(\tau) E_{\alpha,\beta}(p \ln^\alpha(\tau)), \quad \tau \geq 1, \tag{78}$$

where we introduced the logarithmic Mittag-Leffler function

$$E_{\alpha,\beta}(\ln^{\alpha}(\tau)) = \sum_{n=0}^{\infty} \frac{\ln^{n\alpha}(\tau)}{\Gamma(n\alpha+1)}, \quad \tau \geq 1. \tag{79}$$

The solution corresponding to $Re(v) < 0$ is obtained using a similar procedure, accounting for (48).

The $\alpha-$exponential function is more useful, since it allows us to obtain the inversion in the multiple-order pole case:

$$F(v) = \frac{1}{(v^{\alpha} - p)^{n}}, \quad n \in \mathbb{Z}^{+}.$$

This case can be treated easily from (76) through a $n^{th}$-order derivative assuming $p$ as the variable.

**Example 3.** *Consider the simple AR(1) system*

$$\mathfrak{D}y(\tau) + y(\tau) = x(\tau)$$

*Its transfer function is*

$$G(v) = \frac{1}{v+1} = \sum_{n=0}^{\infty} (-1)^{n} v^{-n-1}, \quad Re(v) > 1.$$

*Using (48), the impulse response is*

$$g(\tau) = \sum_{n=0}^{\infty} (-1)^{n} \frac{\ln^{n}(\tau)}{n!} \varepsilon(\tau - 1) = e^{-\ln\tau} \varepsilon(\tau - 1) = \frac{1}{\tau}\, \varepsilon(\tau - 1)$$

*where we used (48) if $Re(v) > 0$. In the $Re(v) < -1$ case, we obtain*

$$g(\tau) = \sum_{n=0}^{\infty} \frac{\ln^{n}(\tau)}{n!} \varepsilon(1 - \tau) = e^{\ln(\tau)} \varepsilon(1 - \tau) = \tau\, \varepsilon(1 - \tau)$$

**Remark 8.** *It is important to note that, according to the theoretical framework that we developed in the previous section, the initial conditions must be taken at $\tau = 1^{\pm}$, not at $\tau = 0^{\pm}$. A similar scheme for introducing those initial conditions as the one in [48] can be found.*

*We could define two "unilateral" MTs with corresponding initial conditions, but that are not necessarily those in any system, as it happens with the one-sided Laplace transform [66]. For instance,*

$$F_{u}(v) = \int_{1}^{\infty} f(\tau) \tau^{-v-1} d\tau.$$

**Example 4.** *In Section 4.4, we introduced the scale conversion operator that reads*

$$S(v) = a^{v} = \sum_{n=0}^{\infty} \frac{v^{n}}{n!} \ln^{n}(a)$$

*Using signal processing language, we can call it an infinite impulse response system. We can obtain an interesting approximation. Assume that $a = 1 + \eta$ with $\eta$ being a small positive number. In such a case, $\ln(a) = \ln(1+\eta) \approx \eta$. Then, we can write*

$$S(v) = \frac{a^{\frac{v}{2}}}{a^{\frac{-v}{2}}} = \frac{e^{\frac{\ln(a)}{2} v}}{e^{\frac{-\ln(a)}{2} v}} \approx \frac{1 + \frac{\eta}{2} v}{1 - \frac{\eta}{2} v}$$

*Therefore, $S(v)$ is the transfer function of a linear system described by the differential equation*

$$\frac{\eta}{2}v\mathfrak{D}y(\tau) + y(\tau) = -\frac{\eta}{2}\mathfrak{D}x(\tau) + x(\tau)$$

*If $\eta$ is not small, we can decompose a into a product; for example, $a = \prod_{k=1}^{n} a^{\frac{1}{n}}$ and use n cascaded systems. We do not go further here.*

## 6. Conclusions

Using the concept of scale-invariant linear systems, a general fractional scale derivative was introduced and studied. A Liouville-like framework for such systems was described. The relationship between scale derivatives and Hadamard's was re-established and reformulated. A new derivative similar to Grünwald–Letnikov's was deduced. Tempered versions were also introduced. Scale-invariant systems with an autoregressive-moving average form differential equation were introduced. For solving it, a new logarithmic Mittag-Leffler series was proposed.

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