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Article

Existence Results for Caputo Tripled Fractional Differential Inclusions with Integral and Multi-Point Boundary Conditions

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Abstract: In this study, based on Coitz and Nadler's fixed point theorem and the non-linear alternative for Kakutani maps, existence results for a tripled system of sequential fractional differential inclusions (SFDIs) with integral and multi-point boundary conditions (BCs) in investigated. A practical examples are given to illustrate the obtained the theoretical results.

Keywords: tripled system; existence; fixed point theorems; inclusions

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1. Introduction

FCs ware first employed in 1695 when L'Hopital summarized his discoveries in a letter to Leibniz. Fractional calculus (FCs) was studied by several twentieth century authors, including Liouville, Grunwald, Letnikov, and Riemann. This field of mathematics, known as fractional differential equations, was invented by mathematicians as a pure branch of mathematics with just a few applications in mathematics. Fractional calculus is a well-established subject with applications in many applied sciences, such as visco-elasticity, medical, and environment, which leads the fractional differential equations to become extremely prevalent. We recommend the monographs [1–8] and the recently mentioned papers [9–17]. It is worth noting that most of the works in the field of fractional differentiation focus mostly on R-L and Caputo types. See [18–22].

In 1772, Russian scientists presented a general idea of stability, such as Lyapunov (1758–1817), where the general theme of his doctoral dissertation was movement stability, and his work soon spread all over Russia and later in the West. With the process of research, scientists entered the time delay, and the first to describe these systems with a time delay was the scientist (Boltzman), who studied its effect but did not refer to the time delay in realistic models.

In the early 1900s, a disagreement arose over the necessity of introducing time delays into systems to predict their future development, but this point of view contradicted the Newtonian traditions, which claimed that knowledge of the current values of all relevant variables should suffice for the prediction. Ulam and Hyers, on the other hand, recognized unknown types of stability known as ulam-stability. Hyer's type of stability study contributes expressively to our understanding of population dynamics and fluid movement, see [23].

In mathematics, differential inclusions relate to one or more functions and their derivatives. In applications, functions generally represent physical quantities, derivatives represent their rates of change, and differential inclusion defines the relationship between the two. Because these relationships are so common, differential equations play a prominent role in many disciplines, including engineering, physics, economics, and biology. The study of differential inclusions mainly consists of studying their solutions (the set of functions that satisfy the equation), and the properties of their solutions. The simplest differential inclusions can be solved by explicit formulas. However, many properties of solutions to particular differential inclusions may be determined without being exactly calculated. If a closed expression is not available for the solutions, the solutions may be numerically approximated using computers. Dynamical systems theory focuses on the qualitative analysis of systems described by differential equations and differential inclusions, while many numerical methods have been developed to determine solutions with a certain degree of precision.

Many of the basic laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations developed first with the sciences in which the equations originated and where the results were put into practice. However, various problems which sometimes arise in quite distinct scientific fields may result in identical differential equations. Whenever this happens, the mathematical theory behind the equations can be seen as a unifying principle behind the various phenomena. For example, consider the propagation of light and sound in the atmosphere, and the waves on the surface of a pond. They can all be described by the same second-order partial differential equation, which is the wave equation, that allows us to think of light and sound as forms of waves, much like the familiar waves in water. Heat conduction, developed by Joseph Fourier, is governed by a second-order partial differential equation, the heat equation. It turns out that many diffusion processes, though apparently different, are described by the same equation; the Black–Scholes equation in finance, for example, is related to the heat equation [24,25]. In [26], the authors were the first who developed the idea of the tripled fixed points. Karakaya et al. [27] introduce tripled fixed points for a class of condensing operators in Banach spaces. In [25], the authors studied the existence results for the following BVP.

$$\begin{cases} {}^{c}\mathcal{D}^{\psi_k}\mathcal{Z}_k(\tau) = \mathfrak{f}_k(\tau, \mathcal{Z}(\tau)), & 1 < \psi_k \leq 2, \\ \mathcal{Z}_k^{(j)}(0) = a_{k,j}\mathcal{Z}_{\varepsilon(k)}^{(j)}(\mathcal{T}), & k = 1,2,3; j = 0,1. \end{cases}$$

where ${}^{c}\mathcal{D}_0^{\psi_k}$ denotes the Caputo fractional derivatives (CFDs) of order ψ_k, $\tau \in J = [0, \mathcal{T}]$, $\mathfrak{f}_k : J \times \mathcal{R}_e^3 \to \mathcal{R}_e$ are continuous functions, $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) \in \mathcal{R}_e^3, \varepsilon = (1,2,3)$ is a cyclic permutation, and $a_{k,j} \in k = 1,2,3, j = 0,1$. In this work, motivated by [28], we consider the following system of sequential fractional differential inclusions:

$$\begin{cases} ({}^{c}\mathfrak{D}^{\psi} + \varphi\, {}^{c}\mathfrak{D}^{\psi-1})\mathcal{Z}(\varpi) \in \mathscr{F}_1(\varpi, \mathcal{Z}(\varpi), \mathscr{Q}(\varpi), \mathcal{Y}(\varpi)), \quad 2 < \psi \leq 3, \\ ({}^{c}\mathfrak{D}^{\phi} + \varphi\, {}^{c}\mathfrak{D}^{\phi-1})\mathscr{Q}(\varpi) \in \mathscr{F}_2(\varpi, \mathcal{Z}(\varpi), \mathscr{Q}(\varpi), \mathcal{Y}(\varpi)), \quad 2 < \phi \leq 3, \\ ({}^{c}\mathfrak{D}^{\omega} + \varphi\, {}^{c}\mathfrak{D}^{\omega-1})\mathcal{Y}(\varpi) \in \mathscr{F}_3(\varpi, \mathcal{Z}(\varpi), \mathscr{Q}(\varpi), \mathcal{Y}(\varpi)), \quad 3 < \omega \leq 4, \\ \mathcal{Z}(0) = 0, \qquad \mathcal{Z}'(0) = 0, \qquad \mathcal{Z}(\mathcal{T}) = \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \mathscr{Q}(\zeta_j) + \Pi_1 \mathcal{I}^{\varsigma}\mathscr{Q}(\vartheta), \\ \mathscr{Q}(0) = 0, \qquad \mathscr{Q}'(0) = 0, \qquad \mathscr{Q}(\mathcal{T}) = \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \mathcal{Y}(\zeta_j) + \Pi_2 \mathcal{I}^{\varrho}\mathcal{Y}(\vartheta), \\ \mathcal{Y}(0) = 0, \qquad \mathcal{Y}'(0) = 0, \qquad \mathcal{Y}''(0) = 0, \qquad \mathcal{Y}(\mathcal{T}) = \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \mathcal{Z}(\zeta_j) + \Pi_3 \mathcal{I}^{\delta}\mathcal{Z}(\vartheta), \end{cases} \tag{1}$$

where ${}^{c}D^{\chi}$ is a CFDs of order $\chi \in \{\psi, \phi, \omega\}, \mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3 : [0, \mathcal{T}] \times \mathcal{R}_e \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{P}(\mathcal{R}_e)$ are given continuous functions, $\mathcal{P}(\mathcal{R}_e)$ is the family of all non-empty subset of $\mathcal{R}_e$, $\zeta_j \in \mathcal{R}_e, j = 1, \ldots, k-2$ and $\vartheta \in [0, \mathcal{T}]$.

The Caputo SFDEs with multi-point and integral boundary conditions discussed in this work are the most widely used Caputo fractional derivatives. The novelty and originality of this work is summarized by using Covitz and Nadler's fixed point theorem and the non-linear alternative for Kakutani maps in showing the existence results for a tripled system of sequential fractional differential inclusions.

Preliminaries are introduced in the second section, main results are shown in the third section. Finally, in Section 4, we give some numerical examples to show the effectiveness of the obtained theoretical results.

2. Preliminaries

This portion introduces basic fractional calculus concepts, definitions, and tentative results [1–3].

Let $\hat{\mathcal{J}} = \mathcal{C}([0,\mathcal{T}],\mathcal{R}_e)$ be a Banach space endowed with the norm $\|\mathcal{Z}\| = \sup\{|\mathcal{Z}(\varpi)|, \varpi \in [0,\mathcal{T}]\}$. Then $(\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}, \|(\mathcal{Z},\mathcal{Q},\mathcal{Y})\|_{\hat{\mathcal{J}}})$ is also a Banach space equipped with the norm $\|(\mathcal{Z},\mathcal{Q},\mathcal{Y})\|_{\hat{\mathcal{J}}} = \|\mathcal{Z}\| + \|\mathcal{Q}\| + \|\mathcal{Y}\|, \mathcal{Z},\mathcal{Q},\mathcal{Y} \in \hat{\mathcal{J}}$.

Let $(\mathcal{M}_1, \|\cdot\|)$ be a normed space and that $\mathcal{U}_{cl}(\mathcal{M}_1) = \{\mathcal{A} \in \mathcal{U}(\mathcal{M}_1) : \mathcal{A} \text{ is closed}\}$, $\mathcal{U}_{c,cp}(\mathcal{M}_1) = \{\mathcal{A} \in \mathcal{U}(\mathcal{M}_1) : \mathcal{A} \text{ is convex and compact}\}$.

A multi-valued map $\mathcal{Q} : \mathcal{M}_1 \to \mathcal{U}(\mathcal{M}_1)$ is

(a) Convex valued if $\mathcal{Q}(s)$ is convex $\forall s \in \mathcal{M}_1$;
(b) Upper semi-continuous (U.S.C.) on $\mathcal{M}_1$ if, for each $s_0 \in \mathcal{M}_1$; the set $\mathcal{Q}(s_0)$ is a non-empty closed subset of $\mathcal{M}_1$ and if, for each open set $\mathcal{V}$ of $\mathcal{M}_1$ containing $\mathcal{Q}(s_0)$, there exists an open neighborhood $\mathcal{V}_0$ of s_0, such that $\mathcal{Q}(\mathcal{V}_0) \subset \mathcal{V}$;
(c) Lower semi-continuous (L.S.C.) if the set $\{s \in \mathcal{M}_1 : \mathcal{Q}(s) \cap \mathcal{E} \neq \oslash\}$ is open for any open set $\mathcal{E}$ in $\mathcal{H}$;
(d) Completely continuous (C.C) if $\mathcal{Q}(\mathcal{E})$ is relatively compact (r.c) for every $\mathcal{E} \in \mathcal{U}_b(\mathcal{M}_1) = \{\mathcal{A} \in \mathcal{U}(\mathcal{M}_1) : \mathcal{A} \text{ is bounded}\}$.

A map $\mathcal{Q} : [c,d] \to \mathcal{U}_{cl}(\mathcal{R}_e)$ of multi-valued is said to be measurable if, for every $s \in \mathcal{R}_e$, the function $t \longmapsto d(s,\mathcal{Q}(t)) = \inf\{|s-l| : l \in \mathcal{Q}(t)\}$ is measurable.

A multi-valued map $\mathcal{Q} : [c,d] \times \mathcal{R}_e \to \mathcal{U}(\mathcal{R}_e)$ is said to be Caratheodory if

(i) $t \longmapsto \mathcal{Q}(t,q,s)$ is measurable for each $q,s \in \mathcal{R}_e$;
(ii) $(q,s) \longmapsto \mathcal{Q}(t,q,s)$ is U.S.C for almost all $t \in [c,d]$.

Further, a Caratheodory function $\mathcal{Q}$ is called $\mathcal{L}^1$-Caratheodory if

(i) For each $\varepsilon > 0$, $\exists\ \phi_\varepsilon \mathcal{L}^1([c,d],\mathcal{R}_e^+) \ni \|\mathcal{Q}(t,q,s)\| = \sup\{|q| : q \in \mathcal{Q}(t,q,s) \leq \phi_\varepsilon(t)\}$ $\forall\ q,s \in \mathcal{R}_e$ with $\|q\|, \|s\| \leq \varepsilon$ and for a.e. $t \in [c,d]$.

Lemma 1. *Let $\mathcal{M}$ a closed convex subset of a Banach space $\mathcal{M}_1$ and $\mathcal{W}$ be an open subset of $\mathcal{K}$ with $0 \in \mathcal{W}$. In addition, $\mathcal{H} : \hat{\mathcal{W}} \to \mathcal{Z}_{c,cp}(\mathcal{K})$ is an u.s.c compact map. Then either*

- *$\mathcal{H}$ has fixed point in $\hat{\mathcal{W}}$ or*
- *$\exists\ w \in \partial\mathcal{W}$ and $\lambda \in (0,1)$, such that $w \in \lambda\mathcal{H}(w)$.*

Lemma 2 ([29])**.** *Let $\mathcal{Y} : \mathcal{M}_1 \to \mathcal{M}_1$ be a completely continuous operator in Banach Space $\mathcal{M}_1$ and the set $\Psi = \{s \in \mathcal{M}_1 | s = \delta\mathcal{Y}s, 0 < \delta < 1\}$ is bounded. Then $\mathcal{Y}$ has a fixed point in $\mathcal{M}_1$.*

Definition 1. *The fractional integral of order ψ with the lower limit zero for a function k is defined as*

$$I^{\psi}k(\tau) = \frac{1}{\Gamma(\psi)}\int_0^{\tau} \frac{k(\rho)}{(\tau-\rho)^{1-\psi}}ds, \tau > 0, \psi > 0, \tag{2}$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\psi) = \int_0^{\infty} \tau^{\psi-1}e^{-\tau}d\tau$.

Definition 2. *The R-L fractional derivative of order $\psi > 0, n-1 < \psi < n, n \in \mathbb{N}$ is defined as*

$$D_{0+}^{\psi}k(\tau) = \frac{1}{\Gamma(n-\psi)}\left(\frac{d}{d\tau}\right)^n \int_0^{\tau} (\tau-\rho)^{n-\psi-1}k(\rho)d\rho, \tau > 0, \tag{3}$$

where the function k has absolutely continuous derivative up to order $(n-1)$.

Definition 3. *The Caputo derivative of order $\psi \in [n-1, n)$ for a function $k : [0, \infty) \to (\mathbb{R})$ can be written as*

$$ {}^{c}D_{0+}^{\psi} k(\tau) = D_{0+}^{\psi}\left(k(\tau) - \sum_{m=0}^{n-1} \frac{\tau^m}{m!} f^{(m)}(0)\right), \tau > 0, n-1 < r < n. \tag{4}$$

Note that the CFDs of order $\psi \in [n-1, n)$ almost everywhere on $[0, \infty)$ if $k \in \mathcal{AC}^n([0, \infty), (\mathbb{R}))$.

Next, we state and prove the auxiliary lemma, which will help us in constructing the existence results for our proposed system.

Lemma 3. *Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathcal{C}[0, \mathcal{T}]$ and $\Delta \neq 0$. Then the solution of the linear fractional differential system,*

$$\begin{cases} ({}^{c}\mathfrak{D}^{\psi} + \varphi {}^{c}\mathfrak{D}^{\psi-1})\mathcal{Z}(\varpi) = \mathcal{G}_1, \quad 2 < \psi \leq 3, \\ ({}^{c}\mathfrak{D}^{\phi} + \varphi {}^{c}\mathfrak{D}^{\phi-1})\mathscr{Q}(\varpi) = \mathcal{G}_2, \quad 2 < \phi \leq 3, \\ ({}^{c}\mathfrak{D}^{\omega} + \varphi {}^{c}\mathfrak{D}^{\omega-1})\mathcal{Y}(\varpi) = \mathcal{G}_3, \quad 3 < \omega \leq 4, \\ \mathcal{Z}(0) = 0, \qquad \mathcal{Z}'(0) = 0, \qquad \mathcal{Z}(\mathcal{T}) = \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \mathscr{Q}(\zeta_j) + \Pi_1 \mathcal{I}^{\varsigma} \mathscr{Q}(\vartheta), \\ \mathscr{Q}(0) = 0, \qquad \mathscr{Q}'(0) = 0, \qquad \mathscr{Q}(\mathcal{T}) = \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \mathcal{Y}(\zeta_j) + \Pi_2 \mathcal{I}^{\varrho} \mathcal{Y}(\vartheta), \\ \mathcal{Y}(0) = 0, \qquad \mathcal{Y}'(0) = 0, \qquad \mathcal{Y}''(0) = 0, \qquad \mathcal{Y}(\mathcal{T}) = \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \mathcal{Z}(\zeta_j) + \Pi_3 \mathcal{I}^{\delta} \mathcal{Z}(\vartheta), \end{cases} \tag{5}$$

is given by

$$\begin{aligned} \mathcal{Y}(\varpi) = & \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2 \mathcal{E}_1} \Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j - \rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathcal{G}_2(\tau) d\tau \right) d\rho \\ & + \Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathcal{G}_2(m) dm \right) d\tau \right) d\rho \\ & - \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathcal{G}_1(\tau) d\tau \right) d\rho \\ & + \frac{1}{\Delta} \Bigg(\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_5 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathcal{G}_3(\tau) d\tau \right) d\rho \\ & + \Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \mathcal{G}_3(m) dm \right) d\tau \right) d\rho \\ & - \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathcal{G}_2(\tau) d\tau \right) d\rho \Bigg\} \\ & + \mathcal{E}_2 \mathcal{E}_4 \mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathcal{G}_2(\tau) d\tau \right) d\rho \\ & + \Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathcal{G}_2(m) dm \right) d\tau \right) d\rho \\ & - \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathcal{G}_1(\tau) d\tau \right) d\rho \Bigg\} \\ & + \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_4 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathcal{G}_1(\tau) d\tau \right) d\rho \\ & + \Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathcal{G}_1(m) dm \right) d\tau \right) d\rho \end{aligned} \tag{6}$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho\Big\}\Big)\Big]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(\tau)d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{Q}(\varpi)\\
&=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(\tau)d\tau\right)d\rho,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\mathcal{Y}(\varpi)\\
&=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathcal{G}_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathcal{G}_2(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathcal{G}_3(\tau)d\tau\right)d\rho,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\mathcal{E}_1 =& \frac{(\varphi\mathcal{T} - 1 + e^{-\varphi\mathcal{T}})}{\varphi^2}, \mathcal{E}_3 = \frac{(\varphi\mathcal{T} - 1 + e^{-\varphi\mathcal{T}})}{\varphi^2}, \mathcal{E}_5 = \frac{(\varphi^2\mathcal{T}^2 - 2\varphi\mathcal{T} + 2 - 2e^{-\varphi\mathcal{T}})}{\varphi^3}, \\
\mathcal{E}_2 =& \frac{1}{\varphi^2}\left[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j(\varphi\zeta_j - 1 + e^{-\varphi\zeta_j}) + \Pi_1 \int_0^{\vartheta} \frac{(\vartheta-\rho)^{\varsigma-1}}{\Gamma(\varsigma)}(\varphi s - 1 + e^{-\varphi s})d\rho\right], \\
\mathcal{E}_4 =& \frac{1}{\varphi^3}\left[\Upsilon_2 \sum_{j=1}^{k-2} \nu_j(\varphi^2\zeta_j{}^2 - 2\varphi\zeta_j + 2 - e^{-\varphi\zeta_j}) + \Pi_2 \int_0^{\vartheta} \frac{(\vartheta-\rho)^{\varrho-1}}{\Gamma(\varrho)}(\varphi^2 s^2 - 2\varphi s + 2 - 2e^{-\varphi s})d\rho\right], \qquad (9) \\
\mathcal{E}_6 =& \frac{1}{\varphi^2}\left[\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j(\varphi\zeta_j - 1 + e^{-\varphi\zeta_j}) + \Pi_3 \int_0^{\vartheta} \frac{(\vartheta-\rho)^{\delta-1}}{\Gamma(\delta)}(\varphi s - 1 + e^{-\varphi s})d\rho\right], \\
\Delta =& (\mathcal{E}_1\mathcal{E}_3\mathcal{E}_5 - \mathcal{E}_2\mathcal{E}_4\mathcal{E}_6).
\end{aligned}
$$

3. Multi-Valued System

Definition 4. *A function $(\mathcal{Z}, \mathcal{Q}, \mathcal{Y}) \in \mathcal{C}^1([0,\mathcal{T}],\mathcal{R}_e) \times \mathcal{C}^1([0,\mathcal{T}],\mathcal{R}_e) \times \mathcal{C}^1([0,\mathcal{T}],\mathcal{R}_e)$ satisfying the boundary conditions and for which there $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} = \mathcal{L}^1([0,\mathcal{T}],\mathcal{R}_e)$, such that*

$\mathfrak{f}(\varpi) \in \mathscr{F}_1(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)), \mathfrak{g}(\varpi) \in \mathscr{F}_2(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)), \mathfrak{h}(\varpi) \in \mathscr{F}_3(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$ a.e. on $\varpi \in [0,\mathcal{T}]$ and

With the help of Lemma 3, we define an operator $\mathscr{F} : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ by

$$
\begin{aligned}
&\mathscr{F}_1\big(\mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)\big) \\
=& \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m, \mathcal{Z}(m), \mathcal{Q}(m), \mathcal{Y}(m))dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho \\
&+ \frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m, \mathcal{Z}(m), \mathcal{Q}(m), \mathcal{Y}(m))dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m, \mathcal{Z}(m), \mathcal{Q}(m), \mathcal{Y}(m))dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m, \mathcal{Z}(m), \mathcal{Q}(m), \mathcal{Y}(m))dm\right)d\tau\right)d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho\Bigg\}\Bigg)\Bigg] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau, \mathcal{Z}(\tau), \mathcal{Q}(\tau), \mathcal{Y}(\tau))d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{F}_2\big(\mathcal{Z}(\varpi),\mathcal{Q}(\varpi),\mathcal{Y}(\varpi)\big)\\
=&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathscr{F}_3\big(\mathcal{Z}(\varpi),\mathcal{Q}(\varpi),\mathcal{Y}(\varpi)\big)=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\\
&\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m,\mathcal{Z}(m),\mathcal{Q}(m),\mathcal{Y}(m))dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau,\mathcal{Z}(\tau),\mathcal{Q}(\tau),\mathcal{Y}(\tau))d\tau\bigg)d\rho.
\end{aligned}
$$

For easy calculations, we set

$$
\begin{aligned}
\mathcal{P}_1 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\left[\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\right.\\
&+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\\
&\left.\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\right]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\psi)},\\
\mathcal{Q}_1 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\left[\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\right.\\
&+\frac{\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6}{\Delta}\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\\
&\left.+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right],\\
\mathcal{O}_1 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\mathcal{E}_1}\left[\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4}{\Delta}\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5}{\Delta}\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\right.\\
&\left.\left.+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\right\}\right],\\
\mathcal{P}_2 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\left[\mathcal{E}_4\mathcal{E}_6\left\{\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\right\}+\mathcal{E}_1\mathcal{E}_4\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\right.\\
&\left.\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\right], \qquad (10)\\
\mathcal{Q}_2 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\left[\mathcal{E}_4\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}\right.\\
&\left.+\mathcal{E}_1\mathcal{E}_5\left\{\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right\}\right]+\frac{(1-e^{-\varphi})}{\varphi\Gamma(\phi)},\\
\mathcal{O}_2 =& \frac{(\varphi-1+e^{-\varphi})}{\varphi^2\Delta}\left[\mathcal{E}_1\mathcal{E}_4\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\mathcal{E}_1\mathcal{E}_5\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\right.\\
&\left.\left.+\Pi_2\frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi+e^{-\varphi\varrho}-1)\right\}\right],\\
\mathcal{P}_3 =& \frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\left[\mathcal{E}_3\mathcal{E}_6\left\{\frac{\mathcal{T}^{\psi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\psi)}\right\}+\mathcal{E}_1\mathcal{E}_3\left\{\Upsilon_3\sum_{j=1}^{k-2}|\sigma_j|\left(\frac{\zeta_j^{\psi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\psi)}\right)\right.\right.\\
&\left.\left.+\Pi_3\frac{\delta^{\psi+\vartheta-1}}{\varphi^2\Gamma(\psi)\Gamma(\vartheta)}(\delta\varphi+e^{-\varphi\delta}-1)\right\}\right],\\
\mathcal{Q}_3 =& \frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\left[\mathcal{E}_3\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}|\xi_j|\left(\frac{\zeta_j^{\phi}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\phi)}\right)\right.\right.\\
&\left.\left.+\Pi_1\frac{\varsigma^{\phi+\vartheta-1}}{\varphi^2\Gamma(\phi)\Gamma(\vartheta)}(\varsigma\varphi+e^{-\varphi\varsigma}-1)\right\}+\mathcal{E}_2\mathcal{E}_6\left\{\frac{\mathcal{T}^{\phi-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\phi)}\right\}\right],\\
\mathcal{O}_3 =& \frac{(\varphi^2-2\varphi+2-2e^{-\varphi})}{\varphi^3\Delta}\left[\mathcal{E}_1\mathcal{E}_3\left\{\frac{\mathcal{T}^{\omega-1}(1-e^{-\varphi\mathcal{T}})}{\varphi\Gamma(\omega)}\right\}+\mathcal{E}_2\mathcal{E}_6\left\{\Upsilon_2\sum_{j=1}^{k-2}|\nu_j|\left(\frac{\zeta_j^{\omega}(1-e^{-\varphi\zeta_j})}{\varphi\Gamma(\omega)}\right)\right.\right.
\end{aligned}
$$

$$+\Pi_2 \frac{\varrho^{\omega+\vartheta-1}}{\varphi^2\Gamma(\omega)\Gamma(\vartheta)}(\varrho\varphi + e^{-\varphi\varrho} - 1)\Bigg\}\Bigg] + \frac{(1-e^{-\varphi})}{\varphi\Gamma(\omega)}.$$

$$\begin{aligned}\mathcal{M}_1 =& (\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||P_1||(\mathcal{U}_1(\mathcal{N})+\mathcal{V}_1(\mathcal{N})+\mathcal{W}_1(\mathcal{N}))+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||P_2||(\mathcal{U}_2(\mathcal{N})+\mathcal{V}_2(\mathcal{N})+\mathcal{W}_2(\mathcal{N}))\\ &+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||P_3||(\mathcal{U}_3(\mathcal{N})+\mathcal{V}_3(\mathcal{N})+\mathcal{W}_3(\mathcal{N}))\end{aligned}$$

Next, we define the operators $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \mathcal{P}(\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}})$ as follows: Then, we define an operator $\mathcal{K} : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \mathcal{P}(\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}})$ by

$$\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) = \begin{pmatrix} \mathcal{K}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) \\ \mathcal{K}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) \\ \mathcal{K}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) \end{pmatrix},$$

$$\begin{aligned}\mathcal{K}_1 =& \{h_1 \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} : \text{ there exist } \mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})},\\ &\text{such that } h_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) = \mathcal{Q}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi), \forall \tau \in [0,\mathcal{T}]\},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{K}_2 =& \{h_2 \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} : \text{ there exist } \mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})},\\ &\text{such that } h_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) = \mathcal{Q}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi), \forall \tau \in [0,\mathcal{T}]\},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{K}_3 =& \{h_3 \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} : \text{ there exist } \mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})},\\ &\text{such that } h_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) = \mathcal{Q}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi), \forall \tau \in [0,\mathcal{T}]\},\end{aligned} \tag{13}$$

3.1. The Caratheodory Case

Our first result dealing with convex values $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ is proved via the Leray–Schauder non-linear alternative for multi-valued maps.

Theorem 1. *Suppose that the following conditions are satisfied:*

$(\mathcal{B}_1)$ $\mathfrak{f}, \mathfrak{g}, \mathfrak{h} : [0,\mathcal{T}] \times \mathcal{R}^3 \to \mathcal{P}(\mathcal{R})$ *are* L^1 *Caratheodory and have convex values;*

$(\mathcal{B}_2)$ *There exist continuous non-decreasing functions* $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 : [0,\infty) \to (0,\infty)$ *functions* $P_1(\varpi), P_2(\varpi), P_3(\varpi) \in (\mathcal{C}[0,\mathcal{T}] \times \mathcal{R}_+)$, *such that*

$$||\mathscr{F}_1(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})||_{\mathcal{P}} := \sup\{|\mathfrak{f}| : \mathfrak{f} \in \mathscr{F}_1(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})\} \le P_1[\mathcal{U}_1(||\mathcal{Z}||) + \mathcal{V}_1(||\mathcal{Q}||) + \mathcal{W}_1(||\mathcal{Y}||)],$$
$$||\mathscr{F}_2(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})||_{\mathcal{P}} := \sup\{|\mathfrak{g}| : \mathfrak{g} \in \mathscr{F}_2(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})\} \le P_2[\mathcal{U}_2(||\mathcal{Z}||) + \mathcal{V}_2(||\mathcal{Q}||) + \mathcal{W}_2(||\mathcal{Y}||)],$$

and

$$||\mathscr{F}_3(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})||_{\mathcal{P}} := \sup\{|\mathfrak{h}| : \mathfrak{h} \in \mathscr{F}_3(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y})\} \le P_3[\mathcal{U}_3(||\mathcal{Z}||) + \mathcal{V}_3(||\mathcal{Q}||) + \mathcal{W}_3(||\mathcal{Y}||)],$$

for each $(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}) \in [0,\mathcal{T}] \times \mathcal{R}_e^3$; $(\mathcal{B}_3)$ *there exists a number* $\mathcal{N} > 0$, *such that*

$$\frac{\mathcal{N}}{\mathcal{M}_1} > 1,$$

where $\mathcal{P}_i$, $\mathcal{Q}_i$, *and* $\mathcal{O}_i$ *are given by (10). The tripled system has at least one solution on* $[0,\mathcal{T}]$.

Proof. Consider the operator $\mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3 : \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}} \to \mathcal{P}(\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}})$ defined by (11)–(13). From $(\mathcal{B}_1)$, it follows that sets $\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ and $\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ are non-empty for each $(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}$. Then, for $\mathfrak{f}\in\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathfrak{g}\in\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathfrak{h}\in\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ for $(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}$, we have

$$
\begin{aligned}
\mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[&\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(&\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\right)d\tau\right)d\rho
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{H}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)\\
&=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho,
\end{aligned}
$$

where $\mathfrak{H}_1 \in \mathcal{K}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau), \mathfrak{H}_2 \in \mathcal{K}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau), \mathfrak{H}_3 \in \mathcal{K}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau)$, and $(\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3) \in \mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)$.

For the applicability of Leray–Schauder non-linear alternative we split our proof into several steps.

Claim 1. The operator $\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ is convex. Let $(\mathfrak{H}_i,\bar{\mathfrak{H}}_i,\hat{\mathfrak{H}}_i) \in (\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3), i=1,2$. Then there exist $\mathfrak{f}_i \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{g}_i \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, \mathfrak{h}_i \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}, i=1,2$, such that, for each $\tau \in [0,\mathcal{T}]$, we have

$$
\begin{aligned}
\mathfrak{H}_i(\varpi)=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}&\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho} e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathfrak{H}}_i(\varpi)=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}&\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\bigg)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\mathfrak{H}}_i(\varpi)=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}&\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\bigg)d\rho
\end{aligned}
$$

$$
\begin{aligned}
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho,
\end{aligned}
$$

Let $0\leq\omega\leq 1$. Then, for each $\tau\in[0,\mathcal{T}]$, we have $[\omega\mathcal{F}_1(\mathfrak{a})+(1-\omega)\mathcal{F}_2(\mathfrak{a})]$

$$
\begin{aligned}
&[\omega\mathfrak{H}_1+(1-\omega)\mathfrak{H}_2](\tau)\\
&=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(m)+(1-\omega)\mathfrak{g}_2(m)]dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(m)+(1-\omega)\mathfrak{h}_2(m)]dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(m)+(1-\omega)\mathfrak{g}_2(m)]dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(m)+(1-\omega)\mathfrak{f}_2(m)]dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
&[\omega\bar{\mathfrak{H}}_1+(1-\omega)\bar{\mathfrak{H}}_2](\tau)\\
&=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho
\end{aligned}
$$

$$+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(m)+(1-\omega)\mathfrak{h}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(m)+(1-\omega)\mathfrak{g}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(m)+(1-\omega)\mathfrak{f}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho,$$

$$[\omega\hat{\mathfrak{H}}_1+(1-\omega)\hat{\mathfrak{H}}_2](\tau)=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}$$
$$\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(m)+(1-\omega)\mathfrak{g}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(\tau)+(1-\omega)\mathfrak{f}_2(\tau)]d\tau\right)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}[\omega\mathfrak{f}_1(m)+(1-\omega)\mathfrak{f}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho$$
$$+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(m)+(1-\omega)\mathfrak{h}_2(m)]dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}[\omega\mathfrak{g}_1(\tau)+(1-\omega)\mathfrak{g}_2(\tau)]d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}[\omega\mathfrak{h}_1(\tau)+(1-\omega)\mathfrak{h}_2(\tau)]d\tau\right)d\rho,$$

We deduce that $\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})},\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ and $\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ are convex valued, since $\mathfrak{f},\mathfrak{g},\mathfrak{h}$ are convex valued. obviously, $[\omega\mathfrak{H}_1+(1-\omega)\mathfrak{H}_2](\tau)\in\mathcal{K}_1,[\omega\bar{\mathfrak{H}}_1+(1-\omega)\bar{\mathfrak{H}}_2](\tau)\in\mathcal{K}_2$, and $[\omega\bar{\mathfrak{H}}_1+(1-\omega)\bar{\mathfrak{H}}_2](\tau)\in\mathcal{K}_3$ hence $[\omega(\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3)+(1-\omega)(\bar{\mathfrak{h}}_1,\bar{\mathfrak{h}}_2,\hat{\mathfrak{H}_3})](\tau)\in\mathcal{K}$

Claim 2. We show that the operator $\mathcal{K}$ maps bounded sets into bounded sets in $\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}$. Let $\tau>0$, define $\mathcal{B}_{\mathfrak{r}}=\{(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}:||(\mathcal{Z},\mathcal{Q},\mathcal{Y})||\leq\mathfrak{r}\}$ be a bounded set in $\hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$. Then, there exist $\mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ and $\mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, such that

$$
\begin{aligned}
&|\mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau)| \\
&= \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} ||P_2||[\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)]d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} ||P_2||[\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)]dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} ||P_1||[\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)]d\tau\right)d\rho \\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} ||P_3||[\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r)]d\tau\right)d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} ||P_3||[\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r)]dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} ||P_2||[\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)]d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} ||P_2||[\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)]d\tau\right)d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} ||P_2||[\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)]dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} ||P_1||[\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)]d\tau\right)d\rho\Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} ||P_1||[\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)]d\tau\right)d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} ||P_1||[\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)]dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} ||P_3||[\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r)]d\tau\right)d\rho\Bigg\}\Bigg)\Bigg] \\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} ||P_1||[\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)]d\tau\right)d\rho, \\
&\leq \mathcal{P}_1||P_1||\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)+\mathcal{Q}_1||P_2||\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)+\mathcal{O}_1||P_3||\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau)| \\
&\leq \mathcal{P}_2||P_1||\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)+\mathcal{Q}_2||P_2||\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)+\mathcal{O}_2||P_3||\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r).
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathfrak{H}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\tau)| \\
&\leq \mathcal{P}_3||P_1||\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)+\mathcal{Q}_3||P_2||\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)+\mathcal{O}_3||P_3||\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r).
\end{aligned}
$$

Hence we obtain

$$
||(\mathfrak{H}_1,\mathfrak{H}_2,\mathfrak{H}_3)|| = ||\mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})|| + ||\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})|| + ||\mathfrak{H}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})||
$$

$$
\begin{aligned}
\leq &(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||P_1||(\mathcal{U}_1(\mathcal{N})+\mathcal{V}_1(\mathcal{N})+\mathcal{W}_1(\mathcal{N})) + (\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||P_2||(\mathcal{U}_2(\mathcal{N})+\mathcal{V}_2(\mathcal{N})+\mathcal{W}_2(\mathcal{N})) \\
&+ (\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||P_3||(\mathcal{U}_3(\mathcal{N})+\mathcal{V}_3(\mathcal{N})+\mathcal{W}_3(\mathcal{N})).
\end{aligned}
$$

Claim 3. We show the equi-continuity of the operator $\mathcal{K}$. Let $\varpi_1, \varpi_2 \in [0, \mathcal{T}]$ with $\varpi_1 < \varpi_2$. Then there exist $\mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, and $\mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, such that

$$
\begin{aligned}
&|\mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_2) - \mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_1)| \\
&\leq \Bigg| \frac{||P_1||\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)}{\Gamma(\psi)} \left(\int_0^{\varpi_1} [(\varpi_2-\mathfrak{s})^{\psi-1} - (\varpi_1-\mathfrak{s})^{\psi-1}]d\mathfrak{s}\right.
\end{aligned}
$$

$$+\int_0^{\varpi_1}[(\varpi_2-\mathfrak{s})^{\psi-1}-(\varpi_1-\mathfrak{s})^{\psi-1}]d\mathfrak{s}\Big)\Big|$$
$$\leq ||P_1||\mathcal{U}_1(r)+\mathcal{V}_1(r)+\mathcal{W}_1(r)\left(\frac{2(\varpi_2-\varpi_1)^{\psi}+|\varpi_2^{\psi}-\varpi_1^{\psi}|}{\Gamma(\psi+1)}\right),$$

and,

$$|\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_2)-\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_1)|\leq ||P_2||\mathcal{U}_2(r)+\mathcal{V}_2(r)+\mathcal{W}_2(r)\left(\frac{2(\varpi_2-\varpi_1)^{\phi}+|\varpi_2^{\phi}-\varpi_1^{\phi}|}{\Gamma(\phi+1)}\right).$$

Similar,

$$|\mathfrak{H}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_2)-\mathfrak{H}_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi_1)|\leq ||P_3||\mathcal{U}_3(r)+\mathcal{V}_3(r)+\mathcal{W}_3(r)\left(\frac{2(\varpi_2-\varpi_1)^{\omega}+|\varpi_2^{\omega}-\varpi_1^{\omega}|}{\Gamma(\omega+1)}\right).$$

Therefore, the operator $\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ is equi-continuous, based on Arzela–Ascoli $\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ is completely continuous. We know that a completely continuous operator is upper semi-continuous if it has a closed graph. Thus, we need to prove that $\mathcal{K}$ has a closed graph.

Claim 4. We show that the operator $\mathcal{K}$ has closed graph. As it is known that a completely continuous operator is upper semi-continuous if it has a closed graph. For this we take $(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)\to(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)$, $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n})\in\mathcal{K}(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)$ and $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n})\to(\mathfrak{H}_*,\bar{\mathfrak{H}}_*,\hat{\mathfrak{H}}_*)$, then we need to show $(\mathfrak{H}_*,\bar{\mathfrak{H}}_*,\hat{\mathfrak{H}}_*)\in\mathcal{K}(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)$. Observe that $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n})\in\mathcal{K}(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)$ implies that exist $\mathfrak{f}_n\in\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)}$, $\mathfrak{g}_n\in\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)}$, and $\mathfrak{h}_n\in\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)}$ such that

$$\begin{aligned}
\mathfrak{H}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\right)d\rho,
\end{aligned}$$

and

$$\begin{aligned}
&\bar{\mathfrak{H}}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)\\
&=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho,
\end{aligned}$$

$$\begin{aligned}
\hat{\mathfrak{H}}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=&\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\\
&\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho,
\end{aligned}$$

Let us consider the continuous linear operator $\Phi_1, \Phi_2, \Phi_3 : \mathfrak{L}^1([0,\mathcal{T}], \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}) \to C([0,\mathcal{T}], \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}})$ given by

$$
\begin{aligned}
\Phi_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) = \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[& \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho \\
& +\Pi_1 \int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho \\
& +\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho \\
& +\Pi_2 \int_0^\varrho \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\} \\
& +\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho \\
& +\Pi_1 \int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\Bigg\} \\
& +\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho \\
& +\Pi_3 \int_0^\delta \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg] \\
& +\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
& \Phi_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) \\
& = \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(\tau)d\tau\right)d\rho \\
& +\Pi_2 \int_0^\varrho \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}\mathfrak{h}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho\Bigg\} \\
& +\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(\tau)d\tau\right)d\rho \\
& +\Pi_1 \int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}\mathfrak{g}(m)dm\right)d\tau\right)d\rho \\
& -\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho\Bigg\} \\
& +\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}\mathfrak{f}(\tau)d\tau\right)d\rho
\end{aligned}
$$

$$
\begin{aligned}
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau)d\tau \right) d\rho \Big\} \Big) \Big] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau)d\tau \right) d\rho,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi_3(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi) \\
&= \frac{(\varphi^2\varpi^2 - 2\varphi\varpi + 2 - e^{-\varphi\varpi})}{\varphi^3\Delta} \Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6 \Bigg\{ \Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau)d\tau \right) d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau)d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_3 \Bigg\{ \Upsilon_3 \sum_{j=1}^{k-2} \sigma_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(\tau)d\tau \right) d\rho \\
&+\Pi_3 \int_0^{\delta} \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)} \mathfrak{f}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau)d\tau \right) d\rho \Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_6 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau)d\tau \right) d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} \mathfrak{g}(\tau)d\tau \right) d\rho \Bigg\} \Bigg) \Bigg] \\
&+ \int_0^{\varpi} e^{-\varphi(\varpi-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} \mathfrak{h}(\tau)d\tau \right) d\rho,
\end{aligned}
$$

From, we know that $(\Phi_1,\Phi_2,\Phi_3)\circ(\mathcal{S}_{\mathscr{F}_1},\mathcal{S}_{\mathscr{F}_2},\mathcal{S}_{\mathscr{F}_3})$ is closed graph operator. Further, we have $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n}) \in (\Phi_1,\Phi_2,\Phi_3)\circ(\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z}_n,\mathcal{Q}_n\mathcal{Y}_n)},\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z}_n,\mathcal{Q}_n\mathcal{Y}_n)},\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)})$ for all n. Since $(\mathcal{Z}_n,\mathcal{Q}_n\mathcal{Y}_n)\to(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)$, $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n})\to(\mathfrak{H}_*,\bar{\mathfrak{H}}_*,\hat{\mathfrak{H}_n})$ it follows that $\mathfrak{f}_*\in\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ and $\mathfrak{g}_*\in\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, and $\mathfrak{h}_*\in\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, such that

$$
\begin{aligned}
\mathfrak{H}_*(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)(\varpi) =& \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1} \Bigg[\Upsilon_1 \sum_{j=1}^{k-2} \xi_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} (\mathfrak{g})_*(\tau)d\tau \right) d\rho \\
&+\Pi_1 \int_0^{\varsigma} \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)} (\mathfrak{g})_*(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)} (\mathfrak{f})_*(\tau)d\tau \right) d\rho \\
&+\frac{1}{\Delta} \Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5 \Bigg\{ \Upsilon_2 \sum_{j=1}^{k-2} \nu_j \int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)} (\mathfrak{h})_*(\tau)d\tau \right) d\rho \\
&+\Pi_2 \int_0^{\varrho} \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\rho} e^{-\varphi(\rho-\tau)} \left(\int_0^{\tau} \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)} (\mathfrak{h})_*(m)dm \right) d\tau \right) d\rho \\
&- \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)} \left(\int_0^{\rho} \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)} (\mathfrak{g})_*(\tau)d\tau \right) d\rho \Bigg\}
\end{aligned}
$$

$$+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho,$$

and

$$\bar{\mathfrak{H}}_*(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)(\varpi)$$
$$=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho,$$

$$\hat{\mathfrak{H}}_*(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)(\varpi)=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}$$
$$\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(m)dm\right)d\tau\right)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho\Bigg\}$$

$$
\begin{aligned}
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_*(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_*(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_*(\tau)d\tau\right)d\rho,
\end{aligned}
$$

that is, $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}_n})\in\mathcal{K}(\mathcal{Z}_*,\mathcal{Q}_*,\mathcal{Y}_*)$.

Exists with: let $(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\nu\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$. this implies that the function $\mathfrak{f}\in\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, $\mathfrak{g}\in\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$, and $\mathfrak{h}\in\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ exists with:

$$
\begin{aligned}
||\mathcal{Z}||\leq&\mathcal{P}_1||P_1||[\mathcal{U}_1(||\mathcal{Z}||)+\mathcal{V}_1(||\mathcal{Q}||)+\mathcal{W}_1(||\mathcal{Y}||)]\\
&+\mathcal{Q}_1||P_2||[\mathcal{U}_2(||\mathcal{Z}||)+\mathcal{V}_2(||\mathcal{Q}||)+\mathcal{W}_2(||\mathcal{Y}||)]\\
&+\mathcal{O}_1||P_3||[\mathcal{U}_3(||\mathcal{Z}||)+\mathcal{V}_3(||\mathcal{Q}||)+\mathcal{W}_3(||\mathcal{Y}||)],
\end{aligned}
$$

$$
\begin{aligned}
||\mathcal{Q}||\leq&\mathcal{P}_2||P_1||[\mathcal{U}_1(||\mathcal{Z}||)+\mathcal{V}_1(||\mathcal{Q}||)+\mathcal{W}_1(||\mathcal{Y}||)]\\
&+\mathcal{Q}_2||P_2||[\mathcal{U}_2(||\mathcal{Z}||)+\mathcal{V}_2(||\mathcal{Q}||)+\mathcal{W}_2(||\mathcal{Y}||)]\\
&+\mathcal{O}_2||P_3||[\mathcal{U}_3(||\mathcal{Z}||)+\mathcal{V}_3(||\mathcal{Q}||)+\mathcal{W}_3(||\mathcal{Y}||)],
\end{aligned}
$$

and

$$
\begin{aligned}
||\mathcal{Y}||\leq&\mathcal{P}_3||P_1||[\mathcal{U}_1(||\mathcal{Z}||)+\mathcal{V}_1(||\mathcal{Q}||)+\mathcal{W}_1(||\mathcal{Y}||)]\\
&+\mathcal{Q}_3||P_2||[\mathcal{U}_2(||\mathcal{Z}||)+\mathcal{V}_2(||\mathcal{Q}||)+\mathcal{W}_2(||\mathcal{Y}||)]\\
&+\mathcal{O}_3||P_3||[\mathcal{U}_3(||\mathcal{Z}||)+\mathcal{V}_3(||\mathcal{Q}||)+\mathcal{W}_3(||\mathcal{Y}||)],
\end{aligned}
$$

following the same arguments

$$
\begin{aligned}
||(\mathcal{Z},\mathcal{Q},\mathcal{Y})||=&||\mathcal{Z}||+||\mathcal{Q}||+||\mathcal{Y}||\\
\leq&(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||P_1||[\mathcal{U}_1(||\mathcal{Z}||)+\mathcal{V}_1(||\mathcal{Q}||)+\mathcal{W}_1(||\mathcal{Y}||)]\\
&+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||P_2||[\mathcal{U}_2(||\mathcal{Z}||)+\mathcal{V}_2(||\mathcal{Q}||)+\mathcal{W}_2(||\mathcal{Y}||)]\\
&+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||P_3||[\mathcal{U}_3(||\mathcal{Z}||)+\mathcal{V}_3(||\mathcal{Q}||)+\mathcal{W}_3(||\mathcal{Y}||)].
\end{aligned}
$$

which implies that

$$
\frac{||(\mathcal{Z},\mathcal{Q},\mathcal{Y})||}{(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||P_1||[\mathcal{U}_1(||\mathcal{Z}||)+\mathcal{V}_1(||\mathcal{Q}||)+\mathcal{W}_1(||\mathcal{Y}||)]+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||P_2||[\mathcal{U}_2(||\mathcal{Z}||)+\mathcal{V}_2(||\mathcal{Q}||)+\mathcal{W}_2(||\mathcal{Y}||)]+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||P_3||[\mathcal{U}_3(||\mathcal{Z}||)+\mathcal{V}_3(||\mathcal{Q}||)+\mathcal{W}_3(||\mathcal{Y}||)]}\leq 1.
$$

In the light of $\mathcal{B}_3$ we can find $\mathcal{N}$ with $||(\mathcal{Z},\mathcal{Q},\mathcal{Y})||\neq\mathcal{N}$. Consider

$$
\Lambda=\{(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\hat{\mathcal{J}}\times\hat{\mathcal{J}}\times\hat{\mathcal{J}}:||(\mathcal{Z},\mathcal{Q},\mathcal{Y})||<\mathcal{N}\}.
$$

Here, $\mathcal{K}:\bar{\Lambda}\to\mathcal{P}_{cp,cv}(\hat{\mathcal{J}})\times\mathcal{P}_{cp,cv}(\hat{\mathcal{J}})\times\mathcal{P}_{cp,cv}(\hat{\mathcal{J}})$ is completely continuous and upper semi-continuous. There is no $(\mathcal{Z},\mathcal{Q},\mathcal{Y})\in\nu\mathcal{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ for some $\nu\in(0,1)$ depending on choosing of Λ.

So, by the non-linear alternative of Leray–Schauder type, we conclude that $\mathcal{K}$ has it least one fixed point$(\mathcal{Z},\mathcal{Q},\mathcal{Y}) \in \bar{\Lambda}$, this solution of problem (1). By this, we finalize the proof. □

3.2. The Case of Libschitz

Here, we consider the situation where there are non-convex values in the multi valued maps of system (1).

Consider the metric space $(\mathcal{E},d)$ which is induced from the normed space $(\mathcal{E};||.||)$, and consider $\mathscr{H}_d : \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{E}) \to \mathcal{R}_e\infty$ be given by $\mathscr{H}_d(\mathcal{U},\mathcal{V}) = \max\{\sup_{\mathcal{Y}\in\mathcal{U}} d(\mathcal{Y},\mathcal{V}), \sup_{\nu\in\mathcal{V}} d(\mathcal{U},\nu)\}$, where $d(\mathcal{U},\nu) = \inf_{\mathcal{Y}\in\mathcal{U}} d(\mathcal{Y},\nu)$ and $d(\nu,\mathcal{V}) = \inf_{\nu\in\mathcal{V}} d(\mathcal{Y},\nu)$.

So $(\mathcal{P}_{b,cl}(\mathcal{E}),\mathscr{H}_d)$ is a metric space and $(\mathcal{P}_{cl}(\mathcal{E}),\mathscr{H}_d)$ is a generalized one.

In the upcoming result, we take advantage of Covitz and Nadler's fixed point theorem for multi-valued maps.

Theorem 2. *$(\mathcal{B}_4)\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3 : [0,\mathcal{T}] \times \mathcal{R}_e^3 \to \mathcal{P}_{cp}(\mathcal{R}_e)$ are such that $\mathscr{F}_1(.,\mathcal{Z},\mathcal{Q},\mathcal{Y}) : [0,\mathcal{T}] \to \mathcal{P}_{cp}(\mathcal{R}_e)$, $\mathscr{F}_2(.,\mathcal{Z},\mathcal{Q},\mathcal{Y}) : [0,\mathcal{T}] \to \mathcal{P}_{cp}(\mathcal{R}_e)$, and $\mathscr{F}_3(.,\mathcal{Z},\mathcal{Q},\mathcal{Y}) : [0,\mathcal{T}] \to \mathcal{P}_{cp}(\mathcal{R}_e)$ are measurable for each $\mathcal{Z},\mathcal{Q},\mathcal{Y} \in \mathcal{R}_e$;*

$(\mathcal{B}_5)$

$$\mathscr{H}_d(\mathscr{F}_1(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_1(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}})) \le m_1(\varpi)(|\mathcal{Z}-\bar{\mathcal{Z}}| + |\mathcal{Q}-\bar{\mathcal{X}}| + |\mathcal{Y}-\bar{\mathcal{Y}}|),$$

$$\mathscr{H}_d(\mathscr{F}_2(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_2(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}})) \le m_2(\varpi)(|\mathcal{Z}-\bar{\mathcal{Z}}| + |\mathcal{Q}-\bar{\mathcal{X}}| + |\mathcal{Y}-\bar{\mathcal{Y}}|),$$

and

$$\mathscr{H}_d(\mathscr{F}_3(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_3(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}})) \le m_3(\varpi)(|\mathcal{Z}-\bar{\mathcal{Z}}| + |\mathcal{Q}-\bar{\mathcal{X}}| + |\mathcal{Y}-\bar{\mathcal{Y}}|),$$

for the majority of $\varpi \in [0,\mathcal{T}]$ and $\mathcal{Z},\mathcal{Q},\mathcal{Y},\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}} \in \mathcal{R}_e$ with $m_1,m_2,m_3 \in C([0,\mathcal{T}],\mathcal{R}_e^+)$ and $d(0,\mathscr{F}_1(\varpi,0,0)) \le m_1(\varpi), d(0,\mathscr{F}_2(\varpi,0,0)) \le m_2(\varpi), d(0,\mathscr{F}_3(\varpi,0,0)) \le m_3(\varpi)$ for almost $\varpi \in [0,\mathcal{T}]$ hold, this implies the existence of solution for system (1) on $[0,\mathcal{T}]$ given that

$$(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||m_1|| + (\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||m_2|| + (\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||m_3|| < 1. \tag{14}$$

Proof. The sets $\mathcal{S}_{\mathfrak{f},(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$,$\mathcal{S}_{\mathfrak{g},(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ and $\mathcal{S}_{\mathfrak{h},(\mathcal{Z},\mathcal{Q},\mathcal{Y})}$ are non-empty for each $(\mathcal{Z},\mathcal{Q},\mathcal{Y}) \in \mathcal{E}\times\mathcal{E}\times\mathcal{E}$ by assumption $\mathcal{B}_4$, so $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ have measurable selections. We now demonstrate that the operator $\mathfrak{K}$ meets the criteria of Covitz and Nadler's fixed point theorem.

We start by $\mathfrak{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y}) \in \mathcal{P}_{cl}(\mathcal{E})\times\mathcal{P}_{cl}(\mathcal{E})\times\mathcal{P}_{cl}(\mathcal{E})$ for each $(\mathcal{Z},\mathcal{Q},\mathcal{Y}) \in \mathcal{E}\times\mathcal{E}\times\mathcal{E}$. Let $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}}) \in \mathfrak{K}(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)$, such that $(\mathfrak{H}_n,\bar{\mathfrak{H}}_n,\hat{\mathfrak{H}}_n) \to (\mathfrak{H},\bar{\mathfrak{H}},\hat{\mathfrak{H}})$ in $\mathcal{E}\times\mathcal{E}\times\mathcal{E}$. Then $(\mathfrak{H},\bar{\mathfrak{H}},\hat{\mathfrak{H}}) \in \mathcal{E}\times\mathcal{E}\times\mathcal{E}$ and there exists $\mathfrak{f}_n \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z}_n,\mathcal{Y}_n,\mathcal{Q}_n)}$, $\mathfrak{g}_n \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z}_n,\mathcal{Y}_n,\mathcal{Q}_n)}$, and $\mathfrak{h}_n \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z}_n,\mathcal{Y}_n,\mathcal{Q}_n)}$, such that

$$\begin{aligned}
\mathfrak{H}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi) =& \frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho \\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\right)d\rho \\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho \\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\right)d\tau\right)d\rho \\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho\Bigg\}
\end{aligned}$$

$$+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho,$$

and

$$\bar{\mathfrak{H}}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho\Bigg\}$$
$$+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg]$$
$$+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho,$$

$$\hat{\mathfrak{H}}_n(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}$$
$$\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\bigg)d\rho$$
$$+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(m)dm\bigg)d\tau\bigg)d\rho$$
$$-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho\Bigg\}$$
$$+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(\tau)d\tau\bigg)d\rho$$

$$
\begin{aligned}
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_n(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_n(\tau)d\tau\right)d\rho,
\end{aligned}
$$

because of the compact values $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$, we take the sub-sequences to show that $\mathfrak{f}_n$, $\mathfrak{g}_n$ and $\mathfrak{h}_n$ tends to $\mathfrak{f}$, $\mathfrak{g}$, and $\mathfrak{h}$ in $\mathfrak{L}^1([0,\mathcal{T}],\mathcal{R}_e)$, respectively. Thus, $\mathfrak{f}\in\mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$, $\mathfrak{g}\in\mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$, and $\mathfrak{h}\in\mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$ for each $\varpi\in[0,\mathcal{T}]$ and that

$$
\begin{aligned}
&\mathfrak{H}_n(\mathcal{Y}_n,\mathcal{Q}_n)(\varpi)\to\mathfrak{H}(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)\\
=&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho,
\end{aligned}
$$

and

$$\bar{\mathfrak{H}}_n(\mathcal{Y}_n,\mathcal{Q}_n)(\varpi)\to\bar{\mathfrak{H}}(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)$$

$$
\begin{aligned}
&= \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\Delta}\left[\left(\mathcal{E}_1\mathcal{E}_5\left\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\right.\right.\right.\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(m)dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\right\}\\
&+\mathcal{E}_4\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\right.\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(m)dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\right\}\\
&+\mathcal{E}_1\mathcal{E}_4\left\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\right.\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(m)dm\right)d\tau\right)d\rho\\
&\left.\left.\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\right\}\right)\right]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
&\hat{\mathfrak{H}}_n(\mathcal{Y}_n,\mathcal{Q}_n)(\varpi)\to\hat{\mathfrak{H}}(\mathcal{Z},\mathcal{Q},\mathcal{Y})(\varpi)\\
&=\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\\
&\times\left[\left(\mathcal{E}_3\mathcal{E}_6\left\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\right.\right.\right.\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(m)dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\right\}\\
&+\mathcal{E}_1\mathcal{E}_3\left\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(\tau)d\tau\right)d\rho\right.\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})(m)dm\right)d\tau\right)d\rho\\
&\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\right\}\\
&+\mathcal{E}_2\mathcal{E}_6\left\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho\right.\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\rho e^{-\varphi(\rho-\tau)}\left(\int_0^\tau\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(m)dm\right)d\tau\right)d\rho\\
&\left.\left.\left.-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})(\tau)d\tau\right)d\rho\right\}\right)\right]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^\rho\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})(\tau)d\tau\right)d\rho,
\end{aligned}
$$

So $(\mathfrak{H}, \bar{\mathfrak{H}}, \hat{\mathfrak{H}_n}) \in \mathcal{K}$, which guarantees that $\mathcal{K}$ is closed. After that, we prove the existence of $\bar{\theta} < 1$, such that

$$\mathscr{H}_d\big(\mathfrak{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathfrak{K}(\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}})\big) \le \bar{\theta}(||\mathcal{Z}-\bar{\mathcal{Z}}||+||\mathcal{Q}-\bar{\mathcal{X}}||+||\mathcal{Y}-\bar{\mathcal{Y}}||) \text{ for each } \mathcal{Z},\bar{\mathcal{Z}},\mathcal{Q},\bar{\mathcal{X}},\mathcal{Y},\bar{\mathcal{Y}} \in \hat{\mathcal{J}}.$$

Let $(\mathcal{Z},\bar{\mathcal{Z}}),(\mathcal{Q},\bar{\mathcal{X}}),(\mathcal{Y},\bar{\mathcal{Y}}) \in \hat{\mathcal{J}} \times \hat{\mathcal{J}} \times \hat{\mathcal{J}}$ and $(\mathfrak{H}_1, \bar{\mathfrak{H}}_1, \hat{\mathfrak{H}_1}) \in \mathfrak{K}(\mathcal{Z},\mathcal{Q},\mathcal{Y})$. then there exist $\mathfrak{f} \in \mathcal{S}_{\mathscr{F}_1,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$, $\mathfrak{g} \in \mathcal{S}_{\mathscr{F}_2,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$, and $\mathfrak{h} \in \mathcal{S}_{\mathscr{F}_3,(\mathcal{Z},\mathcal{Y},\mathcal{Q})}$ such that, $\forall\ \varpi \in [0,\mathcal{T}]$, this gives

$$\begin{aligned}
\mathfrak{H}_1(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi) =& \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_1\int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\bigg)d\rho \\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_2\int_0^\varrho \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\bigg)d\rho\Bigg\} \\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_1\int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\bigg)d\rho\Bigg\} \\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_3\int_0^\delta \frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\bigg)d\rho\Bigg\}\Bigg)\Bigg] \\
&+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\bigg)d\rho,
\end{aligned}$$

$$\begin{aligned}
\bar{\mathfrak{H}}_1(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi) =& \frac{(\varphi\varpi - 1 + e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_2\int_0^\varrho \frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\bigg)d\rho\Bigg\} \\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j} e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\bigg)d\rho \\
&+\Pi_1\int_0^\varsigma \frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^\rho e^{-\varphi(\rho-\tau)}\bigg(\int_0^\tau \frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(m)dm\bigg)d\tau\bigg)d\rho \\
&-\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^\rho \frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\bigg)d\rho\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathfrak{H}}_1(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=&\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\\
&\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_1(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_1(\tau)d\tau\right)d\rho,
\end{aligned}
$$

By $(\mathcal{B}_5)$, we have

$$\mathscr{H}_d(\mathscr{F}_1(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_1(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}}))\leq m_1(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

$$\mathscr{H}_d(\mathscr{F}_2(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_2(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}}))\leq m_2(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

and

$$\mathscr{H}_d(\mathscr{F}_3(\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathscr{F}_3(\varpi,\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}}))\leq m_3(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

So, there $\mathfrak{f}\in\mathscr{F}_1((\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}))$, $\mathfrak{g}\in\mathscr{F}_2((\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}))$ and $\mathfrak{h}\in\mathscr{F}_3((\varpi,\mathcal{Z},\mathcal{Q},\mathcal{Y}))$ such that

$$|\mathfrak{f}_1(\varpi)-\gamma_1|\leq m_1(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

$$|\mathfrak{g}_1(\varpi)-\gamma_2|\leq m_2(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

and

$$|\mathfrak{h}_1(\varpi)-\gamma_3| \leq m_3(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)$$

Defined $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 : [0,\mathcal{T}] \to \mathcal{P}(\mathcal{R}_e)$ by

$$\begin{aligned}\mathcal{V}_1(\varpi) =&\{\mathfrak{f} \in \mathfrak{L}^1([0,\mathcal{T}],\mathcal{R}_e) : |\mathfrak{f}_1(\varpi)-\gamma_1|\\ &\leq m_1(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)\},\end{aligned}$$

$$\begin{aligned}\mathcal{V}_2(\varpi) =&\{\mathfrak{g} \in \mathfrak{L}^1([0,\mathcal{T}],\mathcal{R}_e) : |\mathfrak{g}_1(\varpi)-\gamma_2|\\ &\leq m_2(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{V}_3(\varpi) =&\{\mathfrak{h} \in \mathfrak{L}^1([0,\mathcal{T}],\mathcal{R}_e) : |\mathfrak{h}_1(\varpi)-\gamma_3|\\ &\leq m_3(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|)\}.\end{aligned}$$

Since the multi-valued operators $\mathcal{V}_1(\varpi) \cap \mathscr{F}_1(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$, $\mathcal{V}_2(\varpi) \cap \mathscr{F}_2(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$ and $\mathcal{V}_3(\varpi) \cap \mathscr{F}_3(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$ are measurable, there exist functions $\mathfrak{f}_2(\varpi), \mathfrak{g}_2(\varpi), \mathfrak{h}_2(\varpi)$ which are a measurable selection for $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ and $\mathfrak{f}_2(\varpi) \in \mathscr{F}_1(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$, $\mathfrak{g}_2(\varpi) \in \mathscr{F}_2(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$, $\mathfrak{h}_2(\varpi) \in \mathscr{F}_3(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi))$ such that, for a.e. $\varpi \in [0,\mathcal{T}]$, we have

$$|\mathfrak{f}_1(\varpi)-\mathfrak{f}_2(\varpi)| \leq m_1(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|),$$

$$|\mathfrak{g}_1(\varpi)-\mathfrak{g}_2(\varpi)| \leq m_2(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|),$$

and

$$|\mathfrak{h}_1(\varpi)-\mathfrak{h}_2(\varpi)| \leq m_3(\varpi)(|\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)|+|\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)|+|\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)|).$$

Let

$$\begin{aligned}\mathfrak{H}_2(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi) =&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\mathcal{E}_1}\Bigg[\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\bigg)d\rho\\
&+\frac{1}{\Delta}\Bigg(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\bigg)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\bigg)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\bigg(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\bigg(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(m)dm\bigg)d\tau\bigg)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\bigg)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_2\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\bigg(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\bigg)d\rho\end{aligned}$$

$$
\begin{aligned}
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho\Big\}\Big)\Big]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathfrak{H}}_2(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=&\frac{(\varphi\varpi-1+e^{-\varphi\varpi})}{\varphi^2\Delta}\Bigg[\Bigg(\mathcal{E}_1\mathcal{E}_5\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_4\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_4\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]\\
&+\int_0^{\varpi}e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\right)d\rho,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathfrak{H}}_2(\mathcal{Z}_n,\mathcal{Q}_n,\mathcal{Y}_n)(\varpi)=&\frac{(\varphi^2\varpi^2-2\varphi\varpi+2-e^{-\varphi\varpi})}{\varphi^3\Delta}\\
&\times\Bigg[\Bigg(\mathcal{E}_3\mathcal{E}_6\Bigg\{\Upsilon_1\sum_{j=1}^{k-2}\xi_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\right)d\rho\\
&+\Pi_1\int_0^{\varsigma}\frac{(\varsigma-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_1\mathcal{E}_3\Bigg\{\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(\tau)d\tau\right)d\rho\\
&+\Pi_3\int_0^{\delta}\frac{(\delta-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\psi-2}}{\Gamma(\psi-1)}(\mathfrak{f})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho\Bigg\}\\
&+\mathcal{E}_2\mathcal{E}_6\Bigg\{\Upsilon_2\sum_{j=1}^{k-2}\nu_j\int_0^{\zeta_j}e^{-\varphi(\zeta_j-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho\\
&+\Pi_2\int_0^{\varrho}\frac{(\varrho-\rho)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\rho}e^{-\varphi(\rho-\tau)}\left(\int_0^{\tau}\frac{(\tau-m)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(m)dm\right)d\tau\right)d\rho\\
&-\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\phi-2}}{\Gamma(\phi-1)}(\mathfrak{g})_2(\tau)d\tau\right)d\rho\Bigg\}\Bigg)\Bigg]
\end{aligned}
$$

$$+\int_0^{\varpi} e^{-\varphi(\varpi-\rho)}\left(\int_0^{\rho}\frac{(\rho-\tau)^{\omega-2}}{\Gamma(\omega-1)}(\mathfrak{h})_2(\tau)d\tau\right)d\rho,$$

Hence

$$\begin{aligned}||\mathfrak{H}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})-\mathfrak{H}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})|| \leq &(\mathcal{P}_1||m_1||+\mathcal{Q}_1||m_2||++\mathcal{O}_1||m_3||)\\ &\times(||\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)||+||\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)||+||\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)||)\end{aligned}$$

$$\begin{aligned}||\bar{\mathfrak{H}}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})-\bar{\mathfrak{H}}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})|| \leq &(\mathcal{P}_2||m_1||+\mathcal{Q}_2||m_2||++\mathcal{O}_2||m_3||)\\ &\times(||\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)||+||\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)||+||\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)||)\end{aligned}$$

Similar, we set

$$\begin{aligned}||\hat{\mathfrak{H}}_1(\mathcal{Z},\mathcal{Q},\mathcal{Y})-\hat{\mathfrak{H}}_2(\mathcal{Z},\mathcal{Q},\mathcal{Y})|| \leq &(\mathcal{P}_3||m_1||+\mathcal{Q}_3||m_2||++\mathcal{O}_3||m_3||)\\ &\times(||\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)||+||\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)||+||\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)||)\end{aligned}$$

Thus

$$\begin{aligned}||(\mathfrak{H}_1,\bar{\mathfrak{H}}_1,\hat{\mathfrak{H}}_1),(\mathfrak{H}_2,\bar{\mathfrak{H}}_2,\hat{\mathfrak{H}}_2)|| \leq &(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||m_1||+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||m_2||+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||m_3||\\ &\times(||\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)||+||\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)||+||\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)||)\end{aligned}$$

likewise, by reversing the roles of $(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ and $(\bar{\mathcal{Z}},\bar{\mathcal{Y}},\bar{\mathcal{X}})$, it is possible to obtain

$$\begin{aligned}\mathscr{H}_d[\mathcal{T}(\mathcal{Z},\mathcal{Q},\mathcal{Y}),\mathcal{T}(\bar{\mathcal{Z}},\bar{\mathcal{X}},\bar{\mathcal{Y}})] \leq &(\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3)||m_1||+(\mathcal{Q}_1+\mathcal{Q}_2+\mathcal{Q}_3)||m_2||+(\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3)||m_3||\\ &\times(||\mathcal{Z}(\varpi)-\bar{\mathcal{Z}}(\varpi)||+||\mathcal{Q}(\varpi)-\bar{\mathcal{X}}(\varpi)||+||\mathcal{Y}(\varpi)-\bar{\mathcal{Y}}(\varpi)||).\end{aligned}$$

In light of the assumption, $\mathfrak{K}$ is a contraction. Therefore, according to Covitz and Nadler's fixed point theorem, it has a fixed point $(\mathcal{Z},\mathcal{Q},\mathcal{Y})$ that is a solution to problem (1). This concludes the proof. □

4. Example

Example 1. *In consistence with systems (1) the along with the main mentioned theorems, we present an example in this section.*

$$\begin{cases}({}^c\mathfrak{D}^{\psi}+\varphi{}^c\mathfrak{D}^{\psi-1})\mathcal{Z}(\varpi)\in\mathscr{F}_1(\varpi,\mathcal{Z}(\varpi),\mathcal{Q}(\varpi),\mathcal{Y}(\varpi)), & 2<\psi\leq 3,\\ ({}^c\mathfrak{D}^{\phi}+\varphi{}^c\mathfrak{D}^{\phi-1})\mathcal{Q}(\varpi)\in\mathscr{F}_2(\varpi,\mathcal{Z}(\varpi),\mathcal{Q}(\varpi),\mathcal{Y}(\varpi)), & 2<\phi\leq 3,\\ ({}^c\mathfrak{D}^{\omega}+\varphi{}^c\mathfrak{D}^{\omega-1})\mathcal{Y}(\varpi)\in\mathscr{F}_3(\varpi,\mathcal{Z}(\varpi),\mathcal{Q}(\varpi),\mathcal{Y}(\varpi)), & 3<\omega\leq 4,\\ \mathcal{Z}(0)=0,\quad \mathcal{Z}'(0)=0,\quad \mathcal{Z}(\mathcal{T})=\Upsilon_1\sum_{j=1}^{k-2}\xi_j\mathcal{Q}(\zeta_j)+\Pi_1\mathcal{I}^{\varsigma}\mathcal{Q}(\vartheta),\\ \mathcal{Q}(0)=0,\quad \mathcal{Q}'(0)=0,\quad \mathcal{Q}(\mathcal{T})=\Upsilon_2\sum_{j=1}^{k-2}\nu_j\mathcal{Y}(\zeta_j)+\Pi_2\mathcal{I}^{\varrho}\mathcal{Y}(\vartheta),\\ \mathcal{Y}(0)=0,\quad \mathcal{Y}'(0)=0,\quad \mathcal{Y}''(0)=0,\quad \mathcal{Y}(\mathcal{T})=\Upsilon_3\sum_{j=1}^{k-2}\sigma_j\mathcal{Z}(\zeta_j)+\Pi_3\mathcal{I}^{\delta}\mathcal{Z}(\vartheta),\end{cases} \tag{15}$$

Here $\psi=\frac{7}{3},\phi=\frac{5}{2},\omega=\frac{10}{3};\varsigma=9/20;\varrho=11/20;\delta=13/20;\vartheta=93/50;\zeta_j=36/25;$ $\Upsilon_1=17/400;\Upsilon_2=15/300;\Upsilon_3=13/200;\Pi_1=17/200;\Pi_2=8/125;\Pi_3=6/68;$ $\mathcal{T}=1;\zeta_1=1/20;\zeta_2=2/20;\nu_1=1/100;\nu_2=1/50;\sigma_1=1/1000;\sigma_2=1/500;k=4;$ $\Delta=0.067506818609056$ *with the given data, it is found that*

$$\mathcal{P}_1=1.79143780787545,\mathcal{P}_2=0.83611149394660,\mathcal{P}_3=0.522203861964576,$$
$$\mathcal{Q}_1=0.400530445936702,\mathcal{Q}_2=1.56029202354379,\mathcal{Q}_3=0.109316638513044,$$
$$\mathcal{O}_1=0.190264056004677,\mathcal{O}_2=0.748929077567517,\mathcal{O}_3=0.745842301275926.$$

To illustrate the use of Theorem 1, we will consider

$$\begin{aligned}
\mathscr{F}_1(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)) &= \left[0, \frac{1}{12+\varpi^2}\left(\frac{|\mathcal{Z}|}{1+|\mathcal{Z}|} + \frac{|\sin(\mathcal{Q})|}{1+|\sin(\mathcal{Q})|} + \tan^{-1}(\mathcal{Y}(\varpi))\right), \frac{1}{15+\varpi}\right] \\
\mathscr{F}_2(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)) &= \left[0, \frac{e^{-2\varpi}}{2\sqrt{1600+\varpi}}\left(\frac{1}{270}\sin \mathcal{Z}(\varpi) + \frac{|\mathcal{Q}|}{1+|\mathcal{Q}|} + \frac{|\cos(\mathcal{Y})|}{1+|\cos(\mathcal{Y})|}\right), \frac{1}{15+\varpi}\right] \\
\mathscr{F}_3(\varpi, \mathcal{Z}(\varpi), \mathcal{Q}(\varpi), \mathcal{Y}(\varpi)) &= \left[0, \frac{1}{2\sqrt{400+\varpi^2}}\left(\frac{1}{270}\sin(\mathcal{Z}) + \frac{|\tan(\mathcal{Q})|}{1+|\tan(\mathcal{Q})|} + \frac{|\mathcal{Y}|}{1+|\mathcal{Y}|}\right), \frac{1}{15+\varpi}\right].
\end{aligned}$$

Clearly, utilizing the above data, we obtain

$$(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)||m_1|| + (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3)||m_2|| + (\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3)||m_3|| \approx 0.3304803666 < 1. \quad (16)$$

All the conditions of Theorem 1, we will consider.

5. Conclusions

In this study, we investigated the existence of solutions for tripled fractional differential inclusions with fractional derivatives of different orders and non-local boundary conditions featuring fractional derivatives and integrals. The existence of both convex and non-convex multi-valued maps was established using the non-linear alternative of Kakutani maps and Covitz and Nadler's fixed point theorem, see [30].

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; p. 204.
2. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993; Volume 1.
4. Burton, T.A. A fixed-point theorem of Krasnoselskii. *Appl. Math. Lett.* **1998**, *11*, 85–88. [CrossRef]
5. Covitz, H.; Nadler, S.B. Multi-valued contraction mappings in generalized metric spaces. *Isr. J. Math.* **1970**, *8*, 5–11. [CrossRef]
6. Smart, D.R. *Fixed Point Theorems*; Cambridge University Press: Cambridge, UK, 1980; Volume 66.
7. Abuasbeh, K.; Mahmudov, N.I.; Awadalla, M. Fractional Stochastic Integro-Differential Equations with Nonintantaneous Impulses: Existence, Approximate Controllability and Stochastic Iterative Learning Control. *Fractal Fract.* **2023**, *7*, 87. [CrossRef]
8. Salem, A.; Almaghamsi, L. Existence solution for coupled system of Langevin fractional differential equations of caputo type with Riemann–Stieltjes integral boundary conditions. *Symmetry* **2021**, *13*, 2123. [CrossRef]
9. Zhang, Y. Existence results for a coupled system of nonlinear fractional multi-point boundary value problems at resonance. *J. Inequal. Appl.* **2018**, *2018*, 198. [CrossRef]
10. Cui, M.; Zhu, Y.; Pang, H. Existence and uniqueness results for a coupled fractional order systems with the multi-strip and multi-point mixed boundary conditions. *Adv. Differ. Equ.* **2017**, *2017*, 224. [CrossRef]
11. Manigandan, M.; Muthaiah, S.; Nandhagopal, T.; Vadivel, R.; Unyong, B.; Gunasekaran, N. Existence results for coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order. *AIMS Math.* **2022**, *7*, 723–755. [CrossRef]
12. Su, X. Boundary value problem for a coupled system of nonlinear fractional differential equations. *Appl. Math. Lett.* **2009**, *22*, 64–69. [CrossRef]
13. Subramanian, M.; Manigandan, M.; Tunç, C.; Gopal, T.N.; Alzabut, J. On system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. *J. Taibah Univ. Sci.* **2022**, *16*, 1–23. [CrossRef]

14. Hamoud, A.A. Existence and uniqueness of solutions for fractional neutral volterra-fredholm integro differential equations. *Adv. Theory Nonlinear Anal. Its Appl.* **2020**, *4*, 321–331. [CrossRef]
15. Jung, C.Y.; Farid, G.; Yasmeen, H.; Lv, Y.P.; Pečarić, J. Refinements of some fractional integral inequalities for refined $(\psi, h-m)$-convex function. *Adv. Differ. Equ.* **2021**, *2021*, 391. [CrossRef]
16. Liu, Y. A new method for converting boundary value problems for impulsive fractional differential equations to integral equations and its applications. *Adv. Nonlinear Anal.* **2019**, *8*, 386–454. [CrossRef]
17. Guechi, S.; Dhayal, R.; Debbouche, A.; Malik, M. Analysis and optimal control of ϕ-Hilfer fractional semilinear equations involving nonlocal impulsive conditions. *Symmetry* **2021**, *13*, 2084. [CrossRef]
18. Heymans, N.; Podlubny, I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol. Acta* **2006**, *45*, 765–771. [CrossRef]
19. Ahmad, B.; Nieto, J.J. Sequential fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2012**, *64*, 3046–3052. [CrossRef]
20. Al Elaiw, A.; Manigandan, M.; Awadalla, M.; Abuasbeh, K. Existence results by Mönch's fixed point theorem for a tripled system of sequential fractional differential equations. *AIMS Math.* **2023**, *8*, 3969–3996. [CrossRef]
21. Luca, R. On a system of Riemann–Liouville fractional differential equations with coupled nonlocal boundary conditions. *Adv. Differ. Equ.* **2021**, *2021*, 134. [CrossRef]
22. Awadalla, M.; Manigandan, M. Existence and Stability Results for Caputo-Type Sequential Fractional Differential Equations with New Kind of Boundary Conditions. *Math. Probl. Eng.* **2022**, *2022*, 3999829. [CrossRef]
23. Etemad, S.; Tellab, B.; Alzabut, J.; Rezapour, S.; Abbas, M.I. Approximate solutions and Hyers-Ulam stability for a system of the coupled fractional thermostat control model via the generalized differential transform. *Adv. Differ. Equ.* **2021**, *2021*, 428. [CrossRef]
24. Baeza, L.; Ouyang, H. A railway track dynamics model based on modal substructuring and a cyclic boundary condition. *J. Sound Vib.* **2011**, *330*, 75–86. [CrossRef]
25. Matar, M.M.; Amra, I.A.; Alzabut, J. Existence of solutions for tripled system of fractional differential equations involving cyclic permutation boundary conditions. *Bound. Value Probl.* **2020**, *2020*, 140. [CrossRef]
26. Berinde, V.; Borcut, M. Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 4889–4897. [CrossRef]
27. Karakaya, K.; Bouzara, N.E.; DoLan, K.; Atalan, Y. Existence of tripled fixed points for a class of condensing operators in Banach spaces. *Sci. World J.* **2014**, *2014*, 541862. [CrossRef] [PubMed]
28. Ahmad, B.; Hamdan, S.; Alsaedi, A.; Ntouyas, S.K. A study of a nonlinear coupled system of three fractional differential equations with nonlocal coupled boundary conditions. *Adv. Differ. Equ.* **2021**, *2021*, 278. [CrossRef]
29. Sitho, S.; Ntouyas, S.K.; Samadi, A.; Tariboon, J. Boundary value problems for ψ-Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions. *Mathematics* **2021**, *9*, 1001. [CrossRef]
30. Hakkar, N.; Dhayal, R.; Debbouche, A.; Torres, D.F. Approximate Controllability of Delayed Fractional Stochastic Differential Systems with Mixed Noise and Impulsive Effects. *Fractal Fract.* **2023**, *7*, 104. [CrossRef]