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Infinitely Many Small Energy Solutions to Schrödinger-Kirchhoff Type Problems Involving the Fractional $r(\cdot)$-Laplacian in $\mathbb{R}^N$

Yun-Ho Kim

Department of Mathematics Education, Sangmyung University, Seoul 110-743, Republic of Korea; kyh1213@smu.ac.kr

Abstract: This paper is concerned with the existence result of a sequence of infinitely many small energy solutions to the fractional $r(\cdot)$-Laplacian equations of Kirchhoff–Schrödinger type with concave–convex nonlinearities when the convex term does not require the Ambrosetti–Rabinowitz condition. The aim of the present paper, under suitable assumptions on a nonlinear term, is to discuss the multiplicity result of non-trivial solutions by using the dual fountain theorem as the main tool.

Keywords: fractional $r(x)$-Laplacian; variable exponent fractional Sobolev spaces; weak solution; dual fountain theorem

MSC: 35B33; 35D30; 35J20; 35J60; 35J66

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1. Introduction

The investigation into problems of differential equations and variational problems with nonstandard growth conditions has gained a great deal of attention in recent decades because they can be developed in light of a pure or applied mathematical perspective to illustrate some concrete phenomena arising from elastic mechanics, electro-rheological fluid (“smart fluids”), plasticity theory, plasma physics, and image processing, etc. We refer the readers to [1–5] and references therein for more details.

On the other hand, recently, a great number of works have been devoted to the fractional nonlocal problem, because the appearance of non-local terms in equations is important for many physical applications as well as causing some difficulties and challenges from a mathematical perspective; see [6–10].

Very recently, many authors in [11–24] have studied non-local elliptic problems involving the fractional $r(\cdot)$-Laplacian. Kaufmann et al. [22] has provided some properties on a class of fractional variable exponent Sobolev spaces that is associated with a fractional variable exponent operator. The authors in [14] gave further elementary properties, both on the functional corresponding to a nonlocal operator with variable exponent and its solution space. As applications, they presented that equations involving the fractional $r(\cdot)$-Laplacian admits at least one nontrivial solution. Based on these recent works, Ho–Kim [20] provided further fundamental embedding results for the fractional variable exponent Sobolev space. Using these consequences, they obtained the L^∞-bound of weak solutions to problems driven by the fractional $r(\cdot)$-Laplacian and, in particular, derived the existence of a sequence of infinitely many weak solutions whose L^∞-norms converge to zero. Additionally, they obtained the existence of many solutions for a class of critical non-local problems with variable exponents; see [21]. We refer the interested reader to [25,26] for the existence results of critical nonlocal equations with variable exponents. A remarkable recent work on fractional equations with variable exponents can be found in [15]. More precisely, for subcritical nonlinearity as well as the critical exponent, the existence and multiplicity of solutions and existence of ground-state solutions to the fractional double-phase Robin problem with logarithm-type nonlinearity have been obtained.

The present paper is devoted to a problem of Schrödinger-Kirchhoff type driven by the fractional $r(\cdot)$-Laplacian

$$K\Big([\omega]_{s,r(\cdot,\cdot)}\Big)\mathcal{L}\omega(x)+b(x)|\omega|^{r(x)-2}\omega=\lambda\kappa(x)|\omega|^{q(x)-2}\omega+g(x,\omega) \quad \text{in} \quad \mathbb{R}^N, \tag{1}$$

where

$$[\omega]_{s,r(\cdot,\cdot)}:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)}}{r(x,y)|x-y|^{N+sr(x,y)}}\,dx\,dy,$$

$r\in C(\mathbb{R}^N\times\mathbb{R}^N)$ satisfying $1<\inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N}r(x,y)\le\sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N}r(x,y)<\frac{N}{s}$, $r(x,y)=r(y,x)$ for all $x,y\in\mathbb{R}^N$, $r(x)=r(x,x)$ for all $x\in\mathbb{R}^N$, $q:\mathbb{R}^N\to(1,\infty)$ is a continuous function with $1<\inf_{x\in\mathbb{R}^N}q(x)\le\sup_{x\in\mathbb{R}^N}r(x)$, $b:\mathbb{R}^N\to(0,\infty)$ is a continuous potential function, and $g:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with the subcritical and $r(\cdot)$-superlinear nonlinearity, and $K\in C((0,\infty))$ is a function of Kirchhoff type which will be specified. Additionally, the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\omega(x)=2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|\omega(x)-\omega(y)|^{r(x,y)-2}(\omega(x)-\omega(y))}{|x-y|^{N+sr(x,y)}}\,dy, \quad x\in\mathbb{R}^N,$$

where $s\in(0,1)$ and $B_\varepsilon(x):=\{y\in\mathbb{R}^N:|y-x|\le\varepsilon\}$. Let us set

$$C_+(\mathbb{R}^N)=\Big\{f\in C(\mathbb{R}^N):\inf_{x\in\mathbb{R}^N}f(x)>1\Big\}.$$

For any $f\in C_+(\mathbb{R}^N)$, we define

$$f^+=\sup_{x\in\mathbb{R}^N}f(x) \quad \text{and} \quad f^-=\inf_{x\in\mathbb{R}^N}f(x).$$

Then, we suppose that the Kirchhoff function $K:[0,\infty)\to(0,\infty)$ satisfies the following conditions:

(K1) $K\in C([0,\infty),(0,\infty))$ satisfies $\inf_{t\in\mathbb{R}_0^+}K(t)\ge\tau_0>0$, where τ_0 is a constant.

(K2) There exists $\vartheta\in[1,\frac{N}{N-sr^+})$ such that $\vartheta\mathcal{K}(t)=\vartheta\int_0^t K(q)dq\ge K(t)t$ for any $t\ge 0$.

The Kirchhoff-type problem was initially introduced in [27] as a generalization of the classical D'Alembert's wave equation for free vibrations of elastic strings. Elliptic problems of Kirchhoff type have a strong background in several applications in Physics and have been widely studied by many researchers in recent years; see [28–32].

The purpose of this paper is devoted to the existence result of a sequence of infinitely many small energy solutions to the fractional $r(\cdot)$-Laplacian equations of Kirchhoff–Schrödinger type with concave–convex nonlinearities when the convex term g does not require the Ambrosetti–Rabinowitz condition as follows; namely, there exists a constant $\theta>0$ such that $\theta>\vartheta r^+$ and

$$0<\theta G(x,t)\le g(x,t)t \text{ for all } (x,t)\in\mathbb{R}^N\times\mathbb{R}\setminus\{0\}, \quad \text{where} \quad G(x,t)=\int_0^t g(x,s)\,ds.$$

As we know, this condition is crucial in guaranteeing the boundedness of Palais–Smale sequences of the Euler–Lagrange functional corresponding to the problem (1). The key tool to obtain this multiplicity result is the dual fountain theorem. This result of multiple solutions to nonlinear elliptic problems is motivated by the contributions in recent works [23,30,33–39], and the references therein. In particular, Alves–Liu [33] obtained the existence and multiplicity results to the superlinear $r(x)$-Laplacian problems:

$$-\mathrm{div}(|\nabla\omega|^{r(x)-2}\nabla\omega)+b(x)|\omega|^{r(x)-2}\omega=g(x,\omega) \text{ in } \mathbb{R}^N.$$

Here, the potential function $b \in C(\mathbb{R}^N)$ satisfies suitable conditions and the Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ fulfills the following assumptions:

$(f1)$ $G(x,s) = o(|s|^{r(x)})$ as $s \to 0$ uniformly for all $x \in \mathbb{R}^N$.

$(f2)$ $\lim_{|s|\to\infty} \frac{G(x,s)}{|s|^{r^+}} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$.

$(f3)$ There exists a constant $\theta \geq 1$ such that

$$\theta\mathcal{G}(x,s) \geq \mathcal{G}(x,ts)$$

for $(x,s) \in \mathbb{R}^N \times \mathbb{R}$ and $t \in [0,1]$, where $\mathcal{G}(x,s) = g(x,s)s - r^+ G(x,s)$.

The condition $(f3)$ was originally provided by the works of Jeanjean [40]. In the last few decades, there were extensive studies dealing with the r-Laplacian problem by assuming $(f2)$; see [34,35], and see also [3,36] for the case of variable exponents $r(\cdot)$. Recently, Lin–Tang [37] established the various theorems on the existence of solutions of r-Laplacian equations with mild conditions for the superlinear term g, which is deeply different from those investigated in [34,35,37,40]. Additionally, the authors in [38] established the existence results of infinitely many weak solutions to the $r(\cdot)$-Laplacian-like equations under the following condition:

$(f4)$ There is a positive function $C \in L^1(\mathbb{R}^N)$ such that

$$sg(x,s) - r^+ G(x,s) \leq \varsigma g(x,\varsigma) - r^+ G(x,\varsigma) + C(x)$$

for any $x \in \Omega$ and $0 < s < \varsigma$ or $\varsigma < s < 0$,

which was first provided by Miyagaki and Souto [41]. Let us consider the function

$$g(x,s) = \sigma(x)\left(|s|^{r(x)-2}s\ln(1+|s|) + \frac{|s|^{r(x)-1}s}{1+|s|}\right)$$

with its primitive function

$$G(x,s) = \frac{\sigma(x)}{r(x)}|s|^{r(x)}\ln(1+|s|)$$

for all $s \in \mathbb{R}$ and $r \in C_+(\mathbb{R}^N)$, where $\sigma \in C(\mathbb{R}^N, \mathbb{R})$ with $0 < \inf_{x\in\mathbb{R}^N} \sigma(x) \leq \sup_{x\in\mathbb{R}^N} \sigma(x) < \infty$. Thus, this example fulfils the conditions $(f1)$–$(f4)$. However, for instance, if we consider function

$$g(x,s) = \sigma(x)\left(\nu(x)|s|^{m(x)-2}s + |s|^{r^- -2}s + \frac{2}{r^-}\sin s\right)$$

with its primitive function

$$G(x,s) = \sigma(x)\left(\frac{\nu(x)}{m(x)}|s|^{m(x)} + \frac{1}{r^-}|s|^{r^-} - \frac{2}{r^-}\cos s + \frac{2}{r^-}\right),$$

where $r, \tau, m \in C_+(\mathbb{R}^N)$, $1 < m^- \leq m^+ < r^- \leq r^+$, $\tau(x) > 1$ with $r(x) \leq \tau'(x)m(x) \leq r^*(x)$ for all $x \in \mathbb{R}^N$ and $0 < \nu \in L^{\tau(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then it is clear that this does not satisfy the conditions $(f1)$–$(f4)$. However, this example holds a condition (B2), which will be introduced in Section 3.

From this perspective, on a new class of superlinear term g, we give the existence result of a sequence of infinitely many small energy solutions by employing a variational method. However, our proof for obtaining this result is slightly different from those of previous related works [17,23,30,38,42–47]. Roughly speaking, in view of [23,38,44,48], the conditions $(f1)$ and $(f2)$ play an important role in ensuring assumptions in the dual fountain theorem; however, we verify them when $(f1)$ and $(f2)$ are not assumed. To the best of our belief, although this work is inspired by the papers [17,23,39,42–45,48], and many authors have an interest in the investigation of elliptic problems with variable exponents, the present paper is the first attempt to obtain the multiplicity result for the Schrödinger–Kirchhoff-type problem driven by the non-local fractional $r(\cdot)$-Laplacian because we deduce our result on a new class of the convex term g.

This paper is organized as follows. In Section 2, we shortly introduce the definition of the Lebesgue spaces with variable exponents and the fractional variable exponent Lebesgue–Sobolev space, and collect some elementary properties. Section 3 provides the existence result of infinitely many small energy solutions to the problem (1) by applying as the primary tool the variational principle.

2. Preliminaries

In this section, we briefly recall the definition and some basic properties of the variable exponent Lebesgue–Sobolev space of fractional type, which were systematically studied in [20].

For any $r \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{r(\cdot)}(\mathbb{R}^N) := \left\{\omega : \omega \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} |\omega(x)|^{r(x)}\,dx < \infty\right\},$$

endowed with the Luxemburg norm

$$\|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{\omega(x)}{\lambda}\right|^{r(x)} dx \le 1\right\}.$$

The dual space of $L^{r(\cdot)}(\mathbb{R}^N)$ is $L^{r'(\cdot)}(\mathbb{R}^N)$, where $1/r(x) + 1/r'(x) = 1$.

Let $0 < s < 1$ and let $r \in C(\mathbb{R}^N \times \mathbb{R}^N, (1,\infty))$ be such that r is symmetric, i.e., $r(x,y) = r(y,x)$ for all $x, y \in \mathbb{R}^N$ and

$$1 < r^- := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} r(x,y) \le r^+ := \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} r(x,y) < +\infty.$$

For $r \in C_+(\mathbb{R}^N)$, define

$$W^{s,r(\cdot),r(\cdot,\cdot)}(\mathbb{R}^N) := \left\{\omega \in L^{r(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dxdy < +\infty\right\},$$

and we set

$$|\omega|_{s,r(\cdot,\cdot)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{r(x,y)}}{\lambda^{r(x,y)}|x-y|^{N+sr(x,y)}}\,dxdy < 1\right\}.$$

Then, $W^{s,r(\cdot),r(\cdot,\cdot)}(\mathbb{R}^N)$ endowed with the norm

$$\|\omega\|_{s,r,\mathbb{R}^N} := \|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)} + |\omega|_{s,r(\cdot,\cdot)}$$

is a reflexive and separable Banach space (see [13,14,22]). It is immediate that

$$|\omega|_{s,r,\mathbb{R}^N} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N}\left|\frac{\omega}{\lambda}\right|^{r(x)} dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{r(x,y)}}{\lambda^{r(x,y)}|x-y|^{N+sr(x,y)}}\,dx\,dy < 1\right\}$$

is an equivalent norm of $\|\cdot\|_{s,r,\mathbb{R}^N}$ with the relation

$$\frac{1}{2}\|\omega\|_{s,r,\mathbb{R}^N} \le |\omega|_{s,r,\mathbb{R}^N} \le 2\|\omega\|_{s,r,\mathbb{R}^N}.$$

Throughout the present paper, in some cases, we write $r(x)$ instead of $r(x,x)$ for brevity, and thus, $r \in C_+(\mathbb{R}^N)$. Further, we write $W^{s,r(\cdot,\cdot)}(\mathbb{R}^N)$ instead of $W^{s,r(\cdot),r(\cdot,\cdot)}(\mathbb{R}^N)$.

Lemma 1 ([49,50]). *The space $L^{r(\cdot)}(\mathbb{R}^N)$ is a uniformly convex and separable Banach space, and its conjugate space is $L^{r'(\cdot)}(\mathbb{R}^N)$ where $1/r(\cdot)+1/r'(\cdot)=1$. For any $u\in L^{r(\cdot)}(\mathbb{R}^N)$ and $v\in L^{r'(\cdot)}(\mathbb{R}^N)$, one has*

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \le \left(\frac{1}{r_-}+\frac{1}{(r')_-}\right)\|u\|_{L^{r(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{r'(\cdot)}(\mathbb{R}^N)} \le 2\|u\|_{L^{r(\cdot)}(\mathbb{R}^N)}\|v\|_{L^{r'(\cdot)}(\mathbb{R}^N)}.$$

Lemma 2 ([49]). *If $1/r(\cdot)+1/\ell(\cdot)+1/q(\cdot)=1$, then for any $v\in L^{r(\cdot)}(\mathbb{R}^N), z\in L^{\ell(\cdot)}(\mathbb{R}^N)$ and $w\in L^{\tau(\cdot)}(\mathbb{R}^N)$,*

$$\begin{aligned}\left|\int_{\mathbb{R}^N} vzw\,dx\right| &\le \left(\frac{1}{r^-}+\frac{1}{\ell^-}+\frac{1}{q^-}\right)\|v\|_{L^{r(\cdot)}(\mathbb{R}^N)}\|z\|^{L^{\ell(\cdot)}(\mathbb{R}^N)}\|w\|^{L^{q(\cdot)}(\mathbb{R}^N)}\\ &\le 3\|v\|^{L^{r(\cdot)}(\mathbb{R}^N)}\|z\|^{L^{\ell(\cdot)}(\mathbb{R}^N)}\|w\|^{L^{q(\cdot)}(\mathbb{R}^N)}.\end{aligned}$$

Lemma 3 ([49]). *Denote*

$$\kappa(\omega)=\int_{\mathbb{R}^N}|\omega|^{r(x)}\,dx, \quad \text{for all } \omega\in L^{r(\cdot)}(\mathbb{R}^N).$$

Then

(1) $\kappa(\omega)>1\ (=1;\ <1)$ *if and only if* $\|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)}>1\ (=1;\ <1)$, *respectively;*
(2) *if* $\|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)}>1$, *then* $\|\omega\|^{r^-}_{L^{r(\cdot)}(\mathbb{R}^N)}\le\kappa(\omega)\le\|\omega\|^{r^+}_{L^{r(\cdot)}(\mathbb{R}^N)}$;
(3) *if* $\|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)}<1$, *then* $\|\omega\|^{r^+}_{L^{r(\cdot)}(\mathbb{R}^N)}\le\kappa(\omega)\le\|\omega\|^{r^-}_{L^{r(\cdot)}(\mathbb{R}^N)}$.

Lemma 4 ([51]). *Let $\ell\in L^\infty(\mathbb{R}^N)$ be such that $1\le r(\cdot)\ell(\cdot)\le\infty$ in $\mathbb{R}^N$. If $\omega\in L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)$ with $\omega\ne0$, then*

(1) *If* $\|\omega\|_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}>1$, *then* $\|\omega\|^{\ell^-}_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}\le\||\omega|^{\ell(x)}\|_{L^{r(\cdot)}(\mathbb{R}^N)}\le\|\omega\|^{\ell^+}_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}$;
(2) *If* $\|\omega\|_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}<1$, *then* $\|\omega\|^{\ell^+}_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}\le\||\omega|^{\ell(x)}\|_{L^{r(\cdot)}(\mathbb{R}^N)}\le\|\omega\|^{\ell^-}_{L^{r(\cdot)\ell(\cdot)}(\mathbb{R}^N)}$.

Lemma 5 ([49]). *Suppose that $r:\mathbb{R}^N\to\mathbb{R}$ is Lipschitz-continuous with $1<r^-\le r^+<N$. Let $\ell\in L^\infty(\mathbb{R}^N)$ and $r(\cdot)\le\ell(\cdot)\le r_s^*(\cdot):=\frac{Nr(\cdot)}{N-sr(\cdot)}$ almost everywhere in $\mathbb{R}^N$. Then, there is a continuous embedding $W^{1,r(\cdot)}(\mathbb{R}^N)\hookrightarrow L^{\ell(\cdot)}(\mathbb{R}^N)$.*

We present the embedding theorem for the fractional variable exponent Sobolev space below:

Lemma 6 (Subcrtitical imbeddings, [20]). *It holds that*

(1) $W^{s,r(\cdot,\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$, *if Ω is a bounded Lipschitz domain and $q\in C_+(\overline{\Omega})$ such that $q(\cdot)<r_s^*(\cdot)$ in $\overline{\Omega}$;*
(2) $W^{s,r(\cdot,\cdot)}(\mathbb{R}^N)\hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ *for any uniformly continuous function $q\in C_+(\mathbb{R}^N)$ with $r(\cdot)\le q(\cdot)$ in $x\in\mathbb{R}^N$ and $\inf_{\mathbb{R}^N}(r_s^*(\cdot)-q(\cdot))>0$;*
(3) $W^{s,r(\cdot,\cdot)}(\mathbb{R}^N)\hookrightarrow\hookrightarrow L^{q(\cdot)}_{loc}(\mathbb{R}^N)$ *for any $r\in C_+(\mathbb{R}^N)$ satisfying $q(\cdot)<r_s^*(\cdot)$ in $\mathbb{R}^N$.*

Next, we assume that the potential function b satisfies

(P) $b\in L^1_{loc}(\mathbb{R}^N)$, ${}_{x\in\mathbb{R}^N}b(x)>0$, and $\lim_{|x|\to\infty}b(x)=+\infty$.

On the linear subspace

$$X:=\left\{\omega\in W^{s,r(\cdot,\cdot)}(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dxdy+\int_{\mathbb{R}^N}b(x)|\omega(x)|^{r(x)}\,dx<+\infty\right\},$$

we equip the norm

$$\|\omega\|_X := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} b(x)\left|\frac{\omega}{\lambda}\right|^{r(x)} dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{r(x,y)}}{\lambda^{r(x,y)}|x-y|^{N+sr(x,y)}}\,dx\,dy < 1\right\}.$$

Then, $(X, \|\cdot\|_X)$ is continuously embedded into $W^{s,r(\cdot,\cdot)}$ as a closed subspace. Therefore, $(X, \|\cdot\|_X)$ is also a reflexive and separable Banach space. Let X^* be a dual space of X. Furthermore, $\langle\cdot,\cdot\rangle$ denotes the pairing of X and its dual X^*.

Remark 1 ([33])**.** *Denote*

$$\kappa(\omega) = \int_{\mathbb{R}^N}\left(|\nabla\omega|^{r(x)} + b(x)|\omega|^{r(x)}\right)dx, \quad \textit{for all } \omega \in X.$$

Then,

(1) $\kappa(\omega) > 1$ $(= 1;\ < 1)$ if and only if $\|\omega\|_X > 1$ $(= 1;\ < 1)$, respectively;
(2) if $\|\omega\|_X > 1$, then $\|\omega\|_X^{r^-} \le \kappa(\omega) \le \|\omega\|_X^{r^+}$;
(3) if $\|\omega\|_X < 1$, then $\|\omega\|_X^{r^+} \le \kappa(\omega) \le \|\omega\|_X^{r^-}$.

With the help of Lemma 6, the proof of the following consequence is absolutely the same as in those of Lemma 2.6 in [33].

Lemma 7. *If the potential function b satisfies the assumption* (P)*, then*

(1) there is a compact embedding $X \hookrightarrow L^{r(\cdot)}(\mathbb{R}^N)$;
(2) for any measurable function $\ell : \mathbb{R}^N \to \mathbb{R}$ with $r(\cdot) < \ell(\cdot)$ in $\mathbb{R}^N$, there is a compact embedding $X \hookrightarrow L^{\ell(\cdot)}(\mathbb{R}^N)$ if $\inf\limits_{x\in\mathbb{R}^N}(r^*(x) - \ell(x)) > 0$.

3. Existence of Solutions

In this section, the existence of nontrivial weak solutions for (1) is provided by applying the dual fountain theorem under suitable assumptions.

Definition 1. *We say that $\omega \in X$ is a weak solution of* (1) *if*

$$K\left([\omega]_{s,r(\cdot,\cdot)}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)-2}(\omega(x)-\omega(y))(u(x)-u(y))}{|x-y|^{N+sr(x,y)}}\,dx\,dy$$
$$+\int_{\mathbb{R}^N} b(x)|\omega|^{r(x)-2}\omega u\,dx = \lambda\int_{\mathbb{R}^N}\kappa(x)|\omega|^{q(x)-2}\omega u\,dx + \int_{\mathbb{R}^N} h(x,\omega)u\,dx$$

for all $u \in X$, where

$$[\omega]_{s,r(\cdot,\cdot)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)}}{r(x,y)|x-y|^{N+sr(x,y)}}\,dx\,dy.$$

Let us define the functional $A : X \to \mathbb{R}$ by

$$A(\omega) = \mathcal{K}\left([\omega]_{s,r(\cdot,\cdot)}\right) + \int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|\omega|^{r(x)}\,dx.$$

Then, the following assertion can be found in [3].

Lemma 8. *Suppose that* (K1)–(K2) *are fulfilled. Then, $A : X \to \mathbb{R}$ is slightly less semicontinuous and convex on X. In addition, A' is a mapping of type (S_+), i.e., if $y_n \rightharpoonup y$ in X and* $\limsup_{n\to\infty}\langle A'(y_n) - A'(y), y_n - y\rangle \le 0$*, then $y_n \to y$ in X as $n \to \infty$.*

Let $G(x,t) = \int_0^t g(x,s)\,ds$. Suppose that

(A1) $q, r, \ell \in C_+(\mathbb{R}^N)$ and $1 < q^- \le q^+ < r^- \le r^+ < \ell^- \le \ell^+ < r_s^*(x)$ for all $x \in \mathbb{R}^N$.

(A2) $0 \le \kappa \in L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying that $meas\{x \in \mathbb{R}^N : \kappa(x) \neq 0\} > 0$.

(B1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist a positive constant ρ_2 and a nonnegative function $\rho_1 \in L^{\ell'(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|g(x,t)| \le \rho_1(x) + \rho_2 |t|^{\ell(x)-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(B2) There are $\mu > \vartheta r^+$, $T > 0$ such that

$$\mu G(x,t) \le t g(x,t)$$

for all $x \in \mathbb{R}^N$ and $|t| \ge T$, where $G(x,t) = \int_0^t g(x,s)\,ds$, where ϑ is given in (K2).

(B3) There exist $C > 0$, $1 < m^- \le m^+ < r^- \le r^+$, $\tau(x) > 1$ with $r(x) \le \tau'(x)m(x) \le r^*(x)$ for all $x \in \mathbb{R}^N$ and a positive function $\nu \in L^{\tau(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\liminf_{|t| \to 0} \frac{g(x,t)}{\nu(x)|t|^{m(x)-2}t} \ge C$$

uniformly for almost all $x \in \mathbb{R}^N$.

Let the functional $B_\lambda : X \to \mathbb{R}$ be defined by

$$B_\lambda(\omega) = \lambda \int_{\mathbb{R}^N} \frac{\kappa(x)}{q(x)} |\omega|^{q(x)}\,dx + \int_{\mathbb{R}^N} G(x,\omega)\,dx.$$

Then, it is immediate that $B_\lambda \in C^1(X,\mathbb{R})$, and its Fréchet derivative is

$$\langle B'_\lambda(\omega), u \rangle = \lambda \int_{\mathbb{R}^N} \kappa(x)|\omega|^{q(x)-2}\omega u\,dx + \int_{\mathbb{R}^N} g(x,\omega)u\,dx$$

for any $u, \omega \in X$. Subsequently, the functional $J_\lambda : X \to \mathbb{R}$ is defined by

$$J_\lambda(\omega) = A(\omega) - B_\lambda(\omega). \tag{2}$$

Then, by virtue of Lemma 3.2 in [3], one has that the functional $J_\lambda \in C^1(X,\mathbb{R})$, and its Fréchet derivative is

$$\begin{aligned}\langle J'_\lambda(\omega), u \rangle =& K\left([\omega]_{s,r(\cdot,\cdot)}\right) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\omega(x)-\omega(y)|^{r(x,y)-2}(\omega(x)-\omega(y))(u(x)-u(y))}{|x-y|^{N+sr(x,y)}}\,dx\,dy \\ &+ \int_{\mathbb{R}^N} b(x)|\omega|^{r(x)-2}\omega u\,dx - \lambda \int_{\mathbb{R}^N} \kappa(x)|\omega|^{q(x)-2}\omega u\,dx - \int_{\mathbb{R}^N} g(x,\omega)u\,dx\end{aligned}$$

for any $\omega, u \in X$.

Lemma 9. *Suppose that* (P), (A1)–(A2) *and* (B1)–(B2) *hold. Then, B_λ and B'_λ are weakly strongly continuous on E for any $\lambda > 0$.*

Definition 2. *Let E be a Banach space, $I \in C^1(X,\mathbb{R})$. We say that I ensures the Cerami condition ((C)-condition for short) in E if any (C)-sequence $\{z_n\}_n \subset E$, i.e., $\{I(z_n)\}$ is bounded and $\|I'(z_n)\|_{E^*}(1+\|z_n\|_E) \to 0$ as $n \to \infty$ has a convergent subsequence in E.*

The basic idea of proofs of this assertion follows similar arguments to those in [39]; see also [52].

Lemma 10. *Assume that* (P), (A1)–(A2), (K1)–(K2) *and* (B1)–(B2) *hold. Then, the functional* J_λ *satisfies the* (C)*-condition for any* $\lambda > 0$.

Proof. Let $\{\omega_n\}$ be a (C)-sequence in X, *i.e.*,

$$\sup_{n\in\mathbb{N}} |J_\lambda(\omega_n)| \le \mathfrak{K}_1 \text{ and } \langle J'_\lambda(\omega_n), \omega_n\rangle = o(1) \to 0, \text{ as } n \to \infty, \tag{3}$$

where $\mathfrak{K}_1$ is a positive constant. Firstly, we will prove that the sequence $\{\omega_n\}$ is bounded in X. Denote $\{a \le |\omega_n| \le c\}| := \{x \in \mathbb{R}^N : a \le |\omega_n(x)| \le c\}$ for any real number a and c. Since $b(x) \to +\infty$ as $|x| \to \infty$, we have

$$\begin{aligned}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}dx - C_1 \int_{\{|\omega_n|\le T\}} \left(|\omega_n|^{r(x)} + \rho_1(x)|\omega_n| + \rho_2|\omega_n|^{q(x)}\right)dx \\ \ge \frac{1}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}dx - \mathfrak{K}_0\end{aligned}$$

for any positive constant C_1 and for some positive constant $\mathfrak{K}_0$. In fact, by Young's inequality, we know that

$$\begin{aligned}
&\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}dx - C_1 \int_{\{|\omega_n|\le T\}} \left(|\omega_n|^{r(x)} + \rho_1(x)|\omega_n| + \rho_2|\omega_n|^{\ell(x)}\right) dx \\
&\ge \left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}dx - C_1 \int_{\{|\omega_n|\le T\}} \left(|\omega_n|^{r(x)} + (\rho_1(x))^{\ell'(x)} + |\omega_n|^{\ell(x)} + \rho_2|\omega_n|^{\ell(x)}\right) dx \\
&\ge \frac{1}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \left[\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx + \int_{\{|\omega_n|\le T\}} b(x)|\omega_n|^{r(x)}\,dx\right] \\
&- C_1 \int_{\{|\omega_n|\le 1\}} \left(|\omega_n|^{r(x)} + |\omega_n|^{\ell(x)} + \rho_2|\omega_n|^{\ell(x)}\right) dx \\
&- C_1 \int_{\{1<|\omega_n|\le T\}} \left(|\omega_n|^{r(x)} + |\omega_n|^{\ell(x)} + \rho_2|\omega_n|^{\ell(x)}\right) dx - C_1(1 + \|\rho_1\|^{(\ell')^+}_{L^{\ell'(\cdot)}(\mathbb{R}^N)}) \\
&\ge \frac{1}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \left[\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx + \int_{\{|\omega_n|\le T\}} b(x)|\omega_n|^{r(x)}\,dx\right] \\
&- C_1(2+\rho_2) \int_{\{|\omega_n|\le 1\}} |\omega_n|^{r(x)}\,dx - C_1 T^{\ell^+-r^-}(2+\rho_2) \int_{\{1<|\omega_n|\le T\}} |\omega_n|^{r(x)}\,dx - \widetilde{C}_1 \\
&\ge \frac{1}{2}\left(\frac{1}{\vartheta r^+} - \frac{1}{\mu}\right) \left[\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx + \int_{\{|\omega_n|\le T\}} b(x)|\omega_n|^{r(x)}\,dx\right] \\
&- C_1 T^{\ell^+-r^-}(2+\rho_2) \int_{\{|\omega_n|\le T\}} |\omega_n|^{r(x)}\,dx - \widetilde{C}_1,
\end{aligned} \tag{4}$$

where $\widetilde{C}_1$ is a positive constant. Since $b(x) \to +\infty$ as $|x| \to \infty$, there is $\gamma_0 > 0$ such that $|x| \ge \gamma_0$ implies $b(x) \ge \frac{2\vartheta r^+ \mu C_1 T^{\ell^+-r^-}(2+\rho_2)}{\mu - \vartheta r^+}$. Then, we know that

$$b(x)|\omega_n|^{r(x)} \ge \frac{2\vartheta r^+ \mu C_1 T^{\ell^+-r^-}(2+\rho_2)}{\mu - \vartheta r^+}|\omega_n|^{r(x)} \tag{5}$$

for $|x| \ge \gamma_0$. Set $B_\gamma := \{x \in \mathbb{R}^N : |x| < \gamma\}$. Then, since $b \in L^1_{loc}(\mathbb{R}^N)$, we infer

$$\int_{\{|\omega_n|\le T\}\cap B_{\gamma_0}} b(x)|\omega_n|^{r(x)}\,dx \le \widetilde{C}_2 \quad \text{and} \quad \int_{\{|\omega_n|\le T\}\cap B_{\gamma_0}} |\omega_n|^{r(x)}\,dx \le \widetilde{C}_3$$

for some positive constants $\widetilde{C}_2$ and $\widetilde{C}_3$. This, together with (4) and (5), yields

$$
\begin{aligned}
&\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx - C_1\int_{\{|\omega_n|\le T\}}\Big(|\omega_n|^{r(x)}+\rho_1(x)|\omega_n|+\rho_2|\omega_n|^{\ell(x)}\Big)\,dx\\
&\ge \frac{\mu-\vartheta r^{+}}{2\vartheta r^{+}\mu}\left[\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx+\int_{\{|\omega_n|\le T\}\cap B^c_{\gamma_0}} b(x)|\omega_n|^{r(x)}\,dx+\int_{\{|\omega_n|\le T\}\cap B_{\gamma_0}} b(x)|\omega_n|^{r(x)}\,dx\right]\\
&-\widetilde{C}_0\left[\int_{\{|\omega_n|\le T\}\cap B^c_{\gamma_0}}|\omega_n|^{r(x)}\,dx+\int_{\{|\omega_n|\le T\}\cap B_{\gamma_0}}|\omega_n|^{r(x)}\,dx\right]-\widetilde{C}_1\\
&\ge \frac{\mu-\vartheta r^{+}}{2\vartheta r^{+}\mu}\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx+\frac{\mu-\vartheta r^{+}}{2\vartheta r^{+}\mu}\int_{\{|\omega_n|\le T\}\cap B^c_{\gamma_0}} b(x)|\omega_n|^{r(x)}\,dx\\
&-\widetilde{C}_0\int_{\{|\omega_n|\le T\}\cap B^c_{\gamma_0}}|\omega_n|^{r(x)}\,dx-\mathfrak{K}_0\\
&\ge \frac{1}{2}\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx-\mathfrak{K}_0,
\end{aligned}
$$

as required. Combining this with (K1)–(K2) and (B3), one has

$$
\begin{aligned}
\mathfrak{K}_1+1 &\ge J_\lambda(\omega_n)-\frac{1}{\mu}\langle J'_\lambda(\omega_n),\omega_n\rangle\\
&=\mathcal{K}\Big([\omega_n]_{s,r(\cdot,\cdot)}\Big)+\int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|\omega_n|^{r(x)}\,dx-\lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|\omega_n|^{q(x)}dx-\int_{\mathbb{R}^N} G(x,\omega_n)\,dx\\
&\quad-\frac{1}{\mu}K\Big([\omega_n]_{s,r(\cdot,\cdot)}\Big)[\omega_n]_{s,r(\cdot,\cdot)}-\frac{1}{\mu}\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx\\
&\quad+\frac{\lambda}{\mu}\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}\,dx+\frac{1}{\mu}\int_{\mathbb{R}^N} g(x,\omega_n)\omega_n\,dx\\
&=\frac{1}{\vartheta}K\Big([\omega_n]_{s,r(\cdot,\cdot)}\Big)[\omega_n]_{s,r(\cdot,\cdot)}+\int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|\omega_n|^{r(x)}\,dx-\lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|\omega_n|^{q(x)}dx\\
&\quad-\int_{\mathbb{R}^N} G(x,\omega_n)\,dx-\frac{1}{\mu}K\Big([\omega_n]_{s,r(\cdot,\cdot)}\Big)[\omega_n]_{s,r(\cdot,\cdot)}\\
&\quad-\frac{1}{\mu}\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx+\frac{\lambda}{\mu}\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}\,dx+\frac{1}{\mu}\int_{\mathbb{R}^N} g(x,\omega_n)\omega_n\,dx\\
&\ge \tau_0\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_n(x)-\omega_n(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dx\,dy+\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx\\
&\qquad+\frac{1}{\mu}\int_{\mathbb{R}^N}(g(x,\omega_n)\omega_n-\mu G(x,\omega_n))\,dx+\lambda\left(\frac{1}{\mu}-\frac{1}{q^{-}}\right)\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}\,dx\\
&\ge \tau_0\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_n(x)-\omega_n(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dx\,dy+\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx\\
&\qquad+\frac{1}{\mu}\int_{|\omega_n|\le T}(g(x,\omega_n)\omega_n-\mu G(x,\omega_n))\,dx+\frac{1}{\mu}\int_{|\omega_n|\ge T}(g(x,\omega_n)\omega_n-\mu G(x,\omega_n))\,dx\\
&\qquad+\lambda\left(\frac{1}{\mu}-\frac{1}{q^{-}}\right)\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}\,dx\\
&\ge \tau_0\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_n(x)-\omega_n(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dx\,dy+\left(\frac{1}{\vartheta r^{+}}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} b(x)|\omega_n|^{r(x)}\,dx\\
&\qquad-\frac{C_2}{\mu}\int_{|\omega_n|\le T}\Big(|\omega_n|^{r(x)}+\rho_1(x)|\omega_n|+\rho_2(x)|\omega_n|^{\ell(x)}\Big)dx\\
&\qquad+\lambda\left(\frac{1}{\mu}-\frac{1}{q^{-}}\right)\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \tau_0\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_n(x)-\omega_n(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dx\,dy+\frac{1}{2}\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}b(x)|\omega_n|^{r(x)}\,dx\\
&\qquad+\lambda\left(\frac{1}{\mu}-\frac{1}{q^-}\right)\int_{\mathbb{R}^N}\kappa(x)|\omega_n|^{q(x)}dx-\mathfrak{K}_0\\
&\geq \frac{\min\{\tau_0,1\}}{2}\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_n(x)-\omega_n(y)|^{r(x,y)}}{|x-y|^{N+sr(x,y)}}\,dx\,dy+\int_{\mathbb{R}^N}b(x)|\omega_n|^{r(x)}\,dx\right)\\
&\qquad+2\lambda\left(\frac{1}{\mu}-\frac{1}{q^-}\right)\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\max\left\{\|\omega_n\|^{q^-}_{L^{r(\cdot)}(\mathbb{R}^N)},\|\omega_n\|^{q^+}_{L^{r(\cdot)}(\mathbb{R}^N)}\right\}-\mathfrak{K}_0\\
&\geq \frac{\min\{\tau_0,1\}}{2}\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\|\omega_n\|_X^{r^-}\\
&\qquad+2\lambda\left(\frac{1}{\mu}-\frac{1}{q^-}\right)\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\max\left\{\|\omega_n\|^{q^-}_{L^{r(\cdot)}(\mathbb{R}^N)},\|\omega_n\|^{q^+}_{L^{r(\cdot)}(\mathbb{R}^N)}\right\}-\mathfrak{K}_0\\
&\geq \frac{\min\{\tau_0,1\}}{2}\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\|\omega_n\|_X^{r^-}\\
&\qquad+2\lambda C_3\left(\frac{1}{\mu}-\frac{1}{q^-}\right)\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\|\omega_n\|_X^{q^+}-\mathfrak{K}_0
\end{aligned}
$$

for some positive constants C_2 and C_3. From this, we infer

$$
\begin{aligned}
\mathfrak{K}_1+1+\mathfrak{K}_0+&2\lambda C_3\left(\frac{1}{\mu}-\frac{1}{q^-}\right)\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\|\omega_n\|_X^{q^+}\\
&\geq \frac{\min\{\tau_0,1\}}{2}\left(\frac{1}{\vartheta r^+}-\frac{1}{\mu}\right)\|\omega_n\|_X^{r^-}.
\end{aligned}
$$

Since $r^- > q^+$, we determine that $\{\omega_n\}$ is a bounded sequence in X, and thus $\{\omega_n\}$ has a weakly convergent subsequence in X. Without loss of generality, we suppose that

$$\omega_n \rightharpoonup \omega_0 \text{ in } X \text{ as } n \to \infty.$$

By Lemma 9, we infer that B'_λ is compact, and so $B'_\lambda(\omega_n) \to B'_\lambda(\omega_0)$ in X as $n \to \infty$. Since $J'_\lambda(\omega_n) \to 0$ as $n \to \infty$, we know that

$$\langle J'_\lambda(\omega_n), \omega_n-\omega_0\rangle \to 0 \text{ and } \langle J'_\lambda(\omega_0), \omega_n-\omega_0\rangle \to 0,$$

and thus

$$\langle J'_\lambda(\omega_n)-J'_\lambda(\omega_0), \omega_n-\omega_0\rangle \to 0$$

as $n \to \infty$. From this, we have

$$
\begin{aligned}
&\langle A'(\omega_n)-A'(\omega_0), \omega_n-\omega_0\rangle\\
&=\langle B'_\lambda(\omega_n)-B'_\lambda(\omega_0), \omega_n-\omega_0\rangle+\langle J'_\lambda(\omega_n)-J'_\lambda(\omega_0), \omega_n-\omega_0\rangle \to 0,
\end{aligned}
$$

namely, $\langle A'(\omega_n)-A'(\omega_0), \omega_n-\omega_0\rangle \to 0$ as $n \to \infty$. Since X is reflexive and A' is a mapping of type (S_+) by Lemma 8, we assert that

$$\omega_n \to \omega_0 \text{ in } X \text{ as } n \to \infty.$$

The proof is completed □

Let $\mathfrak{W}$ be a reflexive and separable Banach space. Then, it is known (see [53,54]) that there are $\{e_n\} \subseteq \mathfrak{W}$ and $\{h_n^*\} \subseteq \mathfrak{W}^*$, such that

$$\mathfrak{W} = \overline{\operatorname{span}\{e_n : n = 1, 2, \cdots\}}, \quad \mathfrak{W}^* = \overline{\operatorname{span}\{h_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle h_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $\mathfrak{W}_n = \operatorname{span}\{e_n\}$, $\mathfrak{Y}_k = \bigoplus_{n=1}^{k} \mathfrak{W}_n$, and $\mathfrak{Z}_k = \overline{\bigoplus_{n=k}^{\infty} \mathfrak{W}_n}$.

Definition 3. Suppose that $(\mathfrak{W}, \|\cdot\|)$ is a real reflexive and separable Banach space, $\mathcal{F} \in C^1(\mathfrak{W}, \mathbb{R})$, $c \in \mathbb{R}$. We say that $\mathcal{F}$ fulfills the $(C)_c^*$-condition (with respect to $\mathfrak{Y}_n$) if any sequence $\{v_n\}_{n \in \mathbb{N}} \subset \mathfrak{W}$ for which $v_n \in \mathfrak{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{F}(v_n) \to c \quad \text{and} \quad \|(\mathcal{F}|_{\mathfrak{Y}_n})'(v_n)\|_{\mathfrak{W}^*}(1 + \|v_n\|) \to 0 \text{ as } n \to \infty,$$

has a subsequence converging to a critical point of $\mathcal{F}$.

Proposition 1 ([38])**.** *Supposing that $(\mathfrak{W}, \|\cdot\|)$ is a Banach space, $\mathcal{F} \in C^1(\mathfrak{W}, \mathbb{R})$ is an even functional. If there is $k_0 > 0$ so that, for each $k \geq k_0$, there exist $\beta_k > \alpha_k > 0$, such that*

(D1) $\inf\{\mathcal{F}(v) : \|v\|_{\mathfrak{W}} = \beta_k, v \in \mathfrak{Z}_k\} \geq 0$*;*
(D2) $b_k := \max\{\mathcal{F}(v) : \|v\|_{\mathfrak{W}} = \alpha_k, v \in \mathfrak{Y}_k\} < 0$*;*
(D3) $c_k := \inf\{\mathcal{F}(v) : \|v\|_{\mathfrak{W}} \leq \beta_k, v \in \mathfrak{Z}_k\} \to 0$ *as* $k \to \infty$*;*
(D4) $\mathcal{F}$ *fulfills the* $(C)_c^*$*-condition for every* $c \in [c_{k_0}, 0)$*,*

then $\mathcal{F}$ admits a sequence of negative critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Lemma 11. *Assuming that* (P), (A1)–(A2), (K1)–(K2) *and* (B1)–(B2) *hold, J_λ satisfies the $(C)_c^*$ condition.*

Proof. Since X is a reflexive Banach space, and Φ', Ψ' are of type (S_+), the idea of the proof is absolutely the same as that in ([38], Lemma 3.12). □

Theorem 1. *Suppose that* (P), (A1)–(A2), (K1)–(K2) *and* (B1)–(B3) *hold. If $g(x, -s) = -g(x, s)$ holds for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$, then the problem* (1) *has a sequence of nontrivial solutions $\{\omega_n\}$ in X such that $J_\lambda(\omega_n) \to 0$ as $n \to \infty$ for all $\lambda > 0$.*

Proof. In accordance with the oddness of g and Lemma 11, one has that the functional J_λ is even, and that it satisfies the $(C)_c^*$ condition for every $c \in \mathbb{R}$. Hence, we will prove that conditions (D1), (D2) and (D3) of Proposition 1 hold.

(D1): For convenience, we denote

$$\nu_{1,k} = \sup_{\|\omega\|_X = 1, \omega \in \mathfrak{Z}_k} \|\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)}, \quad \nu_{2,k} = \sup_{\|\omega\|_X = 1, \omega \in \mathfrak{Z}_k} \|\omega\|_{L^{\ell(\cdot)}(\mathbb{R}^N)}.$$

Then, it is easy to verify that $\nu_{1,k} \to 0$ and $\nu_{2,k} \to 0$ as $k \to \infty$(see [38]). Denote $\nu_k = \max\{\nu_{1,k}, \nu_{2,k}\}$. Let $\nu_k < 1$ for k large enough. It follows from Lemmas 1, 5, and Remark 1 that

$$
\begin{aligned}
J_\lambda(\omega) &= \mathcal{K}\Big([\omega]_{s,r(\cdot,\cdot)}\Big) + \int_{\mathbb{R}^N} \frac{b(x)}{r(x)}|\omega|^{r(x)}\,dx - \lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|\omega|^{q(x)} - \int_{\mathbb{R}^N} G(x,\omega)\,dx \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|\omega\|_X^{r^-} - \lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|\omega|^{q(x)}dx - \int_{\mathbb{R}^N} G(x,\omega)\,dx \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|\omega\|_X^{r^+} - \frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} \max\Big\{\|\omega\|^{q^-}_{L^{r(\cdot)}(\mathbb{R}^N)}, \|\omega\|^{q^+}_{L^{r(\cdot)}(\mathbb{R}^N)}\Big\} \\
&\quad - 2\|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\|\omega\|_{L^{r'(\cdot)}(\mathbb{R}^N)} - \frac{\rho_2}{\ell^-}\max\{\|\omega\|^{\ell^-}_{L^{\ell(\cdot)}(\mathbb{R}^N)}, \|\omega\|^{\ell^+}_{L^{\ell(\cdot)}(\mathbb{R}^N)}\} \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|\omega\|_X^{r^+} - \frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} \nu_k^{q^-}\|\omega\|_X^{q^+} - \|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\nu_k\|\omega\|_X - \frac{\rho_2}{\ell^-}\nu_k^{\ell^-}\|\omega\|_X^{\ell^+} \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|\omega\|_X^{r^+} - \left(\frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{\rho_2}{\ell^-}\right)\nu_k^{q^-}\|\omega\|_X^{\ell^+} - \|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\nu_k\|\omega\|_X
\end{aligned}
$$

for sufficiently large k and $\|\omega\|_X \geq 1$. Choose

$$
\beta_k = \left[\left(\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right)\nu_k^{q^-}\right]^{\frac{1}{r^- - 2\ell^+}}. \tag{6}
$$

Let $\omega \in \mathfrak{Z}_k$ with $\|\omega\|_X = \beta_k > 1$ for sufficiently large k. Then, there exists $k_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
J_\lambda(\omega) &\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|\omega\|_X^{r^-} - \left(\frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{\rho_2}{\ell^-}\right)\nu_k^{q^-}\|\omega\|_X^{2\ell^+} - \|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\nu_k\|\omega\|_X \\
&\geq \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\beta_k^{r^-} - \nu_k^{\frac{q^-+r^--2\ell^+}{r^--2\ell^+}}\|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\left[\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right]^{\frac{1}{r^--2\ell^+}} \\
&\geq 0
\end{aligned}
$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, because $\lim_{k\to\infty}\beta_k = \infty$. Therefore,

$$
\inf\{J_\lambda(\omega) : \omega \in \mathfrak{Z}_k, \|\omega\|_X = \beta_k\} \geq 0.
$$

(D2): Since $\mathfrak{Y}_k$ is dimensionally finite, all the norms are equivalent. Then, there are positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$
\varsigma_{1,k}\|\omega\|_X \leq \|\omega\|_{L^{m(\cdot)}(\nu,\mathbb{R}^N)} \text{ and } \|\omega\|_{L^{\ell(\cdot)}(\mathbb{R}^N)} \leq \varsigma_{2,k}\|\omega\|_X
$$

for any $\omega \in \mathfrak{Y}_k$. Let $\omega \in \mathfrak{Y}_k$ with $\|\omega\|_X \leq 1$. From (B1) and (B3), there are $\mathcal{C}_1, \mathcal{C}_2 > 0$ such that

$$
G(x,t) \geq \mathcal{C}_1\nu(x)|t|^{m(x)} - \mathcal{C}_2|t|^{\ell(x)}
$$

for almost all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Observe that

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)}}{r(x,y)|x-y|^{N+sr(x,y)}}\,dx\,dy \leq \mathcal{C}_3
$$

for some positive constant $\mathcal{C}_3$. Then, we have

$$
\begin{aligned}
J_\lambda(\omega) &= \mathcal{K}\Big([\omega]_{s,r(\cdot,\cdot)}\Big) + \int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|\omega|^{r(x)}dx - \lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|\omega|^{q(x)}dx - \int_{\mathbb{R}^N} G(x,\omega)\,dx \\
&\le \left(\sup_{0\le\xi\le\mathcal{C}_3} K(\xi)\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^{r(x,y)}}{r(x,y)|x-y|^{N+sr(x,y)}}\,dx\,dy + \int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|\omega|^{r(x)}dx \\
&\qquad - \mathcal{C}_1\int_{\mathbb{R}^N}\nu(x)|\omega|^{m(x)}dx + \mathcal{C}_2\int_{\mathbb{R}^N}|\omega|^{\ell(x)}\,dx \\
&\le C_4\|\omega\|_X^{r^-} - \mathcal{C}_1\min\{\|\omega\|^{m^+}_{L^{m(\cdot)}(\nu,\mathbb{R}^N)}, \|\omega\|^{m^-}_{L^{m(\cdot)}(\nu,\mathbb{R}^N)}\} + \mathcal{C}_2\max\{\|\omega\|^{\ell^-}_{L^{\ell(\cdot)}(\mathbb{R}^N)}, \|\omega\|^{\ell^+}_{L^{\ell(\cdot)}(\mathbb{R}^N)}\} \\
&\le C_4\|\omega\|_X^{r^-} - \mathcal{C}_1\min\{\varsigma_{1,k}^{m^-},\varsigma_{1,k}^{m^+}\}\|\omega\|_X^{m^+} + \mathcal{C}_2\max\{\varsigma_{2,k}^{\ell_-},\varsigma_{2,k}^{\ell_+}\}\|\omega\|_X^{\ell^-}.
\end{aligned}
$$

Let $h(s) = C_4 s^{r^-} - \mathcal{C}_1\min\{\varsigma_{1,k}^{m^-},\varsigma_{1,k}^{m^+}\}s^{m^+} + \mathcal{C}_2\max\{\varsigma_{2,k}^{\ell_-},\varsigma_{2,k}^{\ell_+}\}s^{\ell^-}$. Since $m^+ < r^- < \ell^-$, we infer $h(s) < 0$ for all $s \in (0, s_0)$ for sufficiently small $s_0 \in (0,1)$. Hence, $J_\lambda(\omega) < 0$ for all $\omega \in \mathfrak{Y}_k$ with $\|\omega\|_X = s_0$. Choosing $\alpha_k = s_0$ for all $k \in \mathbb{N}$, one has

$$
b_k := \max\{J_\lambda(\omega) : \omega \in \mathfrak{Y}_k, \|\omega\|_X = \alpha_k\} < 0.
$$

If necessary, we can change k_0 to a large value, so that $\beta_k > \alpha_k > 0$ for all $k \ge k_0$.

(D3): Because $\mathfrak{Y}_k \cap \mathfrak{Z}_k \neq \phi$ and $0 < \alpha_k < \beta_k$, we have $d_k \le b_k < 0$ for all $k \ge k_0$. For any $\omega \in \mathfrak{Z}_k$ with $\|\omega\|_X = 1$ and $0 < s < \beta_k$, one has

$$
\begin{aligned}
J_\lambda(s\omega) &= \mathcal{K}\Big([s\omega]_{s,r(\cdot,\cdot)}\Big) + \int_{\mathbb{R}^N}\frac{b(x)}{r(x)}|s\omega|^{r(x)}dx - \lambda\int_{\mathbb{R}^N}\frac{\kappa(x)}{q(x)}|s\omega|^{q(x)}dx - \int_{\mathbb{R}^N} G(x,s\omega)\,dx \\
&\ge \frac{\min\{\tau_0,\vartheta\}}{\vartheta r^+}\|s\omega\|_X^{r^+} - \frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\max\Big\{\|s\omega\|^{q^-}_{L^{r(\cdot)}(\mathbb{R}^N)}, \|s\omega\|^{q^+}_{L^{r(\cdot)}(\mathbb{R}^N)}\Big\} \\
&\quad - 2\|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\|s\omega\|_{L^{r(\cdot)}(\mathbb{R}^N)} - \frac{\rho_2}{\ell^-}\max\{\|s\omega\|^{\ell^-}_{L^{\ell(\cdot)}(\mathbb{R}^N)}, \|s\omega\|^{\ell^+}_{L^{\ell(\cdot)}(\mathbb{R}^N)}\} \\
&\ge -\frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{\ell(\cdot)}{\ell(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\beta_k^{q^+}\nu_k^{q^-} - 2\|\rho_1\|_{L^{r'(\cdot)}(\mathbb{R}^N)}\beta_k\nu_k - \frac{\rho_2}{\ell^-}\beta_k^{\ell^+}\nu_k^{\ell^-}
\end{aligned}
$$

for sufficiently large k. Hence, it follows from the definition of β_k that

$$
\begin{aligned}
c_k &\ge -\frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\beta_k^{q^+}\nu_k^{q^-} - \frac{\rho_2}{\ell^-}\beta_k^{\ell^+}\nu_k^{\ell^-} \\
&= -\frac{2\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)}\left[\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right]^{\frac{q^+}{r^- - 2\ell^+}}\nu_k^{\frac{q^-q^+ + q^-(r^- - 2\ell^+)}{r^- - 2\ell^+}} \\
&\quad - 2\|\rho_1\|_{L^{r'(\cdot)}}\left[\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right]^{\frac{1}{r^- - 2\ell^+}}\nu_k^{\frac{q^- + r^- - 2\ell^+}{r^- - 2\ell^+}} \\
&\quad - \frac{\rho_2}{\ell^-}\left[\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right]^{\frac{\ell^+}{r^- - 2\ell^+}}\nu_k^{\frac{q^-\ell^+ + \ell^-(r^- - 2\ell^+)}{r^- - 2\ell^+}}.
\end{aligned} \tag{7}
$$

Because $r^- < \ell^+$, $q^+ + r^- < 2\ell^+$, $q^-\ell^+ + \ell^- r^- < 2\ell^-\ell^+$ and $\nu_k \to 0$ as $k \to \infty$, we conclude that $\lim_{k\to\infty} c_k = 0$.

Consequently, all conditions of Proposition 1 are verified, and we assert that problem (1) admits a sequence of nontrivial solutions $\{\omega_n\}$ in X such that $J_\lambda(\omega_n) \to 0$ as $n \to \infty$ for all $\lambda > 0$. □

Remark 2. *From the viewpoint of [23,38,44,48], the conditions* $(f1)$ *and* $(f2)$ *are essential in obtaining Theorem 1. Under these two assumptions, the existence of two sufficiently large sequences* $0 < \alpha_k < \beta_k$ *is established in papers [23,38,44]. Regrettably, by utilizing the analogous argument* *as in [38], we cannot show the property (D3) in Theorem 1. More precisely, if we change β_k in (6) into*

$$\tilde{\beta}_k = \left[\left(\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right)\nu_k^{q^-}\right]^{\frac{1}{r^- - \ell^+}}$$

and $t^+ + r^- > \ell^+$, then in relation (7),

$$\tilde{\beta}_k^{t^+}\nu_k^{t^-} = \left[\frac{4\lambda}{q^-}\|\kappa\|_{L^{\frac{r(\cdot)}{r(\cdot)-q(\cdot)}}(\mathbb{R}^N)} + \frac{2\rho_2}{\ell^-}\right]^{\frac{t^+}{r^- - \ell^+}}\nu_k^{\frac{t^-(t^+ + r^- - \ell^+)}{r^- - \ell^+}} \to \infty \text{ as } k \to \infty,$$

and thus we cannot obtain the property (D3) in $\tilde{\beta}_k$. However, the authors in [23,44,52] overcome this difficulty with a new setting for β_k. In contrast, the existence of two sequences $0 < \alpha_k < \beta_k \to 0$ as $k \to \infty$ is obtained in [30,46,47] when $(f1)$ is satisfied. On the other hand, we obtain Theorem 1 when $(f1)$ and $(f2)$ are not assumed. From this perspective, the proof of Theorem 1 is different from that of recent works [23,30,38,44,46–48].

4. Conclusions

In this paper, on a new class of nonlinear term g, we give the existence results of a sequence of infinitely many energy solutions by employing the dual fountain theorem. When we verify assumptions in the dual fountain theorem, the conditions on the nonlinear term g near zero and at infinity are essential, however, we obtain our main result without assuming them. This is a novelty of the present paper. Additionally, we address to the readers several comments and perspectives.

I. We point out that with a similar analysis, our main consequences continue to hold when $\mathcal{L}\omega$ in (1) is changed into any non-local integro-differential operator $\mathfrak{L}_{\mathcal{L}}$, defined pointwise as

$$\mathfrak{L}_{\mathcal{L}}\omega(x) = 2\int_{\mathbb{R}^N} |\omega(x) - \omega(y)|^{r(x,y)-2}(\omega(x) - \omega(y))\mathcal{L}(x,y)dy \quad \text{for all } x \in \mathbb{R}^N,$$

where $\mathcal{L} : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ is a kernel function satisfying the properties

(L1) $m\mathcal{L} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $m(x,y) = \min\{|x-y|^{r(x,y)}, 1\}$;
(L2) there exists a positive constant θ_0 such that $\mathcal{L}(x,y)|x-y|^{N+sr(x,y)} \geq \theta_0$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $x \neq y$, where $0 < s < 1$;
(L3) $\mathcal{L}(x,y) = \mathcal{L}(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

II. A new research direction that has a strong relationship with several related applications is the study of critical double-phase-type equations

$$\begin{aligned}(-\Delta)^s_{r(\cdot)}\omega + (-\Delta)^s_{p(\cdot)}\omega + b(x)(|\omega|^{r(x)-2}\omega + |\omega|^{p(x)-2}\omega) \\ = \lambda\kappa(x)|\omega|^{q(x)-2}\omega + g(x,\omega) \quad \text{in} \quad \mathbb{R}^N,\end{aligned}$$

where $r(x) < p(x)$ for all $x \in \mathbb{R}^N$ and $\{x \in \mathbb{R}^N : q(x) = r_s^*(x)\} \neq \emptyset$.

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