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On Discrete Weighted Lorentz Spaces and Equivalent Relations between Discrete ℓ_p-Classes

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Abstract: In this paper, we study some relations between different weights in the classes $\mathcal{B}_p$, $\mathcal{B}_p^*$, $\mathcal{M}_p$ and $\mathcal{M}_p^*$ that characterize the boundedness of the Hardy operator and the adjoint Hardy operator. We also prove that these classes generate the same weighted Lorentz space Λ_p. These results will be proven by using the properties of classes $\mathcal{B}_p$, $\mathcal{B}_p^*$, $\mathcal{M}_p$ and $\mathcal{M}_p^*$, including the self-improving properties and also the properties of the generalized Hardy operator $\mathcal{H}_p$, the adjoint operator $\mathcal{S}_q$ and some fundamental relations between them connecting their composition to their sum.

Keywords: discrete operators; weighted norm inequalities; self-improving properties; $\mathcal{A}^p-$weights; $\mathcal{B}_p-$weights; $\mathcal{B}_p^*-$weights; $\mathcal{M}_p$ weights; $\mathcal{M}_p^*-$weights

1. Introduction

In recent years, the study of regularity and boundedness of discrete operators on ℓ^p analogs for L^p-regularity and boundedness has been considered by some authors; see, for example, [1–3] and the references they cited. One of the reasons for this upsurge of interest in discrete cases is due to the fact that the discrete operators may even behave differently from their continuous counterparts, as is exhibited by the discrete spherical maximal operator [4]. In some special cases, it is possible to translate or adapt the expressions and results almost straightforwardly from the continuous setting to the discrete setting or vice versa; however, in some other cases, that is far from trivial.

For example, in the simplest cases, ℓ^p-bounds for discrete analogs of classical operators, such as Calderón–Zygmund singular integral operators, fractional integral operators, and the maximal Hardy–Littlewood operator follow from known L^p-bounds for the original operators in the Euclidean setting, via elementary comparison arguments (see [5–7]). However, ℓ^p-bounds for discrete analogs of more complicated operators are not implied by results in the continuous setting, and moreover, the discrete analogs are resistant to conventional methods. The main challenge here is to develop a unique approach, as there are no general methods to study these questions. Instead these methods have to be purposely built from the basic concepts and definitions. The discrete weighted Hardy type inequality is given by

$$\left(\sum_{r=1}^{\infty} u(r)\left(\sum_{s=1}^{r} f(s)\right)^{q}\right)^{\frac{1}{q}} \leq C\left(\sum_{r=1}^{\infty} v(r) f^{p}(r)\right)^{\frac{1}{p}}, \tag{1}$$

for nonnegative sequence $\{f(r)\}_{r=1}^{\infty}$, given nonnegative weights $\{u(r)\}_{r=1}^{\infty}$ and $\{v(r)\}_{r=1}^{\infty}$, fixed parameters $0 < p, q < \infty$ and a constant $C > 0$ that is independent of the sequence $\{f(r)\}$. The $\ell^p(v)$ is the Banach space of sequences defined on $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and is given by

$$\ell^p(v) = \left\{ f(r) : \|f(r)\| := \left(\sum_{r=1}^{\infty} |f(r)|^p v(r)\right)^{1/p} < \infty \right\}. \tag{2}$$

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In the unpublished note [8] (problem 92.11), Heinig posed the question of characterizing the weights for which the discrete inequality (1) holds. The answer to this question is transacting the characterizations of the weights for nonincreasing sequences and for unrestricted nonnegative sequences were presented by many authors; we refer the reader to the papers [9–11] and the references cited therein. The characterizations of the weights of integral inequalities similar to the inequality (1) for nonincreasing functions have been established by several authors; see, for example, the papers [12–20] and the references cited therein. The paper by Ariňo and Muckenhoupt [13] was the first paper to consider this problem. In particular, the authors in [13] established the characterizations of the weighted functions in connection with the boundedness of the Hardy operator $\mathcal{H}f(t) = (1/t)\int_0^t f(x)dx$, for $t > 0$ with equal weights on the space $L_u^p(\mathbb{R}^+)$ subject to the case when $p = q > 1$. In paper [21], the authors also considered this problem and established some new characterizations of the weighted functions when $1 < p < q$. The problem when $0 < q < p < 1$ has been studied by Carro and Lorente in paper [16] for decreasing functions. Despite the variety of ideas related to weighted inequalities that appeared with the birth of singular integrals, it was only in the 1970s that a better understanding of the subject was obtained and the full characterization of the weights w for which the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) := \sup_{x\in I} \frac{1}{|I|}\int_I f(y)dy, \tag{3}$$

is bounded on $L_w^p(\mathbb{R})$ by means of the so-called A_p-*condition* was achieved by Muckenhoupt and published in 1972 (see [20]). Muckenhoupt's result became a landmark in the theory of weighted inequalities because most of the previously known results for classical operators had been obtained for special classes of weights (such as power weights) and has been extended to cover several operators, such as Hardy operator, Hilbert operator, Calderón–Zygmund singular integral operators, fractional integral operators, etc. For more details on the structure of the Muckenhoupt weights and the self-improving properties with the applications of extrapolation theory, we refer the reader to the recent paper [22] and the references cited therein.

In [21], the authors mentioned that the study of inequality (1) is not an easy task and more difficult to analyze than its integral counterparts and discovered that the conditions do not correspond, in any natural way, with those that are obtained by the discretization of the results of functions but the reverse is true. This means that what goes for sums goes, with the obvious modifications, for integrals that, in fact, proved the first part of the basic principle of Hardy, Littlewood and Polya [23] (p. 11). Indeed the proofs for series translate immediately and become much simpler when applied to integrals, but the converse is not always true. The famous problem in the stability of discrete logistic equation and the continuous equation reflects this idea. For example, in the continuous case, we know that the equilibrium point is globally stable but in the discrete case, there is a local stability of the fixed point for some values of the net growth rate, and then for different values, the solution will be periodic of period two and then periodic of period four and periodic of period eight, and then for a particular value the solution will be chaotic. This, in fact, explains the differences between the study of continuous and discrete models and hence the authors should be careful when translating the results from the continuous case to the discrete analogy. This, in fact, motivated us to consider the inequality (1) and aim to develop a new technique to prove some new equivalent relations between characterizations of weights, use the new characterizations to formulate some conditions for the boundedness of the discrete Hardy operator $\mathcal{H}(a_n) = (1/n)\sum_{s=1}^n a_s$, for $n \in \mathbb{Z}^+$ and prove some embedding theorems for Lorentz spaces to show the applications of the obtained results.

In the following, for completeness, we give some brief definitions and basic relations of the classes related to our results and show the motivation of our paper. Throughout the paper, we assume that $1 < p < \infty$, and the weights are positive sequences defined on $\mathbb{I} \subset \mathbb{Z}^+ = \{1, 2, 3, \ \ldots\}$, where $\mathbb{I}$ is of the form $\mathbb{I} = \{1, 2, \ldots, r\}$ and v is a discrete nonnegative sequence. Now, we give the definitions of the main two classes that have been used to characterize the weights in connection with the boundedness of Hardy's type operators. A sequence v defined on $\mathbb{I} \subseteq \mathbb{Z}^+$ is said to belong to the Muckenhoupt class $\mathcal{A}^p$, for $p > 1$, if there exists a constant $A > 1$ satisfying the inequality

$$\left(\frac{1}{r}\sum_{s=1}^{r} v(s)\right)\left(\frac{1}{r}\sum_{s=1}^{r} v^{\frac{-1}{p-1}}(s)\right)^{p-1} \leq A, \text{ for all } r \in \mathbb{I}. \tag{4}$$

For a given exponent $p > 1$, we define the $\mathcal{A}^p(A)-$norm of the discrete weight u by the following quantity

$$[\mathcal{A}^p(v)] := \sup_{r\in\mathbb{I}} \frac{1}{r}\sum_{s=1}^{r} v(s)\left(\frac{1}{r}\sum_{s=1}^{r} v^{\frac{1}{1-p}}(s)\right)^{p-1},$$

The Hardy–Littlewood maximal operator $\mathcal{M}f$ of the sequence f is defined by

$$(\mathcal{M}f)(r) := \sup_{r\in\mathbb{I}} \frac{1}{r}\sum_{s=1}^{r} f(s). \tag{5}$$

Observe that $\mathcal{M}$ is merely sublinear rather than linear, and it is a contraction on ℓ^∞. The structure and the properties of the discrete Muckenhoupt weights, including the self-improving properties with applications on extrapolation theory, have been studied in [24–28] and the references cited therein. The boundedness of the discrete Hardy–Littlewood maximal operator was characterized in [29], and it has been proven that $\mathcal{M}$ is bounded on $\ell^p(v)$ if and only if $v \in \mathcal{A}^p$.

A sequence v is said to belong to the class $\mathcal{M}_p$ on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$ for $p > 1$ if there exists a positive constant $A > 1$ such that the inequality

$$\left(\sum_{s=r}^{\infty} \frac{v(s)}{s^p}\right)^{1/p}\left(\sum_{s=1}^{r} v^{\frac{-1}{p-1}}(s)\right)^{(p-1)/p} \leq A, \text{ for all } r \in \mathbb{I}, \tag{6}$$

holds. In [25], the authors proved that the Hardy operator $\mathcal{H}$, defined by the form

$$\mathcal{H}f(r) = \frac{1}{r}\sum_{s=1}^{r} f(s),$$

is bounded on $\ell^p(v)$, for $1 < p < \infty$ if and only if $v \in \mathcal{M}_p$. A sequence v is said to belong to the class $\mathcal{M}_p^*$ on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$ for $p > 1$ and $p' = \frac{p}{p-1}$ is the conjugate of p, if there exists a positive constant $A > 1$ such that

$$\left(\sum_{s=1}^{r} v(s)\right)^{1/p}\left(\sum_{s=r}^{\infty} \frac{v^{-p'/p}(s)}{s^{p'}}\right)^{1/p'} \leq A, \text{ for all } r \in \mathbb{I}. \tag{7}$$

In [25], the authors proved that the adjoint Hardy operator $\mathcal{S}$, which is defined by the form

$$\mathcal{S}f(r) = \sum_{s=r}^{\infty} \frac{f(s)}{s}, \quad \text{for all } r \in \mathbb{I},$$

is bounded on $\ell^p(v)$, for $1 < p < \infty$, if and only if $v \in \mathcal{M}_p^*$. A sequence v is said to belong to the class $\mathcal{B}_p$ in the interval $\mathbb{I} \subseteq \mathbb{Z}^+$, for $p > 0$, if there exists a positive constant $A > 1$ satisfying the inequality

$$\sum_{s=r}^{\infty} \frac{v(s)}{s^p} \leq \frac{A}{r^p} \sum_{s=1}^{r} v(s), \text{ for all } r \in \mathbb{I}. \tag{8}$$

In [30], Heing and Kufner proved that the Hardy operator $\mathcal{H}$ is bounded on $\ell^p(v)^d$; that is $\mathcal{H} : \ell^p(v)^d \to \ell^p(v)$, and

$$\sum_{r=1}^{\infty} v(r) \left(\frac{1}{r} \sum_{s=1}^{r} f(s) \right)^p \leq C \sum_{r=1}^{\infty} v(r) f^p(r), \tag{9}$$

for $1 < p < \infty$ if and only if $v \in \mathcal{B}_p$ and $\lim_{r \to \infty}(v(r+1)/v(r)) = c$ and $\sum_{r=1}^{\infty} v(r) = \infty$. In [21], Bennett and Grosse-Erdmann improved the result of Heing and Kufner by excluding the conditions that have been posed on v. A discrete nonnegative sequence v is said to belong to the discrete class $\mathcal{B}_p^*$ of weights on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$, for $p > 0$, if there exists a positive constant $A > 1$ such that the inequality

$$\sum_{s=1}^{r} \frac{v(s)}{s^p} \leq \frac{A}{r^p} \sum_{s=1}^{r} v(s), \text{ for all } r \in \mathbb{I}, \tag{10}$$

holds.

A discrete nonnegative sequence v belongs to the discrete class $\mathcal{M}_1$ on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$ if there exists a positive constant $A > 0$ such that the inequality

$$\mathcal{S}v(r) \leq Av(r), \text{ for all } r \in \mathbb{I}. \tag{11}$$

A discrete nonnegative sequence v belongs to the discrete class $\mathcal{M}_1^*$ on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$ if there exists a positive constant $A > 0$ such that the inequality

$$\mathcal{H}v(r) \leq Av(r), \text{ for all } r \in \mathbb{I}. \tag{12}$$

A discrete nonnegative sequence v belongs to the discrete class $\mathcal{B}_\infty^*$ on the interval $\mathbb{I} \subseteq \mathbb{Z}^+$ if there exists a positive constant $A > 1$ such that the inequality

$$\mathcal{H}(\mathcal{H}v(r)) \leq A\mathcal{H}v(r), \text{ for all } r \in \mathbb{I}. \tag{13}$$

In [21], Bennett and Grosse-Erdmann developed a new approach to prove, and even to formulate, their main results in order to improve the result of Heinig and Kufner by excluding the conditions posed on v. They proved that (1) holds if and only if

$$A_1 := \sup_{r \geq 1} \left(\sum_{s=r}^{\infty} \frac{v(s)}{s^p} \right)^{1/p} \left(\sum_{s=1}^{r} \frac{s^{\frac{p}{p-1}} v(s+1)}{V^{\frac{1}{p-1}}(s) V(s+1)} \right)^{(p-1)/p} < \infty, \; V(r) = \sum_{s=1}^{r} v(s). \tag{14}$$

Inequality (9) states that for a bounded sequence f on $\ell^p(v)$, $1 < p < \infty$, the Hardy operator $\mathcal{H}(f)$ is also bounded on $\ell^p(v)$, if and only if (6) holds. In [29], the authors proved that the Hardy operator is bounded on $\ell^p(v)$ and (9) holds if and only if the discrete Muckenhoupt condition

$$A_2 := \sup_{r \geq 1} \left(\sum_{s=1}^{r} v(s) \right)^{1/p} \left(\sum_{s=1}^{r} (v(s))^{\frac{-1}{p-1}} \right)^{p-1/p} < \infty, \tag{15}$$

holds. In [25], the authors proved that inequality (9) holds for $1 < p < \infty$ if and only if

$$A_3 := \sup_{r\geq 1}\left(\sum_{s=r}^{\infty}\frac{v(s)}{s^p}\right)^{1/p}\left(\sum_{s=1}^{r}(v(s))^{\frac{-1}{p-1}}\right)^{(p-1)/p} < \infty. \tag{16}$$

In [12] (Theorem 4.1), the authors studied the boundedness of the operator in two different spaces with two different weights, u and v. They proved that if $1 < p \leq q < \infty$, and

$$A_4 := \sup_{r\geq 1}\left(\sum_{s=r}^{\infty}u(s)\right)^{\frac{1}{q}}\left(\sum_{s=1}^{r}(v(s))^{1-p^*}\right)^{\frac{1}{p^*}} < \infty, \tag{17}$$

then inequality (1) holds where $p^* = p/(p-1)$. Note that when $p = q$, then condition (17) becomes condition (6) with $v(r)$ replaced by $\frac{v(r)}{r^p}$. In [10,11], Bennett established two more different characterizations of the weights $u(r)$ and $v(r)$ such that inequality (1) holds when $1 < p \leq q$. He proved that inequality (1) holds for a nonnegative sequence f and $1 < p \leq q < \infty$, if and only if

$$A_5 := \sup_{r\geq 1}\left(\sum_{s=1}^{r}(u(s))^{1-p^*}\right)^{\frac{-1}{p}}\left(\sum_{s=1}^{r}v(s)\left(\sum_{m=1}^{s}(v(m))^{1-p^*}\right)^{q}\right)^{\frac{1}{q}} < \infty, \tag{18}$$

or

$$A_6 := \sup_{r\geq 1}\left(\sum_{s=r}^{\infty}u(s)\right)^{\frac{-1}{q^*}}\left(\sum_{s=r}^{\infty}v^{1-p^*}(s)\left(\sum_{m=s}^{\infty}v(m)\right)^{p^*}\right)^{\frac{1}{p^*}} < \infty. \tag{19}$$

Okpoti in [31] proved, for $1 \leq p \leq q < \infty$, some characterizations of weights in (1) in the forms:

$$A_7(\delta) := \sup_{r\geq 1}\left(\sum_{s=1}^{r}v^{1-p^*}(s)\right)^{\frac{(\delta-1)}{p}}\left(\sum_{s=r}^{\infty}u(s)\left(\sum_{m=1}^{s}v^{1-p^*}(m)\right)^{\frac{q(p-\delta)}{p}}\right)^{\frac{1}{q}} < \infty, \tag{20}$$

for some δ, $1 < \delta \leq p$. In [21], Bennett and Grosse-Erdmann proved that the two conditions (8) and (14) are equivalent and in [31], the authors proved that the equivalence of conditions (15)–(20).

Motivated by the results in [21,31], there is a main question: *Is it possible to prove some equivalence relations between the classes* $\mathcal{B}_p$, $\mathcal{B}_p^*$, $\mathcal{M}_p$ and $\mathcal{M}_p^*$?

Our main purpose in this paper is to give the affirmative answer to this question by using the notion of the weighted Lorentz space $\Lambda_p(v)$. In the following, we give brief definitions and some properties of the classical Lorentz and its discrete space. The classical Lorentz space $L^{p,q}(\mathbb{R}^+)$ for $1 < p < \infty$ and $1 \leq q < \infty$ was introduced by G. Lorentz in [32] and was defined by

$$L^{p,q}(\mathbb{R}^+) = \left\{f : \|f\|_{p,q} = \left(\frac{q}{p}\int_0^{\infty}(t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q} < \infty\right\},$$

where f^* is the nonincreasing rearrangement of f. In [33], the authors proved that $L^{p,q}$ is a linear space and $\|.\|_{p,q}$ is a quasi-norm if and only if $1 \leq q < p \leq \infty$. Since then, there has been a wide interest in studying the normability and duality properties of $L^{p,q}(\mathbb{R}^+)$; see [13,34–46] and the references cited therein. We shall denote by $A = 2^{\mathbb{Z}^+}$, the power set of $\mathbb{Z}^+$, and by μ, a counting measure. The notion X^d denotes the set of all nonincreasing and nonnegative sequences of X. The distribution sequence of any real sequence $\{v(r)\}_{r\geq 1}$ is defined by

$$D_v(\lambda) = \mu\{r \in \mathbb{Z}^+ : |v(r)| > \lambda\} = \sum_{\{r:|v(r)|>\lambda\}} \mu(r), \text{ for } \lambda \geq 0.$$

The nonincreasing rearrangement of v with respect to the counting measure μ is given by

$$v^*(r) = \inf\{\lambda > 0 : D_v(\lambda) \leq r\}, \text{ for } r \geq 0.$$

For $E \subset \mathbb{Z}^+$, and $v(r) \geq 0$, Proposition 7.6.2 [47] implies, for $r \in E$, that

$$\sum_{r\in E} v(r) \leq \sum_{r=1}^{\mu(E)} v^*(r),$$

where the equality holds when v is a nonincreasing sequence. The classical Lorentz sequence space $\ell^{p,q}(\mathbb{Z}^+)$ (or simply $\ell^{p,q}$), for $1 < p < \infty$ and $1 \leq q < \infty$ is defined by

$$\ell^{p,q}(\mathbb{Z}^+) = \left\{ v : \|v\|_{p,q} = \left(\sum_{r=1}^{\infty} (r^{1/p} v^*(r))^q \frac{1}{r} \right)^{1/q} < \infty \right\}, \tag{21}$$

where v^* is the nonincreasing rearrangement of v, which is obtained by rearranging the sequence $|v(r)|$ in nonincreasing order. The Lorentz sequence space $\ell^{p,q}$, $1 < p < \infty$ and $1 \leq q < \infty$ is a linear space and $\|.\|_{p,q}$ is a quasi-norm. Moreover $\ell^{p,q}$, $1 < p \leq \infty$ and $1 \leq q \leq \infty$, is complete with respect to the quasi-norm $\|.\|_{p,q}$ and $\ell^{p,q}$, $1 \leq q \leq p < \infty$ is a complete normed linear space with respect to $\|.\|_{p,q}$. The ℓ^p-spaces for $1 \leq p < \infty$ are equivalent to the $\ell^{p,p}$-spaces. The weighted Lorentz sequence space $\Lambda_q(v)$ for $0 < q < \infty$, is defined by

$$\Lambda_q(v) = \left\{ f : \|f\|_{q,v} = \left(\sum_{r=1}^{\infty} v(r)(f^*(r))^q \right)^{1/q} < \infty \right\}, \tag{22}$$

where f^* is the nonincreasing arrangement of f and v is a positive weight on $\mathbb{Z}^+$.

This paper is organized as follows: In Section 2, we present some basic lemmas that prove the relation between the composition of operators $\mathcal{S}_q$ and $\mathcal{H}_p$ and their sum $\mathcal{S}_q + \mathcal{H}_p$ and then use this to prove some properties of the classes of weights. These results are essential in the proof of our main results. In Section 3, we prove the main results, which give the equivalence relations between different weights in the classes $\mathcal{B}_p$, $\mathcal{B}_p^*$, $\mathcal{M}_p$ and $\mathcal{M}_p^*$. We will also prove that although the two different weights v and $\widetilde{v}$ (not necessarily monotone) do not belong to the same class, they generate the same weighted Lorentz space, i.e., $\Lambda_p(v) \simeq \Lambda_p(\widetilde{v})$. This paper will be ended with a conclusion.

2. Preliminaries and Basic Lemmas

In this section, we present the basic lemmas that prove the relation between the composition of operators $\mathcal{S}_q$ and $\mathcal{H}_p$ and their sum $\mathcal{S}_q + \mathcal{H}_p$ and then use them to prove some properties of the classes of weights. Recall that two positive quantities A and B are said to be *equivalent*, written $A \simeq B$, if there exist two constants c and C such that the inequality $cB \leq A \leq CB$ holds. Furthermore, $A \lesssim B$ is satisfied if there exists a constant C such that the inequality $A \leq CB$ holds. Clearly, the relation $\lesssim$ is transitive; that is, if $A \lesssim B$ and $B \lesssim C$ hold, then $A \lesssim C$ also holds. Throughout this paper, we assume that v is a positive real-valued weight defined on $\mathbb{Z}^+ = \{1, 2, \ldots\}$. The forward difference operator, denoted by Δ, is defined by $\Delta u(s) = u(s+1) - u(s)$. Define the generalized Hardy operator $\mathcal{H}_p$ and adjoint Hardy operator $\mathcal{S}_q$, for $0 \leq p$ and $q > 0$, by

$$\mathcal{H}_p f(r) := \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^p}, \quad \text{and} \quad \mathcal{S}_q f(r) := \frac{1}{r^{1-q}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q}. \tag{23}$$

Clearly, for $p = 0$, the operator $\mathcal{H}_p$ gives the Hardy operator, and for $q = 1$, the operator $\mathcal{S}_q$ gives the adjoint Hardy operator. We can easily prove for any $\delta > 0$ that

$$\begin{aligned} \mathcal{H}_\delta f(r) &= \frac{1}{r^{1-\delta}} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} = \frac{1}{r} \sum_{s=1}^{r} \left(\frac{r}{s}\right)^\delta f(s) \\ &\geq \frac{1}{r} \sum_{s=1}^{r} f(s) = \mathcal{H}_0 f(r) = \mathcal{H} f(r), \end{aligned} \tag{24}$$

and for all $1 < q \leq p$, we have

$$\begin{aligned} \mathcal{S}_p f(r) &= r^{p-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^p} = r^{q-1} \sum_{s=r}^{\infty} \left(\frac{r}{s}\right)^{p-q} \frac{f(s)}{s^q} \\ &\leq r^{q-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} = \mathcal{S}_q f(r). \end{aligned} \tag{25}$$

The next lemmas, which are adapted from [21,48,49], are essential in the proof of the main results.

Lemma 1. *Assume that v and ω are two weights such that*

$$\sum_{s=1}^{r} v(s) \simeq \sum_{s=1}^{r} \omega(s),$$

for every $r \geq 1$, then for any nonincreasing sequence u, it satisfies that

$$\sum_{r=1}^{\infty} u(r) v(r) \simeq \sum_{r=1}^{\infty} u(r) \omega(r).$$

Lemma 2 (Fubini's Theorem). *Let u and $v : \mathbb{Z}^+ \to \mathbb{R}$ be two nonnegative sequences, then*

$$\sum_{r=1}^{N} v(r) \left(\sum_{s=r}^{N} u(s) \right) = \sum_{r=1}^{N} u(r) \left(\sum_{s=1}^{r} v(s) \right), \tag{26}$$

and as $N \to \infty$, we obtain the inequality

$$\sum_{r=1}^{\infty} v(r) \left(\sum_{s=r}^{\infty} u(s) \right) = \sum_{r=1}^{\infty} u(r) \left(\sum_{s=1}^{r} v(s) \right). \tag{27}$$

Lemma 3. *Let v be a nonnegative weight and $1 \leq p < \infty$. Then $v \in \mathcal{B}^*_\infty$ if and only if $v \in \mathcal{B}^*_p$.*

Lemma 4. *If $v \in \mathcal{B}_p$, for $0 < p < \infty$, then $v \in \mathcal{B}_{p-\epsilon}$ for some $0 < \epsilon < p$.*

Lemma 5. *If $v \in \mathcal{M}^*_p$ and $1 \leq p \leq q$, then $v \in \mathcal{M}^*_q$ and $[v]_{\mathcal{M}^*_p} \leq [v]_{\mathcal{M}^*_q}$.*

Lemma 6. *If $v \in \mathcal{M}_p$, and $1 \leq p \leq q$, then $v \in \mathcal{M}_q$.*

Lemma 7. *Let $\mathcal{H}_p$ and $\mathcal{S}_q$ be operators defined as in (23). Then for $0 \leq p < q$, the equivalence relations*

$$\mathcal{S}_q \circ \mathcal{H}_p \simeq (\mathcal{S}_q + \mathcal{H}_p) \text{ and } \mathcal{H}_p \circ \mathcal{S}_q \simeq (\mathcal{H}_p + \mathcal{S}_q),$$

hold.

Proof. For any nonnegative sequence u, and the definitions of the operators $\mathcal{S}_q$ and $\mathcal{H}_p$ and by re-writing the summation (switching the order of summation), we have that

$$\begin{aligned}
(\mathcal{S}_q \circ \mathcal{H}_p)f(r) &= \mathcal{S}_q(\mathcal{H}_p f(r)) \\
&= r^{q-1}\sum_{s=r}^{\infty}\frac{1}{s^q}\left(\frac{1}{s^{1-p}}\sum_{k=1}^{s}\frac{f(k)}{k^p}\right) \\
&= r^{q-1}\sum_{s=r}^{\infty}s^{p-q-1}\sum_{k=1}^{s}\frac{f(k)}{k^p} \\
&= r^{q-1}\sum_{k=r}^{\infty}\frac{f(k)}{k^p}\sum_{s=k}^{\infty}s^{p-q-1} + r^{q-1}\sum_{k=1}^{r}\frac{f(k)}{k^p}\sum_{s=r}^{\infty}s^{p-q-1}.
\end{aligned} \tag{28}$$

Since $q > p$, then by employing inequality (Hardy and Littlewood [23])

$$\gamma y^{\gamma-1}(x-y) \leq x^{\gamma} - y^{\gamma} \leq \gamma x^{\gamma-1}(x-y), x \geq y \geq 0 \text{ and } \gamma < 0 \text{ or } \gamma \geq 1, \tag{29}$$

with $\gamma = p - q < 0$, we obtain

$$\sum_{s=r}^{\infty}s^{p-q-1} \geq \sum_{s=r}^{\infty}\frac{1}{p-q}\Delta s^{p-q} = \frac{1}{q-p}r^{p-q}, \tag{30}$$

and also, by the fact that $s \geq (s+1)/2$ for all $s \geq 1$ and employing (29), we have

$$\begin{aligned}
\sum_{s=r}^{\infty}s^{p-q-1} &\leq \sum_{s=r}^{\infty}2^{-p+q+1}(s+1)^{p-q-1} \leq \sum_{s=r}^{\infty}\frac{2^{-p+q+1}}{p-q}\Delta s^{p-q} \\
&= \frac{2^{-p+q+1}}{q-p}r^{p-q}.
\end{aligned} \tag{31}$$

By using (28) and (30), we obtain

$$\begin{aligned}
(\mathcal{S}_q \circ \mathcal{H}_p)f(r) &\geq \frac{1}{q-p}r^{q-1}\sum_{s=r}^{\infty}\frac{f(s)}{s^p}s^{p-q} + \frac{1}{q-p}r^{q-1}\sum_{s=1}^{r}\frac{f(s)}{s^p}r^{p-q} \\
&= \frac{1}{q-p}r^{q-1}\sum_{s=r}^{\infty}\frac{f(s)}{s^q} + \frac{1}{q-p}r^{p-1}\sum_{s=1}^{r}\frac{f(s)}{s^p} \\
&= \frac{1}{q-p}\left[\mathcal{S}_q f(r) + \mathcal{H}_p f(r)\right],
\end{aligned}$$

and by using (28) and (31), we obtain

$$\begin{aligned}
(\mathcal{S}_q \circ \mathcal{H}_p)f(r) &\leq \frac{2^{-p+q+1}}{q-p}r^{q-1}\sum_{s=1}^{r}\frac{f(s)}{s^p}r^{p-q} + \frac{2^{-p+q+1}}{q-p}r^{q-1}\sum_{s=r}^{\infty}\frac{f(s)}{s^p}s^{p-q} \\
&= \frac{2^{-p+q+1}}{q-p}r^{p-1}\sum_{s=1}^{r}\frac{f(s)}{s^p} + \frac{2^{-p+q+1}}{q-p}r^{q-1}\sum_{s=r}^{\infty}\frac{f(s)}{s^q} \\
&= \frac{2^{-p+q+1}}{q-p}\left[\mathcal{S}_q f(r) + \mathcal{H}_p f(r)\right].
\end{aligned}$$

This proves that $\mathcal{S}_q \circ \mathcal{H}_p \gtrsim \mathcal{S}_q + \mathcal{H}_p$ and $\mathcal{S}_q \circ \mathcal{H}_p \lesssim \mathcal{S}_q + \mathcal{H}_p$, which is equivalent to $\mathcal{S}_q \circ \mathcal{H}_p \simeq \mathcal{S}_q + \mathcal{H}_p$. This is the required result. Furthermore, by the definitions of operators $\mathcal{S}_q$ and $\mathcal{H}_p$ and re-writing the summation (switching the order of summation), we have

$$\begin{aligned} (\mathcal{H}_p \circ \mathcal{S}_q) f(r) &= \mathcal{H}_p(\mathcal{S}_q(f(r))) = \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{1}{s^p} \left(s^{q-1} \sum_{k=s}^{\infty} \frac{f(k)}{k^q} \right) \\ &= \frac{1}{r^{1-p}} \sum_{s=1}^{r} s^{q-p-1} \left(\sum_{k=s}^{\infty} \frac{f(k)}{k^q} \right) \\ &= \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^q} \left(\sum_{k=1}^{s} k^{q-p-1} \right) \\ &\quad + \frac{1}{r^{1-p}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} \left(\sum_{k=1}^{r} k^{q-p-1} \right). \end{aligned} \tag{32}$$

We have two cases for p and q, which are $q - p < 1$ and $q - p > 1$. Let us first consider the case $q - p < 1$, and by employing inequality (Hardy and Littlewood [23])

$$\gamma y^{\gamma-1}(x-y) \geq x^\gamma - y^\gamma \geq \gamma x^{\gamma-1}(x-y), x \geq y \geq 0, 0 < \gamma < 1, \tag{33}$$

with $0 < \gamma = q - p < 1$, we obtain

$$\sum_{s=1}^{r} s^{q-p-1} \leq \sum_{s=1}^{r} \frac{1}{q-p} \Delta(s-1)^{q-p} = \frac{1}{q-p} r^{q-p}, \tag{34}$$

and also we have

$$\sum_{s=1}^{r} s^{q-p-1} \geq r^{q-p-1} \sum_{s=1}^{r} 1 = r^{q-p}. \tag{35}$$

By using (32) and (34), we obtain

$$\begin{aligned} (\mathcal{H}_p \circ \mathcal{S}_q) f(r) &\leq \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^q} s^{q-p} + \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} r^{q-p} \\ &= \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^p} + \frac{1}{q-p} r^{q-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} \\ &= \frac{1}{q-p} [\mathcal{H}_p f(r) + \mathcal{S}_q f(r)], \end{aligned}$$

and by using (32) and (35), we obtain

$$\begin{aligned} (\mathcal{H}_p \circ \mathcal{S}_q) f(r) &\geq \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^q} s^{q-p} + \frac{1}{r^{1-p}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} r^{q-p} \\ &= \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^p} + r^{q-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} \\ &= \mathcal{H}_p f(r) + \mathcal{S}_q f(r). \end{aligned}$$

This proves that $\mathcal{H}_p \circ \mathcal{S}_q \lesssim \mathcal{H}_p + \mathcal{S}_q$ and $\mathcal{H}_p \circ \mathcal{S}_q \gtrsim \mathcal{H}_p + \mathcal{S}_q$ for $0 < q - p < 1$, which is equivalent to $\mathcal{H}_p \circ \mathcal{S}_q \simeq \mathcal{H}_p + \mathcal{S}_q$. Now, if $q - p > 1$, by employing inequality (29) with $\gamma = q - p > 1$, we obtain

$$\sum_{s=1}^{r} s^{q-p-1} \geq \sum_{s=1}^{r} \frac{1}{q-p} \Delta(s-1)^{q-p} = \frac{1}{q-p} r^{q-p}, \tag{36}$$

and also, we have that

$$\sum_{s=1}^{r} s^{q-p-1} \le r^{q-p-1} \sum_{s=1}^{r} 1 = r^{q-p}. \tag{37}$$

By using (32) and (36), we obtain

$$\begin{aligned}
&(\mathcal{H}_p \circ \mathcal{S}_q) f(r) \\
\ge\ & \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^q} s^{q-p} + \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} r^{q-p} \\
=\ & \frac{1}{q-p} \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^p} + \frac{1}{q-p} r^{q-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} \\
=\ & \frac{1}{q-p} \left[\mathcal{H}_p f(r) + \mathcal{S}_q f(r) \right],
\end{aligned}$$

and by using (32) and (37), we obtain

$$\begin{aligned}
(\mathcal{H}_p \circ \mathcal{S}_q) f(r) &\le \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^q} s^{q-p} + \frac{1}{r^{1-p}} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} r^{q-p} \\
&= \frac{1}{r^{1-p}} \sum_{s=1}^{r} \frac{f(s)}{s^p} + r^{q-1} \sum_{s=r}^{\infty} \frac{f(s)}{s^q} \\
&= \mathcal{H}_p f(r) + \mathcal{S}_q f(r).
\end{aligned}$$

This proves that $\mathcal{H}_p \circ \mathcal{S}_q \lesssim \mathcal{H}_p + \mathcal{S}_q$ and $\mathcal{H}_p \circ \mathcal{S}_q \gtrsim \mathcal{H}_p + \mathcal{S}_q$ for $q - p > 1$, which is equivalent to $\mathcal{H}_p \circ \mathcal{S}_q \simeq \mathcal{H}_p + \mathcal{S}_q$. This is the required result. This completes our proof. □

Lemma 8. *Assume that f is a positive weight and $0 < \delta < 1$. Then inequality*

$$\mathcal{H}(\mathcal{H}_\delta f(r)) \le \frac{1}{\delta} \mathcal{H}_\delta f(r), \tag{38}$$

holds for all $r \ge 1$.

Proof. By the definition of operator $\mathcal{H}_\delta$ and by applying Fubini's Theorem, we have that

$$\mathcal{H}(\mathcal{H}_\delta f(r)) = \frac{1}{r} \sum_{s=1}^{r} \frac{1}{s^{1-\delta}} \sum_{k=1}^{s} \frac{f(k)}{k^\delta} = \frac{1}{r} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} \sum_{k=s}^{r} k^{\delta-1},$$

and by employing inequality (33) with $\delta < 1$, we obtain

$$\begin{aligned}
\mathcal{H}(\mathcal{H}_\delta f(r)) &\le \frac{1}{r} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} \sum_{k=s}^{r} \frac{1}{\delta} \Delta (k-1)^\delta \\
&= \frac{1}{\delta} \frac{1}{r} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} \left[r^\delta - (s-1)^\delta \right] \\
&\le \frac{1}{\delta} \frac{1}{r} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} r^\delta \le \frac{1}{\delta} \frac{1}{r^{1-\delta}} \sum_{s=1}^{r} \frac{f(s)}{s^\delta} = \frac{1}{\delta} \mathcal{H}_\delta f(r).
\end{aligned}$$

This is required inequality (38). This completes our proof. □

3. The Relations between Classes $\mathcal{M}_p$, $\mathcal{M}_p^*$, $\mathcal{B}_p$, $\mathcal{B}_p^*$ and $\Lambda_p(v)$

In this section, we study the relation between classes $\mathcal{M}_p$, $\mathcal{M}_p^*$, $\mathcal{B}_p$ and $\mathcal{B}_p^*$ defined on the space ℓ^p for a general weight v (not necessarily monotone) and the weighted Lorentz space $\Lambda_p(v)$.

Lemma 9. *Assume that v is a positive weight and $0 < \delta < 1$. If $v \in \mathcal{B}_\infty^*$, then there exists a constant $C > 1$ such that the inequality*

$$\mathcal{H}(\mathcal{H}_\delta v(r)) \leq C\mathcal{H}v(r), \tag{39}$$

holds for all $r \geq 1$.

Proof. Assume that $v \in \mathcal{B}_\infty^*$, then there exists a constant B such that

$$\mathcal{H}(\mathcal{H}v(r)) \leq B\mathcal{H}v(r), \text{ for all } r \geq 1. \tag{40}$$

By applying Hölder's inequality twice with exponents $1/\delta > 1$ and $1/(1-\delta)$, we have

$$\begin{aligned}
\sum_{s=1}^{r} \mathcal{H}_\delta v(s) &= \sum_{s=1}^{r} \frac{1}{s^{1-\delta}} \sum_{k=1}^{s} \frac{v(k)}{k^\delta} = \sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} \left(\frac{s}{k}\right)^\delta v^\delta(k) v^{1-\delta}(k) \\
&\leq \sum_{s=1}^{r} \left(\frac{1}{s} \sum_{k=1}^{s} \frac{s}{k} v(k)\right)^\delta \left(\frac{1}{k} \sum_{k=1}^{s} v(k)\right)^{1-\delta} \\
&\leq \left[\sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} \frac{s}{k} v(k)\right]^\delta \left[\sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} v(k)\right]^{1-\delta} \\
&= \left[\sum_{s=1}^{r} \sum_{k=1}^{s} \frac{v(k)}{k}\right]^\delta \left[\sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} v(k)\right]^{1-\delta}.
\end{aligned} \tag{41}$$

By applying Lemma 3 for $p = 1$ and (40) into (41), we have

$$\begin{aligned}
\sum_{s=1}^{r} (\mathcal{H}_\delta v(s)) &\leq \left[A \sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} v(k)\right]^\delta \left[\sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} v(k)\right]^{1-\delta} \\
&= A^\delta \sum_{s=1}^{r} \frac{1}{s} \sum_{k=1}^{s} v(k) \leq A^\delta B \sum_{s=1}^{r} v(s),
\end{aligned}$$

or equivalently,

$$\mathcal{H}(\mathcal{H}_\delta v(r)) \leq C\mathcal{H}v(r),$$

where $C = A^\delta B > 1$. This is the required result. The proof is complete. □

Theorem 1. *Let v be a positive weight and $p \geq 1$. If $v \in \mathcal{B}_p$, then there exists $\widetilde{v} \in \mathcal{M}_p$ such that v and $\widetilde{v}$ generate equivalent weighted Lorentz spaces.*

Proof. Assume that $v \in \mathcal{B}_p$, for $p \geq 1$, then by Lemma 4, we have that $v \in \mathcal{B}_{p-\delta}$ for some $0 < \delta < p$. Fix $0 < \delta < p$ such that $v \in \mathcal{B}_{p-\delta}(B)$ and define

$$\widetilde{v}(r) = r^{p-1-\delta} \sum_{s=r}^{\infty} \frac{v(s)}{s^{p-\delta}} = \mathcal{S}_{p-\delta} v(r). \tag{42}$$

Now, by using the fact that $f(x) = x/(x+1)$ is an increasing function for $x \geq 1$ and bounded below by $1/2$, we have, by applying Fubini's Theorem 2, that

$$\begin{aligned}
\mathcal{S}_p \widetilde{v}(r) &= r^{p-1} \sum_{s=r}^{\infty} \frac{1}{s^p} \left(s^{p-1-\delta} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right) \\
&= r^{p-1} \sum_{s=r}^{\infty} s^{-1-\delta} \left(\sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right) \\
&= r^{p-1} \sum_{s=r}^{\infty} \left(\frac{s}{s+1} \right)^{-1-\delta} (s+1)^{-1-\delta} \left(\sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right) \\
&\leq 2^{1+\delta} r^{p-1} \sum_{s=r}^{\infty} (s+1)^{-1-\delta} \left(\sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right) \\
&= 2^{1+\delta} r^{p-1} \sum_{s=r}^{\infty} \frac{v(s)}{s^{p-\delta}} \left(\sum_{k=r}^{s} (k+1)^{-1-\delta} \right),
\end{aligned} \tag{43}$$

and by employing inequality (29) with $\gamma = -\delta < 0$, we obtain that

$$\sum_{k=r}^{s} (k+1)^{-1-\delta} \leq \sum_{k=r}^{s} \frac{-1}{\delta} \Delta k^{-\delta} = \frac{-1}{\delta} (s+1)^{-\delta} + \frac{1}{\delta} r^{-\delta} \leq \frac{1}{\delta} r^{-\delta}. \tag{44}$$

Then, by using (43) and (44), we have

$$\mathcal{S}_p \widetilde{v}(r) \leq \frac{2^{1+\delta}}{\delta} r^{p-\delta-1} \sum_{s=r}^{\infty} \frac{v(s)}{s^{p-\delta}} = \frac{2^{1+\delta}}{\delta} \mathcal{S}_{p-\delta} v(r), \tag{45}$$

that is

$$\mathcal{S}_p \widetilde{v}(r) \leq C \widetilde{v}(r), \tag{46}$$

where $C = 2^{1+\delta}/\delta > 1$. For $p = 1$, inequality (46) reads $\mathcal{S}\widetilde{v}(r) \leq C\widetilde{v}(r)$, which is the $\mathcal{M}_1$ condition. If $p > 1$, then by using the nonincreasing property of $\sum_{s=r}^{\infty} v(s)/s^{p-\delta}$, we have that

$$\begin{aligned}
\left(\sum_{s=1}^{r} \left(\widetilde{v}(s) \right)^{1-p'} \right)^{p-1} &= \left(\sum_{s=1}^{r} \left(s^{p-1-\delta} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right)^{1-p'} \right)^{p-1} \\
&= \left(\sum_{s=1}^{r} s^{-1+\delta/(p-1)} \left(\sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right)^{1-p'} \right)^{p-1} \\
&\leq \left(\sum_{s=r}^{\infty} \frac{v(s)}{s^{p-\delta}} \right)^{-1} \left(\sum_{s=1}^{r} s^{-1+\delta/(p-1)} \right)^{p-1}.
\end{aligned} \tag{47}$$

If $\delta/(p-1) < 1$, by applying inequality (33) with $0 < \gamma = \delta/(p-1) < 1$, we obtain

$$\sum_{s=1}^{r} s^{-1+\delta/(p-1)} \leq \sum_{s=1}^{r} \frac{p-1}{\delta} \Delta (s-1)^{\delta/(p-1)} = \frac{p-1}{\delta} r^{\delta/(p-1)},$$

and if $\delta/(p-1) > 1$, by applying inequality (29) with $\gamma = \delta/(p-1) > 1$, we obtain

$$\begin{aligned}
\sum_{s=1}^{r} s^{-1+\delta/(p-1)} &\leq \sum_{s=1}^{r} \frac{p-1}{\delta} \Delta s^{\delta/(p-1)} = \frac{p-1}{\delta} \left[(r+1)^{\delta/(p-1)} - 1 \right] \\
&\leq \frac{p-1}{\delta} (r+1)^{\delta/(p-1)} \leq 2^{\delta/(p-1)} \frac{p-1}{\delta} r^{\delta/(p-1)}.
\end{aligned}$$

Then (47) implies that

$$\begin{aligned}&\left(\sum_{s=1}^{r}\left(\widetilde{v}(s)\right)^{1-p'}\right)^{p-1}\\ \leq\ &\left(\frac{p-1}{\delta}\right)^{p-1}\left(\frac{1}{r^{\delta}}\sum_{s=r}^{\infty}\frac{v(s)}{s^{p-\delta}}\right)^{-1}=\left(\frac{p-1}{\delta}\right)^{p-1}\left(r^{1-p}\mathcal{S}_{p-\delta}v(r)\right)^{-1},\end{aligned}\tag{48}$$

when $\delta/(p-1)<1$, and then we have that

$$\begin{aligned}&\left(\sum_{s=1}^{r}\left(\widetilde{v}(s)\right)^{1-p'}\right)^{p-1}\\ \leq\ &2^{\delta}\left(\frac{p-1}{\delta}\right)^{p-1}\left(\frac{1}{r^{\delta}}\sum_{s=r}^{\infty}\frac{v(s)}{s^{p-\delta}}\right)^{-1}=2^{\delta}\left(\frac{p-1}{\delta}\right)^{p-1}\left(r^{1-p}\mathcal{S}_{p-\delta}v(r)\right)^{-1},\end{aligned}\tag{49}$$

when $\delta/(p-1)<1$. From (46) and (48) together with (45), we can easily see that

$$\left(\sum_{s=r}^{\infty}\frac{\widetilde{v}(s)}{s^{p}}\right)^{1/p}\left(\sum_{s=1}^{r}\left(\widetilde{v}(s)\right)^{1-p'}\right)^{(p-1)/p}\leq A,$$

for $\delta/(p-1)<1$, where the constant A is given by

$$A=\left(\frac{2^{1+\delta}}{\delta}\right)^{1/p}\left(\frac{p-1}{\delta}\right)^{(p-1)/p}.$$

From (46) and (49) together with (45), we can easily see that

$$\left(\sum_{s=r}^{\infty}\frac{\widetilde{v}(s)}{s^{p}}\right)^{1/p}\left(\sum_{s=1}^{r}\left(\widetilde{v}(s)\right)^{1-p'}\right)^{(p-1)/p}\leq A,$$

for $\delta/(p-1)>1$, where the constant A, in this case, is given by

$$A=\left(\frac{2^{1+2\delta}}{\delta}\right)^{1/p}\left(\frac{p-1}{\delta}\right)^{(p-1)/p}.$$

This proves that $\widetilde{v}\in\mathcal{M}_p$. Now, by applying summation by parts formulae

$$\sum_{s=1}^{r}\Delta u(s)w(s+1)=u(s)w(s)|_{s=1}^{r+1}-\sum_{s=1}^{r}u(s)\Delta w(s),\tag{50}$$

with $w(s)=\sum_{k=s}^{\infty}v(k)/k^{p-\delta}$ and $\Delta u(s)=s^{p-1-\delta}$, we obtain

$$\begin{aligned}\sum_{s=1}^{r}\widetilde{v}(s)&=\sum_{s=1}^{r}s^{p-1-\delta}\left(\sum_{k=s}^{\infty}\frac{v(k)}{k^{p-\delta}}\right)\\ &=\left(\sum_{k=s}^{\infty}\frac{v(k)}{k^{p-\delta}}\right)\left(\sum_{k=1}^{s-1}k^{p-1-\delta}\right)\Bigg|_{s=1}^{r+1}-\sum_{s=1}^{r}\left(-\frac{v(s)}{s^{p-\delta}}\right)\left(\sum_{k=1}^{s}k^{p-1-\delta}\right)\\ &=\left(\sum_{s=r+1}^{\infty}\frac{v(s)}{s^{p-\delta}}\right)\left(\sum_{s=1}^{r}s^{p-1-\delta}\right)+\sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(\sum_{k=1}^{s}k^{p-1-\delta}\right)\\ &\geq\sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(\sum_{k=1}^{s}k^{p-1-\delta}\right).\end{aligned}\tag{51}$$

If $p-\delta>1$, by applying inequality (29) with $\gamma=p-\delta>1$, then we have from (51) that

$$\begin{aligned}\sum_{s=1}^{r}\widetilde{v}(s) &\geq \sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(\sum_{k=1}^{s}\frac{1}{p-\delta}\Delta(k-1)^{p-\delta}\right)\\ &= \sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(\frac{s^{p-\delta}}{p-\delta}\right)=\frac{1}{p-\delta}\sum_{s=1}^{r}v(s),\end{aligned}$$

and if $p-\delta<1$, since $k^{p-1-\delta}\geq s^{p-1-\delta}$ for all $k\leq s$, then (51) becomes

$$\sum_{s=1}^{r}\widetilde{v}(s)\geq\sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(s^{p-1-\delta}\sum_{k=1}^{s}1\right)=\sum_{s=1}^{r}\frac{v(s)}{s^{p-\delta}}\left(s^{p-\delta}\right)=\sum_{s=1}^{r}v(s),$$

This proves that

$$\sum_{s=1}^{r}\widetilde{v}(s)\gtrsim\sum_{s=1}^{r}v(s).$$

Furthermore, by using Lemma 7 for $p=0$ and q replaced by $p-\delta$ satisfying $0<q-p$ and since $v\in\mathcal{B}_{p-\delta}$, we see that there exists a positive constant $C>0$ such that

$$\begin{aligned}\sum_{s=1}^{r}\widetilde{v}(s) &= \sum_{s=1}^{r}s^{p-1-\delta}\left(\sum_{k=s}^{\infty}\frac{v(k)}{k^{p-\delta}}\right)=r\left(\mathcal{H}_0\circ\mathcal{S}_{p-\delta}\right)v(r)\\ &\leq Cr\left(\mathcal{H}_0+\mathcal{S}_{p-\delta}\right)v(r)\\ &\leq C[V(r)+BV(r)]=C^*V(r),\end{aligned}$$

where $C^*=(B+1)/C$. That is, $\sum_{s=1}^{r}\widetilde{v}(s)\lesssim\sum_{s=1}^{r}v(s)$. Hence, we have that

$$\sum_{s=1}^{r}\widetilde{v}(s)\simeq\sum_{s=1}^{r}v(s),$$

and then by applying Lemma 1, we have that

$$\sum_{r=1}^{\infty}(u^*(r))^p\widetilde{v}(r)\simeq\sum_{r=1}^{\infty}(u^*(r))^p v(r).$$

This proves that $\Lambda_p(v)$ is equivalent to $\Lambda_p(\widetilde{v})$, which is the required result. This completes our proof. □

Theorem 2. *Let v be a positive weight and $p\geq1$. If $v\in\mathcal{B}_\infty^*$, then there exists $\widetilde{v}\in\mathcal{M}_p^*$ such that v and $\widetilde{v}$ generate equivalent weighted Lorentz spaces.*

Proof. Assume that $v\in\mathcal{B}_\infty^*$. Now, fix $0<\delta<1$ and define

$$\widetilde{v}(r)=v(r)+\mathcal{H}_\delta v(r)=v(r)+\frac{1}{r^{1-\delta}}\sum_{s=1}^{r}\frac{v(s)}{s^\delta}.$$

Now, we have

$$\mathcal{H}\widetilde{v}(r)=\mathcal{H}[v(r)+\mathcal{H}_\delta v(r)]=\mathcal{H}v(r)+\mathcal{H}(\mathcal{H}_\delta v(r)),$$

and by using (24), (38) and the nonnegative nature of v, we have that

$$\mathcal{H}\widetilde{v}(r)\leq\mathcal{H}_\delta v(r)+C\mathcal{H}_\delta v(r)\leq(1+\frac{1}{\delta})\mathcal{H}_\delta v(r)\leq(1+\frac{1}{\delta})\widetilde{v}(r),$$

where $C \geq 1$. This proves that $\widetilde{v} \in \mathcal{M}_1^*$, and then Lemma 5 implies that $v \in \mathcal{M}_p^*$. Furthermore, we have

$$\sum_{s=1}^{r} \widetilde{v}(s) = \sum_{s=1}^{r} [v(s) + \mathcal{H}_\delta v(s)] \geq \sum_{s=1}^{r} v(s). \tag{52}$$

That is $\sum_{s=1}^{r} \widetilde{v}(s) \gtrsim \sum_{s=1}^{r} v(s)$. Further, since $v \in \mathcal{B}_\infty^*$, then by using (39), we have that

$$\begin{aligned}
\sum_{s=1}^{r} \widetilde{v}(s) &= \sum_{s=1}^{r} [v(s) + \mathcal{H}_\delta v(s)] = \sum_{s=1}^{r} v(s) + r\mathcal{H}(\mathcal{H}_\delta v)(r), \\
&\leq \sum_{s=1}^{r} v(s) + Cr\mathcal{H}v(r) = (1+C) \sum_{s=1}^{r} v(s).
\end{aligned}$$

That is, we have

$$\sum_{s=1}^{r} \widetilde{v}(s) \lesssim \sum_{s=1}^{r} v(s), \tag{53}$$

Then, from (52) and (53), we see that

$$\sum_{s=1}^{r} \widetilde{v}(s) \simeq \sum_{s=1}^{r} v(s),$$

holds for all $r \geq 1$. Then by applying Lemma 1, we have that

$$\sum_{r=1}^{\infty} (u^*(r))^p \widetilde{v}(r) \simeq \sum_{r=1}^{\infty} (u^*(r))^p v(r),$$

and then $\Lambda_p(v)$ is equivalent to $\Lambda_p(\widetilde{v})$, which is the required result. This completes our proof. □

Theorem 3. *Let v be a positive weight and $p \geq 1$. If $v \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$, then there exists $\widetilde{v} \in \mathcal{M}_p \cap \mathcal{M}_p^*$ such that v and $\widetilde{v}$ generate equivalent weighted Lorentz spaces.*

Proof. Assume that $v \in \mathcal{B}_p \cap \mathcal{B}_\infty^*$, then $v \in \mathcal{B}_p(B)$ and $v \in \mathcal{B}_\infty^*(C)$. Without loss of generality, we choose $0 < \delta < p$ such that $\delta \leq p-1, v \in \mathcal{B}_{p-\delta}(B)$ and define

$$\widetilde{v}(r) = \mathcal{S}_{p-\delta} v(r) + \mathcal{H}_\delta v(r).$$

Now, we have

$$\mathcal{S}_p \widetilde{v}(r) = (\mathcal{S}_p \circ \mathcal{S}_{p-\delta}) v(r) + (\mathcal{S}_p \circ \mathcal{H}_\delta) v(r), \tag{54}$$

and by applying Lemma 7 for $q - p = p - \delta > 0$, then (54) becomes

$$\mathcal{S}_p \widetilde{v}(r) \leq (\mathcal{S}_p \circ \mathcal{S}_{p-\delta}) v(r) + \frac{2^{p-\delta+1}}{p-\delta} (\mathcal{S}_p + \mathcal{H}_\delta) v(r). \tag{55}$$

By using inequality (25) for $p - \delta < p$ and inequality (45), then (55) becomes

$$\begin{aligned}
\mathcal{S}_p \widetilde{v}(r) &\leq \frac{2^{1+\delta}}{\delta} \mathcal{S}_{p-\delta} v(r) + \frac{2^{p-\delta-1}}{p-\delta} (\mathcal{S}_{p-\delta} + \mathcal{H}_\delta) v(r) \\
&\leq \left(\frac{2^{1+\delta}}{\delta} + \frac{2^{p-\delta-1}}{p-\delta} \right) (\mathcal{S}_{p-\delta} + \mathcal{H}_\delta) v(r) = L\widetilde{v}(r),
\end{aligned} \tag{56}$$

where

$$L = \frac{2^{1+\delta}}{\delta} + \frac{2^{p-\delta-1}}{p-\delta} > 1.$$

If $p = 1$, then (56) proves that $\widetilde{v} \in \mathcal{M}_1$. If $p > 1$, then by using Lemma 7 for $q - p = (p-\delta) - \delta \geq 1 - \delta > 0$, we have that

$$\begin{aligned}
\sum_{s=1}^{r} \left(\widetilde{v}(s)\right)^{-1/(p-1)} &= \sum_{s=1}^{r} \left(\mathcal{S}_{p-\delta}v(s) + \mathcal{H}_\delta v(s)\right)^{-1/(p-1)} \\
&\leq \left(\frac{p-2\delta}{2^{p-2\delta-1}}\right)^{-1/(p-1)} \sum_{s=1}^{r} \left((\mathcal{S}_{p-\delta} \circ \mathcal{H}_\delta)v(s)\right)^{-1/(p-1)} \\
&= \left(\frac{p-2\delta}{2^{p-2\delta-1}}\right)^{-1/(p-1)} \sum_{s=1}^{r} \left(s^{p-\delta-1} \sum_{k=s}^{\infty} \frac{\mathcal{H}_\delta v(k)}{k^{p-\delta}}\right)^{-1/(p-1)} \\
&\leq \left(\frac{p-2\delta}{2^{p-2\delta-1}}\right)^{-1/(p-1)} \left(\sum_{s=r}^{\infty} \frac{1}{s^{p-\delta}} \mathcal{H}_\delta v(s)\right)^{-1/(p-1)} \sum_{s=1}^{r} s^{-1+\delta/(p-1)}. \qquad (57)
\end{aligned}$$

Since $\delta \leq p-1$, then by employing inequality (33) with $0 < \gamma = \delta/(p-1) < 1$, we have that

$$\sum_{s=1}^{r} s^{-1+\delta/(p-1)} \leq \sum_{s=1}^{r} \frac{p-1}{\delta} \Delta(s-1)^{\delta/(p-1)} = \frac{p-1}{\delta} r^{\delta/(p-1)}, \qquad (58)$$

By using (58) and (57) and using Lemma 7, we have that

$$\begin{aligned}
&\sum_{s=1}^{r} \left(\widetilde{v}(s)\right)^{-1/(p-1)} \\
&\leq \left(\frac{p-2\delta}{2^{p-2\delta-1}}\right)^{-1/(p-1)} \frac{p-1}{\delta} r^{\delta/(p-1)} \left(\sum_{s=r}^{\infty} \frac{1}{s^{p-\delta}} \mathcal{H}_\delta(s)\right)^{-1/(p-1)} \\
&= \left(\frac{p-2\delta}{2^{p-2\delta-1}}\right)^{-1/(p-1)} \frac{p-1}{\delta} \left[r^{1-p}(\mathcal{S}_{p-\delta} \circ \mathcal{H}_\delta)v(r)\right]^{-1/(p-1)} \\
&\leq C^* \left[r^{1-p}(\mathcal{S}_{p-\delta} + \mathcal{H}_\delta)v(r)\right]^{-1/(p-1)} = C^* \left[r^{1-p}\widetilde{v}(r)\right]^{-1/(p-1)}, \qquad (59)
\end{aligned}$$

where $C^* = \frac{p-1}{\delta} 2^{1-2\delta/(p-1)}$. Then (56) and (59) imply that

$$\begin{aligned}
&\left(\sum_{s=r}^{\infty} \frac{\widetilde{v}(s)}{s^p}\right)^{1/p} \left(\sum_{s=1}^{r} \left(\widetilde{v}(s)\right)^{-1/(p-1)}\right)^{(p-1)/p} \\
&= \left[r^{1-p}\mathcal{S}_p \widetilde{v}(r)\right]^{1/p} \left(\sum_{s=1}^{r} \left(\widetilde{v}(s)\right)^{-1/(p-1)}\right)^{(p-1)/p} \\
&\leq \left[r^{1-p} L \widetilde{v}(r)\right]^{1/p} (C^*)^{(p-1)/p} \left[r^{1-p}\widetilde{v}(r)\right]^{-1/p} \\
&= C^{**},
\end{aligned}$$

where $C^{**} = L^{1/p}(C^*)^{(p-1)/p}$. This proves that $\widetilde{v} \in \mathcal{M}_p$. Furthermore, by using the fact that $\mathcal{H} = \mathcal{H}_0$, applying Lemma 7 with $q - p = p - \delta > 0$ and using inequalities (24) and (38), then we have that

$$\begin{aligned}
\mathcal{H}\widetilde{v}(r) &= \mathcal{H}\left(\mathcal{S}_{p-\delta}v(r) + \mathcal{H}_\delta v(r)\right) \\
&= (\mathcal{H}_0 \circ \mathcal{S}_{p-\delta})v(r) + (\mathcal{H} \circ \mathcal{H}_\delta)v(r) \\
&\leq (\mathcal{H}_0 + \mathcal{S}_{p-\delta})v(r) + \frac{1}{\delta}\mathcal{H}_\delta v(r) \\
&\leq (\mathcal{H}_\delta + \mathcal{S}_{p-\delta})v(r) + \frac{1}{\delta}\mathcal{H}_\delta v(r) \leq D\widetilde{v}(r),
\end{aligned}$$

where $D = 1 + (1/\delta)$. This proves that $\widetilde{v} \in M_1^*$ and then Lemma 5 implies that $v \in M_p^*$. Finally, by the fact that $v \in \mathcal{B}_{p-\delta}(B)$, $v \in \mathcal{B}_\infty^*(C)$ and inequality (39), then we have that

$$\begin{aligned}
\sum_{s=1}^{r} \widetilde{v}(s) &= \sum_{s=1}^{r} (\mathcal{H}_\delta + \mathcal{S}_{p-\delta}) v(s) = r\mathcal{H}(\mathcal{H}_\delta)v(r) + \sum_{s=1}^{r} s^{p-\delta-1} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \\
&\leq Cr\mathcal{H}v(r) + \sum_{s=1}^{r} \frac{B}{s} \sum_{k=1}^{s} v(k) = Cr\mathcal{H}v(r) + Br\mathcal{H}(\mathcal{H}v(r)) \\
&\leq Cr\mathcal{H}v(r) + BAr\mathcal{H}v(r) = (C + BA) \sum_{s=1}^{r} v(s).
\end{aligned}$$

This proves that $\sum_{s=1}^{r} \widetilde{v}(s) \lesssim \sum_{s=1}^{r} v(s)$. Furthermore, by using relation (24) and the fact that $v \in \mathcal{B}_{p-\delta}(B)$, we have that

$$\begin{aligned}
\sum_{s=1}^{r} \widetilde{v}(s) &= \sum_{s=1}^{r} [\mathcal{S}_{p-\delta}v(s) + \mathcal{H}_\delta v(s)] \geq \sum_{s=1}^{r} [\mathcal{S}_{p-\delta}v(s) + \mathcal{H}v(s)] \\
&= \sum_{s=1}^{r} \left[s^{p-\delta-1} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} + \frac{1}{s} \sum_{k=1}^{s} v(k) \right] \\
&\geq \sum_{s=1}^{r} \left[s^{p-\delta-1} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} + \frac{1}{s} \frac{s^{p-\delta}}{B} \sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right] \\
&= (1 + \frac{1}{B}) \sum_{s=1}^{r} s^{p-\delta-1} \left(\sum_{k=s}^{\infty} \frac{v(k)}{k^{p-\delta}} \right) \\
&\geq (1 + \frac{1}{B}) \sum_{s=1}^{r} s^{p-\delta-1} \left(\sum_{k=s}^{r} \frac{v(k)}{k^{p-\delta}} \right),
\end{aligned}$$

which by Fubini's Theorem and inequality (29) with $\gamma = p - \delta > 1$, implies that

$$\begin{aligned}
\sum_{s=1}^{r} \widetilde{v}(s) &\geq (1 + \frac{1}{B}) \sum_{s=1}^{r} \frac{v(s)}{s^{p-\delta}} \left(\sum_{k=1}^{s} k^{p-\delta-1} \right) \\
&\geq (1 + \frac{1}{B}) \sum_{s=1}^{r} \frac{v(s)}{s^{p-\delta}} \left(\sum_{k=1}^{s} \frac{1}{p-\delta} \Delta(k-1)^{p-\delta} \right) \\
&= (1 + \frac{1}{B}) \frac{1}{p-\delta} \sum_{s=1}^{r} v(s).
\end{aligned}$$

This proves that

$$\sum_{s=1}^{r} \widetilde{v}(s) \gtrsim \sum_{s=1}^{r} v(s).$$

Hence $\sum_{s=1}^{r} \widetilde{v}(s) \simeq \sum_{s=1}^{r} v(s)$. By applying Lemma 1, we have

$$\sum_{r=1}^{\infty} (u^*(r))^p \widetilde{v}(r) \simeq \sum_{r=1}^{\infty} (u^*(r))^p v(r),$$

and then $\Lambda_p(v)$ is equivalent to $\Lambda_p(\widetilde{v})$, which is the required result. This completes our proof. □

4. Conclusions

In this paper, we were able to prove some equivalence relations between different weights in the classes $\mathcal{B}_p$, $\mathcal{B}_p^*$, $\mathcal{M}_p$ and $\mathcal{M}_p^*$—classes that have been used in proving the boundedness of Hardy's operator and its adjoint form. We also proved that although the two different weights, v and $\widetilde{v}$ (not necessarily monotone), do not belong to the same class, they generate the same weighted Lorentz space, i.e., $\Lambda_p(v) \simeq \Lambda_p(\widetilde{v})$.

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References

1. Bober, J.; Hughes, E.C.K.; Pierce, L.B. On a discrete version of Tanaka's theorem for maximal functions. *Proc. Amer. Math. Soc.* **2012**, *140*, 1669–1680. [CrossRef]
2. Liu, F. Endpoint regularity of discrete multisublinear fractional maximal operators associated with l^1-balls. *J. Ineq. Appl.* **2018**, *2018*, 33. [CrossRef] [PubMed]
3. Madrid, J. Sharp inequalities for the variation of the discrete maximal function. *Bull. Austr. Math. Soc.* **2017**, *95*, 94–107. [CrossRef]
4. Magyar, A.; Stein, E.M.; Wainger, S. Discrete analogues in harmonic analysis: Spherical averages. *Ann. Math.* **2002**, *155*, 189–208. [CrossRef]
5. Stein, E.M.; Wainger, S. Discrete analogues in harmonic analysis I: l^2-estimates for singular Radon transforms. *Am. J. Math.* **1999**, *121*, 1291–1336. [CrossRef]
6. Stein, E.M.; Wainger, S. Discrete analogues in harmonic analysis II: Fractional integration. *J. d'Analyse Math.* **2000**, *80*, 335–355. [CrossRef]
7. Stein, E.M.; Wainger, S. Two discrete fractional integral operators revisited. *J. d'Analyse Math.* **2002**, *87*, 451–479. [CrossRef]
8. Connor, J. Open Problems in Sequence Spaces. 1992, *Unpublished notes*.
9. Bennett, G. Some elementary inequalities. *Quart. J. Math. Oxf. Ser.* **1987**, *38*, 401–425. [CrossRef]
10. Bennett, G. Some Elementary Inequalities II. *Quart. Math. Oxf.* **1988**, *3*, 385–400. [CrossRef]
11. Bennett, G. Some elementary inequalities III. *Quart. Math. Oxf. Ser.* **1991**, *42*, 149–174. [CrossRef]
12. Andersen, K.F.; Heinig, H.P. Weighted norm inequalities for certain integral operators. *SIAM J. Math. Anal.* **1983**, *14*, 834–844. [CrossRef]
13. Ariňo, M.; Muckenhoupt, B. Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non-increasing functions. *Trans. Amer. Math. Soc.* **1990**, *320*, 727–735.
14. Bradely, J.S. Hardy inequalities with mixed norms. *Canad. Math. Bull.* **1920**, *21*, 405–408. [CrossRef]
15. Carro, J.M.; Soria, J. Boundedness of some integral operators. *Canad. J. Math.* **1993**, *45*, 1155–1166. [CrossRef]
16. Carro, J.M.; Lorente, M. Rubio De Francia's extrapolation theorem for B_p-weights. *Proc. Amer. Math. Soc.* **2010**, *138*, 629–640. [CrossRef]
17. Gogatishvili, A.; Kufner, A.; Persson, L.E. Some new scales of characterizations of Hardy's inequality. *Proc. Est. Acad. Sci.* **2010**, *59*, 7–18. [CrossRef]
18. Kufner, A.; Persson, L.E.; Wedestig, A. Astudy of Some Constants Characterizing the Weighted Hardy Inequality. *Banach Cent. Publ.* **2004**, *64*, 135–146.
19. Sinnamon, G.J.; Stepanov, W.D. The weighted Hardy inequality: New proofs and the case $p = 1$. *J. Lond. Math. Soc.* **1996**, *54*, 89–101. [CrossRef]
20. Muckenhoupt, B. Hardy's inequality with weights. *Studia Math.* **1972**, *44*, 31–83. [CrossRef]
21. Bennett, G.; Grosse-Erdmann, K.-G. Weighted Hardy inequalities for decreasing sequences and functions. *Math. Ann.* **2006**, *334*, 489–531. [CrossRef]
22. Saker, S.H.; O'Regan, D.; Agarwal, R.P. A higher integrability theorem from a reverse weighted inequality. *Bull. Lond. Math. Soc.* **2019**, *51*, 967–977. [CrossRef]
23. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1934.
24. Saker, S.H.; Baleanu, D.; Krnić, M. Some further properties of discrete Muckenhoupt and Gehring weights. *J. Math. Ineq.* **2022**, *16*, 1–18. [CrossRef]

25. Saker, S.H.; Rabie S.S.; Agarwal, R.P. Boundedness of discrete operators and equivalence and embedding relations between ℓ^p-classes. *Appl. Anal. Discrete Math. submitted*.
26. Saker, S.H.; Agarwal, R.P. Theory of discrete Muckenhoupt weights and discrete Rubio de Francia extrapolation theorems. *Appl. Anal. Discret. Math.* **2021**, *15*, 295–316. [CrossRef]
27. Saker, S.H.; Alzabut, J.; O'Regan, D.; Agarwal, R.P. Self-improving properties of weighted Gehring classes with applications to partial differential equations. *Adv. Differ. Eqns.* **2021**, *2021*, 397. [CrossRef]
28. Saker, S.H.; Krnić, M. The weighted discrete Gehring classes, Muckenhoupt classes and their basic properties. *Proc. Amer. Math. Soc.* **2021**, *149*, 231–243. [CrossRef]
29. Saker, S.H.; Mahmoud, R.R. Boundedness of both discrete Hardy and Hardy-Littlewood Maximal operators via Muckenhoupt weights. *Rocky Mount. J. Math.* **2021**, *51*, 733–746. [CrossRef]
30. Heing, H.P.; Kufner, A. Hardy operators of monotone functions and sequences in Orlicz spaces. *J. Lond. Math. Soc.* **1996**, *53*, 256–270. [CrossRef]
31. Okpoti, C.A.; Persson, L.E.; Wedestig, A. Scales of Weight Characterizations for the Discrete Hardy and Carleman Inequalities. In Proceedings of the Function Spaces, Differential Operators and Nonlinear Analysis (FSDONA '04), Milovy, Czech Republic, 28 May–2 June 2004; Academy of Sciences of the Czech Republic, Milovy: Sněžné, Czech Republic, 2004; pp. 236–258.
32. Lorentz, G.G. Some new functional spaces. *Ann. Math. II* **1950**, *51*, 37–55. [CrossRef]
33. Lorentz, G.G. On the theory of spaces Λ. *Pacific J. Math.* **1951**, *1*, 411–429. [CrossRef]
34. Barza, S.; Soria, J. Sharp constants between equivalent norms in weighted Lorentz spaces. *J. Aust. Math. Soc.* **2010**, *88*, 19–27. [CrossRef]
35. Barza, S.; Kolyada, V.; Soria, J. Sharp constants related to the triangle inequality in Lorentz spaces. *Trans. Amer. Math. Soc.* **2009**, *361*, 5555–5574. [CrossRef]
36. Barza, S.; Marcocli, A.N.; Persson, L.-E. Best constants between equivalent norms in Lorentz sequence spaces. *J. Funct. Spaces Appl.* **2012**, *2012*, 713534. [CrossRef]
37. Barza, S.; Marcoci, A.N.; Marcoci, L.G.; Persson, L.-E. Optimal estimates in Lorentz spaces of sequences with an increasing weight. *Proc. Rom. Acad. Ser. A* **2013**, *14*, 20–27.
38. Bennett, C.; Sharpley, R. *Interpolation of Operators*; Academic Press: Boston, MA, USA, 1988.
39. Bennett, G. *Factorizing the Classical Inequalities*; American Mathematical Society: Providence, RI, USA, 1996; p. 576.
40. Carro, M.J.; Raposo, J.A.; Soria, J. Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities. *Mem. Am. Math. Soc.* **2007**, *187*. [CrossRef]
41. Carro, M.; Pick, L.; Soria, J.; Stepanov, V.D. On embeddings between classical Lorentz spaces. *Math. Ineq. Appl.* **2001**, *4*, 397–428. [CrossRef]
42. Carro, M.; Soria, J. Weighted Lorentz spaces and the Hardy operator. *J. Funct. Anal.* **1993**, *112*, 480–494. [CrossRef]
43. Gogatishvili, A.; Soudsk, F. Normability of Lorentz Spaces-an alternative approach. *Czechoslov. Math. J.* **2014**, *64*, 581–597. [CrossRef]
44. Jameson, G.J.O.; Lashkaripour, R. Norms of certain operators on weighted l_p spaces and Lorentz sequences spaces. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 6.
45. Kolyada, V. Inequalities of Gagliardo-Nirenberg type and estimates for the moduli of continuity. *Russ. Math. Surv.* **2005**, *60*, 1147–1164. [CrossRef]
46. Sawyer, E. Boundedness of classical operators on classical Lorentz spaces. *Stud. Math.* **1990**, *96*, 145–158. [CrossRef]
47. Garling, D.J.H. *Inequalities: A Journey into Linear Analysis*; Cambridge University Press: Cambridge, UK, 2007.
48. Saker, S.H.; Krnić, M.; Baleanu, D. On structure of discrete Muckenhoupt and discrete Gehring classes. *J. Ineq. Appl.* **2020**, *2020*, 233. [CrossRef]
49. Saker, S.H.; Rabie, S.S.; Alzabut, J.; O'Regan, D.; Agarwal, R.P. Some basic properties and fundamental relations for discrete Muckenhoupt and Gehring classes. *Adv. Differ. Eqns.* **2021**, *2021*, 8. [CrossRef]