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An Application of the Homotopy Analysis Method for the Time- or Space-Fractional Heat Equation

Rafał Brociek [1,*], Agata Wajda [2], Marek Błasik [1] and Damian Słota [1]

[1] Department of Mathematics Applications and Methods for Artificial Intelligence, Faculty of Applied Mathematics, Silesian University of Technology, 44-100 Gliwice, Poland
[2] Institute of Energy and Fuel Processing Technology, 41-803 Zabrze, Poland
* Correspondence: rafal.brociek@polsl.pl

Abstract: This paper focuses on the usage of the homotopy analysis method (HAM) to solve the fractional heat conduction equation. In the presented mathematical model, Caputo-type fractional derivatives over time or space are considered. In the HAM, it is not necessary to discretize the considered domain, which is its great advantage. As a result of the method, a continuous function is obtained, which can be used for further analysis. For the first time, for the considered equations, we proved that if the series created in the method converges, then the sum of the series is a solution of the equation. A sufficient condition for this convergence is provided, as well as an estimation of the error of the approximate solution. This paper also presents examples illustrating the accuracy and stability of the proposed algorithm.

Keywords: fractional derivative; heat equation; homotopy analysis method

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1. Introduction

Mathematical models, in the form of various types of differential equations containing fractional derivatives, have been used in the description and modeling of many phenomena occurring in physics, biology and technology. The anomalous diffusion equation, which contains a fractional derivative with respect to time, is related to the continuous-time random walk process and can describe anomalous diffusion processes, for example, impurity diffusion processes in porous materials [1]. Mathematical modeling and simulations of heat conduction using fractional calculus are presented, among others, in [2–4], where models with derivatives of the Riemann–Liouville and Caputo types are considered. For heat conduction problems, models with fractional derivatives are particularly useful for modeling phenomena occurring in porous materials and composites due to the occurrence of the anomalous diffusion phenomenon [2,5,6]. The use of models with fractional derivatives, instead of integer derivatives, allow for better modeling of heat conduction in the before-mentioned porous media. Some examples can be found in [5,6]. Applying fractional calculus allows the simulation of various processes, for example, the release of drugs from a polymer matrix [7], the sediment transport problem [8] and heat conduction in porous materials [2,6,9].

At the end of the twentieth century, methods were developed to solve different types of operator equations. These include, among others, the following: Adomian's decomposition method, the variational iterative method and the homotopy analysis method along with its variant, the homotopy perturbation method. Shijun Liao suggested the homotopy analysis method in 1992 [10–12]. This method has found numerous applications. It has been used to solve many problems described by classical differential equations [13–15], integral equations [16–18], differential–integral equations [19–21], as well as others. Theoretical results for integer-order differential equations can be found in the papers [11,12,15,22,23], whereas for integral equations the results can be found in [17,18]. In the homotopy analysis method, it is not necessary to discretize the considered domain, which is its great advantage. The obtained solution is a continuous function, which we can use for further analysis.

The homotopy analysis method has already been applied to solving various fractional differential equations. Abdulaziz et al. [24] used the method for the fractional modified KdV equation. The study showed that for a convergent series, its sum satisfies the considered equation. The use of the homotopy analysis transform method for solving time-fractional Korteweg-de Vries and Korteweg-de Vries-Burgers equations is presented by Saad et al. in the paper [25]. Different versions of the presented method can be applied to solve fractional heat- and wave-like equations. Examples are shown in [26–29]. Odibat and Baleanu, in the paper [30], present a linearization-based approach of the homotopy analysis method and applied it in order to solve the time-fractional parabolic differential equation. The method can also be used for finding solutions of the time-fractional inverse Stefan problem [31]. Arafa et al. [32] provided examples where the homotopy analysis method was applied to solve the Schnakenberg model with fractional order derivatives. In turn, applying the method to an equation with modified Caputo–Fabrizio fractional derivatives is described in [33]. The possibilities of using the q-homotopy analysis method on the fractional diffusion equation over time are presented in [34]. The same version of the method is used to solve the fractional diffusion equation together with the Atangana–Baleanu derivative in the paper [35]. Odibat [36] describes how to properly select the linear operator used in the homotopy analysis method in the case of the fractional ordinary differential equation. Examples of using the HAM for the time–space fractional diffusion equation can be found in [37]. An equation defined on the entire plane, with only an initial condition, is considered. The Caputo derivative is used over time, while the Riesz derivative is used over space. In turn, examples of the application of the homotopy analysis method for time- and space-fractional diffusion equations with Caputo derivatives are presented in [38]. In the considered examples, the initial condition is provided, while the boundary conditions are omitted. In one case, an equation in a bounded space is considered. In [39], the authors present an application of the HAM for solving the following fractional models: the Cauchy–Riemann equations, the acoustic wave equations and time-fractional diffusion. The considered equations were defined in bounded areas with given boundary conditions and an initial condition. Most of these studies present computational examples. Theoretical results are contained only in [24,31].

So far, research presenting the usefulness of the HAM for the fractional heat equation are focused mainly on the case where the fractional derivative is considered over time. Studies on the space-fractional equation contain only examples that are mostly defined in an unbounded area and have only the initial condition given. In addition, with the exception of one paper, they do not contain any theoretical results regarding convergence and error estimation. Therefore, this study presents and describes the HAM for solving the time- or space-fractional heat equation with Caputo derivatives. The article also provides proof of the method's convergence, where the sum of the series is a solution of the considered equation. A sufficient condition for convergence of the series and an estimate of the error of the approximate solution are given in this consideration, as well as examples illustrating the accuracy of the method.

2. Formulation of the Problems

The first of the considered equations is the time-fractional heat equation [40,41]:

$$D_t^{\alpha} u(x,t) = a\,\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad (x,t) \in \Omega = (b_1,b_2) \times (0,t^*), \tag{1}$$

where a Caputo-order derivative is denoted by $\alpha \in (0,1)$; $a = \overline{w}\,\overline{a} > 0$ is the scaled (fractional) thermal diffusivity $[\mathrm{m}^2/\mathrm{s}^{\alpha}]$, that is the thermal diffusivity $\overline{a}$ $[\mathrm{m}^2/\mathrm{s}]$ multiplied by the scaling constant of value one $\overline{w}$ and a proper unit $[\mathrm{s}^{1-\alpha}]$; $f = \overline{w}\,\overline{f}$ is the scaled (fractional) source function $[\mathrm{K}/\mathrm{s}^{\alpha}]$, that is the source function $\overline{f}$ $[\mathrm{K}/\mathrm{s}]$ multiplied by the scaling constant $\overline{w}$ [$s^{1-\alpha}$]; and u, t and x refer to the temperature, time and spatial location, respectively.

The second of the considered equations is the space-fractional heat equation [40,41]:

$$\frac{\partial u(x,t)}{\partial t} = a\, D_x^{\beta} u(x,t) + f(x,t), \qquad (x,t) \in \Omega = (b_1, b_2) \times (0, t^*), \tag{2}$$

where $\beta \in (1,2)$ is order of the Caputo fractional derivative; $a = \widehat{w}\,\overline{a} > 0$ is the scaled (fractional) thermal diffusivity [m^{β}/s], that is the thermal diffusivity $\overline{a}$ [m^2/s] properly scaled by a constant $\widehat{w}$ with the numerical value of one and the unit [$m^{\beta-2}$]; f is the source function [K/s]; and u, t and x refer to the temperature, time and spatial variable, respectively.

The above equations are supplemented by an initial condition [40]:

$$u(x,0) = \varphi_0(x), \qquad x \in [b_1, b_2]. \tag{3}$$

The following boundary conditions of the form in [40] are given:

$$\gamma_1\, u(b_1,t) + \gamma_2\, \frac{\partial u(b_1,t)}{\partial x} = \varphi_1(t), \qquad t \in [0, t^*], \tag{4}$$

$$\gamma_3\, u(b_2,t) + \gamma_4\, \frac{\partial u(b_2,t)}{\partial x} = \varphi_2(t), \qquad t \in [0, t^*], \tag{5}$$

where $\gamma_i \in \mathbb{R}$, $i \in \{1,2,3,4\}$, $\gamma_1^2 + \gamma_2^2 \neq 0$ and $\gamma_3^2 + \gamma_4^2 \neq 0$. By appropriate selection of the values of the constants γ_i, we can obtain boundary conditions of the first, second or third order.

In this paper, we use the Caputo fractional derivative [42]. For $\eta \in (n-1, n]$, $n \in \mathbb{N}$, this derivaitive is defined as follows:

$$D_z^{\eta} f(z) = \frac{1}{\Gamma(n-\eta)} \int_0^z \frac{f^{(n)}(s)}{(z-s)^{\eta-n+1}}\, ds,$$

where $\Gamma(\cdot)$ is the gamma function.

3. The Homotopy Analysis Method

The homotopy analysis method is described in [10–12,23,31]. This method is used to solve the considered problem and, in general, it can be used to solve the following operator equations:

$$N(u(x,t)) = 0, \qquad (x,t) \in \Omega, \tag{6}$$

where N is an operator, and u is an unknown state function. We begin by defining the homotopy operator

$$\mathcal{H}(\Phi, p) := (1-p)\, L\big(\Phi(x,t;p) - u_0(x,t)\big) - p\, h\, N\big(\Phi(x,t;p)\big), \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, $h \neq 0$ is the convergence control parameter, u_0 is an initial solution of the problem (6) and L is an auxiliary linear operator.

Next, let us focus on the equation $\mathcal{H}(\Phi, p) = 0$, hence, we receive the so-called zero-order deformation equation

$$(1-p)\, L\big(\Phi(x,t;p) - u_0(x,t)\big) = p\, h\, N\big(\Phi(x,t;p)\big). \tag{8}$$

For $p = 1$ we have $N(\Phi(x,t;1)) = 0$, so $u(x,t) = \Phi(x,t;1)$ is the solution of the considered equation.

Expanding the function Φ into the Maclaurin series with respect to the parameter p and rearranging, we receive the following:

$$\Phi(x,t;p) = \sum_{m=0}^{\infty} u_m(x,t)\, p^m, \tag{9}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \Phi(x,t;p)}{\partial p^m}\Big|_{p=0} =: \mathcal{D}_m\big(\Phi(x,t;p)\big), \qquad m = 1,2,3,\ldots. \tag{10}$$

The operator $\mathcal{D}_m$ is the so-called mth-order homotopy-derivative operator [12]. If the series in dependence (9) converges in the appropriate region, then for $p = 1$ we obtain the exact solution

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t). \tag{11}$$

If we are unable to determine the sum of the above series, we can choose its partial sum as an approximate solution:

$$\widehat{u}_n(x,t) = \sum_{m=0}^{n} u_m(x,t). \tag{12}$$

The step that needs to be taken to obtain the function u_m is calculating the derivative m times, with respect to the parameter p, of the left and right sides of Formula (8). Then, we divide the result by $m!$ and substitute in $p = 0$, obtaining the so-called mth-order deformation equation ($m \geqslant 1$):

$$L\big(u_m(x,t) - \chi_m\, u_{m-1}(x,t)\big) = h\,\mathcal{D}_{m-1}\Big[N\Big(\sum_{i=0}^{\infty} u_i(x,t)\, p^i\Big)\Big], \tag{13}$$

where

$$\chi_m = \begin{cases} 0 & m \leqslant 1, \\ 1 & m \geqslant 2. \end{cases} \tag{14}$$

By the appropriate selection of the parameter h, it is possible to influence the area and rate of convergence of the series (11) [12,23]. To calculate the value of this parameter, the residual function is defined:

$$E_n(h) = \int_{\Omega} \Big(N\big[\widehat{u}_n(x,t)\big]\Big)^2\, dx\, dt. \tag{15}$$

Then, by calculating the minimum of this function, the optimal value of the parameter h is found.

In the case of Equations (1) and (2) considered in the study, the appropriate operator N can be defined as follows:

$$N_1(u) = a\,\frac{\partial^2 u}{\partial x^2} - D_t^{\alpha} u + f(x,t), \tag{16}$$

and

$$N_2(u) = \frac{\partial u}{\partial t} - a\, D_x^{\beta} u - f(x,t). \tag{17}$$

An operator of the following form can be taken as a linear operator:

$$L(u) = \frac{\partial^2 u}{\partial x^2}. \tag{18}$$

Assuming that the series converges and appropriate integrals exist, for the Caputo derivative $\alpha \in (0,1)$, we obtain the following:

$$
\begin{aligned}
D_t^{\alpha}\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial}{\partial t}\Big(\sum_{i=0}^{\infty} u_i(x,s)\,p^i\Big)\frac{1}{(t-s)^{\alpha}}\,ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t \Big(\sum_{i=0}^{\infty} \frac{\partial u_i(x,s)}{\partial t}\,p^i\Big)\frac{1}{(t-s)^{\alpha}}\,ds \\
&= \sum_{i=0}^{\infty} p^i\Big(\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u_i(x,s)}{\partial t}\frac{1}{(t-s)^{\alpha}}\,ds\Big) = \sum_{i=0}^{\infty} D_t^{\alpha} u_i(x,t)\,p^i.
\end{aligned}
$$

Similarly for $\beta \in (1,2)$ we have

$$
D_x^{\beta}\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big) = \sum_{i=0}^{\infty} D_x^{\beta} u_i(x,t)\,p^i.
$$

Using this one we get the following:

$$
\begin{aligned}
\mathcal{D}_{m-1}\left(D_t^{\alpha}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right)\right) &= \mathcal{D}_{m-1}\left(\sum_{i=0}^{\infty} D_t^{\alpha} u_i(x,t)\,p^i\right) \\
&= \frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}}\Big(\sum_{i=0}^{\infty} D_t^{\alpha} u_i(x,t)\,p^i\Big)\right)\Bigg|_{p=0} \\
= \frac{1}{(m-1)!}\left((m-1)!\,D_t^{\alpha}\,u_{m-1}(x,t) + \sum_{i=m}^{\infty} w(i)\,D_t^{\alpha}\,u_i(x,t)\,p^{i-m+1}\right)\Bigg|_{p=0}& \\
&= D_t^{\alpha}\,u_{m-1}(x,t),
\end{aligned}
\tag{19}
$$

for $m = 1, 2, \ldots$ and where $w(i) \in \mathbb{N}$ for $i = m, m+1, \ldots$ Similarly, we obtain

$$
\mathcal{D}_{m-1}\left(a\,D_x^{\beta}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right)\right) = a\,D_x^{\beta}\,u_{m-1}(x,t), \tag{20}
$$

$$
\mathcal{D}_{m-1}\left(a\,\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right)\right) = a\,\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}, \tag{21}
$$

$$
\mathcal{D}_{m-1}\left(\frac{\partial}{\partial t}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right)\right) = \frac{\partial u_{m-1}(x,t)}{\partial t}, \tag{22}
$$

$$
\mathcal{D}_{m-1}\left(f(x,t)\right) = \begin{cases} f(x,t), & m = 1, \\ 0, & m \geqslant 2. \end{cases} \tag{23}
$$

Using the above correlations, we yield the following:

$$
\begin{aligned}
&\mathcal{D}_{m-1}\Big[N_1\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big)\Big] \\
&\quad = \mathcal{D}_{m-1}\Big[a\,\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right) - D_t^{\alpha}\left(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\right) + f(x,t)\Big] = \\
&\quad = \begin{cases} a\,\frac{\partial^2 u_0(x,t)}{\partial x^2} - D_t^{\alpha} u_0(x,t) + f(x,t), & m = 1, \\ a\,\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - D_t^{\alpha} u_{m-1}(x,t), & m \geqslant 2, \end{cases}
\end{aligned}
\tag{24}
$$

and

$$\mathcal{D}_{m-1}\Big[N_2\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big)\Big]$$
$$= \mathcal{D}_{m-1}\Big[\frac{\partial}{\partial t}\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big) - a\,D_x^{\beta}\Big(\sum_{i=0}^{\infty} u_i(x,t)\,p^i\Big) - f(x,t)\Big]$$
$$= \begin{cases} \frac{\partial u_0(x,t)}{\partial t} - a\,D_x^{\beta} u_0(x,t) - f(x,t), & m = 1, \\ \frac{\partial u_{m-1}(x,t)}{\partial t} - a\,D_x^{\beta} u_{m-1}(x,t), & m \geqslant 2. \end{cases} \tag{25}$$

We begin by considering the application of the method to solve Equation (1). With the use of the above-mentioned correlation (24), and with the mth-order deformation equation, we obtain for $m = 1$ the following:

$$L\big(u_1(x,t)\big) = h\,\Big(a\,\frac{\partial^2 u_0(x,t)}{\partial x^2} - D_t^{\alpha}\,u_0(x,t) + f(x,t)\Big);$$

and the following for $m \geqslant 2$:

$$L\big(u_m(x,t)\big) = L\big(u_{m-1}(x,t)\big) + h\,\Big(a\,\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - D_t^{\alpha}\,u_{m-1}(x,t)\Big).$$

Considering the form of the L operator, we obtain partial differential equations from which we can determine the function u_m. For $m = 1$ we have the followng:

$$\frac{\partial^2 u_1(x,t)}{\partial x^2} = h\,\Big(a\,\frac{\partial^2 u_0(x,t)}{\partial x^2} - D_t^{\alpha}\,u_0(x,t) + f(x,t)\Big); \tag{26}$$

and the following for $m \geqslant 2$:

$$\frac{\partial^2 u_m(x,t)}{\partial x^2} = (1 + a\,h)\,\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - h\,D_t^{\alpha}\,u_{m-1}(x,t). \tag{27}$$

While in the case of Equation (2), we obtain the following for $m = 1$:

$$L\big(u_1(x,t)\big) = h\,\Big(\frac{\partial u_0(x,t)}{\partial t} - a\,D_x^{\beta}\,u_0(x,t) - f(x,t)\Big);$$

and the following for $m \geqslant 2$:

$$L\big(u_m(x,t)\big) = L\big(u_{m-1}(x,t)\big) + h\,\Big(\frac{\partial u_{m-1}(x,t)}{\partial t} - a\,D_x^{\beta}\,u_{m-1}(x,t)\Big).$$

Then, taking into account the form of the operator L, for $m = 1$ we have the following:

$$\frac{\partial^2 u_1(x,t)}{\partial x^2} = h\,\Big(\frac{\partial u_0(x,t)}{\partial t} - a\,D_x^{\beta}\,u_0(x,t) - f(x,t)\Big), \tag{28}$$

and the following for $m \geqslant 2$:

$$\frac{\partial^2 u_m(x,t)}{\partial x^2} = \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + h\,\Big(\frac{\partial u_{m-1}(x,t)}{\partial t} - a\,D_x^{\beta}\,u_{m-1}(x,t)\Big). \tag{29}$$

For the uniqueness of the solution, the partial differential equation, Equation (29), must be supplemented with additional conditions. Using the given boundary conditions (4) and (5), we define the conditions, ensuring unique solutions of the partial differential equations obtained above. Equations (26) and (28) are supplemented with conditions of the form

$$\gamma_1 \,(u_0 + u_1)(b_1, t) + \gamma_2 \,\frac{\partial(u_0 + u_1)}{\partial x}(b_1, t) = \varphi_1(t), \qquad t \in [0, t^*], \tag{30}$$

$$\gamma_3 \,(u_0 + u_1)(b_2, t) + \gamma_4 \,\frac{\partial(u_0 + u_1)}{\partial x}(b_2, t) = \varphi_2(t), \qquad t \in [0, t^*]. \tag{31}$$

In turn, Equations (27) and (29) for $m \geqslant 2$, are supplemented with conditions of the form

$$\gamma_1 \, u_m(b_1, t) + \gamma_2 \,\frac{\partial u_m(b_1, t)}{\partial x} = 0, \qquad t \in [0, t^*], \tag{32}$$

$$\gamma_3 \, u_m(b_2, t) + \gamma_4 \,\frac{\partial u_m(b_2, t)}{\partial x} = 0, \qquad t \in [0, t^*]. \tag{33}$$

This way of specifying the conditions for new equations ensures that an approximation formed as a partial sum of the series (11), satisfies the given boundary conditions. As an initial approximation, a function defining the initial condition can be taken:

$$u_0(x, t) = \varphi_0(x). \tag{34}$$

The sum of the series built in the method is a solution to the considered equation because the following theorem holds.

Theorem 1. *Let the functions u_m, $m \geqslant 1$, be solutions of Equations (26) and (27) (or Equations (28) and (29)), respectively, with conditions (30)–(33). Then, if the series $\sum_{m=0}^{\infty} u_m(x, t)$ converges in the domain Ω and an appropriate integrals exist, then its sum is the solution of Equation (1) (Equation (2), respectively).*

Proof. We show the proof for Equation (1) (using the operator N_1).

Let the series $\sum_{m=0}^{\infty} u_m$ be convergent and

$$s(x, t) = \sum_{m=0}^{\infty} u_m(x, t).$$

From the necessary condition for series convergence, we obtain that for any $(x, t) \in \Omega$:

$$\lim_{m \to \infty} u_m(x, t) = 0.$$

According to the method, we have

$$L\big(u_m(x, t) - \chi_m \, u_{m-1}(x, t)\big) = h \, \mathcal{D}_{m-1}\Big[N_1\Big(\sum_{i=0}^{\infty} u_i(x, t)\, p^i\Big)\Big].$$

Hence, we obtain

$$\begin{aligned} h \sum_{m=1}^{\infty} \mathcal{D}_{m-1}\Big[N_1\Big(\sum_{i=0}^{\infty} u_i(x, t)\, p^i\Big)\Big] &= \lim_{n \to \infty} \sum_{m=1}^{n} L\big(u_m(x, t) - \chi_m \, u_{m-1}(x, t)\big) \\ &= \lim_{n \to \infty} L\Big(\sum_{m=1}^{n} \big(u_m(x, t) - \chi_m \, u_{m-1}(x, t)\big)\Big) = \lim_{n \to \infty} L\big(u_n(x, t)\big) \\ &= L\big(\lim_{n \to \infty} u_n(x, t)\big) = L(0) = 0, \end{aligned}$$

where we used the linearity and continuity of the operator L. Since $h \neq 0$, we have

$$\sum_{m=1}^{\infty} \mathcal{D}_{m-1}\Big[N_1\Big(\sum_{i=0}^{\infty} u_i(x,t)\, p^i\Big)\Big] = 0.$$

Then, using the correlation in (24), we obtain

$$\begin{aligned}
0 = \sum_{m=1}^{\infty} \mathcal{D}_{m-1}\Big[N_1\Big(\sum_{i=0}^{\infty} u_i(x,t)\, p^i\Big)\Big] \\
&= a \sum_{m=1}^{\infty} \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - \sum_{m=1}^{\infty} D_t^{\alpha} u_{m-1}(x,t) + f(x,y) \\
&= a \frac{\partial^2}{\partial x^2} \sum_{m=1}^{\infty} u_{m-1}(x,t) - D_t^{\alpha} \sum_{m=1}^{\infty} u_{m-1}(x,t) + f(x,y) \\
&= a \frac{\partial^2}{\partial x^2} s(x,t) - D_t^{\alpha} s(x,t) + f(x,y) = N_1(s(x,t)).
\end{aligned}$$

In the case of the second equation, the proof is similar. □

Theorem 2 concerns a sufficient condition for the convergence of the series created in the described method.

Theorem 2. *Let the functions u_m, $m \geq 1$ be solutions of Equations (26) and (27) (or Equations (28) and (29)), respectively with the conditions (30)–(33). Then, if parameter h is selected in such a way that there exist constants $\xi_h \in (0,1)$ and $m_0 \in \mathbb{N}$ such that for each $m \geqslant m_0$, the following inequality*

$$\|u_{m+1}\| \leqslant \xi_h \, \|u_m\| \tag{35}$$

is satisfied in the area Ω, then the series $\sum_{m=0}^{\infty} u_m(x,t)$ is uniformly convergent in Ω.

The last theorem provides an estimate of the error of the approximate solution, which we obtain using the partial sum of the series.

Theorem 3. *If the assumptions of Theorem 2 are satisfied and additionally $n \in \mathbb{N}$ and $n \geqslant m_0$, then we get the following estimation of error of approximate solution:*

$$\|u - \widehat{u}_n\| \leqslant \frac{\xi_h^{n+1-m_0}}{1-\gamma_h} \|u_{m_0}\|. \tag{36}$$

The proofs of the last two theorems proceed similarly to the classical equation of heat conduction (see [15]).

4. Examples

4.1. Example 1

In the first example, Equation (1) with boundary conditions of the first order is considered. In calculations it is assumed that $(b_1, b_2) = (0,1)$, $(0, t^*) = (0,1)$, $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = \gamma_4 = 0$, $a = 1$ and $\alpha = \frac{3}{4}$. An initial condition and boundary conditions are described by functions $\varphi_0(x) = 0$, $\varphi_1(t) = 0$, $\varphi_2(t) = t^{3/4}$. The source function is defined as follows:

$$f(x,t) = \frac{3\,\pi\, x^2}{2\sqrt{2}\,\Gamma(\frac{1}{4})} - 2\, t^{\frac{3}{4}}.$$

According to the described method, we take a function describing the initial condition as the initial approximation

$$u_0(x,t) = \varphi_0(x) = 0.$$

Solving Equation (26) with conditions (30)–(31) we obtain the function

$$u_1(x,t) = t^{\frac{3}{4}}\, x\, (1 + h - h\, x) + \frac{\pi\, h\, x\, (x^3 - 1)}{8\sqrt{2}\,\Gamma\big(\frac{1}{4}\big)}.$$

The following functions u_m, $m \geqslant 2$, are obtained by solving Equation (27) with conditions (32) and (33). Then, we obtain successively the following:

$$\begin{aligned}
u_2(x,t) &= \frac{h\,(x^2 - x)}{16\,\Gamma\big(\frac{1}{4}\big)} \Big(\sqrt{2}\,\pi\,(x^2 - x - 1 + 2\,h\,x^2) - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\tfrac{1}{4}\big)\Big),\\
u_3(x,t) &= \frac{h\,(h+1)\,(x^2 - x)}{16\,\Gamma\big(\frac{1}{4}\big)} \Big(\sqrt{2}\,\pi\,(x^2 - x - 1 + h\,(3\,x^2 - x - 1))\\
&\quad - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\tfrac{1}{4}\big)\Big),\\
u_4(x,t) &= \frac{h\,(h+1)^2\,(x^2 - x)}{16\,\Gamma\big(\frac{1}{4}\big)} \Big(\sqrt{2}\,\pi\,(x^2 - x - 1 + h\,(4\,x^2 - 2\,(x+1)))\\
&\quad - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\tfrac{1}{4}\big)\Big),\\
u_5(x,t) &= \frac{h\,(h+1)^3\,(x^2 - x)}{16\,\Gamma\big(\frac{1}{4}\big)} \Big(\sqrt{2}\,\pi\,(x^2 - x - 1 + h\,(5\,x^2 - 3\,(x+1)))\\
&\quad - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\tfrac{1}{4}\big)\Big).
\end{aligned}$$

In general, for $m \geqslant 2$ we have the following:

$$\begin{aligned}
u_m(x,t) = \frac{h\,(h+1)^{m-2}\,(x^2 - x)}{16\,\Gamma\big(\frac{1}{4}\big)} \Big(\sqrt{2}\,\pi\,\Big[x^2 - x - 1\\
+ h\,(m\,x^2 - (m-2)\,(x+1))\Big] - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\tfrac{1}{4}\big)\Big).
\end{aligned}$$

Calculating the sum of the series for $m \geqslant 2$, we obtain

$$\sum_{m=2}^{\infty} u_m(x,t) = \frac{(x - x^2)\Big(\sqrt{2}\,\pi\,h\,(x^2 + x + 1) - 16\,(h+1)\,t^{\frac{3}{4}}\,\Gamma\big(\frac{1}{4}\big)\Big)}{16\,\Gamma\big(\frac{1}{4}\big)}.$$

Adding functions to u_0 and u_1, we obtain the exact solution

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = t^{\frac{3}{4}}\, x^2.$$

4.2. Example 2

In the next example, the same Equation (1) is considered, but with boundary conditions of the first and second order. In the example, it is assumed $(b_1, b_2) = (0,1)$, $(0, t^*) = (0,1)$, $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = 1$, $\gamma_4 = 0$, $a = 1$ and $\alpha = \frac{1}{2}$. Initial and boundary conditions are described by functions $\varphi_0(x) = 0$, $\varphi_1(t) = t$, $\varphi_2(t) = t$. The source function is defined as follows:

$$f(x,t) = \frac{2\sqrt{t}\,x}{\sqrt{\pi}}.$$

Then, beginning from the zero function $u_0(x,t) = 0$, we obtain successively

$$
\begin{aligned}
u_1(x,t) &= t\,x + \frac{1}{3\sqrt{\pi}}\left(h\sqrt{t}\,(x^3-1)\right),\\
u_2(x,t) &= \frac{1}{120\sqrt{\pi}}\,h^2\left(40\sqrt{t}\,(x^3-1) - \sqrt{\pi}\,(x^5 - 10\,x^2 + 9)\right),\\
u_3(x,t) &= \frac{1}{120\sqrt{\pi}}\,h^2\left(40\,(h+1)\sqrt{t}\,(x^3-1) - \sqrt{\pi}\,(2\,h+1)\,(x^5 - 10\,x^2 + 9)\right),\\
u_4(x,t) &= \frac{1}{120\sqrt{\pi}}\,h^2\,(h+1)\left(40\,(h+1)\sqrt{t}\,(x^3-1) - \sqrt{\pi}\,(3\,h+1)\,(x^5 - 10\,x^2 + 9)\right),\\
u_5(x,t) &= \frac{1}{120\sqrt{\pi}}\,h^2\,(h+1)^2\left(40\,(h+1)\sqrt{t}\,(x^3-1) - \sqrt{\pi}\,(4\,h+1)\,(x^5 - 10\,x^2 + 9)\right).
\end{aligned}
$$

In general, for $m \geqslant 2$ we have the following:

$$
\begin{aligned}
u_m(x,t) = \frac{1}{120\sqrt{\pi}}\,h^2\,(h+1)^{m-3}\Big(&40\,(h+1)\sqrt{t}\,(x^3-1)\\
&- \sqrt{\pi}\,((m-1)\,h+1)\,(x^5 - 10\,x^2 + 9)\Big).
\end{aligned}
$$

Calculating the sum of the series for $m \geqslant 2$, we obtain

$$\sum_{m=2}^{\infty} u_m(x,t) = \frac{h\sqrt{t}(1-x^3)}{3\sqrt{\pi}}.$$

Adding functions to u_0 and u_1, we obtain the exact solution

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = t\,x.$$

4.3. Example 3

This time we consider Equation (2) with boundary conditions of the first kind. It is assumed $(b_1, b_2) = (0,1)$, $(0, t^*) = (0,1)$, $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = \gamma_4 = 0$, $a = 1$ and $\beta = \frac{3}{2}$. Initial and boundary conditions are described by functions $\varphi_0(x) = 0$, $\varphi_1(t) = 0$, $\varphi_2(t) = t^2$. The source function is defined as follows

$$f(x,t) = 2\,t\,x^{\frac{3}{2}} - \frac{3}{4}\sqrt{\pi}\,t^2.$$

The exact solution in this case is a function $u_e(x,t) = t^2\,x^{\frac{3}{2}}$.
Assuming $u_0(x,t) = 0$ and following the procedure, we can obtain successively

$$
\begin{aligned}
u_1(x,t) &= t^2\,x - \frac{8}{35}\,h\,t\,x\,(x^{5/2} - 1),\\
u_2(x,t) &= \frac{h\,x}{27720}\Big(32\,h\,(8\,x^{9/2} - 33\,x^2 + 25) - 33\,t\,\Big[315\sqrt{\pi}\,t\,(x-1)\\
&\quad + 8\,(24\,x^{5/2} + 35\,x^2 - 59)\Big]\Big),\\
&\ \ \vdots \quad .
\end{aligned}
$$

This time, no exact solution is found. Therefore, it is limited to an approximate solution (12). In order to determine it, the value of the parameter h has to be calculated. Figure 1 presents the chart of the logarithm of the squared residual for $n = 10$. Determining the minimum of the function E_{10}, the optimal value of the convergence control parameter equal to -0.016365 is found.

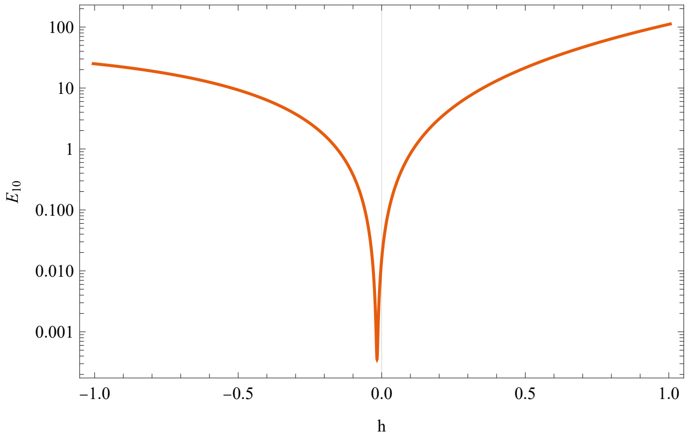

Figure 1. Logarithm of the squared residual E_{10}.

Figure 2 presents a plot of the exact solution u_e and approximate solution $\widehat{u}_{10}$. The error of the approximations, $\widehat{u}_{10}$ and $\widehat{u}_5$, are shown in Figure 3. For the $\widehat{u}_5$ solution, the maximum error was 0.107012. Whereas in the case of $\widehat{u}_{10}$ it was more than five times smaller and amounted to 0.0208275. On the other hand, the error calculated for the entire area, using the norm in $L^2(\Omega)$, for $\widehat{u}_1$ it was 0.0481733, for $\widehat{u}_5$ it was equal to 0.0337288, for $\widehat{u}_8$ it was 0.0146278 and for $\widehat{u}_{10}$ it was equal to 0.0085601. The relative values of these errors were, respectively, 5.44294%, 3.81091%, 1.65274% and 0.967176%.

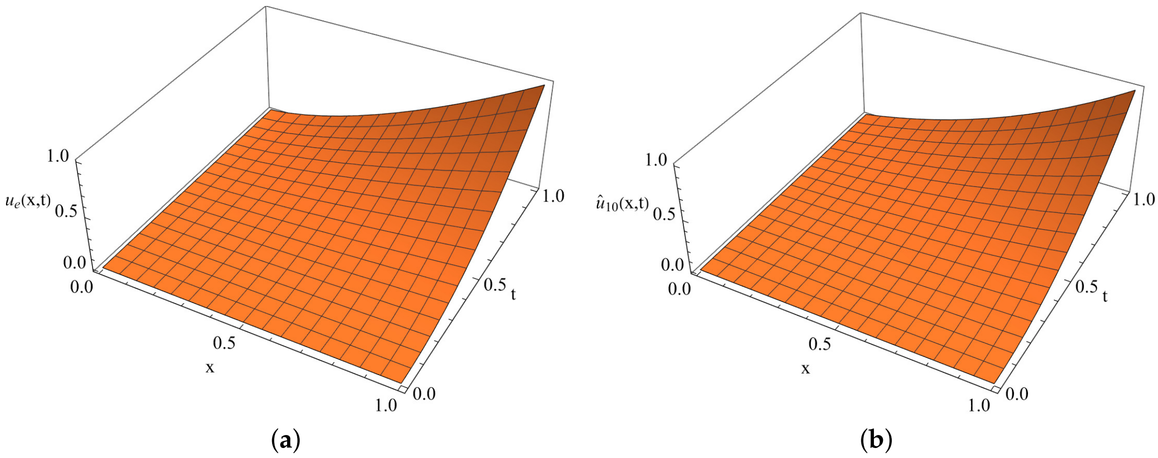

Figure 2. The exact solution u_e (**a**) and the approximated solution $\widehat{u}_{10}$ (**b**).

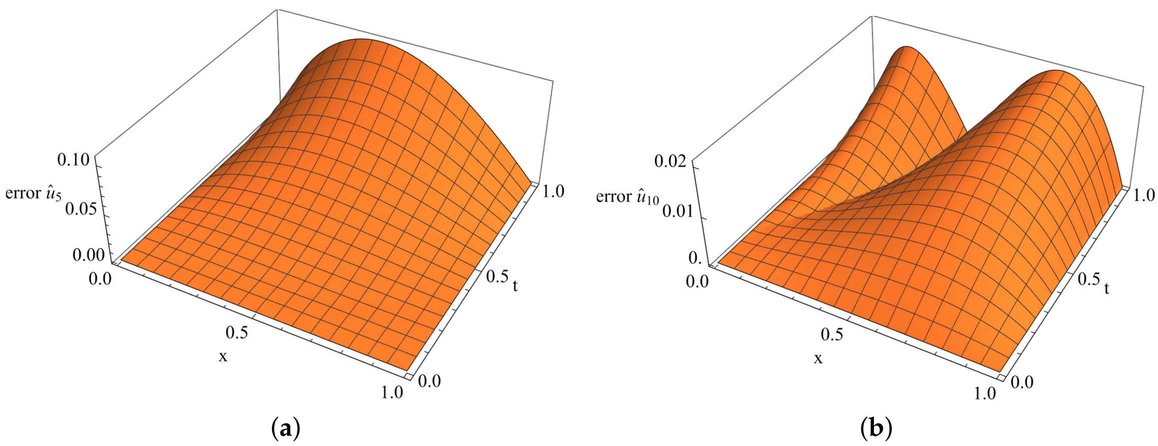

Figure 3. Approximation errors of $\widehat{u}_5$ (**a**) and $\widehat{u}_{10}$ (**b**).

Figure 4 presents the errors, which are the difference $|u_e(x,t_i) - \widehat{u}_{10}(x,t_i)|$, for solution $\widehat{u}_{10}$ and selected times t_i. The errors increased as time increased, but the growth was not very quick (see also Figure 3). During the initial time, the error was different from zero, but its value did not exceed 0.00025. Thus, the initial condition was not met exactly. Admittedly, in the considered case, the zero approximation was a function of the initial condition. However, during the calculations, we added more components to it that were non-zero for $t = 0$. Only after taking into consideration all additional components, according to Theorem 1, their sum for $t = 0$ would be zero and we would obtain an exact solution. The largest error value we obtained was for $t = 1$ and it was 0.0208275, which was reached for $x = 0.74462$.

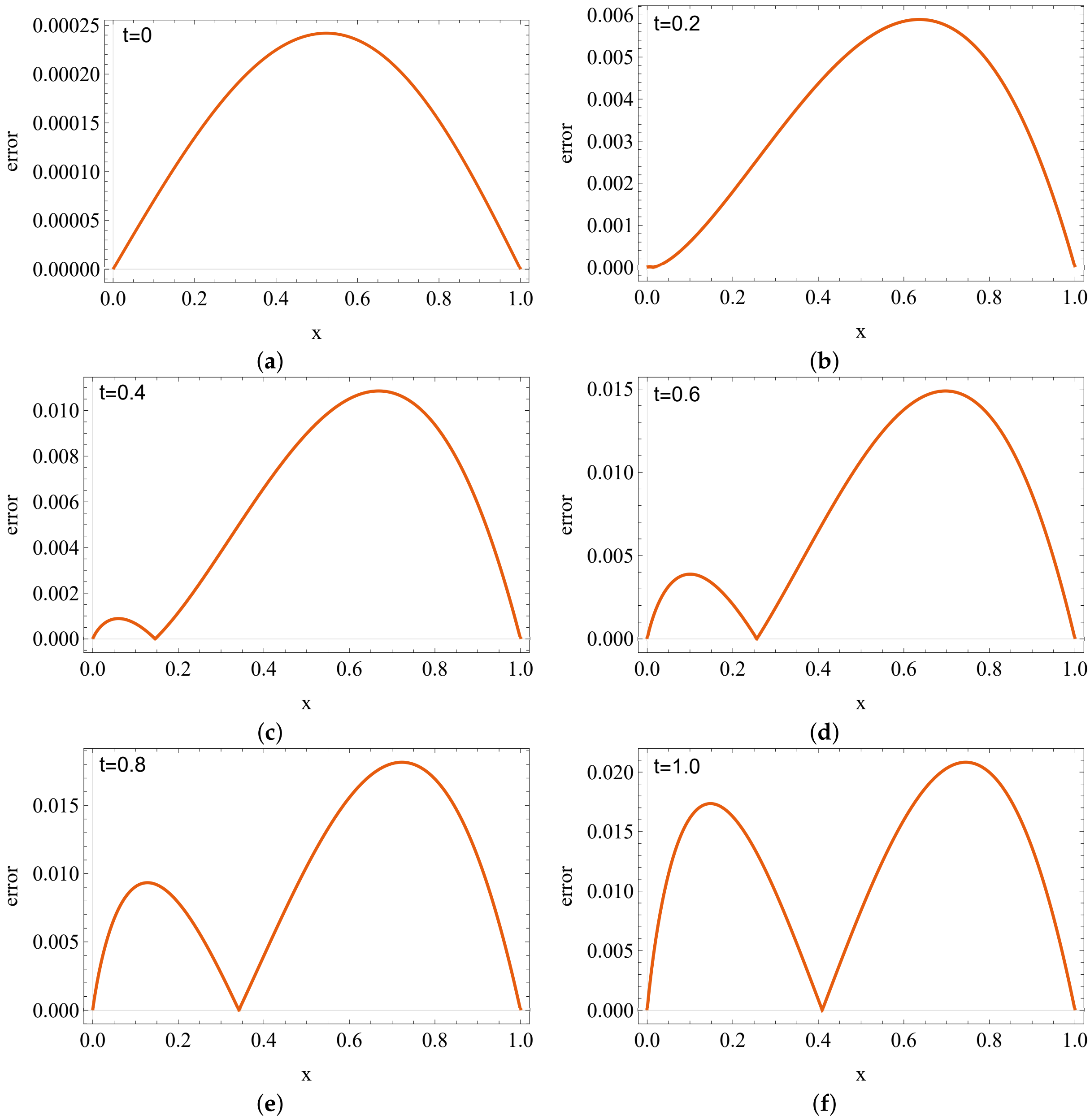

Figure 4. Errors $|u_e(x,t) - \widehat{u}_{10}(x,t)|$ for differnt times: (**a**) $t = 0$, (**b**) $t = 0.2$, (**c**) $t = 0.4$, (**d**) $t = 0.6$, (**e**) $t = 0.8$, (**f**) $t = 1$.

Figure 5 shows the error of the solution $\widehat{u}_{10}$ for the selected variable values x. At the ends of the interval, the errors were zero. This was a consequence of the construction of the conditions (30)–(33). They ensured that any approximate solution would satisfy the given boundary conditions (4) and (5). In addition, since we had boundary conditions of the first kind in the example, the function $\widehat{u}_{10}$ at the ends of the interval $(b_1, b_2) = (0, 1)$ coincided with the exact solution.

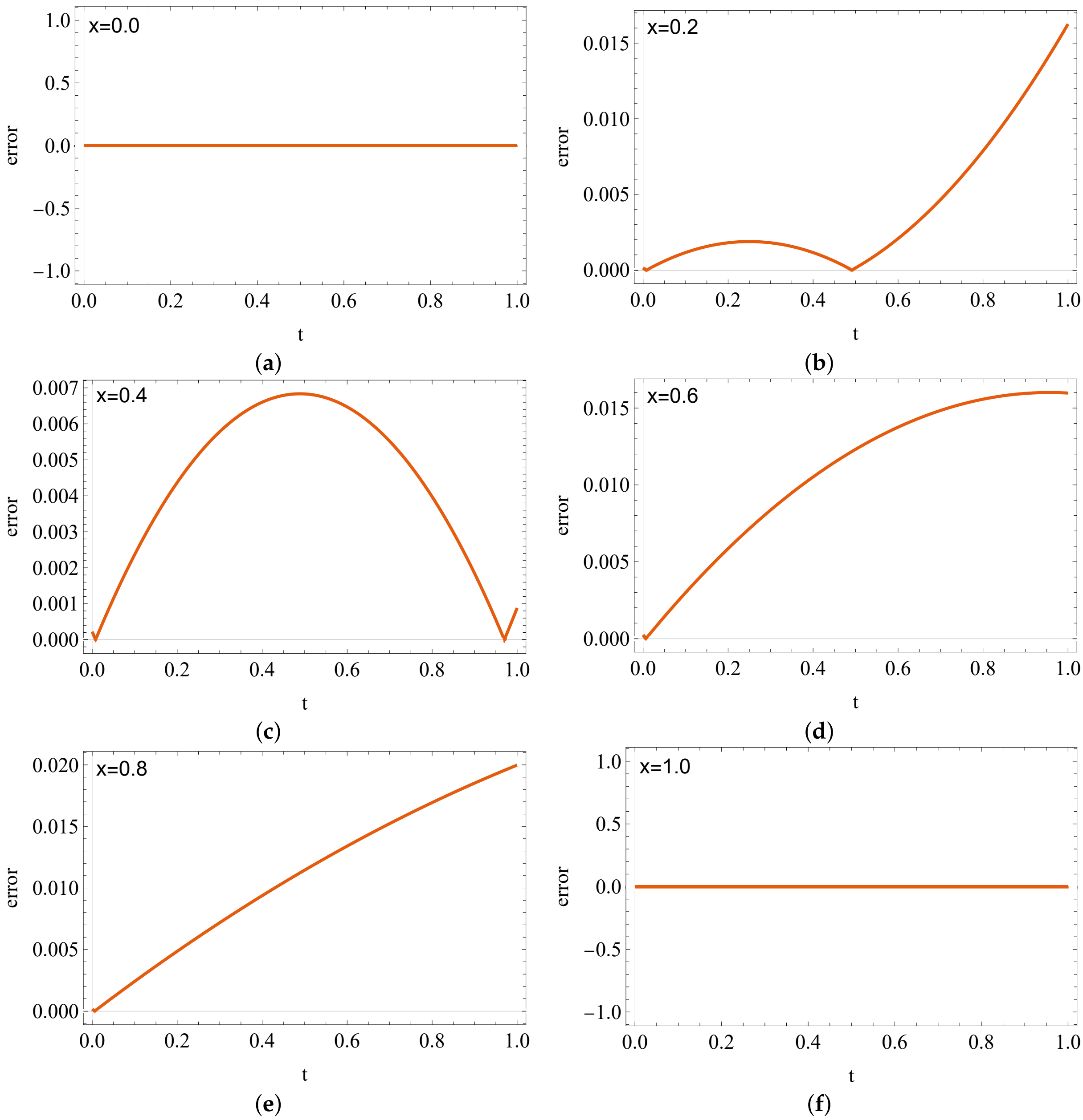

Figure 5. Errors $|u_e(x,t) - \widehat{u}_{10}(x,t)|$ for different values of variable x: (**a**) $x = 0$, (**b**) $x = 0.2$, (**c**) $x = 0.4$, (**d**) $x = 0.6$, (**e**) $x = 0.8$, (**f**) $x = 1$.

5. Conclusions

In this article, the fractional heat conduction equation was considered. Two models were considered: one with the Caputo derivative over time and the other with the Caputo derivative over space. In order to solve the relevant differential equations, the homotopy analysis method was used. The idea behind this approach was to transform the initial equation into a series whose elements satisfied the differential equation. When the series converged, the sum of the series was a solution to the corresponding differential equation. The article gives three theorems: one regarding the convergence of the series, a second providing the condition when the series is convergent and a third estimating the error of the approximate solution (in the case of taking a finite number of terms of the series).

Numerical examples are presented in which it can be observed that the considered series converged quickly. For the first two examples, an exact solution was obtained. In the third example, an approximate solution was obtained (taking 10 terms of the series). The errors of the solution obtained in this way were insignificant, which proved the effectiveness of the method. The advantage of the presented method is that it allows the discretization of the domain to be avoided and, thus, one can obtain a solution in the form of a continuous function, which can be used for further analysis.

In the future, we plan to apply the method to fractional order equations with other definitions of fractional derivatives (for example, Riemann–Liouville and Riesz). We also plan to use it in solving various types of inverse problems.

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