fractal and fractional

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Brief Report

An Application of the Prabhakar Fractional Operator to a Subclass of Analytic Univalent Function

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Abstract: Fractional differential operators have recently been linked with numerous other areas of science, technology and engineering studies. For a real variable, the class of fractional differential and integral operators is evaluated. In this study, we look into the Prabhakar fractional differential operator, which is the most applicable fractional differential operator in a complex domain. In terms of observing a group of normalized analytical functions, we express the operator. In the open unit disc, we deal with its geometric performance. Applying the Prabhakar fractional differential operator ${}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta}$ to a subclass of analytic univalent function results in the creation of a new subclass of mathematical functions: $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta$,m,c,z,p,q). We obtain the characteristic, neighborhood and convolution properties for this class. Some of these properties are extensions of defined results.

Keywords: analytic function; fractional calculus operator; Hadmard product (convolution); hypergeometric function; neighborhood; Prabhakar operator; Prabhakar fractional differential operator; univalent function

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1. Introduction

Let $\mathcal{A}$ be the class of analytical functions in the open unit disc

$$U = \{ z \in C : |z| < 1 \}$$

of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

The subclass of $\mathcal{A}$ that consists of functions of the type is also denoted as $\mathcal{W}$:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \tag{2}$$

These functions are normalized in U and are univalent.
Let us say $f(z) \in \mathcal{A}$ is of the form in Equation (1), and $g(z) \in \mathcal{A}$ is given of the form

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{3}$$

The convolution or Hadamard product of two mathematical functions f, g is denoted by the notation $(f * g)$ and is defined as follows:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \tag{4}$$

Tilak R. Prabhakar [1] introduced the entire function in 1971.The three-parameter Mittag-Leffler function can be another name for the Prabhakar function $\Xi^{\wp}_{\rho,\mu}(z)$ [2]. The generalized Mittag-Leffler function is defined by [3–5]

$$\Xi^{\wp}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\wp)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!}$$

where $\rho, \mu, \wp \in C, \Re(\wp) > 0, (\wp)_k$ is the Pochammer symbol, which is defined by the following equation:

$$(\wp)_k = \begin{cases} 1(\mathrm{k{=}0}) \\ \wp(\wp+1)......(\wp+k-1)\ (\mathrm{k} \in \mathrm{N}) \end{cases} \tag{5}$$

The following is the derivation of the generalized hypergeometric function $_{\eta}F_{\delta}$:

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_\eta; & \\ & & & z \\ \beta_1 & \cdots & \beta_\delta; & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_\eta)_n}{(\beta_1)_n \cdots (\beta_\delta)_n} \frac{z^k}{k!}$$

$$= {}_{\eta}F_{\delta}(\alpha_1, ...\alpha_\eta; \beta_1.....\beta_\delta; z) \tag{6}$$

We assume (for simplicity) that the variable z is the numerator parameters $\alpha_1 \cdots \alpha_\eta$ and the denominator parameters $\beta_1 \cdots \beta_\delta$ take on complex values, provided that there exist positive integers for η and δ or zero (interpreting an empty product as one). The special case $_{\eta}F_{\delta}$ of (Gauss) hypergeometric series is referred to by Equation (5).

Srivastava et al. [6] explored the class of complex fractional operators (differential and integral) geometrically and generalized it to two-dimensional fractional operators. Ibrahim's parameters are for a group of analytical functions in the disc of the open unit [7]. We use these operators' fractions to express many sorts of analytical functions and complicated differential equations [8], named fractional algebraic differential variables and the Ulam stability, using equations [9,10].

In the area of complex fractional differential operators, we continue our research. Using the well-known Prabhakar fractional differential operator, we formulate an arrangement of the fractional differential operator in the open unit disc in this inquiry. As a result, we examine the classes via the lens of geometric function theory.

The normalized complex Prabhakar operator in the open unit disc is denoted by the symbols ${}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta}$. Since ${}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta} \in \mathcal{A}$, we may investigate it from the perspective of geometric function theory.

The series in [11] expands the integral operator that corresponds to the fractional differential operator ${}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta}$:

$${}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} [\frac{\varphi_k \Xi^{-\gamma}[\omega z^\alpha]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,n+1-\beta}[\omega z^\alpha]}] z^k \tag{7}$$

It is obvious that

$$\left({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta} * {}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta}\right) f(z) = \left({}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta} * {}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta}\right) f(z) = f(z)$$

The operators ${}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta}$ and ${}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta}$ are combined to form a linear convex combination:

$${}^{c}_{d}\sum^{\gamma,\omega}_{\alpha,\beta} = C {}^{c}_{d}\varsigma^{\gamma,\omega}_{\alpha,\beta} f(z) + (1-C) {}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta} f(z),$$

C is in the range of [0,1]. Clearly, ${}_{d}^{c}\sum_{\alpha,\beta}^{\gamma,\omega} \in \mathcal{A}$.

If Equation (1) defines the function *f(z)*, we can see from Equation (7) that

$$ {}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega} f(z) = z + \sum_{k=2}^{\infty} [\frac{\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]}{\Gamma(k+1) E_{\alpha,k+1-\beta}^{-\gamma}[\omega z^{\alpha}]}] z^k a_k \tag{8} $$

If there is a Schwarz function w(z), which is by definition analytic in U with w(0) = 0 and $|w(z)| < 1$ for every z $\in$ U such that *f(z)* = *g*(w(z)), z $\in$ U, then we say that *f* is subordinate to *g*, written as *f(z)* $\prec$ *g(z)*, for two analytic functions *f*, *g* $\in \mathcal{W}$.

Similarly, we have the following equivalence if the function *g(z)* is univalent in U: *f(z)* $\prec$ *g(z)* $\leftrightarrow$ *f(0)* = *g(0)* and *f*(U) $\subset$ *g*(U).

2. Preliminaries

Definition 1. *The ϕ neighborhood for any function f $\in \mathcal{W}$ and $\phi \geq 0$ is defined as follows [9]:*

$$ N_{k,\phi}(f) = [g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \in \mathcal{W}; \sum_{k=2}^{\infty} k||a_k| - |b_k|| \leq \phi] \tag{9} $$

For the function e(z) = z, in particular, we observe that

$$ N_{k,\phi}(e) = [g(z) = z - \sum_{k=2}^{\infty} |b_k| z^k \in \mathcal{W}; \sum_{k=2}^{\infty} k|b_k| \leq \phi] \tag{10} $$

Definition 2. *If the following subordination criteria are met, then we may state that [11] f $\in \mathcal{W}$ is in class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$ for the given parameters p and q with $-1 \leq q \leq p \leq 1$:*

$$ 1 + \frac{z({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})''}{({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})'} \prec \frac{1 + p\omega(z)}{1 + q\omega(z)} $$

Applying the meaning of subordination is equal to the following circumstance:

$$ \left| \frac{\frac{z({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})''}{({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})'}}{q\frac{z({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})''}{({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})'} + (q-p)} \right| < 1 \tag{11} $$

where (z $\in$ U), and then the class of functions in $\mathcal{W}$ meeting the inequality is denoted as $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$:

$$ \Re\left[1 + \frac{z({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})''}{({}_{d}^{c}\theta_{\alpha,\beta}^{\gamma,\omega})'}\right] > \mu, $$

where $0 \leq \mu < 1$ and f, z $\in$ U.

3. Characteristic Property for the Class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$

We now examine the characterization property for the function *f* belonging to the class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$ by determining the coeffiecient limits:

Theorem 1. *A function $f \in \mathcal{W}$ belongs to the class $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q)$ if and only if the following conditions are satisfied:*

$$\frac{k[(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_k| \leq (p-q), \tag{12}$$

for $c,\gamma,\omega,s \in N_0, \gamma \leq s+1, \alpha \geq 0$ and $-1 \leq q < p \leq 1$.

Proof. Assume that $f \in \mathcal{W}$ belongs to the class $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q)$. Then, we have

$$1 + \frac{z({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta})''}{({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta})'} \prec \frac{1+p\omega(z)}{1+q\omega(z)}$$

$$\frac{({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta})' + z({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta})''}{({}^{c}_{d}\theta^{\gamma,\omega}_{\alpha,\beta})'} \prec \frac{1+p\omega(z)}{1+q\omega(z)}$$

$$\frac{1+\sum_{k=2}^{\infty}[\frac{\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}]}]kz^{k-1}a_k + \sum_{k=2}^{\infty}[\frac{\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}]}]k(k-1)z^{k-2}a_k}{1+\sum_{k=2}^{\infty}[\frac{\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}]}]kz^{k-1}a_k} \prec \frac{1+p\omega(z)}{1+q\omega(z)}$$

$$\frac{1+p\omega(z)}{1+q\omega(z)} = \frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] + \sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}a_k] + \sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}] k(k-1)z^{k-1}a_k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] + \sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}a_k]}$$

$$\frac{-\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k(k-1)z^{k-1}a_k]}{\Big[(q-p)\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] - p\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}a_k] + q\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2z^{k-1}a_k]\Big]} = \omega(z) \tag{13}$$

$$|\omega(z)| = \left|\frac{-\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k(k-1)z^{k-1}a_k]}{\Big[(q-p)\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] - p\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}a_k] + q\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2z^{k-1}a_k]\Big]}\right|$$

Thus, we obtain

$$\left| \frac{\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k(k-1)z^{k-1}a_k]}{\Big[(p-q)\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] + p\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}a_k] - q\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2 z^{k-1}a_k]\Big]} \right| < 1$$

Therefore, we have

$$\Re\left\{ \frac{\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k(k-1)z^{k-1}|a_k|]}{\Big[(p-q)\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}[\omega z^{\alpha}] + p\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]kz^{k-1}|a_k|] - q\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2 z^{k-1}|a_k|]\Big]} \right\} < 1$$

With $|z|$ = r, for sufficiently small r values with $0 < r < 1$, since $\omega(z)$ is analytic for $|z| = 1$, then the inequality yields

$$(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2 r^k|a_k|] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]kr^k|a_k|] < (p-q)r\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}$$

$r \to 1$

$$\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_k| \le (p-q)$$

$$\frac{k[(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_k| \le (p-q)$$

□

4. Neighborhoods for the Class $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q)$

Next is the determination of the inclusion relation involving (n,δ) neighborhoods. We define the (n,δ) neighborhood of a function *f(z)* $\in \mathcal{W}$ in accordance with prior studies on the neighbors of analytic functions by Goodman [12], which were generalized by Ruscheweyh [13] and examined by other writers including Atshan [14] and Atshan and Kulkarni [15]:

Theorem 2. *If*

$$\phi = \frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}{2(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]} \tag{14}$$

then $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q) \subset N_{n,\phi}(e)$.

Proof. From Equation (12), it is clear that if $f \in \mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$, then

$$\frac{k[(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_k| \leq (p-q)$$

$$k[2(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]]|a_k| \leq (p-q)\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}$$

$$\sum_{k=2}^{\infty} k|a_k| \leq \frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}{2(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]]} = \phi$$

□

5. Convolution Property of Class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$

This section examines a few generalized convolution features for the functions *f(z)* belonging to the class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$:

$$(f_1 * f_2 * ...f_i)(z) = z + \sum_{k=2}^{\infty}\left(\prod_{j=1}^{i} a_{k,j}\right)z^k, (j = 1, 2, 3....)$$

Theorem 3. *Let the function f_j = (j = 1, 2, ...) defined by*

$$f_j(z) = z + \sum_{k=2}^{\infty}|a_k, j|z^k. \tag{15}$$

be in the class $\mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q)$. Then, $f_1 * f_2 \in \mathcal{W}(\gamma, \omega, \alpha, \beta, \theta, m, c, z, p, q, \sigma)$:

$$\sigma \leq 1 - \frac{\begin{array}{c}\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}(1+q)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - \Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}(1+p)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k] + \\ \{(1+p)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]\}\left\{(1+q)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]\right\}\end{array}}{\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2\{(1+q)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]\}} \tag{16}$$

Proof. We need to determine the highest σ value such that

$$\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_k| \leq (p-q)$$

$$\frac{(1+\sigma)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_{k,1}||a_{k,2}| \leq (p-q)$$

$$\frac{(1+\sigma)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k \Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\left|a_{k,j}\right| \leq (p-q)$$

By the Cauchy–Schwartz inequality, we have

$$\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\sqrt{|a_{k,1}||a_{k,2}|}\leq 1$$

We only want to show that

$$\frac{(1+\sigma)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_{k,1}||a_{k,2}|\leq$$
$$\frac{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\sqrt{|a_{k,1}||a_{k,2}|}$$

This is equivalent to

$$\sqrt{|a_{k,1}||a_{k,2}|}\leq\frac{(1+\sigma)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}$$
$$\sqrt{|a_{k,1}||a_{k,2}|}\leq\frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}$$

This is sufficient to show that

$$\frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}\leq$$
$$\frac{(1+\sigma)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}$$

which implies

$$\left\{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}\right\}\left\{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\right\}\leq$$
$$\left\{(1+\sigma)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k\right\}$$
$$\left\{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\right\}$$

We then obtain Equation (16):

$$\sigma\leq 1-\frac{\begin{array}{c}\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]+\\ \{(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\}\left\{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\right\}\end{array}}{\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2\{(1+q)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\}}$$

□

Theorem 4. *Let the function f_j = (j = 1, 2, ...) defined as being in the class $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q)$*

$$h(z)=z+\sum_{k=2}^{\infty}(|a_{k,1}|^2+|a_{k,2}|^2)z^k \tag{17}$$

belong to the class $\mathcal{W}(\gamma,\omega,\alpha,\beta,\theta,m,c,z,p,q,\epsilon)$:

$$\epsilon \leq 1 - \frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}((1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k])^2 + \{(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\}2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2}{\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2} \tag{18}$$

Proof. We need to determine the highest ϵ value such that

$$\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}(|a_{k,1}|^2 + |a_{k,2}|^2) \leq 1$$

Since

$$\left\{\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\right\}^2 |a_{k,1}|^2 \leq$$

$$\left\{\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_{k,1}|\right\}^2 \leq 1$$

and

$$\left\{\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\right\}^2 |a_{k,2}|^2 \leq$$

$$\left\{\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}|a_{k,2}|\right\}^2 \leq 1$$

then by combining these inequalities, we obtain

$$\frac{1}{2}\left\{\frac{(1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\right\}^2 |a_{k,1}|^2 + |a_{k,2}|^2 \leq 1$$

$$\frac{(1+\epsilon)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2] - (1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}(|a_{k,1}|^2 + |a_{k,2}|^2) \leq 1$$

The inequality will be satisfied if

$$\frac{(1+\epsilon)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]}{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}}\leq$$

$$\frac{((1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k])^2}{2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2}$$

such that

$$2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2(1+\epsilon)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\leq$$
$$\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}((1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k])^2$$

We then obtain Equation (18):

$$\epsilon\leq 1-\frac{\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta}((1+q)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k^2]-(1+p)\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k])^2+\left\{(1+p)[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]\right\}2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2}{\sum_{k=2}^{\infty}[\varphi_k\Xi^{-\gamma}[\omega z^{\alpha}]k]2(\Gamma(k+1)\Xi^{-\gamma}_{\alpha,k+1-\beta})^2}$$

□

6. Application

Fractional calculus is extensively used in engineering science and technology, especially in the areas of heat and mass transfer, nonlinear differential equations, and fuzzy differential equations. Fractional differential calculus now plays a significant part in the diagnosis of diseases in the medical field.

7. Conclusions

The generalized Prabhakar fractional differential operator, which will be utilized in the future to deal with a variety of real-world application problems, has presented us with a number of major implications. This study suggests that this new finding can be used to resolve a variety of issues in mathematical methods and other new fields.

8. Future and Outlook

The characteristic, neighborhood, and convolution properties will be included within future fuzzy subordination.

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