



Article

Abstract Univariate Neural Network Approximation Using a q -Deformed and λ -Parametrized Hyperbolic Tangent Activation Function

George A. Anastassiou

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA;
ganastss@memphis.edu

Abstract: In this work, we perform univariate approximation with rates, basic and fractional, of continuous functions that take values into an arbitrary Banach space with domain on a closed interval or all reals, by quasi-interpolation neural network operators. These approximations are achieved by deriving Jackson-type inequalities via the first modulus of continuity of the on hand function or its abstract integer derivative or Caputo fractional derivatives. Our operators are expressed via a density function based on a q -deformed and λ -parameterized hyperbolic tangent activation sigmoid function. The convergences are pointwise and uniform. The associated feed-forward neural networks are with one hidden layer.

Keywords: q -deformed ; λ -parameterized hyperbolic tangent activation function; abstract neural network approximation; abstract quasi-interpolation operator; modulus of continuity; abstract Caputo fractional derivative; fractional approximation

MSC: 26A33, 41A17, 41A25, 41A30, 46B25



Citation: Anastassiou, G.A. Abstract Univariate Neural Network Approximation Using a q -Deformed and λ -Parametrized Hyperbolic Tangent Activation Function. *Fractal Fract.* **2023**, *7*, 208. <https://doi.org/10.3390/fractalfract7030208>

Academic Editors: Predrag Vuković, Dumitru Baleanu and Devendra Kumar

Received: 23 January 2023

Revised: 8 February 2023

Accepted: 16 February 2023

Published: 21 February 2023



Copyright: © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The author of [1,2], see Chapters 2–5, was the pioneer to start neural network quantitative approximation to continuous functions by precisely defined neural network operators of Cardaliaguet–Euvrard and “Squashing” types, by using the modulus of continuity of the given function or its high order derivative, and deriving almost sharp Jackson-type inequalities. He dealt with both the univariate and multivariate cases. The defining these operators “bell-shaped” and “squashing” functions were taken with a compact support. Furthermore in [2], he provides the N th order asymptotic expansion for the error of weak approximation of these two operators to a particular natural class of smooth functions, for more visit Chapters 4–5 therein.

Coming back the author motivated by [3], who continued his research on neural networks approximation by employing and using the appropriate quasi-interpolation operators of sigmoidal and hyperbolic tangent type, which resulted in [4–8], by treating both the univariate and multivariate cases. He also completed the corresponding fractional cases [9–11].

Let h be a general sigmoid function with $h(0) = 0$, and $y = \pm 1$ the horizontal asymptotes. Of course h is strictly increasing over $\mathbb{R}$. Let the parameter $0 < r < 1$ and $x > 0$. Then clearly $-x < x$ and $-x < -rx < rx < x$, furthermore it holds $h(-x) < h(-rx) < h(rx) < h(x)$. Consequently the sigmoid $y = h(rx)$ has a graph inside the graph of $y = h(x)$, of course with the same asymptotes $y = \pm 1$. Therefore $h(rx)$ has derivatives (gradients) at more points x than $h(x)$ has different than zero or not as close

to zero, thus killing a smaller number of neurons! Furthermore, of course $h(rx)$ is more distant from $y = \pm 1$, than $h(x)$ it is. A highly desired fact in neural networks theory.

The brain non-symmetry has been clearly discovered in animals and humans in terms of structure, function and behaviors. This observation reflects evolutionary, hereditary, developmental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this paper is a specific study under this philosophy of approaching reality as close as possible.

Consequently the author here performs q -deformed and λ -parameterized hyperbolic tangent function activated neural network approximations to continuous functions over closed intervals of reals or over the whole $\mathbb{R}$ with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he deals with the related X -valued fractional approximation. All convergences here are quantitative expressed via the first modulus of continuity of the on hand function or its X -valued high order derivative, or X -valued fractional derivatives and given by almost attained Jackson-type inequalities.

Our closed intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to derive our results we describe important properties of the basic density function defining our operators which is induced by a q -deformed and λ -parameterized hyperbolic tangent sigmoid function.

Feed-forward X -valued neural networks (FNNs) with one hidden layer, the only type of networks we use in this work, are mathematically expressed by

$$S_n(x) = \sum_{j=0}^n d_j k(\langle c_j \cdot x \rangle + f_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $f_j \in \mathbb{R}$ are the thresholds, $c_j \in \mathbb{R}^s$ are the connection weights, $d_j \in X$ are the coefficients, $\langle c_j \cdot x \rangle$ is the inner product of c_j and x , and k is the activation function of the network. For more in neural networks read [12–14].

2. About q -Deformed and λ -Parameterized Hyperbolic Tangent Function $g_{q,\lambda}$

Here all this background comes from [15], [16].

We use $g_{q,\lambda}$, see (1), and exhibit that it is a sigmoid function and we will present several of its properties related to the approximation by neural network operators. It will act as activation function.

So, let us consider the function

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \quad \lambda, q > 0, \quad x \in \mathbb{R}. \quad (1)$$

We have that

$$g_{q,\lambda}(0) = \frac{1-q}{1+q}.$$

We notice also that

$$g_{q,\lambda}(-x) = \frac{e^{-\lambda x} - qe^{\lambda x}}{e^{-\lambda x} + qe^{\lambda x}} = \frac{\frac{1}{q}e^{-\lambda x} - e^{\lambda x}}{\frac{1}{q}e^{-\lambda x} + e^{\lambda x}} = -\frac{\left(e^{\lambda x} - \frac{1}{q}e^{-\lambda x}\right)}{e^{\lambda x} + \frac{1}{q}e^{-\lambda x}} = -g_{\frac{1}{q},\lambda}(x). \quad (2)$$

That is

$$g_{q,\lambda}(-x) = -g_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}, \quad (3)$$

and

$$g_{\frac{1}{q},\lambda}(x) = -g_{q,\lambda}(-x),$$

hence

$$g'_{\frac{1}{q},\lambda}(x) = g'_{q,\lambda}(-x). \quad (4)$$

It is

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} = \frac{1 - \frac{q}{e^{2\lambda x}}}{1 + \frac{q}{e^{2\lambda x}}} \xrightarrow{(x \rightarrow +\infty)} 1,$$

i.e.,

$$g_{q,\lambda}(+\infty) = 1, \quad (5)$$

Furthermore,

$$g_{q,\lambda}(x) = \frac{e^{2\lambda x} - q}{e^{2\lambda x} + q} \xrightarrow{(x \rightarrow -\infty)} \frac{-q}{q} = -1,$$

i.e.,

$$g_{q,\lambda}(-\infty) = -1. \quad (6)$$

We find that

$$g'_{q,\lambda}(x) = \frac{4q\lambda e^{2\lambda x}}{(e^{2\lambda x} + q)^2} > 0, \quad (7)$$

therefore $g_{q,\lambda}$ is strictly increasing.

Next we obtain ($x \in \mathbb{R}$)

$$g''_{q,\lambda}(x) = 8q\lambda^2 e^{2\lambda x} \left(\frac{q - e^{2\lambda x}}{(e^{2\lambda x} + q)^3} \right) \in C(\mathbb{R}). \quad (8)$$

We observe that

$$q - e^{2\lambda x} \geq 0 \Leftrightarrow q \geq e^{2\lambda x} \Leftrightarrow \ln q \geq 2\lambda x \Leftrightarrow x \leq \frac{\ln q}{2\lambda}.$$

So, in case of $x < \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave up, with $g''_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = 0$.

Furthermore, in case of $x > \frac{\ln q}{2\lambda}$, we have that $g_{q,\lambda}$ is strictly concave down.

Clearly, $g_{q,\lambda}$ is a shifted sigmoid function with $g_{q,\lambda}(0) = \frac{1-q}{1+q}$, and $g_{q,\lambda}(-x) = -g_{q^{-1},\lambda}(x)$, (a semi-odd function), see also [17].

By $1 > -1$, $x+1 > x-1$, we consider the function

$$M_{q,\lambda}(x) := \frac{1}{4}(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)) > 0, \quad (9)$$

$\forall x \in \mathbb{R}; q, \lambda > 0$. Notice that $M_{q,\lambda}(\pm\infty) = 0$, so the x -axis is horizontal asymptote.

We have that

$$\begin{aligned} M_{q,\lambda}(-x) &= \frac{1}{4}(g_{q,\lambda}(-x+1) - g_{q,\lambda}(-x-1)) = \\ &= \frac{1}{4}(g_{q,\lambda}(-(x-1)) - g_{q,\lambda}(-(x+1))) = \\ &= \frac{1}{4}\left(-g_{\frac{1}{q},\lambda}(x-1) + g_{\frac{1}{q},\lambda}(x+1)\right) = \\ &= \frac{1}{4}\left(g_{\frac{1}{q},\lambda}(x+1) - g_{\frac{1}{q},\lambda}(x-1)\right) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}. \end{aligned} \quad (10)$$

Thus,

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \quad \forall x \in \mathbb{R}; q, \lambda > 0, \quad (11)$$

a deformed symmetry.

Next, we have that

$$M'_{q,\lambda}(x) = \frac{1}{4} \left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1) \right), \quad \forall x \in \mathbb{R}. \quad (12)$$

Let $x < \frac{\ln q}{2\lambda} - 1$, then $x-1 < x+1 < \frac{\ln q}{2\lambda}$ and $g'_{q,\lambda}(x+1) > g'_{q,\lambda}(x-1)$ (by $g_{q,\lambda}$ being strictly concave up for $x < \frac{\ln q}{2\lambda}$), that is $M'_{q,\lambda}(x) > 0$. Hence, $M_{q,\lambda}$ is strictly increasing over $(-\infty, \frac{\ln q}{2\lambda} - 1)$.

Let now $x-1 > \frac{\ln q}{2\lambda}$, then $x+1 > x-1 > \frac{\ln q}{2\lambda}$, and $g'_{q,\lambda}(x+1) < g'_{q,\lambda}(x-1)$, that is $M'_{q,\lambda}(x) < 0$.

Therefore $M_{q,\lambda}$ is strictly decreasing over $(\frac{\ln q}{2\lambda} + 1, +\infty)$.

Let us next consider, $\frac{\ln q}{2\lambda} - 1 \leq x \leq \frac{\ln q}{2\lambda} + 1$. We have that

$$M''_{q,\lambda}(x) = \frac{1}{4} \left(g''_{q,\lambda}(x+1) - g''_{q,\lambda}(x-1) \right) = 2q\lambda^2 \left[e^{2\lambda(x+1)} \left(\frac{q - e^{2\lambda(x+1)}}{(e^{2\lambda(x+1)} + q)^3} \right) - e^{2\lambda(x-1)} \left(\frac{q - e^{2\lambda(x-1)}}{(e^{2\lambda(x-1)} + q)^3} \right) \right]. \quad (13)$$

By $\frac{\ln q}{2\lambda} - 1 \leq x \Leftrightarrow \frac{\ln q}{2\lambda} \leq x+1 \Leftrightarrow \ln q \leq 2\lambda(x+1) \Leftrightarrow q \leq e^{2\lambda(x+1)} \Leftrightarrow q - e^{2\lambda(x+1)} \leq 0$.

By $x \leq \frac{\ln q}{2\lambda} + 1 \Leftrightarrow x-1 \leq \frac{\ln q}{2\lambda} \Leftrightarrow 2\lambda(x-1) \leq \ln q \Leftrightarrow e^{2\lambda(x-1)} \leq q \Leftrightarrow q - e^{2\lambda(x-1)} \geq 0$.

Clearly by (13) we obtain that $M''_{q,\lambda}(x) \leq 0$, for $x \in [\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1]$.

More precisely $M_{q,\lambda}$ is concave down over $[\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1]$, and strictly concave down over $(\frac{\ln q}{2\lambda} - 1, \frac{\ln q}{2\lambda} + 1)$.

Consequently $M_{q,\lambda}$ has a bell-type shape over $\mathbb{R}$.

Of course it holds $M''_{q,\lambda}(\frac{\ln q}{2\lambda}) < 0$.

At $x = \frac{\ln q}{2\lambda}$, we have

$$M'_{q,\lambda}(x) = \frac{1}{4} \left(g'_{q,\lambda}(x+1) - g'_{q,\lambda}(x-1) \right) = q\lambda \left(\frac{e^{2\lambda(x+1)}}{(e^{2\lambda(x+1)} + q)^2} - \frac{e^{2\lambda(x-1)}}{(e^{2\lambda(x-1)} + q)^2} \right). \quad (14)$$

Thus,

$$\begin{aligned} M'_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) &= q\lambda \left(\frac{e^{2\lambda(\frac{\ln q}{2\lambda}+1)}}{(e^{2\lambda(\frac{\ln q}{2\lambda}+1)} + q)^2} - \frac{e^{2\lambda(\frac{\ln q}{2\lambda}-1)}}{(e^{2\lambda(\frac{\ln q}{2\lambda}-1)} + q)^2} \right) = \\ &= q\lambda \left(\frac{qe^{2\lambda}}{(qe^{2\lambda} + q)^2} - \frac{qe^{-2\lambda}}{(qe^{-2\lambda} + q)^2} \right) = \\ &= \lambda \left(\frac{e^{2\lambda}}{(e^{2\lambda} + 1)^2} - \frac{e^{-2\lambda}}{(e^{-2\lambda} + 1)^2} \right) = \\ &= \lambda \left(\frac{e^{2\lambda}(e^{-2\lambda} + 1)^2 - e^{-2\lambda}(e^{2\lambda} + 1)^2}{(e^{2\lambda} + 1)^2(e^{-2\lambda} + 1)^2} \right) = 0. \end{aligned} \quad (15)$$

That is, $\frac{\ln q}{2\lambda}$ is the only critical number of $M_{q,\lambda}$ over $\mathbb{R}$. Hence at $x = \frac{\ln q}{2\lambda}$, $M_{q,\lambda}$ achieves its global maximum, which is

$$\begin{aligned} M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) &= \frac{1}{4} \left[g_{q,\lambda}\left(\frac{\ln q}{2\lambda} + 1\right) - g_{q,\lambda}\left(\frac{\ln q}{2\lambda} - 1\right) \right] = \\ &= \frac{1}{4} \left[\left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda} + 1\right)} - qe^{-\lambda\left(\frac{\ln q}{2\lambda} + 1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda} + 1\right)} + qe^{-\lambda\left(\frac{\ln q}{2\lambda} + 1\right)}} \right) - \left(\frac{e^{\lambda\left(\frac{\ln q}{2\lambda} - 1\right)} - qe^{-\lambda\left(\frac{\ln q}{2\lambda} - 1\right)}}{e^{\lambda\left(\frac{\ln q}{2\lambda} - 1\right)} + qe^{-\lambda\left(\frac{\ln q}{2\lambda} - 1\right)}} \right) \right] = \\ &= \frac{1}{4} \left[\left(\frac{\sqrt{q}e^{\lambda} - qq^{-\frac{1}{2}}e^{-\lambda}}{\sqrt{q}e^{\lambda} + qq^{-\frac{1}{2}}e^{-\lambda}} \right) - \left(\frac{\sqrt{q}e^{-\lambda} - qq^{-\frac{1}{2}}e^{\lambda}}{\sqrt{q}e^{-\lambda} + qq^{-\frac{1}{2}}e^{\lambda}} \right) \right] = \\ &= \frac{1}{4} \left[\left(\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}} \right) - \left(\frac{e^{-\lambda} - e^{\lambda}}{e^{-\lambda} + e^{\lambda}} \right) \right] = \\ &= \frac{1}{4} \left[\frac{2(e^{\lambda} - e^{-\lambda})}{e^{\lambda} + e^{-\lambda}} \right] = \frac{1}{2} \left(\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}} \right) = \frac{\tanh(\lambda)}{2}. \end{aligned} \quad (16)$$

Conclusion: The maximum value of $M_{q,\lambda}$ is

$$M_{q,\lambda}\left(\frac{\ln q}{2\lambda}\right) = \frac{\tanh(\lambda)}{2}, \quad \lambda > 0. \quad (17)$$

We mention

Theorem 1 ([16]). *We have that*

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \quad \forall x \in \mathbb{R}, \forall \lambda, q > 0. \quad (18)$$

Thus,

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(nx-i) = 1, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}. \quad (19)$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \quad \forall x \in \mathbb{R}. \quad (20)$$

However, $M_{\frac{1}{q},\lambda}(x-i) \stackrel{(11)}{=} M_{q,\lambda}(i-x), \forall x \in \mathbb{R}$.

Hence,

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i-x) = 1, \quad \forall x \in \mathbb{R}, \quad (21)$$

and

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(i+x) = 1, \quad \forall x \in \mathbb{R}. \quad (22)$$

It follows

Theorem 2 ([16]). *It holds*

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)dx = 1, \quad \lambda, q > 0. \quad (23)$$

So that $M_{q,\lambda}$ is a density function on $\mathbb{R}; \lambda, q > 0$.

We need the following result

Theorem 3 ([16]). Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2; q, \lambda > 0$. Then,

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = T e^{-2\lambda n^{(1-\alpha)}}, \quad (24)$$

where $T := \max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

Theorem 4 ([16]). Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0, \lambda > 0$, we consider the number $\lambda_q > z_0 > 0$ with $M_{q,\lambda}(z_0) = M_{q,\lambda}(0)$ and $\lambda_q > 1$. Then,

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{M_{q,\lambda}(\lambda_q)}, \frac{1}{M_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Delta(q). \quad (25)$$

We also mention

Remark 1 ([16]). (i) We have that

$$\lim_{n \rightarrow +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \neq 1, \text{ for at least some } x \in [a, b], \quad (26)$$

where $\lambda, q > 0$.

(ii) Let $[a, b] \subset \mathbb{R}$. For large n we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Furthermore, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k) \leq 1. \quad (27)$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 1. Let $f \in C([a, b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X -valued linear neural network operators

$$H_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx-k)}, \quad x \in [a, b]; q > 0, q \neq 1. \quad (28)$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Furthermore, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. The same H_n is used for real valued functions. We study here the pointwise and uniform convergence of $H_n(f, x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx-k), \quad (29)$$

(the same H_n^* can be defined for real valued functions) that is

$$H_n(f, x) := \frac{H_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k)}. \quad (30)$$

So that

$$\begin{aligned} H_n(f, x) - f(x) &= \frac{H_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k)} - f(x) = \\ &= \frac{H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k)}. \end{aligned} \quad (31)$$

Consequently, we derive that

$$\begin{aligned} \|H_n(f, x) - f(x)\| &\leq \Delta(q) \left\| H_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) \right) \right\| = \\ &= \Delta(q) \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) M_{q,\lambda}(nx - k) \right\|, \end{aligned} \quad (32)$$

where $\Delta(q)$ as in (25).

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a, b], X)$ the first modulus of continuity

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in [a, b] \\ |x - y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \quad (33)$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X -valued), and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a, b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \rightarrow 0} \omega_1(f, \delta) = 0$, see [18].

We make

Definition 2. When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define

$$\overline{H}_n(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx - k), \quad (34)$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X -valued quasi-interpolation neural network operator.

We give

Remark 2. We have that

$$\left\| f\left(\frac{k}{n}\right) \right\| \leq \|f\|_{\infty, \mathbb{R}} < +\infty,$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| M_{q,\lambda}(nx - k) \leq \|f\|_{\infty, \mathbb{R}} M_{q,\lambda}(nx - k) \quad (35)$$

and

$$\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| M_{q,\lambda}(nx-k) \leq \|f\|_{\infty, \mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} M_{q,\lambda}(nx-k) \right),$$

and, finally,

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| M_{q,\lambda}(nx-k) \leq \|f\|_{\infty, \mathbb{R}}, \quad (36)$$

a convergent series in $\mathbb{R}$.

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| M_{q,\lambda}(nx-k)$ is absolutely convergent in X , hence it is convergent in X and $\overline{H_n}(f, x) \in X$. We denote by $\|f\|_{\infty} := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a, b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3. Main Results

We present a set of X -valued neural network approximations to a function given with rates.

Theorem 5. Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q > 0$, $q \neq 1$, $x \in [a, b]$. Then,

$$(i) \quad \|H_n(f, x) - f(x)\| \leq \Delta(q) \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_{\infty} T e^{-2\lambda n^{(1-\alpha)}} \right] =: \tau, \quad (37)$$

where T as in (24),

and

(ii)

$$\|H_n(f) - f\|_{\infty} \leq \tau. \quad (38)$$

We obtain that $\lim_{n \rightarrow \infty} H_n(f) = f$, pointwise and uniformly.

Proof. We see that

$$\begin{aligned} & \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) M_{q,\lambda}(nx-k) \right\| \leq \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) = \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) + \\ & \left\{ \begin{array}{l} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{array} \right. \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) \leq \\ & \left\{ \begin{array}{l} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| > \frac{1}{n^\alpha} \end{array} \right. \\ & \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \omega_1\left(f, \left| \frac{k}{n} - x \right| \right) M_{q,\lambda}(nx-k) + \\ & \left\{ \begin{array}{l} k = \lceil na \rceil \\ \left| \frac{k}{n} - x \right| \leq \frac{1}{n^\alpha} \end{array} \right. \end{aligned} \quad (39)$$

$$\begin{aligned}
& 2\|f\|_{\infty} \sum_{\substack{k = \lceil na \rceil \\ |k - nx| > n^{1-\alpha}}}^{\lfloor nb \rfloor} M_{q,\lambda}(nx - k) \leq \\
& \omega_1\left(f, \frac{1}{n^{\alpha}}\right) \sum_{\substack{k = -\infty \\ \left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} M_{q,\lambda}(nx - k) + \\
& 2\|f\|_{\infty} \sum_{\substack{k = -\infty \\ |k - nx| > n^{1-\alpha}}}^{\infty} M_{q,\lambda}(nx - k) \stackrel{\text{(by Theorem 3)}}{\leq} \\
& \omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} Te^{-2\lambda n^{(1-\alpha)}}
\end{aligned} \tag{40}$$

That is

$$\begin{aligned}
& \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) M_{q,\lambda}(nx - k) \right\| \leq \\
& \omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} Te^{-2\lambda n^{(1-\alpha)}}.
\end{aligned} \tag{41}$$

Using the last equality we derive (37). $\square$

Next we give

Theorem 6. Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $q > 0$, $q \neq 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then

$$(i) \quad \|\overline{H}_n(f, x) - f(x)\| \leq \omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} Te^{-2\lambda n^{(1-\alpha)}} =: \gamma, \tag{42}$$

and

$$(ii) \quad \|\overline{H}_n(f) - f\|_{\infty} \leq \gamma. \tag{43}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we obtain $\lim_{n \rightarrow \infty} \overline{H}_n(f) = f$, pointwise and uniformly.

Proof. We observe that

$$\begin{aligned}
\|\overline{H}_n(f, x) - f(x)\| & \stackrel{(18)}{=} \left\| \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) M_{q,\lambda}(nx - k) - f(x) \sum_{k=-\infty}^{\infty} M_{q,\lambda}(nx - k) \right\| = \\
& \left\| \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) M_{q,\lambda}(nx - k) \right\| \leq \\
& \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx - k) = \\
& \sum_{\substack{k = -\infty \\ \left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx - k) +
\end{aligned}$$

$$\begin{aligned}
& \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| M_{q,\lambda}(nx-k) \leq \\
& \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\infty} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) M_{q,\lambda}(nx-k) + \\
& 2\|f\|_\infty \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\infty} M_{q,\lambda}(nx-k) \leq \\
& \omega_1\left(f, \frac{1}{n^\alpha}\right) \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\infty} M_{q,\lambda}(nx-k) + 2\|f\|_\infty T e^{-2\lambda n^{(1-\alpha)}} \leq \\
& \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty T e^{-2\lambda n^{(1-\alpha)}}, \tag{44}
\end{aligned}$$

proving the claim. $\square$

We need the X-valued Taylor's formula in an appropriate form:

Theorem 7 ([19,20]). Let $N \in \mathbb{N}$, and $f \in C^N([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and X is a Banach space. Let any $x, y \in [a, b]$. Then,

$$f(x) = \sum_{i=0}^N \frac{(x-y)^i}{i!} f^{(i)}(y) + \frac{1}{(N-1)!} \int_y^x (x-t)^{N-1} \left(f^{(N)}(t) - f^{(N)}(y) \right) dt. \tag{46}$$

The derivatives $f^{(i)}$, $i \in \mathbb{N}$, are defined like the numerical ones, see [21], p. 83. The integral $\int_y^x$ in (46) is of Bochner type, see [22].

By [20,23] we have that: if $f \in C([a, b], X)$, then $f \in L_\infty([a, b], X)$ and $f \in L_1([a, b], X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f .

Theorem 8. Let $f \in C^N([a, b], X)$, $n, N \in \mathbb{N}$, $q > 0$, $q \neq 1$, $0 < \alpha < 1$, $x \in [a, b]$ and $n^{1-\alpha} > 2$. Then,

(i)

$$\begin{aligned}
\|H_n(f, x) - f(x)\| & \leq \Delta(q) \left\{ \sum_{j=1}^N \frac{\|f^{(j)}(x)\|}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j T e^{-2\lambda n^{(1-\alpha)}} \right] + \right. \\
& \left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{2\|f^{(N)}\|_\infty (b-a)^N}{N!} T e^{-2\lambda n^{(1-\alpha)}} \right] \right\}, \tag{47}
\end{aligned}$$

(ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, \dots, N$, for some $x_0 \in [a, b]$, it holds

$$\|H_n(f, x_0) - f(x_0)\| \leq \Delta(q).$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N}{N!} Te^{-2\lambda n^{(1-\alpha)}} \right\}, \quad (48)$$

and

(iii)

$$\|H_n(f) - f\|_\infty \leq \Delta(q) \left\{ \sum_{j=1}^N \frac{\|f^{(j)}\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j Te^{-2\lambda n^{(1-\alpha)}} \right] + \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \|f^{(N)}\|_\infty (b-a)^N}{N!} Te^{-2\lambda n^{(1-\alpha)}} \right] \right\}. \quad (49)$$

Again we obtain $\lim_{n \rightarrow \infty} H_n(f) = f$, pointwise and uniformly.

Proof. It is lengthy, and as similar to [24] is omitted. $\square$

All integrals from now on are of Bochner type [22].

We need

Definition 3 ([20]). Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a, b] \rightarrow X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo–Bochner left fractional derivative of order α :

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad \forall x \in [a, b]. \quad (50)$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary X -valued derivative (defined similar to numerical one, see [21], p. 83), and also set $D_{*a}^0 f := f$.

By [19], $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^\alpha f \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_\infty([a, b], X)} < \infty$, then by [23], $D_{*a}^\alpha f \in C([a, b], X)$, hence $\|D_{*a}^\alpha f\| \in C([a, b])$.

We mention

Definition 4 ([19]). Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \rightarrow X$. We call the Caputo–Bochner right fractional derivative of order α :

$$(D_{b-}^\alpha f)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall x \in [a, b]. \quad (51)$$

We observe that $(D_{b-}^m f)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $(D_{b-}^0 f)(x) = f(x)$.

By [19], $(D_{b-}^\alpha f)(x)$ exists almost everywhere on $[a, b]$ and $(D_{b-}^\alpha f) \in L_1([a, b], X)$.

If $\|f^{(m)}\|_{L_\infty([a, b], X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [19], $D_{b-}^\alpha f \in C([a, b], X)$, hence $\|D_{b-}^\alpha f\| \in C([a, b])$.

We make

Remark 3 ([18]). Let $f \in C^{n-1}([a, b])$, $f^{(n)} \in L_\infty([a, b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then,

$$\|D_{*a}^\nu f(x)\| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu}, \quad \forall x \in [a, b]. \quad (52)$$

Thus, we observe

$$\begin{aligned} \omega_1(D_{*a}^\nu f, \delta) &= \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|D_{*a}^\nu f(x) - D_{*a}^\nu f(y)\| \leq \\ &\sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\|f^{(n)}\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu} + \frac{\|f^{(n)}\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(y-a)^{n-\nu} \right) \\ &\leq \frac{2\|f^{(n)}\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \end{aligned} \quad (53)$$

Consequently,

$$\omega_1(D_{*a}^\nu f, \delta) \leq \frac{2\|f^{(n)}\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \quad (54)$$

Similarly, let $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_\infty([a, b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1(D_{b-}^\alpha f, \delta) \leq \frac{2\|f^{(m)}\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \quad (55)$$

So for $f \in C^{m-1}([a, b])$, $f^{(m)} \in L_\infty([a, b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1(D_{*x_0}^\alpha f, \delta)_{[x_0,b]} \leq \frac{2\|f^{(m)}\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}, \quad (56)$$

and

$$\sup_{x_0 \in [a,b]} \omega_1(D_{x_0-}^\alpha f, \delta)_{[a,x_0]} \leq \frac{2\|f^{(m)}\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \quad (57)$$

By [20] we obtain that $D_{*x_0}^\alpha f \in C([x_0, b], X)$, and by [19] we obtain that $D_{x_0-}^\alpha f \in C([a, x_0], X)$.

We present the following X -valued fractional approximation result by neural networks.

Theorem 9. Let $\alpha > 0$, $q > 0$, $q \neq 1$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a, b], X)$, $0 < \beta < 1$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then,

(i)

$$\begin{aligned} &\left\| H(f, x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n((\cdot - x)^j)(x) - f(x) \right\| \leq \\ &\frac{\Delta(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right. \\ &\left. Te^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty,[a,x]}(x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty,[x,b]}(b-x)^\alpha \right) \right\}, \end{aligned} \quad (58)$$

(ii) if $f^{(j)}(x) = 0$, for $j = 1, \dots, N-1$, we have

$$\|H_n(f, x) - f(x)\| \leq \frac{\Delta(q)}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + Te^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\}, \quad (59)$$

(iii)

$$\|H_n(f, x) - f(x)\| \leq \Delta(q) \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}(x)\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j Te^{-2\lambda n^{(1-\beta)}} \right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + Te^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty, [a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty, [x,b]} (b-x)^\alpha \right) \right\} \right\}, \quad (60)$$

$\forall x \in [a, b]$,

and

(iv)

$$\|H_n f - f\|_\infty \leq \Delta(q) \left\{ \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j Te^{-2\lambda n^{(1-\beta)}} \right\} + \frac{1}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + Te^{-2\lambda n^{(1-\beta)}} (b-a)^\alpha \left(\sup_{x \in [a,b]} \|D_{x-}^\alpha f\|_{\infty, [a,x]} + \sup_{x \in [a,b]} \|D_{*x}^\alpha f\|_{\infty, [x,b]} \right) \right\} \right\}. \quad (61)$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X -valued fractionally type pointwise and uniform convergence with rates of $H_n \rightarrow I$ the unit operator, as $n \rightarrow \infty$.

Proof. The proof is very lengthy and similar to [24]; therefore, it is omitted. $\square$

Next we apply Theorem 9 for $N = 1$.

Theorem 10. Let $0 < \alpha, \beta < 1$, $q > 0$, $q \neq 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

(i)

$$\|H_n(f, x) - f(x)\| \leq$$

$$\frac{\Delta(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + Te^{-2\lambda n^{(1-\beta)}} \left(\|D_{x-}^\alpha f\|_{\infty,[a,x]} (x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (62)$$

and
(ii)

$$\|H_n f - f\|_\infty \leq \frac{\Delta(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + (b-a)^\alpha Te^{-2\lambda n^{(1-\beta)}} \left(\sup_{x \in [a,b]} \|D_{x-}^\alpha f\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \|D_{*x}^\alpha f\|_{\infty,[x,b]} \right) \right\}. \quad (63)$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 1. Let $0 < \beta < 1$, $q > 0$, $q \neq 1$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then

(i)

$$\|H_n(f, x) - f(x)\| \leq \frac{2\Delta(q)}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + Te^{-2\lambda n^{(1-\beta)}} \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \quad (64)$$

and
(ii)

$$\|H_n f - f\|_\infty \leq \frac{2\Delta(q)}{\sqrt{\pi}} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \sqrt{(b-a)} Te^{-2\lambda n^{(1-\beta)}} \left(\sup_{x \in [a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \quad (65)$$

We make

Remark 4. Some convergence analysis follows based on Corollary 1.

Let $0 < \beta < 1$, $\lambda > 0$, $f \in C^1([a, b], X)$, $x \in [a, b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (65). Assume that

$$\omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^\beta}\right)_{[a,x]} \leq \frac{R_1}{n^\beta}, \quad (66)$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^\beta}\right)_{[x,b]} \leq \frac{R_2}{n^\beta}, \quad (67)$$

$\forall x \in [a, b]$, $\forall n \in \mathbb{N}$, where $R_1, R_2 > 0$.

Then it holds

$$\frac{\left[\sup_{x \in [a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^\beta}\right)_{[x,b]} \right]}{n^{\frac{\beta}{2}}} \leq \frac{\frac{(R_1+R_2)}{n^\beta}}{n^{\frac{\beta}{2}}} = \frac{(R_1+R_2)}{n^{\frac{3\beta}{2}}} = \frac{R}{n^{\frac{3\beta}{2}}}, \quad (68)$$

where $R := R_1 + R_2 > 0$.

The other summand of the right hand side of (65), for large enough n , converges to zero at the speed $e^{-2\lambda n^{(1-\beta)}}$, so it is about $Ae^{-2\lambda n^{(1-\beta)}}$, where $A > 0$ is a constant.

Then, for large enough $n \in \mathbb{N}$, by (65), (68) and the above comment, we obtain that

$$\|H_n f - f\|_\infty \leq \frac{B}{n^{\frac{3\beta}{2}}}, \quad (69)$$

where $B > 0$, converging to zero at the high speed of $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 5, for $f \in C([a, b], X)$ and for large enough $n \in \mathbb{N}$, the speed is $\frac{1}{n^\beta}$. So by (69), $\|H_n f - f\|_\infty$ converges much faster to zero. The last comes because we assumed differentiability of f . Notice that in Corollary 1 no initial condition is assumed.

Funding: This research received no external funding

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declare no conflict of interest.

References

1. Anastassiou, G.A. Rate of convergence of some neural network operators to the unit-univariate case. *J. Math. Anal. Appl.* **1997**, *212*, 237–262. [\[CrossRef\]](#)
2. Anastassiou, G.A. *Quantitative Approximations*; Chapman & Hall/CRC: Boca Raton, FL, USA; New York, NY, USA, 2001.
3. Chen, Z.; Cao, F. The approximation operators with sigmoidal functions. *Comput. Math. Appl.* **2009**, *58*, 758–765. [\[CrossRef\]](#)
4. Anastassiou, G.A. Univariate hyperbolic tangent neural network approximation. *Math. Comput. Model.* **2011**, *53*, 1111–1132. [\[CrossRef\]](#)
5. Anastassiou, G.A. Multivariate hyperbolic tangent neural network approximation. *Comput. Math.* **2011**, *61*, 809–821.
6. Anastassiou, G.A. Multivariate sigmoidal neural network approximation. *Neural Netw.* **2011**, *24*, 378–386. [\[CrossRef\]](#)
7. Anastassiou, G.A. Intelligent Systems: Approximation by Artificial Neural Networks. In *Intelligent Systems Reference Library*; Springer: Berlin/Heidelberg, Germany, 2011; Volume 19.
8. Anastassiou, G.A. Univariate sigmoidal neural network approximation. *J. Comput. Anal. Appl.* **2012**, *14*, 659–690.
9. Anastassiou, G.A. Fractional neural network approximation. *Comput. Math. Appl.* **2012**, *64*, 1655–1676. [\[CrossRef\]](#)
10. Anastassiou, G.A. *Intelligent Systems II: Complete Approximation by Neural Network Operators*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2016.
11. Anastassiou, G.A. *Nonlinearity: Ordinary and Fractional Approximations by Sublinear and Max-Product Operators*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2018.
12. Haykin, S. *Neural Networks: A Comprehensive Foundation*, 2nd ed.; Prentice Hall: New York, NY, USA, 1998.

13. McCulloch, W.; Pitts, W. A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophys.* **1943**, *7*, 115–133. [[CrossRef](#)]
14. Mitchell, T.M. *Machine Learning*; WCB-McGraw-Hill: New York, NY, USA, 1997.
15. Anastassiou, G.A. q -Deformed and λ -parametrized hyperbolic tangent function based Banach space valued multivariate multi layer neural network approximations. *Ann. Univ. Sci. Bp. Sect. Comp.* **2023**, *in press*.
16. El-Shehawy, S.A.; Abdel-Salam, E.A.-B. The q -deformed hyperbolic Secant family. *Intern. J. Appl. Math. Stat.* **2012**, *29*, 51–62.
17. Anastassiou, G.A. General sigmoid based Banach space valued neural network approximation. *J. Comput. Anal. Appl.* **2023**, *31*, 520–534.
18. Anastassiou, G.A. Vector fractional Korovkin type Approximations. *Dyn. Syst. Appl.* **2017**, *26*, 81–104.
19. Anastassiou, G.A. Strong Right Fractional Calculus for Banach space valued functions. *Rev. Proyec.* **2017**, *36*, 149–186. [[CrossRef](#)]
20. Anastassiou, G.A. A strong Fractional Calculus Theory for Banach space valued functions. *Nonlinear Funct. Anal. Appl.* **2017**, *22*, 495–524.
21. Shilov, G.E. *Elementary Functional Analysis*, Dover Publications, Inc.: New York, NY, USA, 1996.
22. Mikusinski, J. *The Bochner integral*, Academic Press: New York, NY, USA, 1978.
23. Kreuter, M. Sobolev Spaces of Vector-Valued Functions. Master's Thesis. Ulm University, Ulm, Germany, 2015.
24. Anastassiou, G.A.; Karateke, S. Parametrized hyperbolic tangent induced Banach space valued ordinary and fractional neural network approximation. *Progr. Fract. Differ. Appl.* **2023**, *in press*.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.