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Fractional Model of Multiple Trapping with Charge Leakage: Transient Photoconductivity and Transit–Time Dispersion

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Abstract: The model of multiple trapping into energy-distributed states is a successful tool to describe the transport of nonequilibrium charge carriers in amorphous semiconductors. Under certain conditions, the model leads to anomalous diffusion equations that contain time fractional derivatives. From this perspective, the multiple-trapping model can be used to interpret fractional transport equations, formulate initial and boundary conditions for them, and to construct numerical methods for solving fractional kinetic equations. Here, we shortly review the application of fractional multiple-trapping equations to problems of transient photoconductivity relaxation and transit–time dispersion in the time-of-flight experiment and discuss the connection of the multiple-trapping model with generalized fractional kinetic equations. Different types of charge leakage are discussed. The tempered fractional relaxation is obtained for recombination via localized states and distributed order equations arise for the non-exponential density of states presented as a weighted mixture of exponential functions. Analytical solutions for photocurrent decay in transient photoconductivity and time-of-flight experiments are provided for several simplified situations.

Keywords: amorphous semiconductor; fractional derivative; Laplace transform; multiple trapping model; transient photoconductivity

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1. Introduction

The model of multiple trapping in energy-distributed states has been successfully applied in a number of studies (see, e.g., [1–3]) to describe the charge carrier transport in disordered semiconductors. Particularly, it turned out to be very useful in interpreting experimental data on drift mobility of charge carriers in amorphous materials [4–7]. The model implies random walk of electrons or holes via extended states controlled by multiple trapping into energy-distributed localized states and release from them. A number of papers (see, e.g., [8,9]) indicate the formal equivalence of the linear version of multiple trapping model and the Continous Time Random Walk (CTRW) model. The latter model operates with waiting time distribution [10] that can be derived in the multiple trapping model after coarse-grained averaging [11]. This equivalence can also be expressed in the identity of the integral drift–diffusion equation of the CTRW model and the diffusive transport equation derived from the capture–release system. Under certain conditions, the multiple trapping model leads to anomalous diffusion equations with fractional time derivatives [9,12]. In the CTRW model, the time–fractional diffusion equation arises in the case of power law (heavy-tailed) distributions of waiting times in traps for asymptotically large times. The physical reasons for the heavy-tailed distribution of waiting times were discussed in many papers [4,10,13]. In the multiple trapping mechanism, such a distribution can be a consequence of the exponential density of localized states. The exponential form

of band tails has been widely used in the description of experiments on relaxation and transport of charge carriers in disordered semiconductors [4,5,13].

In the present work, we focus on fractional–order kinetic equations derived from the multiple-trapping model. From this perspective, the model can be used to interpret fractional transport equations, formulate corresponding initial and boundary conditions, and to construct numerical methods for solving fractional kinetic equations. Here, we apply the fractional multiple trapping equations with charge leakage to compute transient photoconductivity and photocurrents in the time-of-flight experiment and discuss the relationships of special aspects of the multiple trapping model with generalized fractional operators in the kinetic equation.

2. Anomalous Relaxation of Transient Photoconductivity

In [14], the measurements of transient photoconductivity were used to determine the energy distribution of localized states in amorphous semiconductors. The transient photoconductivity is calculated from the time dependence of a photocurrent response to a short illumination pulse in samples with co-planar electrodes. The analysis of these measurements usually assumes unipolar conductivity in amorphous semiconductors. The photogeneration and recombination rates are assumed to be homogeneous, and, due to homogeneity, there is no diffusive flux of carriers. Equations of multiple trapping with the linear recombination of free carriers, for this case, can be written as follows:

$$\frac{dn}{dt} = -\sum_i \frac{dn_i}{dt} - \frac{n}{\tau} + n_0\delta(t), \tag{1}$$

$$\frac{dn_i}{dt} = \omega_i n - \gamma_i n_i. \tag{2}$$

Here, n is the delocalized carrier density, n_i is the density of carriers trapped at the i-th localized state, n_0 is the injected free carrier density, constant τ is the recombination lifetime, and constants ω_i and $\gamma_i = \nu \exp(-\varepsilon_i/kT)$ are trapping and release rates, respectively, for the i-th localized state. Here, ε_i is the energy for i-th localized state measured from the mobility edge, ν is the attempt-to-escape frequency. The trapping rate $\omega_i = \sigma v \rho(\varepsilon_i)\Delta\varepsilon_i$ is defined by the capture cross section σ, thermal velocity v, and density of states $\rho(\varepsilon)$. The term with the delta function in Equation (1) corresponds to the initial condition, thus implying the instant generation of non-equilibrium charge carriers by a short illumination pulse. The injected density n_0 is small enough that traps are sparsely populated and the linear approximation can be applied.

The solutions of (2) are

$$n_i(t) = \omega_i \int_0^t e^{-\gamma_i(t-t')} n(t')dt'. \tag{3}$$

Now, putting $\gamma_i = \nu e^{-\frac{\varepsilon_i}{kT}}$ and $\omega_i = \sigma v \rho(\varepsilon_i)\Delta\varepsilon_i$ into (1) provides the following expression

$$\frac{dn}{dt} = -\frac{d}{dt}\int_0^t \left(\sum_i \sigma\, v\rho(\varepsilon_i)\Delta\varepsilon_i e^{-\gamma_i(t-t')}\right) n(t')dt' - \frac{n}{\tau} + n_0\delta(t). \tag{4}$$

Assuming continuous spectrum of localized states, we obtain

$$\frac{dn}{dt} = -\frac{d}{dt}\int_0^t \left(\int_0^\infty \sigma v e^{-\nu(t-t')e^{-\frac{\varepsilon}{kT}}} \rho(\varepsilon)d\varepsilon\right) n(t')dt' - \frac{n}{\tau} + n_0\delta(t). \tag{5}$$

Rewrite the latter equation in the form:

$$\frac{dn}{dt} = -\frac{d}{dt}\int_0^t Q(t-t')n(t')dt' - \frac{n}{\tau} + n_0\delta(t). \tag{6}$$

The memory kernel Q is defined as follows,

$$Q(t) = \int_0^\infty \sigma v e^{-\nu t e^{-\frac{\varepsilon}{kT}}} \rho(\varepsilon) d\varepsilon. \tag{7}$$

The Laplace transformation of the latter expression leads to formula

$$\tilde{Q}(s) = \int_0^\infty \frac{\sigma v \rho(\varepsilon) d\varepsilon}{s + \nu e^{-\frac{\varepsilon}{kT}}}. \tag{8}$$

The Laplace transformation of Equation (5) is

$$\left(s[1 + \tilde{Q}(s)] + \frac{1}{\tau} \right) \tilde{n}(s) = n_0, \tag{9}$$

and for $\tilde{n}(s)$, we have

$$\tilde{n}(s) = \frac{n_0}{s\left(1 + \tilde{Q}(s)\right) + \tau^{-1}}. \tag{10}$$

In the experiment on transient photoconductivity, the photocurent is determined by the following relation $I(t) = q\mu F n(t)$, where q is the carrier charge, μ is the microscopic mobility, and F is the applied electric field. The Laplace transformation of this current leads to expression

$$\tilde{I}(s) = q\mu F \tilde{n}(s) = \frac{I_0}{s\left(1 + \tilde{Q}(s)\right) + \tau^{-1}}. \tag{11}$$

For the energy independent product $\omega_0 = \sigma v$ and exponential density of states $\rho(\varepsilon) = \varepsilon_0^{-1} \exp(-\varepsilon/\varepsilon_0)$, the memory kernel adopts the form [12,15]:

$$Q(t) = \frac{\omega_0}{\varepsilon_0} \int_0^\infty \exp\left(-\nu t e^{-\frac{\varepsilon}{kT}} \right) \exp(-\varepsilon/\varepsilon_0) d\varepsilon. \tag{12}$$

Introducing notation $\alpha = kT/\varepsilon_0 \in (0, 1]$ and new variable

$$\xi = \nu t e^{-\frac{\varepsilon}{kT}}, \tag{13}$$

we transform formula (12) as follows:

$$Q(t) = \frac{\alpha \omega_0}{(\nu t)^\alpha} \int_0^{\nu t} e^{-\xi} \xi^{\alpha - 1} d\xi \sim \omega_0 \alpha \Gamma(\alpha) (\nu t)^{-\alpha}, \quad t \to \infty, \tag{14}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1}\, dt$$

is the Gamma function. The longtime asymptotic behavior ($t \to \infty$) physically implies that the charge carrier undergoes a large number of localization–delocalization events during time t.

Using the following relation

$$\Gamma(\alpha)\Gamma(1 - \alpha) = \frac{\pi}{\sin \alpha \pi},$$

we rewrite the memory kernel in the form [12,13]:

$$Q(t) = \omega_0 \alpha \Gamma(\alpha) (\nu t)^{-\alpha} = \omega_0 \frac{\pi \alpha}{\sin \alpha \pi} \frac{(\nu t)^{-\alpha}}{\Gamma(1 - \alpha)} = \omega_0 \frac{(c_\alpha t)^{-\alpha}}{\Gamma(1 - \alpha)}, \quad 0 < \alpha \leq 1, \tag{15}$$

with

$$c_\alpha = \nu \left(\frac{\sin \pi \alpha}{\pi \alpha} \right)^{1/\alpha}.$$

The substitution of expression (15) into Equation (5) yields

$$\frac{dn}{dt} + \omega_0 c_\alpha^{-\alpha}\, {}_0D_t^\alpha n = -\frac{n}{\tau} + n_0\delta(t), \tag{16}$$

where

$${}_0D_t^\alpha n = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{n(\tau)}{(t-\tau)^\alpha}d\tau, \tag{17}$$

is the Riemann–Liouville derivative of fractional order $\alpha = kT/\varepsilon_0$. Equation (16) corresponds to the fractional multiple trapping regime of the photoconductivity relaxation. After the Laplace transformation of this equation, we obtain

$$s\tilde{n}(s) + \omega_0 c_\alpha^{-\alpha} s^\alpha \tilde{n}(s) + \frac{\tilde{n}(s)}{\tau} = n_0.$$

As a result, for the Laplace transform of delocalized carriers concentration, we have

$$\tilde{n}(s) = \frac{n_0}{s\left(1 + \frac{\omega_0}{c_\alpha^\alpha} s^{\alpha-1}\right) + \frac{1}{\tau}}. \tag{18}$$

and, for the transient photocurrent,

$$\tilde{I}(s) = \frac{I_0}{s\left(1 + \frac{\omega_0}{c_\alpha^\alpha} s^{\alpha-1}\right) + \frac{1}{\tau}}, \quad I_0 = q\mu F n_0. \tag{19}$$

In the asymptotics of large time (physically, the time during which a large number of trapping events occur on average), for the Laplace transform, one can put $s \to 0$ according to the Tauberian theorem [16]. So, Equation (18) can be reduced to the asymptotic form:

$$\tilde{n}(s) = \tau n_0 \frac{\chi}{s^\alpha + \chi}, \tag{20}$$

where, we introduced constant $\chi = c_\alpha^\alpha/(\omega_0\tau)$.

The inverse Laplace transformation leads to function

$$n(t) = n_0\ \tau\chi t^{\alpha-1} E_{\alpha,\alpha}(-\chi t^\alpha), \tag{21}$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is the two-parameter Mittag–Leffler function [17].

Note that, even in the absence of free carrier recombination ($\tau \to \infty$), the current relaxation will be observed due to the aging effect associated with the power law distribution of waiting times in localized states. Over time, the number of free carriers will decay for $\alpha < 1$, and the carriers will mostly be trapped. Rewrite Equation (18) for $\tau \to \infty$ in the form:

$$\tilde{n}(s) = \frac{n_0}{s\left(1 + \frac{\omega_0}{c_\alpha^\alpha} s^{\alpha-1}\right)} = n_0 \frac{s^{-\alpha}}{s^{1-\alpha} + \frac{\omega_0}{c_\alpha^\alpha}}.$$

The inverse Laplace transformation leads to the following function well known in theory of non-Debye relaxation [7,18,19]:

$$n(t) = n_0\ E_{1-\alpha}\left(-\frac{\omega_0}{c_\alpha^\alpha} t^{1-\alpha}\right),$$

where

$$E_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)}$$

is the one-parameter Mittag–Leffler function [17].

By inferring the quasi-continuous energy distribution of the localized states, one can derive the energy distribution of the carriers during the relaxation. We use Equation (3) and pass to the density of localized carriers energy,

$$n_i\ \Delta\varepsilon_i \quad \Rightarrow \quad \varphi(t,\varepsilon)\ d\varepsilon,$$

and obtain

$$\varphi(t,\varepsilon) = \sigma v \rho(\varepsilon) \int_0^t e^{-\nu e^{-\frac{\varepsilon}{kT}}(t-t')} n(t') dt'.$$

The Laplace transformation on time provides

$$\tilde{\varphi}(s,\varepsilon) = \sigma v\ \rho(\varepsilon) \frac{\tilde{n}(s)}{s + \nu e^{-\frac{\varepsilon}{kT}}}$$

For the exponential density of states, we obtained solution (21); its substitution leads to

$$\varphi(t,\varepsilon) = n_0\ \tau\chi\ \sigma v \rho(\varepsilon) \int_0^t e^{-\nu e^{-\frac{\varepsilon}{kT}}(t-\theta)}\ \theta^{\alpha-1} E_{\alpha,\alpha}(-\chi\theta^\alpha) d\theta,$$

with the corresponding Laplace transform

$$\tilde{\varphi}(s,\varepsilon) = \frac{\sigma v \tau n_0}{\varepsilon_0} \frac{\chi e^{-\frac{\varepsilon}{\varepsilon_0}}}{\left(s + \nu e^{-\frac{\varepsilon}{kT}}\right)(s^\alpha + \chi)}.$$

2.1. Distributed Order Relaxation

If the density of localized states can be represented as a superposition

$$\rho(\varepsilon) = \sum_i \eta_i\ \varepsilon_i^{-1} \exp(-\varepsilon/\varepsilon_i),$$

the kernel of the integral Equation (6) adopts the form:

$$Q(t) = \sum_i \eta_i\ \omega_0 \frac{(c_{\alpha_i} t)^{-\alpha_i}}{\Gamma(1-\alpha_i)}, \quad 0 < \alpha_i \le 1,$$

and we arrive at the distributed–order fractional relaxation equation:

$$\frac{dn}{dt} + \omega_0 \sum_i b_i\, {}_0D_t^{\alpha_i} n = -\frac{n}{\tau} + n_0 \delta(t), \tag{22}$$

where $b_i = \eta_i c_{\alpha_i}^{-\alpha_i}$.

For the Laplace transform of photocurrent, we have

$$\tilde{I}(s) = \frac{I_0}{s + \omega_0 \sum\limits_i b_i s^{\alpha_i} + \tau^{-1}}. \tag{23}$$

In a quite general case, the non-exponential density of localized states can be presented as weighted mixture

$$\rho(\varepsilon) = \int_0^\infty \eta(\lambda)\ \lambda \exp(-\varepsilon\ \lambda)\ d\lambda,$$

and the distributed-order relaxation equation will contain a continuous spectrum of orders.

The distributed-order fractional kinetic equations allows for considering multi-scale effects [20] and physical phenomena in composite materials [13,21].

2.2. Recombination of Localized Carriers: Tempered Fractional Case

Equations (1) and (2) assume the linear recombination of free (delocalized) carriers, and imply that trapped carriers can only be delocalized. In the case of a possible recombination of trapped carriers [22,23], the system of these equations can be rewritten in the form

$$\frac{dn}{dt} + \sum_i \frac{dn_i}{dt} = -\frac{n}{\tau} - \sum_i \frac{n_i}{\tau_i} + n_0\delta(t),$$

and

$$\frac{dn_i}{dt} = \omega_i n - \gamma_i n_i - \frac{n_i}{\tau_i}.$$

The solution for the concentration of localized carriers now has the form

$$n_i(t) = \omega_i \int_0^t e^{-\gamma_i(t-t')} e^{-(t-t')/\tau_i} n(t')\,dt'.$$

Further, assuming $\tau_i = \tau$ for simplification and passing to the integral equation for concentration of free carriers, we derive

$$\frac{dn}{dt} = -\frac{d}{dt}\int_0^t Q(t-t') e^{-(t-t')/\tau} n(t')\,dt' - \frac{1}{\tau}\int_0^t Q(t-t') e^{-(t-t')/\tau} n(t')\,dt' - \frac{n}{\tau} + n_0\delta(t).$$

Here, the memory kernel $Q(t)$ is determined by the same relationship as above, and, for the exponential density of localized states, we obtain

$$\frac{dn}{dt} + \omega_0 c_\alpha^{-\alpha} \frac{d}{dt}\, e^{-t/\tau}\, {}_0I_t^{1-\alpha}\left(e^{t/\tau} n\right) = -\frac{1}{\tau}\omega_0 c_\alpha^{-\alpha} e^{-t/\tau}\, {}_0I_t^{1-\alpha}\left(e^{t/\tau} n\right) - \frac{n}{\tau} + n_0\delta(t),$$

where $I_t^{1-\alpha}$ is the fractional Riemann–Liouville integral. Furthermore, after simplification, we obtain the following equation:

$$\frac{dn}{dt} + \omega_0 c_\alpha^{-\alpha} e^{-t/\tau}\, {}_0D_t^{\alpha}\left(e^{t/\tau} n\right) = -\frac{n}{\tau} + n_0\delta(t),$$

which contains an operator related to the tempered fractional Riemann–Liouville derivative.

In this case, the Laplace transform of photocurrent decay has the form

$$\tilde{I}(s) = \frac{I_0}{s + \omega_0 c_\alpha^{-\alpha}(s+\tau^{-1})^\alpha + \tau^{-1}}.$$

The asymptotic form

$$\tilde{I}(s) = \frac{I_0}{\omega_0 c_\alpha^{-\alpha}(s+\tau^{-1})^\alpha + \tau^{-1}}$$

after inverse Laplace transformation provides

$$I(t) = I_0\ \tau\chi\ t^{\alpha-1} E_{\alpha,\alpha}(-\chi t^\alpha)\ \exp(-t/\tau), \tag{24}$$

where $\chi = c_\alpha^\alpha/(\omega_0\tau)$.

The inverse Laplace transformation is performed by means of the numerical algorithm described in [24,25]. Figure 1 shows the computed photocurrents in the log–log scale for different situations described above. The parameters are indicated in the figure caption.

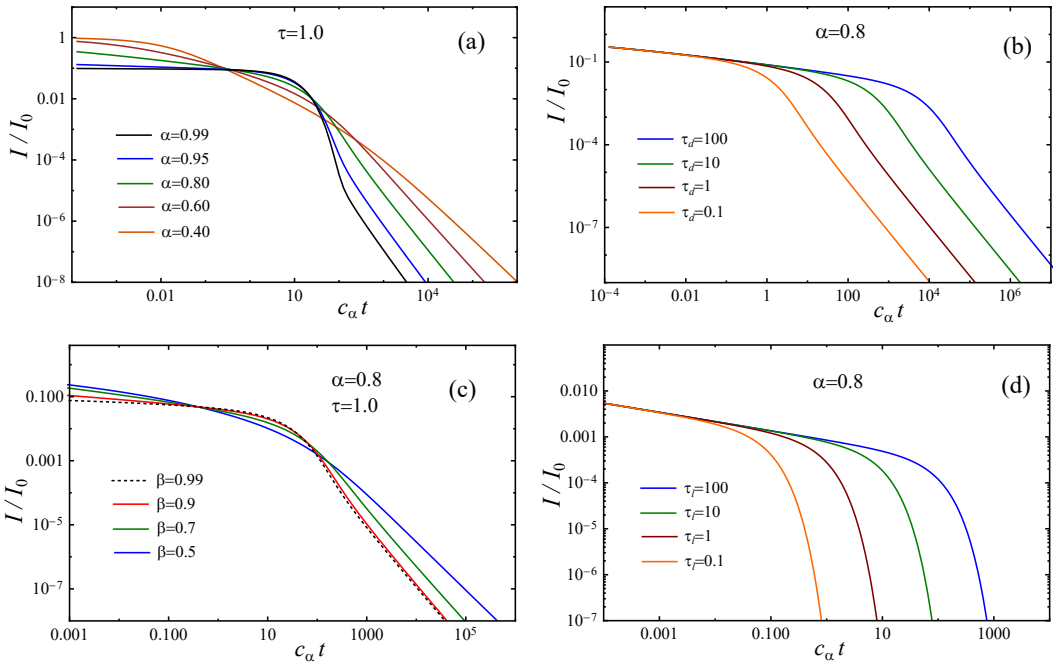


Figure 1. Reduced transient current $I(t)/I_0$ versus t in the log–log scale for different cases: (**a**) inversion of Laplace transform (19) for $c_\alpha = 1$, $\omega_0 = 10$, and $\tau = 1$, and different α values indicated in the plot, (**b**) the same for different τ values at fixed $\alpha = 0.8$, (**c**) inversion of Laplace transform (23) $b_1 = b_2 = 1$, $\omega_0 = 10$, $\alpha_1 = \alpha = 0.8$, and $\alpha_2 = \beta$ indicated in the plot, and (**d**) transient current (24) for $c_\alpha = 1$, $\omega_0 = 10$, $\tau = \tau_l$.

3. Transit–Time Dispersion for Fractional Drift with Leakage

The main properties of dispersive transport in amorphous semiconductors were revealed using the time-of-flight experiment (ToF) [4,10]. The ToF method implies measurement of the transit time t_T required for a packet of carriers to pass through a layer of disordered semiconductor. The sample is placed between two electrodes, one of which is usually transparent. The charges are often generated by photoexcitation with a short laser pulse. These carriers propagate along the electric field, which results in a bias current that flows until all the carriers have reached the opposite electrode. In the ToF method, the external electric field significantly exceeds the field of nonequilibrium charge carriers, and the linear approximation is usually applied.

Using the continuity equation

$$\frac{\partial n(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = -R(x,t) + n_0\,\delta(t)\delta(x)$$

with leakage and generation terms, and the expression for conduction current density in the diffusion approximation

$$j(x,t) = -D\frac{dn(x,t)}{dx} + \mu F\, n(x,t),$$

one can obtain the fractional diffusion–drift equations accounting for the leakage terms discussed in the previous section. Note, however, that usually recombination is not taken into account when describing the time-of-flight experiment in the classical formulation. This is due to the fact that the transfer is monopolar: when electron–hole pairs are generated near one of the electrodes, one type of carrier is absorbed by the nearest electrode. Oppositely charged carriers drift to the opposite electrode. Further, we consider the term $R(x,t)$ as corresponding to the generalized linear leakage process. Particularly, it may be associated with trapping to deep centers or geometric traps characterized by localization times exceeding the registration time of the transient current. This means that the charge

carrier localized to such traps does not contribute to the current events during the ToF experiment. Other physical processes can also be responsible for the presence of leakage term, particularly tunneling through a substrate in the case of planar geometry. Our goal is to consider the fractional drift of charge carriers in the ToF method, taking into account the leakage terms discussed in the previous section.

Following the same procedure as in the previous section, one may derive the following equations for concentrations of delocalized carriers $n(x,t)$:

- Fractional drift equation with leakage via delocalized states:

$$\frac{\partial n(x,t)}{\partial t} + \omega_0 c_\alpha^{-\alpha}\, {}_0D_t^\alpha n(x,t) + \mu F \frac{\partial n(x,t)}{\partial x} = -\frac{n(x,t)}{\tau} + n_0\delta(t)\delta(x) \tag{25}$$

- Fractional drift equation with leakage via localized and delocalized states:

$$\frac{\partial n(x,t)}{\partial t} + \omega_0 c_\alpha^{-\alpha} e^{-t/\tau}\, {}_0D_t^\alpha \left(e^{t/\tau} n(x,t)\right) + \mu F \frac{\partial n(x,t)}{\partial x} = -\frac{n(x,t)}{\tau} + n_0\delta(t)\delta(x). \tag{26}$$

Further we can consider the general case of leakage

$$\frac{\partial n(x,t)}{\partial t} + \omega_0 c_\alpha^{-\alpha} e^{-t/\tau_l}\, {}_0D_t^\alpha \left(e^{t/\tau_l} n(x,t)\right) + \mu F \frac{\partial n(x,t)}{\partial x} = -\frac{n(x,t)}{\tau_d} + n_0\delta(t)\delta(x). \tag{27}$$

Here, τ_l and τ_d are the constant leakage times through localized and delocalized states, respectively. Equation (25) corresponds to the case, when $\tau_l \to \infty$, i.e., there is no leakage through the localized states. Applying the Laplace transformation to Equation (27) and solving it, we obtain

$$\tilde{n}(x,s) = \frac{n_0}{\mu F} \exp\left(-\frac{x\omega_0}{\mu F}\left[\frac{\left(s+\tau_l^{-1}\right)^\alpha}{c_\alpha^\alpha} + \frac{s+\tau_d^{-1}}{\omega_0} \right] \right). \tag{28}$$

The latter expression can be easily inverted. Rewriting it in the form

$$\tilde{n}(x,s) = \frac{n_0}{\mu F} e^{-\frac{x}{\mu F}\frac{\omega_0}{c_\alpha^\alpha}\left(s+\tau_l^{-1}\right)^\alpha} e^{-\frac{x}{\mu F}s} e^{-\frac{x}{\mu F \tau_d}},$$

and using the rules of the inverse Laplace transformation, we arrive at the following expression

$$n(x,t) = \frac{n_0}{\mu F} e^{-\frac{t-x/\mu F}{\tau_l} - \frac{x}{\mu F \tau_d}} \left(\frac{x}{\mu F}\frac{\omega_0}{c_\alpha^\alpha}\right)^{-1/\alpha} g_+^{(\alpha)}\left(\left(t-\frac{x}{\mu F}\right)\left(\frac{x}{\mu F}\frac{\omega_0}{c_\alpha^\alpha}\right)^{-1/\alpha}\right) H(t-x/\mu F).$$

Here, $g_+^{(\alpha)}(t)$ is the one-sided Lévy stable density with Laplace transform $\tilde{g}_+^{(\alpha)}(s) = \exp(-s^\alpha)$, and $H(t)$ is the Heaviside step function.

The transient current $I(t)$ is the spatial average of the flux across the sample,

$$\tilde{I}(s) = \frac{q\mu F}{L}\int_0^L \tilde{n}(x,s)dx, \tag{29}$$

where q is the charge of a carrier. Using Equations (28) and (29), one can derive

$$\tilde{I}(s) = \frac{qn_0}{\omega_0 t_0}\ \frac{1-\exp\left(-\left[\frac{\left(s+\tau_l^{-1}\right)^\alpha}{c_\alpha^\alpha} + \frac{s+\tau_d^{-1}}{\omega_0}\right]\omega_0 t_0\right)}{\frac{\left(s+\tau_l^{-1}\right)^\alpha}{c_\alpha^\alpha} + \frac{s+\tau_d^{-1}}{\omega_0}}, \quad t_0 = \frac{L}{\mu F}. \tag{30}$$

Figure 2 demonstrates transient photocurrent curves $I(t)$ corresponding to the Laplace transform (30) for different values of parameters α, τ_l and τ_d indicated in figure. Charge

leakage via localized states leads to an exponential truncation of the transient current tail. Leakage via delocalized states does not lead to a change in the power law exponents of $I(t)$ before and after time of flight t_T, but reduces t_T and smooths out the break itself.

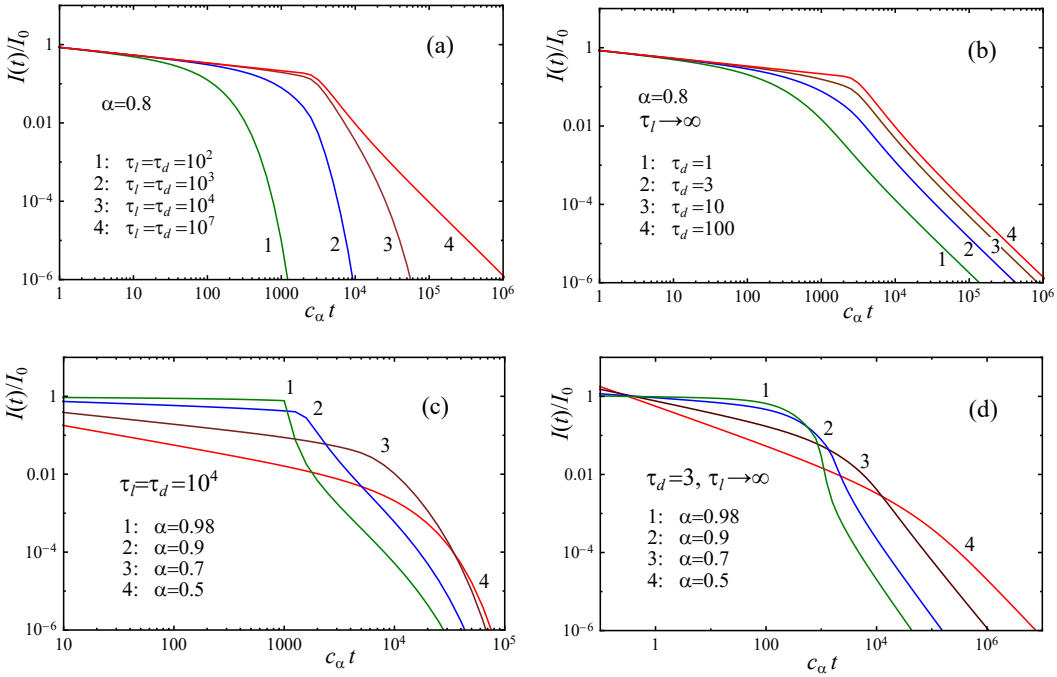


Figure 2. Transient photocurrent $I(t)$ corresponding to the Laplace transform (30) for different values of parameters α, τ_l and τ_d indicated in figure. Parameters τ_l and τ_d are given in units of c_α^{-1}, $\omega_0 = 100c_\alpha$, $t_0 = 10c_\alpha^{-1}$, $I_0 = qn_0/(\omega_0 t_0)$. Recombination via localized states for $\alpha < 1$ leads to the exponential truncation of a transient current tail (**a**). Recombination via delocalized states only does not lead to a change in the power law exponents of $I(t)$ before and after the break, but reduces time of flight t_T and smooths out the break itself (**b**). Effect of charge leakage on transient currents depends on dispersion parameter α and ratio of τ_l, τ_d and t_T (**c**). Variation of α for fixed $\tau_d = 3$ in units of c_α^{-1} and $\tau_l \to \infty$ confirms the indicated effect of leakage via delocalized states (**d**).

As already mentioned, the theory of charge transfer in disordered semiconductors is well developed [26–29]. However, there are a number of applications where the considered kinetic model can be especially useful. In particular, phase-change materials (PCM) that can be reversibly switched between amorphous and crystalline phases with different electrical resistivity [30,31] are potential objects for the proposed theory. The large resistance contrast is used to store information in PCM-based memory cells [32]. As noted in [13], the advantage of fractional diffusion equations is that they allow us to describe dispersive transport in amorphous semiconductors and normal diffusive transport in crystalline semiconductors within the unified formalism. For amorphous PCM, it has been shown that multiple trapping can successfully describe low-field conductivity measurements at temperatures above approximately 200 K, while at lower temperatures hopping dominates (see reviews [30,31]). The density of states in the amorphous phase of PCM is characterized by exponential tails of localized states both for conduction band and valence bands [31]. The degree of amorphization ζ determines the width of exponential density of the localized states. In the fractional model, ζ is directly related to the non-integer order of the time derivative and the index of Lévy stable statistics. Shallow traps usually act as donor or acceptor levels. On the contrary, deep traps reduce the conductivity in most cases by keeping the Fermi energy in the middle of the bandgap and acting as electron–hole recombination centers.

4. Conclusions

We considered the application of fractional multiple trapping equations to problems of transient photoconductivity and transit–time dispersion in the time-of-flight method. We briefly discussed the connection of multiple trapping model with generalized fractional kinetic equations. From this perspective, the multiple trapping model can be used to interpret fractional transport equations, formulate initial and boundary conditions for them, and to construct numerical methods for solving fractional kinetic equations. Different types of charge leakage are discussed. Tempered fractional kinetics are obtained for recombination via localized states and distributed order equations arise for non-exponential density of states presented as the weighted mixture of exponential functions. Analytical solutions for transient photocurrent are provided for several simplified situations. Fractional diffusion equations allow us to describe dispersive transport in amorphous semiconductors and normal diffusive transport in crystalline semiconductors within the unified formalism. This advantage can be used to describe charge carrier transport in chalcogenide phase-change materials that can be reversibly switched between amorphous and crystalline phases.

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