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Article

The Fractional Analysis of a Nonlinear mKdV Equation with Caputo Operator

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Abstract: In this study, we aim to provide reliable methods for the initial value problem of the fractional modified Korteweg–de Vries (mKdV) equations. Fractional differential equations are essential for more precise simulation of numerous processes. The hybrid Yang transformation decomposition method (YTDM) and Yang homotopy perturbation method (YHPM) are employed in a very simple and straightforward manner to handle the current problems. The derivative of fractional order is displayed in a Caputo form operator. To illustrate the conclusion given from the findings, a few numerical cases are taken into account for their approximate analytical solutions. We looked at two cases and contrasted them with the actual result to validate the methodologies. These techniques create recurrence relations representing the proposed problem's solution. It is possible to find the series solutions to the given problems, and these solutions have components that converge to precise solutions more quickly. Tables and graphs are used to describe the new results, which demonstrate the present methods' adequate accuracy. The actual and estimated outcomes are demonstrated in graphs and tables to be quite similar, demonstrating the usefulness of the proposed approaches. The innovation of the current work resides in the application of effective methods that require less calculation and achieve a greater level of accuracy. Additionally, the suggested approaches can be applied in the future to resolve other nonlinear fractional problems, which will be a scientific contribution to the research community.

Keywords: fractional mKdV equation; analytical techniques; Caputo operator; Yang transform

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1. Introduction

The study of fractional order derivatives and integrations is known as fractional calculus (FC). When L'Hospital questioned Leibniz in 1965 about the derivative of fractional order, he first proposed the concept of FC. The theory of FC was initially given as an apparent paradox, but as time went on, it grew in popularity as a topic of study. Many mathematicians were drawn to the field of FC due to its wide range of uses in several fields of study. By employing FC rather than regular calculus, some of the significant physical events in nature have been modeled more precisely. In the literature, the applications of FC can be found in fluid traffic [1], airfoil [2], modeling of earthquake nonlinear oscillation [3], finance [4], cancer chemotherapy [5], electrodynamics [6], Zener model [7], Chaos theory [8], diabetes [9], Poisson–Nernst–Planck diffusion [10], hepatitis B disease model [11], tuberculosis [12], fractional COVID-19 model [13], pine wilt disease [14], hepatitis B virus [15], and other applications in various areas of research [16–18]. Furthermore, the FC can contribute significantly to modeling many nonlinear phenomena that can propagate in different plasma models such as solitary waves, cnoidal waves, shock waves, rogue waves, and so on [19–25].

Fractional partial differential equations (FPDEs) are currently regarded as the most dependable and efficient method for creating the most precise mathematical models of a variety of significant phenomena in physics and other applied sciences [26]. FPDEs are more accurate at simulating many natural processes than simple PDEs, such as tuberculosis [27] and optics [28]. Since many complicated natural events are described using nonlinear FPDEs, the study of FPDEs and the nonlinearity related to each topic is of greater importance. Nehad et al. have provided the solutions to a few nonlinear FPDEs that can be found in [29]. Similar to this, Hilfer and Ray have presented several effective methods for solving specific nonlinear FPDEs in Refs. [30,31], respectively.

The study of this topic has become intriguing for scholars because of the aforementioned valuable uses of FC in real issues. In order to further expand the analysis of this topic, mathematicians understood it was necessary to look into the numerical or analytical solutions of FPDEs and their systems. As we all know, numerous significant mathematical models that describe some physical processes in nature have been solved using numerical and analytical approaches. To solve FPDEs to related systems, mathematicians have put a lot of effort into creating and developing a number of methods. To solve FPDEs and associated systems, significant and effective approaches are addressed, such as the first integral method [32], the extended direct algebraic method [33], the modified Kudryashov method [34], the finite difference method [35], the optimal homotopy asymptotic method (OHAM) [36], the Adomian decomposition method (ADM) [37], the standard reductive perturbation method (RPM) [38], the homotopy perturbation technique (HPT) [39–41], the Elzaki transform decomposition method [42], the Haar wavelet method [43], the fractional sub-equation method [44], the differential transform method (DTM) [45], and the variational iteration method (VIM) with transformation [46].

The KdV equation was developed by Korteweg and de Vries in 1895 and used to examine the waves that occur on shallow water surfaces. Many investigations have been conducted on this precisely solvable model. The creation of acoustic waves in plasma from ions and crystal lattices has several new applications that have been proposed by researchers. A typical KdV equation has the following structure:

$$\mathbb{K}_{\theta} + 6\mathbb{K}\mathbb{K}_{\varsigma} + \mathbb{K}_{\varsigma\varsigma\varsigma} = 0. \tag{1}$$

Numerous scientific fields have benefited from the massive use of the KdV equation, such as the study of acoustic waves (AWs) in different plasma models, shallow water waves, magneto-hydrodynamic waves in warm plasma, and bubble liquid mixes [47–52]. The KdV model also is used to explain particular theoretical physical aspects related to quantum mechanics. The algorithm is employed in the fields of aerodynamics, continuum mechanics, fluid dynamics, and mass transport for the generation of solitons, shock waves, boundary layer behavior, and turbulence. It has long been researched and applied. The basic KdV equation has given rise to a number of revisions and generalizations, as documented in the literature [53,54].

In this paper, we consider the following form of the modified Korteweg–de Vries (mKdV) equation

$$D_{\theta}^{\wp}\mathbb{K}(\varsigma,\theta) + 6\mathbb{K}^{2}(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) + \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) = 0, \quad 0 < \wp \leq 1. \tag{2}$$

Numerous scholars have made efforts to improve the methods already used to solve FPDEs and associated systems by applying various transformations. The Laplace, natural, and Mohand transformations [55–57], among others, are well-known transformations that can be used to reduce the original problem before using VIM, ADM, HPM, DTM, and other techniques to solve the desired problems. The Yang transformation (YT) is essential in resolving FPDEs and associated systems in the same framework. Differential equations with constant coefficients can be solved using this transformation, which was developed by Xiao-Jun Yang. The YT was initially applied to the solution of PDEs before being applied to the solution of ordinary differential equations. Many researchers have now coupled this transformation with other already-used techniques to solve more complex nonlinear issues. The Homotopy perturbation transform technique (HPTM) and Yang transform decomposition method (YTDM), which combines Yang transformation with the HPM and ADM to develop new approaches based on YT, are used to solve the mKdV equation.

The rest of the study is organized as follows. Section 2 compiles some basic definitions. Section 3 introduces the concept of HPTM, whereas Section 4 introduces the concept of YTDM. The proposed methods convergence analysis is provided in Section 5. In Section 6, various approximate solutions to the fractional mKdV equation are derived by using the form of the initial value, and the structure of the solutions is displayed using graphs and tables. In Section 8, the work's conclusion is examined.

2. Preliminaries

Here, some important definitions are discussed which are necessary to complete the present research task.

Definition 1. *The operator in Caputo sense for the fractional derivative is [58]*

$$D_{\theta}^{\wp}\mathbb{K}(\varsigma,\theta) = \frac{1}{\Gamma(k-\wp)}\int_0^{\theta}(\theta-\wp)^{k-\wp-1}\mathbb{K}^{(k)}(\varsigma,\psi)d\psi, \quad k-1<\wp\leq k, \quad k\in\mathbf{N}. \tag{3}$$

Definition 2. *The YT for the given function is [59]*

$$Y\{\mathbb{K}(\theta)\} = M(u) = \int_0^{\infty} e^{\frac{-\theta}{u}}\mathbb{K}(\theta)d\theta, \;\; \theta>0, \;\; u\in(-\theta_1,\theta_2), \tag{4}$$

and the inverse of the YT is

$$Y^{-1}\{M(u)\} = \mathbb{K}(\theta). \tag{5}$$

Definition 3. *The YT of a function having the nth derivative is [59]*

$$Y\{\mathbb{K}^{n}(\theta)\} = \frac{M(u)}{u^{n}} - \sum_{k=0}^{n-1}\frac{\mathbb{K}^{k}(0)}{u^{n-k-1}}, \;\; \forall \;\; n=1,2,3,\cdots \tag{6}$$

Definition 4. *The YT of the function having fractional derivative is [59]*

$$Y\{\mathbb{K}^{\wp}(\theta)\} = \frac{M(u)}{u^{\wp}} - \sum_{k=0}^{n-1}\frac{\mathbb{K}^{k}(0)}{u^{\wp-(k+1)}}, \;\; n-1<\wp\leq n. \tag{7}$$

3. Analysis of HPTM

Here, the general methodology of HPTM is applied for solving the FPDE.

$$D_{\theta}^{\wp}\mathbb{K}(\varsigma,\theta) = \mathcal{P}_1[\varsigma]\mathbb{K}(\varsigma,\theta) + \mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta), \;\; 0<\wp\leq 1, \tag{8}$$

having initial guess

$$\mathbb{K}(\varsigma,0) = \xi(\varsigma).$$

Here, $D_{\theta}^{\wp} = \frac{\partial^{\wp}}{\partial\theta^{\wp}}$ is the Caputo-type operator, $\mathcal{P}_1[\varsigma]$ is linear, and $\mathcal{Q}_1[\varsigma]$ is nonlinear function.

By utilizing the YT, we get

$$Y[D_\theta^\wp \mathbb{K}(\varsigma,\theta)] = Y[\mathcal{P}_1[\varsigma]\mathbb{K}(\varsigma,\theta) + \mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta)], \tag{9}$$

$$\frac{1}{u^\wp}\{M(u) - u\mathbb{K}(0)\} = Y[\mathcal{P}_1[\varsigma]\mathbb{K}(\varsigma,\theta) + \mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta)], \tag{10}$$

where

$$M(\mathbb{K}) = u\mathbb{K}(0) + u^\wp Y[\mathcal{P}_1[\varsigma]\mathbb{K}(\varsigma,\theta) + \mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta)]. \tag{11}$$

When utilizing the inverse of YT, we get

$$\mathbb{K}(\varsigma,\theta) = \mathbb{K}(0) + Y^{-1}[u^\wp Y[\mathcal{P}_1[\varsigma]\mathbb{K}(\varsigma,\theta) + \mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta)]]. \tag{12}$$

In terms of HPM, the basic solution in a power series is:

$$\mathbb{K}(\varsigma,\theta) = \sum_{k=0}^{\infty} \epsilon^k \mathbb{K}_k(\varsigma,\theta) \tag{13}$$

with parameter $\epsilon \in [0,1]$.

The nonlinear term is considered as

$$\mathcal{Q}_1[\varsigma]\mathbb{K}(\varsigma,\theta) = \sum_{k=0}^{\infty} \epsilon^k H_n(\mathbb{K}). \tag{14}$$

Additionally, $H_k(\mathbb{K})$ represents He's polynomials, which reads [60]

$$H_k(\mathbb{K}_0,\mathbb{K}_1,\ldots,\mathbb{K}_n) = \frac{1}{\Gamma(n+1)} D_\epsilon^k \left[\mathcal{Q}_1\left(\sum_{k=0}^{\infty} \epsilon^i \mathbb{K}_i\right)\right]_{\epsilon=0}, \tag{15}$$

where $D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k}$.

By putting (13) and (14) in (12), we have

$$\sum_{k=0}^{\infty} \epsilon^k \mathbb{K}_k(\varsigma,\theta) = \mathbb{K}(0) + \epsilon \times \left(Y^{-1}\left[u^\wp Y\{\mathcal{P}_1 \sum_{k=0}^{\infty} \epsilon^k \mathbb{K}_k(\varsigma,\theta) + \sum_{k=0}^{\infty} \epsilon^k H_k(\mathbb{K})\}\right]\right). \tag{16}$$

By comparing the coefficients of similar orders of ϵ, we obtain

$$\begin{aligned}
&\epsilon^0 : \mathbb{K}_0(\varsigma,\theta) = \mathbb{K}(0),\\
&\epsilon^1 : \mathbb{K}_1(\varsigma,\theta) = Y^{-1}[u^\wp Y(\mathcal{P}_1[\varsigma]\mathbb{K}_0(\varsigma,\theta) + H_0(\mathbb{K}))],\\
&\epsilon^2 : \mathbb{K}_2(\varsigma,\theta) = Y^{-1}[u^\wp Y(\mathcal{P}_1[\varsigma]\mathbb{K}_1(\varsigma,\theta) + H_1(\mathbb{K}))],\\
&.\\
&.\\
&.\\
&\epsilon^k : \mathbb{K}_k(\varsigma,\theta) = Y^{-1}[u^\wp Y(\mathcal{P}_1[\varsigma]\mathbb{K}_{k-1}(\varsigma,\theta) + H_{k-1}(\mathbb{K}))],
\end{aligned} \tag{17}$$

$k > 0, k \in \mathbf{N}$.

Lastly, the solution of $\mathbb{K}_k(\varsigma,\theta)$ is stated as

$$\mathbb{K}(\varsigma,\theta) = \lim_{M\to\infty} \sum_{k=1}^{M} \mathbb{K}_k(\varsigma,\theta). \tag{18}$$

4. Analysis of the YTDM

Here, the general methodology of the YTDM is applied to solve the FPDE

$$D_\theta^\wp \mathbb{K}(\varsigma,\theta) = \mathcal{P}_1(\varsigma,\theta) + \mathcal{Q}_1(\varsigma,\theta), 0 < \wp \leq 1, \tag{19}$$

having initial guess

$$\mathbb{K}(\varsigma,0) = \xi(\varsigma).$$

Here, $D_\theta^\wp = \frac{\partial^\wp}{\partial\theta^\wp}$ is the Caputo type operator, $\mathcal{P}_1$ is linear and $\mathcal{Q}_1$ is nonlinear function.

By utilizing the YT, we get

$$\begin{aligned} &Y[D_\theta^\wp \mathbb{K}(\varsigma,\theta)] = Y[\mathcal{P}_1(\varsigma,\theta) + \mathcal{Q}_1(\varsigma,\theta)], \\ &\frac{1}{u^\wp}\{M(u) - u\mathbb{K}(0)\} = Y[\mathcal{P}_1(\varsigma,\theta) + \mathcal{Q}_1(\varsigma,\theta)]. \end{aligned} \tag{20}$$

where

$$M(\mathbb{K}) = u\mathbb{K}(0) + u^\wp Y[\mathcal{P}_1(\varsigma,\theta) + \mathcal{Q}_1(\varsigma,\theta)], \tag{21}$$

When utilizing the inverse of the YT, we get

$$\mathbb{K}(\varsigma,\theta) = \mathbb{K}(0) + Y^{-1}[u^\wp Y[\mathcal{P}_1(\varsigma,\theta) + \mathcal{Q}_1(\varsigma,\theta)]. \tag{22}$$

The series form solution of $\mathbb{K}(\varsigma,\theta)$ reads:

$$\mathbb{K}(\varsigma,\theta) = \sum_{m=0}^{\infty} \mathbb{K}_m(\varsigma,\theta) \tag{23}$$

and the nonlinear term reads

$$\mathcal{Q}_1(\varsigma,\theta) = \sum_{m=0}^{\infty} \mathcal{A}_m \tag{24}$$

with

$$\mathcal{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\ell^m}\left\{\mathcal{Q}_1\left(\sum_{k=0}^{\infty}\ell^k\varsigma_k, \sum_{k=0}^{\infty}\ell^k\theta_k\right)\right\}\right]_{\ell=0}. \tag{25}$$

Inserting (23) and (24) into (22), we get

$$\sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) = \mathbb{K}(0) + Y^{-1}u^\wp\left[Y\left\{\mathcal{P}_1(\sum_{m=0}^{\infty}\varsigma_m, \sum_{m=0}^{\infty}\theta_m) + \sum_{m=0}^{\infty}\mathcal{A}_m\right\}\right]. \tag{26}$$

Thus, by comparison we have

$$\mathbb{K}_0(\varsigma,\theta) = \mathbb{K}(0), \tag{27}$$

$$\mathbb{K}_1(\varsigma,\theta) = Y^{-1}\left[u^\wp Y^+\{\mathcal{P}_1(\varsigma_0,\theta_0) + \mathcal{A}_0\}\right].$$

Hence, in general, for $m \geq 1$, we have

$$\mathbb{K}_{m+1}(\varsigma,\theta) = Y^{-1}\left[u^\wp Y^+\{\mathcal{P}_1(\varsigma_m,\theta_m) + \mathcal{A}_m\}\right].$$

5. Convergence Analysis

In this part, we give the suggested techniques for convergence analysis.

Theorem 1. *Let us assume that $\mathbb{K}$ and $\mathbb{K}_n(\varsigma,\wp)$ are defined in Banach space. If this is the case, the series solution described by Equation (13) converges to the solution of Equation (8) if $\exists\, \varsigma \in (0,1)$ such that $||\mathbb{K}_{n+1}|| \leq \varsigma||\mathbb{K}_n||$, the convergence condition has been demonstrated [61].*

Theorem 2. *The nonlinear term $\mathcal{Q}_1(\varsigma, \wp)$ described by (24) that satisfies the Lipschitz condition $||\mathbb{K}(\mathcal{Q}_1) - \mathbb{K}(\mathcal{Q}_1^*)|| \leq \varrho||\mathcal{Q}_1 - \mathcal{Q}_1^*||$. By means of the Lipschitz constant $\varrho, 0 \leq \varrho < 1$, for any $\mathcal{Q}_1, \mathcal{Q}_1^* \in C[0,1]$, The sequence leads to the accurate solution $\mathbb{K}$ if $||a_0|| < \infty$.*

Proof. See [62] □

6. Applications

Example 1. *Let us consider the following nonlinear fractional mKdV equation*

$$D_\theta^\wp \mathbb{K}(\varsigma,\theta) + 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) + \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) = 0, \quad 0 < \wp \leq 1, \tag{28}$$

subjected to initial source

$$\mathbb{K}(\varsigma,0) = -\frac{2\kappa \exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1}.$$

By utilizing YT, we get

$$Y\left(\frac{\partial^\wp \mathbb{K}}{\partial\theta^\wp}\right) = Y\left(- 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{29}$$

After that we obtain

$$\frac{1}{u^\wp}\{M(u) - u\mathbb{K}(0)\} = Y\left(- 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right), \tag{30}$$

$$M(u) = u\mathbb{K}(0) + u^\wp\left(- 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{31}$$

When utilizing the inverse of the YT, we get

$$\begin{aligned} &\mathbb{K}(\varsigma,\theta) = \mathbb{K}(0) + Y^{-1}\left[u^\wp\left\{Y\left(- 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right], \\ &\mathbb{K}(\varsigma,\theta) = -\frac{2\kappa \exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} + Y^{-1}\left[u^\wp\left\{Y\left(- 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right]. \end{aligned} \tag{32}$$

Thus, by HPM

$$\sum_{k=0}^{\infty} \epsilon^k \mathbb{K}_k(\varsigma,\theta) = -\frac{2\kappa \exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} + \epsilon\left(Y^{-1}\left[u^\wp Y\left[-\left(\sum_{k=0}^{\infty}\epsilon^k H_k(\mathbb{K})\right) - \left(\sum_{k=0}^{\infty}\epsilon^k \mathbb{K}_k(\varsigma,\theta)\right)_{\varsigma\varsigma\varsigma}\right]\right]\right). \tag{33}$$

Additionally, the nonlinear terms are taken in the form of He's polynomial $H_k(\mathbb{K})$ as

$$\sum_{k=0}^{\infty} \epsilon^k H_k(\mathbb{K}) = \mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta). \tag{34}$$

The first few nonlinear terms are determined as

$$\begin{aligned} &H_0(\mathbb{K}) = \mathbb{K}_0^2(\mathbb{K}_0)_\varsigma, \\ &H_1(\mathbb{K}) = \mathbb{K}_0^2(\mathbb{K}_1)_\varsigma + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_0)_\varsigma, \\ &H_2(\mathbb{K}) = \mathbb{K}_0^2(\mathbb{K}_2)_\varsigma + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_1)_\varsigma + (\mathbb{K}_1^2 + 2\mathbb{K}_0\mathbb{K}_2)(\mathbb{K}_0)_\varsigma, \end{aligned}$$

By comparing the coefficients of similar orders of ϵ, we have

$$
\begin{aligned}
\epsilon^0 &: \mathbb{K}_0(\varsigma,\theta) = -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1},\\
\epsilon^1 &: \mathbb{K}_1(\varsigma,\theta) = Y^{-1}\left(u^{\wp}Y\left[-(\mathbb{K}_0)_{\varsigma\varsigma\varsigma} - H_0(\mathbb{K})\right]\right) = -\frac{2\kappa^4\exp(\kappa\varsigma)(\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^2}\frac{\theta^{\wp}}{\Gamma(\wp+1)},\\
\epsilon^2 &: \mathbb{K}_2(\varsigma,\theta) = Y^{-1}\left(u^{\wp}Y\left[-(\mathbb{K}_1)_{\varsigma\varsigma\varsigma} - H_1(\mathbb{K})\right]\right) = -\frac{\kappa^7\exp(\kappa\varsigma)(\exp(4\kappa\varsigma)-6\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^3}\frac{\theta^{2\wp}}{\Gamma(2\wp+1)},\\
&\vdots
\end{aligned}
$$

The obtained solution can be taken in series form as

$$
\begin{aligned}
\mathbb{K}(\varsigma,\theta) &= \mathbb{K}_0(\varsigma,\theta) + \mathbb{K}_1(\varsigma,\theta) + \mathbb{K}_2(\varsigma,\theta) + \cdots\\
\mathbb{K}(\varsigma,\theta) &= -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} - \frac{2\kappa^4\exp(\kappa\varsigma)(\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^2}\frac{\theta^{\wp}}{\Gamma(\wp+1)} -\\
&\frac{\kappa^7\exp(\kappa\varsigma)(\exp(4\kappa\varsigma)-6\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^3}\frac{\theta^{2\wp}}{\Gamma(2\wp+1)} + \cdots
\end{aligned}
$$

Utilizing the YTDM

By utilizing the YT, we get

$$
Y\left\{\frac{\partial^{\wp}\mathbb{K}}{\partial\theta^{\wp}}\right\} = Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{35}
$$

After that, we have

$$
\frac{1}{u^{\wp}}\{M(u) - u\mathbb{K}(0)\} = Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right), \tag{36}
$$

$$
M(u) = u\mathbb{K}(0) + u^{\wp}Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{37}
$$

When utilizing the inverse of the YT, we get

$$
\begin{aligned}
\mathbb{K}(\varsigma,\theta) &= \mathbb{K}(0) + Y^{-1}\left[u^{\wp}\left\{Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right],\\
\mathbb{K}(\varsigma,\theta) &= -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} + Y^{-1}\left[u^{\wp}\left\{Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right].
\end{aligned} \tag{38}
$$

Thus, the series form solution is taken as

$$
\mathbb{K}(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta). \tag{39}
$$

The Adomian polynomial is used to determine the nonlinear terms as $\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathcal{A}_m$. Hence, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) &= \mathbb{K}(\varsigma,0) + Y^{-1}\left[u^{\wp}Y\left[-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right],\\ \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) &= -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} + Y^{-1}\left[u^{\wp}Y\left[-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right].\end{aligned} \tag{40}$$

The first few nonlinear terms are determined as

$$\begin{aligned}\mathcal{A}_0 &= \mathbb{K}_0^2(\mathbb{K}_0)_\varsigma,\\ \mathcal{A}_1 &= \mathbb{K}_0^2(\mathbb{K}_1)_\varsigma + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_0)_\varsigma,\\ \mathcal{A}_2 &= \mathbb{K}_0^2(\mathbb{K}_2)_\varsigma + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_1)_\varsigma + (\mathbb{K}_1^2 + 2\mathbb{K}_0\mathbb{K}_2)(\mathbb{K}_0)_\varsigma.\end{aligned}$$

When comparing both sides, we have

$$\mathbb{K}_0(\varsigma,\theta) = -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1}.$$

For $m = 0$

$$\mathbb{K}_1(\varsigma,\theta) = -\frac{2\kappa^4\exp(\kappa\varsigma)(\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^2}\frac{\theta^{\wp}}{\Gamma(\wp+1)}.$$

For $m = 1$

$$\mathbb{K}_2(\varsigma,\theta) = -\frac{\kappa^7\exp(\kappa\varsigma)(\exp(4\kappa\varsigma)-6\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^3}\frac{\theta^{2\wp}}{\Gamma(2\wp+1)}.$$

In the same sense, the remaining terms for $(m \geq 3)$ are easily obtained

$$\mathbb{K}(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) = \mathbb{K}_0(\varsigma,\theta) + \mathbb{K}_1(\varsigma,\theta) + \mathbb{K}_2(\varsigma,\theta) + \cdots$$

$$\mathbb{K}(\varsigma,\theta) = -\frac{2\kappa\exp(\kappa\varsigma)}{\exp(2\kappa\varsigma)+1} - \frac{2\kappa^4\exp(\kappa\varsigma)(\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^2}\frac{\theta^{\wp}}{\Gamma(\wp+1)} - \frac{\kappa^7\exp(\kappa\varsigma)(\exp(4\kappa\varsigma)-6\exp(2\kappa\varsigma)-1)}{(\exp(2\kappa\varsigma)+1)^3}\frac{\theta^{2\wp}}{\Gamma(2\wp+1)} + \cdots$$

By taking $\wp = 1$ we get

$$\mathbb{K}(\varsigma,\theta) = -\frac{2\kappa\exp(\kappa(\varsigma-\kappa^2\theta))}{\exp(2\kappa(\varsigma-\kappa^2\theta))+1}. \tag{41}$$

Example 2. *Let us assume the nonlinear fractional mKdV equation*

$$D_\theta^{\wp}\mathbb{K}(\varsigma,\theta) + 6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) + \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) = 0, \quad 0 < \wp \leq 1, \tag{42}$$

subjected to initial source

$$\mathbb{K}(\varsigma,0) = \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1}.$$

By utilizing the YT, we get

$$Y\left(\frac{\partial^{\wp}\mathbb{K}}{\partial\theta^{\wp}}\right) = Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_\varsigma(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{43}$$

After that, we obtain

$$\frac{1}{u^{\wp}}\{M(u)-u\mathbb{K}(0)\}=Y\bigg(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta)-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\bigg), \tag{44}$$

$$M(u)=u\mathbb{K}(0)+u^{\wp}\bigg(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta)-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\bigg). \tag{45}$$

When utilizing the inverse of the YT, we get

$$\begin{aligned}\mathbb{K}(\varsigma,\theta)&=\mathbb{K}(0)+Y^{-1}\bigg[u^{\wp}\bigg\{Y\bigg(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta)-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\bigg)\bigg\}\bigg],\\ \mathbb{K}(\varsigma,\theta)&=\frac{4\exp(\varsigma)}{\exp(2\varsigma)+1}+Y^{-1}\bigg[u^{\wp}\bigg\{Y\bigg(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta)-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\bigg)\bigg\}\bigg].\end{aligned} \tag{46}$$

Thus, by HPM

$$\sum_{k=0}^{\infty}\epsilon^k\mathbb{K}_k(\varsigma,\theta)=\frac{4\exp(\varsigma)}{\exp(2\varsigma)+1}+\epsilon\Bigg(Y^{-1}\Bigg[u^{\wp}Y\Bigg[-\Bigg(\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{K})\Bigg)-\Bigg(\sum_{k=0}^{\infty}\epsilon^k\mathbb{K}_k(\varsigma,\theta)\Bigg)_{\varsigma\varsigma\varsigma}\Bigg]\Bigg]\Bigg). \tag{47}$$

Additionally, the nonlinear terms are taken in the form of He's polynomial $H_k(\mathbb{K})$ as

$$\sum_{k=0}^{\infty}\epsilon^kH_k(\mathbb{K})=\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta). \tag{48}$$

The first few nonlinear terms are determined as

$$\begin{aligned}H_0(\mathbb{K})&=\mathbb{K}_0^2(\mathbb{K}_0)_{\varsigma},\\ H_1(\mathbb{K})&=\mathbb{K}_0^2(\mathbb{K}_1)_{\varsigma}+2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_0)_{\varsigma},\\ H_2(\mathbb{K})&=\mathbb{K}_0^2(\mathbb{K}_2)_{\varsigma}+2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_1)_{\varsigma}+(\mathbb{K}_1^2+2\mathbb{K}_0\mathbb{K}_2)(\mathbb{K}_0)_{\varsigma}.\end{aligned}$$

By comparing the coefficients of similar orders of ϵ, we have

$$\begin{aligned}&\epsilon^0:\mathbb{K}_0(\varsigma,\theta)=\frac{4\exp(\varsigma)}{\exp(2\varsigma)+1},\\ &\epsilon^1:\mathbb{K}_1(\varsigma,\theta)=Y^{-1}\bigg(u^{\wp}Y\bigg[-(\mathbb{K}_0)_{\varsigma\varsigma\varsigma}-H_0(\mathbb{K})\bigg]\bigg)=\frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^4}(\exp(6\varsigma)+73\exp(4\varsigma)-\\ &73\exp(2\varsigma)-1)\frac{\theta^{\wp}}{\Gamma(\wp+1)},\\ &\epsilon^2:\mathbb{K}_2(\varsigma,\theta)=Y^{-1}\bigg(u^{\wp}Y\bigg[-(\mathbb{K}_1)_{\varsigma\varsigma\varsigma}-H_1(\mathbb{K})\bigg]\bigg)=\frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^7}(\exp(12\varsigma)+2158\exp(10\varsigma)+\\ &2863\exp(8\varsigma)-26236\exp(6\varsigma)+2863\exp(4\varsigma)+2158\exp(2\varsigma)+1)\frac{\theta^{2\wp}}{\Gamma(2\wp+1)},\\ &\vdots\end{aligned}$$

The obtained solution in series form will take the form

$$\mathbb{K}(\varsigma,\theta) = \mathbb{K}_0(\varsigma,\theta) + \mathbb{K}_1(\varsigma,\theta) + \mathbb{K}_2(\varsigma,\theta) + \cdots$$
$$\mathbb{K}(\varsigma,\theta) = \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1} + \frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^4}(\exp(6\varsigma) + 73\exp(4\varsigma) - 73\exp(2\varsigma) - 1)\frac{\theta^{\wp}}{\Gamma(\wp+1)} +$$
$$\frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^7}(\exp(12\varsigma) + 2158\exp(10\varsigma) + 2863\exp(8\varsigma) - 26236\exp(6\varsigma) + 2863\exp(4\varsigma) +$$
$$2158\exp(2\varsigma) + 1)\frac{\theta^{2\wp}}{\Gamma(2\wp+1)} + \cdots$$

Utilizing the YTDM

By utilizing the YT, we get

$$Y\left\{\frac{\partial^{\wp}\mathbb{K}}{\partial\theta^{\wp}}\right\} = Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{49}$$

After that, we have

$$\frac{1}{u^{\wp}}\{M(u) - u\mathbb{K}(0)\} = Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right), \tag{50}$$

$$M(u) = u\mathbb{K}(0) + u^{\wp}Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right). \tag{51}$$

When utilizing the inverse of the YT, we get

$$\begin{aligned}\mathbb{K}(\varsigma,\theta) &= \mathbb{K}(0) + Y^{-1}\left[u^{\wp}\left\{Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right],\\ \mathbb{K}(\varsigma,\theta) &= \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1} + Y^{-1}\left[u^{\wp}\left\{Y\left(-6\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) - \mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta)\right)\right\}\right].\end{aligned} \tag{52}$$

Thus, the solution in series form reads

$$\mathbb{K}(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta). \tag{53}$$

The Adomian polynomial is used to determine nonlinear terms as $\mathbb{K}^2(\varsigma,\theta)\mathbb{K}_{\varsigma}(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathcal{A}_m$. Hence, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) &= \mathbb{K}(\varsigma,0) + Y^{-1}\left[u^{\wp}Y\left[-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right],\\ \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) &= \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1} + Y^{-1}\left[u^{\wp}Y\left[-\mathbb{K}_{\varsigma\varsigma\varsigma}(\varsigma,\theta) - \sum_{m=0}^{\infty}\mathcal{A}_m\right]\right].\end{aligned} \tag{54}$$

The first few nonlinear terms are determined as

$$\begin{aligned}\mathcal{A}_0 &= \mathbb{K}_0^2(\mathbb{K}_0)_{\varsigma},\\ \mathcal{A}_1 &= \mathbb{K}_0^2(\mathbb{K}_1)_{\varsigma} + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_0)_{\varsigma},\\ \mathcal{A}_2 &= \mathbb{K}_0^2(\mathbb{K}_2)_{\varsigma} + 2\mathbb{K}_0\mathbb{K}_1(\mathbb{K}_1)_{\varsigma} + (\mathbb{K}_1^2 + 2\mathbb{K}_0\mathbb{K}_2)(\mathbb{K}_0)_{\varsigma}.\end{aligned}$$

When comparing both sides, we have

$$\mathbb{K}_0(\varsigma,\theta) = \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1}.$$

For $m = 0$

$$\mathbb{K}_1(\varsigma,\theta) = \frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^4}(\exp(6\varsigma)+73\exp(4\varsigma)-73\exp(2\varsigma)-1)\frac{\theta^{\wp}}{\Gamma(\wp+1)}.$$

For $m = 1$

$$\mathbb{K}_2(\varsigma,\theta) = \frac{2\exp(\varsigma)}{(\exp(2\varsigma)+1)^7}(\exp(12\varsigma)+2158\exp(10\varsigma)+2863\exp(8\varsigma)-26236\exp(6\varsigma)+2863\exp(4\varsigma)+ 2158\exp(2\varsigma)+1)\frac{\theta^{2\wp}}{\Gamma(2\wp+1)}.$$

and in the same sense, the other terms for $(m \geq 3)$ are easily obtained

$$\mathbb{K}(\varsigma,\theta) = \sum_{m=0}^{\infty}\mathbb{K}_m(\varsigma,\theta) = \mathbb{K}_0(\varsigma,\theta)+\mathbb{K}_1(\varsigma,\theta)+\mathbb{K}_2(\varsigma,\theta)+\cdots$$

$$\mathbb{K}(\varsigma,\theta) = \frac{4\exp(\varsigma)}{\exp(2\varsigma)+1} + \frac{4\exp(\varsigma)}{(\exp(2\varsigma)+1)^4}(\exp(6\varsigma)+73\exp(4\varsigma)-73\exp(2\varsigma)-1)\frac{\theta^{\wp}}{\Gamma(\wp+1)}+ \frac{2\exp(\varsigma)}{(\exp(2\varsigma)+1)^7}(\exp(12\varsigma)+2158\exp(10\varsigma)+2863\exp(8\varsigma)-26236\exp(6\varsigma)+2863\exp(4\varsigma)+ 2158\exp(2\varsigma)+1)\frac{\theta^{2\wp}}{\Gamma(2\wp+1)}+\cdots$$

By taking $\wp = 1$, we get

$$\mathbb{K}(\varsigma,\theta) = \frac{4(\exp(\theta-\varsigma)+3\exp(27\theta-3\varsigma)+3\exp(29\theta-5\varsigma)+\exp(55\theta-7\varsigma))}{1+4\exp(2\theta-2\varsigma)+6\exp(28\theta-4\varsigma)+4\exp(54\theta-6\varsigma)+\exp(56\theta-8\varsigma)}. \tag{55}$$

7. Numerical Simulation Studies

The graphical and numerical results indicate the usefulness of the method, and its accuracy is evaluated in view of exact results. The implementation of our methods gives results with good performance and simplicity. The solution plot of $\mathbb{K}(\varsigma,\theta)$ has been compared to the actual solution plot, which is depicted in Figure 1. Figure 2 illustrates the mathematical representations of $\mathbb{K}(\varsigma,\theta)$ for $\lambda = 0.8$ and 0.6. Similar plots of $\mathbb{K}(\varsigma,\theta)$ are shown in Figure 3 for various values of $\lambda = 0.25, 0.50, 0.75$, and 1. For several values of ς and θ of problem 1, the approximation to the equation $\mathbb{K}(\varsigma,\theta)$ is displayed in Table 1. The solution plot of $\mathbb{K}(\varsigma,\theta)$ has been compared to the actual solution plot, which is depicted in Figure 4. Figure 5 illustrates the mathematical representations of $\mathbb{K}(\varsigma,\theta)$ for $\lambda = 0.8$ and 0.6. Similar plots of $\mathbb{K}(\varsigma,\theta)$ are shown in Figure 6 for various values of $\lambda = 0.25, 0.50, 0.75$, and 1. For several values of ς and θ of problem 2, the approximation to the equation $\mathbb{K}(\varsigma,\theta)$ is displayed in Table 2. There would have been better approximation solutions if we had increased the order of the approximation, which would have increased the number of terms in the solution.

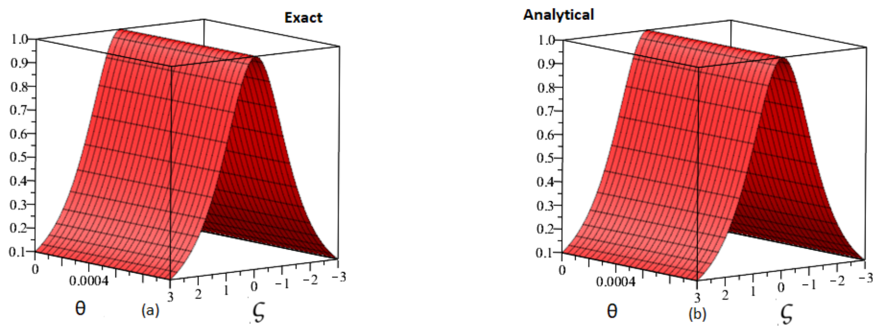

Figure 1. The profile of both the exact solution (**a**) and obtained solution (**b**) are considered.

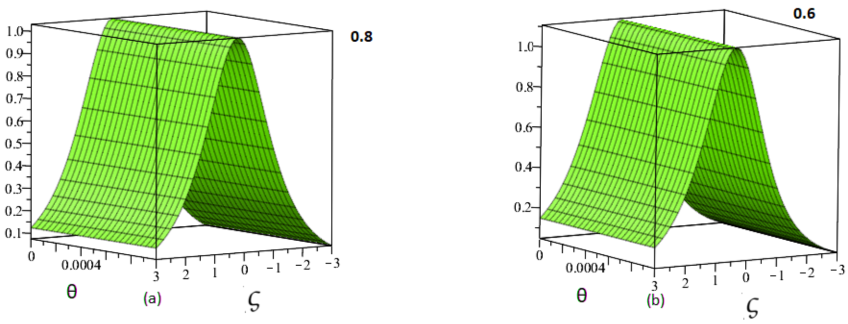

Figure 2. The obtained solution is graphically depicted at (**a**) $\wp = 0.8$ and (**b**) $\wp = 0.6$.

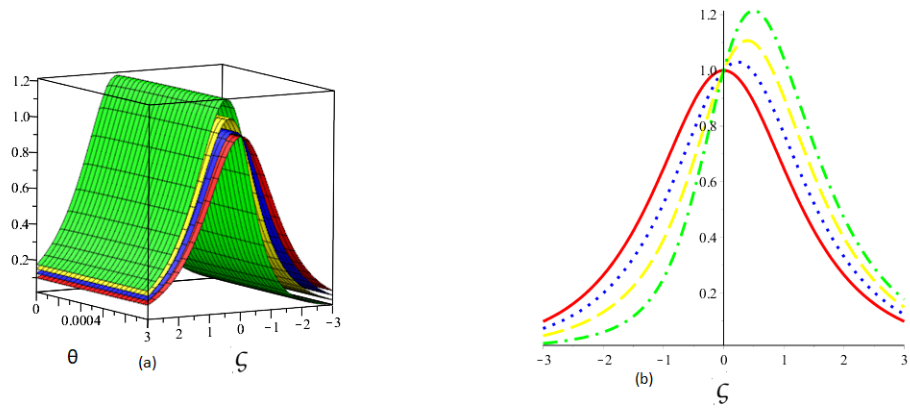

Figure 3. The obtained solution is graphically depicted for various order of $\wp$ in (**a**) three dimensions and (**b**) two dimensions.

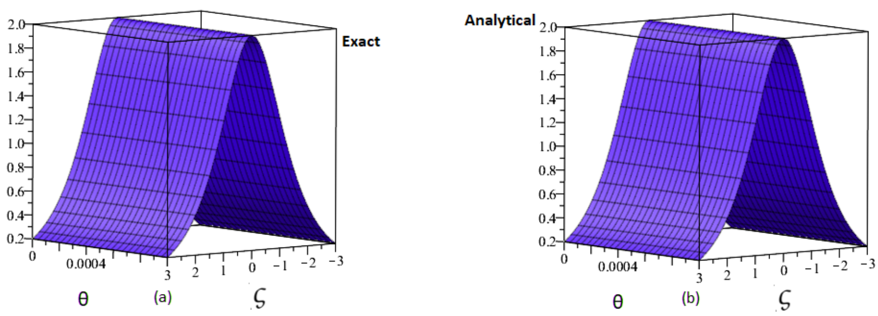

Figure 4. The profile of both exact solution (**a**) and obtained solution (**b**) are considered.

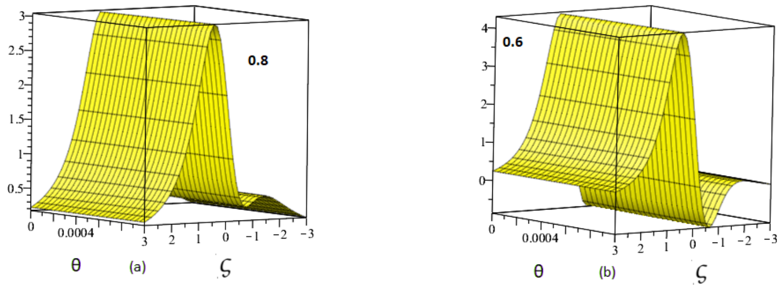

Figure 5. The obtained solution is graphically depicted at (**a**) $\wp = 0.8$ and (**b**) $\wp = 0.6$.

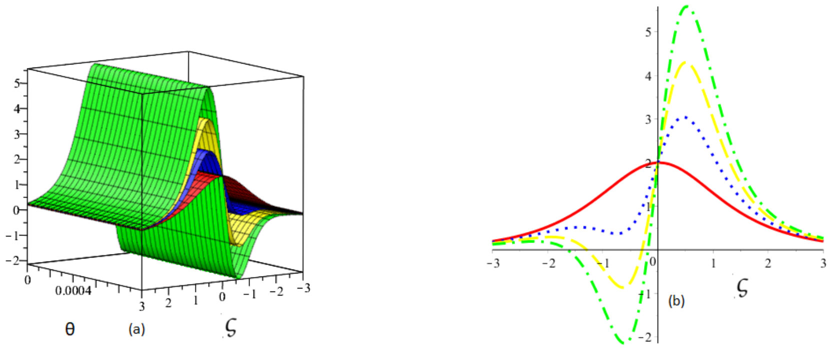

Figure 6. The obtained solution is graphically depicted for various order of $\wp$ in (**a**) three dimensions and (**b**) two dimensions.

Table 1. Behavior of both the obtained approximate solution and exact solution are considered for various orders of $\wp$, for example, 1.

θ	ς	$\wp = 0.4$	$\wp = 0.6$	$\wp = 0.8$	$\wp = 1$ (Approx)	$\wp = 1$ (Exact)
0.01	0.2	0.986366	0.984417	0.982469	0.980521	0.980521
	0.4	0.935975	0.932435	0.928896	0.925358	0.925358
	0.6	0.857688	0.853125	0.848563	0.844003	0.844003
	0.8	0.763194	0.758193	0.753193	0.748196	0.748196
	1	0.663457	0.658485	0.653515	0.648547	0.648547
0.02	0.2	0.986590	0.984630	0.982672	0.980713	0.980713
	0.4	0.936382	0.932823	0.929265	0.925709	0.925709
	0.6	0.858213	0.853625	0.849039	0.844456	0.844456
	0.8	0.763770	0.758741	0.753715	0.748692	0.748692
	1	0.664029	0.659029	0.654034	0.649041	0.649041
0.03	0.2	0.986811	0.984842	0.982874	0.980904	0.980904
	0.4	0.936783	0.933206	0.929632	0.926058	0.926058
	0.6	0.858730	0.854119	0.849512	0.844908	0.844908
	0.8	0.764336	0.759283	0.754234	0.749189	0.749189
	1	0.664591	0.659568	0.654549	0.649535	0.649535
0.04	0.2	0.987029	0.985051	0.983076	0.981094	0.981094
	0.4	0.937180	0.985051	0.929998	0.926407	0.926407
	0.6	0.859241	0.854611	0.849984	0.845359	0.845359
	0.8	0.764896	0.759821	0.754751	0.749685	0.749685
	1	0.665149	0.660104	0.655063	0.650029	0.650029
0.05	0.2	0.987246	0.985260	0.983277	0.981284	0.981284
	0.4	0.937574	0.933967	0.930364	0.926756	0.926756
	0.6	0.859749	0.855099	0.850455	0.845811	0.845811
	0.8	0.765453	0.760357	0.755267	0.750181	0.750181
	1	0.665702	0.660636	0.655576	0.650523	0.650523

Table 2. Behavior of both the obtained approximate solution and the exact solution for various orders of $\wp$ are considered, for example, 2.

θ	ς	$\wp = 0.4$	$\wp = 0.6$	$\wp = 0.8$	$\wp = 1$ (Approx)	$\wp = 1$ (Exact)
0.01	0.2	2.110317	2.039013	1.974863	1.967541	1.967541
	0.4	2.093671	1.977584	1.873145	1.861473	1.861473
	0.6	1.951520	1.825541	1.712203	1.699649	1.699649
	0.8	1.727575	1.616957	1.517440	1.506470	1.506470
	1	1.474678	1.389601	1.313060	1.304643	1.304643
0.02	0.2	2.118130	2.046452	1.981980	1.974024	1.974024
	0.4	2.106391	1.989695	1.884731	1.872782	1.872782
	0.6	1.965324	1.838684	1.724776	1.712267	1.712267
	0.8	1.739696	1.628498	1.528480	1.517709	1.517709
	1	1.484001	1.398477	1.321551	1.313350	1.313350
0.03	0.2	2.125862	2.053852	1.989092	1.980094	1.980094
	0.4	2.118979	2.001743	1.896311	1.883925	1.883925
	0.6	1.978985	1.851759	1.737343	1.724946	1.724946
	0.8	1.751690	1.639979	1.539514	1.529116	1.529116
	1	1.493226	1.407307	1.330038	1.322232	1.322232
0.04	0.2	2.133542	2.061228	1.996203	1.985736	1.985736
	0.4	2.131482	2.013751	1.907887	1.894886	1.894886
	0.6	1.992553	1.864790	1.749905	1.737675	1.737675
	0.8	1.763604	1.651421	1.550545	1.540690	1.540690
	1	1.502389	1.416107	1.338522	1.331291	1.331291
0.05	0.2	2.141182	2.068584	2.003311	1.990940	1.990940
	0.4	2.143921	2.025728	1.919460	1.905650	1.905650
	0.6	2.006053	1.877788	1.762464	1.750447	1.750447
	0.8	1.775457	1.662833	1.561573	1.552427	1.552427
	1	1.511506	1.424885	1.347003	1.340528	1.340528

8. Conclusions

In conclusion, we successfully applied the homotopy perturbation transform method and Yang transform decomposition method to the initial value problem-related mKdV equation with variable coefficients to obtain an analytical approximation. Two stages are taken to finish the numerical solutions. The Yang transformation is used to break down the target issues into simpler forms in the first stage, after which the perturbation method and decomposition method are utilized to get the solutions. The tables and graphs demonstrate that sophisticated methods are more effective in analyzing how successfully the targeted problems are being addressed. The solutions are given at several fractional orders, and it has been shown that fractional solutions quickly converge to integer-order solutions. The graphical representation makes it very clear how the fractional-order and integer-order solutions are related to one another. Due to the approaches' precision and simplicity, they can be used to solve high nonlinear FPDEs and related systems.

Author Contributions: Conceptualization, H.A.A. and R.S.; methodology, N.A.S.; software, R.S.; validation, J.D.C.; formal analysis, N.A.S.; investigation, S.M.E.I.; resources, R.S.; data curation, S.A.E.-T.; writing—original draft preparation, R.S.; writing—review and editing, S.A.E.-T.; visualization, J.D.C.; supervision, J.D.C. and S.A.E.-T.; project administration, H.A.A.; funding acquisition, J.D.C. All authors have read and agreed to the published version of the manuscript.

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