fractal and fractional

MDPI

Article

Nonlinear Piecewise Caputo Fractional Pantograph System with Respect to Another Function

Mohammed S. Abdo [1,2,*], Wafa Shammakh [3], Hadeel Z. Alzumi [3], Najla Alghamd [3] and M. Daher Albalwi [4]

1 Department of Mathematics, Hodeidah University, Al-Hudaydah 3114, Yemen
2 Center of Tropical Medicine and Epidemiology Studies, Hodeidah University, Al-Hudaydah 3114, Yemen
3 Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah 23218, Saudi Arabia
4 Yanbu Industrial College, The Royal Commission for Jubail and Yanbu, Yanbu 30436, Saudi Arabia
* Correspondence: msabdo@hoduniv.net.ye

Abstract: The existence, uniqueness, and various forms of Ulam–Hyers (UH)-type stability results for nonlocal pantograph equations are developed and extended in this study within the frame of novel *psi*-piecewise Caputo fractional derivatives, which generalize the piecewise operators recently presented in the literature. The required results are proven using Banach's contraction mapping and Krasnoselskii's fixed-point theorem. Additionally, results pertaining to UH stability are obtained using traditional procedures of nonlinear functional analysis. Additionally, in light of our current findings, a more general challenge for the pantograph system is presented that includes problems similar to the one considered. We provide a pertinent example as an application to support the theoretical findings.

Keywords: pantograph equation; piecewise fractional derivative; fixed point theorem

MSC: 34A08; 34A12; 47H10

check for updates

Citation: Abdo, M.S.; Shammakh, W.; Alzumi, H.Z.; Alghamd, N.; Albalwi, M.D. Nonlinear Piecewise Caputo Fractional Pantograph System with Respect to Another Function. *Fractal Fract.* **2023**, *7*, 162. https://doi.org/10.3390/fractalfract7020162

Academic Editor: Wei-Shih Du

Received: 22 December 2022
Revised: 2 February 2023
Accepted: 3 February 2023
Published: 6 February 2023

1. Introduction

Fractional calculus (FC) has received great interest from researchers due to its wide range of applications in various scientific fields. The crucial concepts and definitions of FC have been presented [1,2]. In [3,4], the authors introduced some fundamental history of fractional calculus and its applications to engineering and different areas of science.

Many classes of fractional differential equations (FDEs) have been extensively studied and analyzed in the past decades; for example, theories involving the existence of unique solutions have been documented [5–10]. Numerical and analytical methods have been developed with the aim of solving such equations [11–14]. These equations have been tracked as useful in modeling some real-world problems with incredible acheivements.

The qualitative properties of solutions address a vital part of the theory of FDEs. The previously aforementioned region has been studied well for standard differential equations. In any case, for FDEs, there are numerous aspects and perspectives that require further research and surveying. The consideration of the existence and uniqueness has been particularly considered by using Riemann–Liouville (R-L), Caputo, Hilfer, and other FDs (see [15–23] and the references therein).

Recently, many generalizations of classical FDs above have been presented by Kilbas et al. [2], Almeida in [24], and Sousa-Oliveira [25]. These derivatives are called the ψ-Reiman–Liouville, ψ-Caputo, and ψ-Hilfer FDs.

Another significant class of FDEs is the pantograph equations (PEs), which have not been as thoroughly investigated in the frame of novel FDs. PEs are an important category of delay equations that give changes in the dependent worth at a past time [26], and they are applied in deterministic situations. A pantograph is basically a tool used for measuring and drawing. This tool is currently used in electric trains and electric cells [27–29].

In 1971 Ockendon and Taylor [30] discussed how electric flow is collected by the pantograph of an electric train using the following delay equation

$$\begin{cases} v'(\varkappa) = av(\varkappa) + bv(\lambda\varkappa), \ \ \varkappa \in [0,T], \ 0 < \lambda < 1, \\ v(0) = v_0 \end{cases} \tag{1}$$

which is currently called PE. Since that time, many researchers have studied and included it in different mathematical and scientific fields such as number theory, probability, electrodynamics, and medication (see [30–32] and the references therein).

The analytical and numerical methods of (1) have been deliberated by several authors [33–35]. Derfel and Iserels [36,37], extensively studied the PEs. The following type of nonlinear PE

$$\begin{cases} v'(\varkappa) = f(\varkappa, v(\varkappa), v(\lambda_1\varkappa), \ldots, v(\lambda_m\varkappa)), \ \ \varkappa \in [0,T], \ 0 < \lambda_1 < \cdots < \lambda_m < 1, \\ v(0) = v_0 \end{cases} \tag{2}$$

was studied by Liu et al. [38], whereas the nonlinear neutral PE

$$\begin{cases} v'(\varkappa) = f(\varkappa, v(\varkappa), v(\lambda\varkappa), v'(\lambda\varkappa)), \ \ \varkappa > 0, \ 0 < \lambda < 1, \\ v(0) = v_0 \end{cases} \tag{3}$$

was considered by Sezer et al. [39].

Many studied have been conducted on fractional PEs because of their significance to numerous areas of exploration. For example, Balachandran et al. [40], discussed the existence of solutions for the following Caputo-type pantograph problem

$$\begin{cases} {}^{C}\mathbb{D}^{\vartheta}_{0^+} v(\varkappa) = f(\varkappa, v(\varkappa), v(\lambda\varkappa)), \ \ \varkappa \in [0,T], \ 0 < \vartheta < 1, \\ v(0) = v_0 + g(v). \end{cases} \tag{4}$$

In this regard, Atangana and Araz [41] introduced the concept of the piecewise derivative with the aim of modeling real-world problems following multiple processes. Motivated by the above works and by [41], we consider the following piecewise Caputo pantograph problems (PCPPs):

$$\begin{cases} {}^{PC}\mathbb{D}^{\vartheta}_{0^+} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), \ldots, v(\lambda_m\varkappa)), \\ v(0) = v_0 + g(v), \end{cases} \tag{5}$$

and

$$\begin{cases} {}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), \ldots, v(\lambda_m\varkappa)), \\ v(0) = v_0 + g(v), \end{cases} \tag{6}$$

where $0 < \vartheta \leq 1$, $\varkappa \in \mathbb{J} := [0,b]$, $v_0 \in \mathbb{R}$, $0 < \lambda_i < 1$, for $i = 1,2,\ldots,m$, and ${}^{PC}\mathbb{D}^{\vartheta}_{0^+}$, and ${}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+}$ represent the piecewise and ψ-piecewise Caputo FD of order ϑ, respectively, defined by

$${}^{PC}\mathbb{D}^{\vartheta}_{0^+} f(\varkappa) = \begin{cases} \frac{d}{d\varkappa} f(\varkappa) : if \ \ \varkappa \in [0,\varkappa_1], \\ {}^{C}\mathbb{D}^{\vartheta}_{\varkappa_1} f(\varkappa) : if \ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

and

$${}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+} f(\varkappa) = \begin{cases} \left(\frac{1}{\psi'(\varkappa)}\frac{d}{d\varkappa}\right) f(\varkappa) : if \ \ \varkappa \in [0,\varkappa_1], \\ {}^{C}\mathbb{D}^{\vartheta;\psi}_{\varkappa_1} f(\varkappa) : if \ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

where ${}^{PC}\mathbb{D}^{\vartheta}_{0^+}$ and ${}^{PC}\mathbb{D}^{\vartheta;\psi}_{0^+}$ are classical (or generalized) derivative on $0 \leq \varkappa \leq \varkappa_1$ and Caputo (or ψ-Caputo) FD on $\varkappa_1 \leq \varkappa \leq b$, $\varphi : \mathbb{J} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $g : C(\mathbb{J}, \mathbb{R}) \to \mathbb{R}$ are given functions, ($m \in \mathbb{N}$).

It is essential to note that the utilization of nonlinear condition $v(0) = v_0 + g(v)$ in physical issues yields a better impact than the initial condition $v(0) = v_0$ (see [42]).

We pay attention to the topic of novel piecewise operators. To the best of our knowledge, no results in the literature address the qualitative aspects of the aforesaid problems using the ψ-piecewise FC. Consequently, to close this gap and enrich the literature, we developed and extended the existence, uniqueness, and Ulam–Hyers stability results of ψ-piecewise Caputo pantograph problems (5) and (6) based on known fixed-point theorems of the Banach type and Krasnoselskii type. Furthermore, we present a more general problem as a system that covers the problems at hand.

Remark 1.
(i) If $\psi(\varkappa) = \varkappa$, then problem (6) reduces to problem (5).
(ii) If $\psi(\varkappa) = \varkappa$ and $m = 1$, then for $\varkappa \in [\varkappa_1, b]$, the problem (6) reduces to problem (4) considered in [40].
(iii) If $\psi(\varkappa) = \varkappa$, $g \equiv 0$ and $m = 2$, then for $\varkappa \in [0, \varkappa_1]$, the problem (6) reduces to problem (3) for an implicit term [39].
(iv) If $\psi(\varkappa) = \varkappa$ and $g \equiv 0$, then for $\varkappa \in [0, \varkappa_1]$, the problem (6) reduces to (2), see [38].
(v) Our current results for problem (6) stay available for problem (5).

This paper is arranged as follows: Section 2 provides some required results and fundamentals about piecewise FC. Our major outcomes for problems (5) and (6) are proved in Section 3. A comprehensive example verifying the validity of the theories is presented in Section 4. The conclusions of our study are summarized in the final section.

2. Primitive Results

In this section, we provide some notions and basic results of a piecewise FC. Let

$$\mathcal{C} := \mathcal{C}(\mathbb{J}, \mathbb{R}) = \left\{ \varpi : \mathbb{J} \to \mathbb{R};\ \|\varpi\| = \max_{\varkappa \in \mathbb{J}} |\varpi(\varkappa)| \right\}$$

$\mathcal{C}$ is a Banach space under the norm $\|\cdot\|$.

Definition 1 ([41])**.** *Let $\vartheta > 0$ and $\varpi : \mathbb{J} \to \mathbb{R}$ be continuous. Then, the piecewise RL fractional integral is given by*

$$^{PRL}\mathbb{I}^{\vartheta}_{0^+}\varpi(\varkappa) = \begin{cases} \mathbb{I}\varpi(\varkappa), & if \ \ \varkappa \in [0, \varkappa_1], \\ ^{RL}\mathbb{I}^{\vartheta}_{\varkappa_1}\varpi(\varkappa) & if \ \ \varkappa \in [\varkappa_1, b]. \end{cases}$$

where $\mathbb{I}\varpi(\varkappa) = \int_0^{\varkappa_1} \varpi(t)dt$ is the classical integral on $[0, \varkappa_1]$ and $^{RL}\mathbb{I}^{\vartheta}_{\varkappa_1}\varpi(\varkappa) = \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t))^{\vartheta - 1}\varpi(t)dt$ is the RL fractional integral on $[\varkappa_1, b]$.

Definition 2 ([41])**.** *Let $0 < \vartheta \leq 1$ and $\varpi : \mathbb{J} \to \mathbb{R}$ be continuous. Then, the piecewise Caputo FD is given by*

$$^{PC}\mathbb{D}^{\vartheta}_{0^+}\varpi(\varkappa) = \begin{cases} \mathbb{D}\varpi(\varkappa), & if \ \ \varkappa \in [0, \varkappa_1], \\ ^{C}\mathbb{D}^{\vartheta}_{\varkappa_1}\varpi(\varkappa) & if \ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

where $\mathbb{D}\varpi(\varkappa) = \frac{d}{d\varkappa}\varpi(\varkappa)$ is the classical derivative on $[0, \varkappa_1]$ and $^{C}\mathbb{D}^{\vartheta}_{\varkappa_1}\varpi(\varkappa) = \frac{1}{\Gamma(1-\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t))^{-\vartheta}\varpi'(t)dt$ is a Caputo FD on $[\varkappa_1, b]$.

Lemma 1 ([41])**.** *For a given function $\varpi : \mathbb{J} \to \mathbb{R}$, and $0 < \vartheta \leq 1$. Then, the following PC-FDE*

$$\begin{aligned} ^{PC}\mathbb{D}^{\vartheta}_{0^+}\varpi(\varkappa) &= f(\varkappa) \\ \varpi(0) &= \varkappa_0 \end{aligned}$$

has the following solution

$$\varpi(\varkappa) = \begin{cases} \varpi(0) + \int_0^{\varkappa_1} \varpi(\varkappa)d\varkappa, & if \ \ \varkappa \in [0, \varkappa_1], \\ \varpi(\varkappa_1) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} (\varkappa - t))^{\vartheta-1} \varpi(t)dt & if \ \ \varkappa \in [\varkappa_1, b]. \end{cases}$$

The following definition is the mainstay of our results, so we present the piecewise version of the ψ-fractional derivative and integral as follows:

Definition 3. *Let $0 < \vartheta \leq 1$, $\mathbb{I} = [a, b]$ be a finite or infinite interval, $\varpi : \mathbb{I} \to \mathbb{R}$ be an integrable function, and $\psi \in C^1(\mathbb{I})$ be an increasing function such that $\psi(\varkappa) \neq 0$, for all $\varkappa \in \mathbb{I}$. Then, the piecewise version of ψ-Caputo FD is given by*

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} \varpi(\varkappa) = \begin{cases} \mathbb{D}^{1;\psi} \varpi(\varkappa), & if \ \ \varkappa \in [0, \varkappa_1], \\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi} \varpi(\varkappa), & if \ \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{7}$$

where $\mathbb{D}^{1;\psi}\varpi(\varkappa) = \frac{\varpi'(\varkappa)}{\psi'(\varkappa)}$, and ${}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi}\varpi(\varkappa)$ is a ψ-Caputo FD defined by Almeida [24]), that is

$$ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi} \varpi(\varkappa) = \frac{1}{\Gamma(1-\vartheta)} \int_{\varkappa_1}^{\varkappa} \frac{\psi'(t)}{(\psi(\varkappa) - \psi(t))^{\vartheta}} \varpi'_{\psi}(t)dt.$$

The associated ψ-piecewise fractional integral by

$$ {}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi} \varpi(\varkappa) = \begin{cases} \mathbb{I}^{1;\psi} \varpi(t), & if \ \ \varkappa \in [0, \varkappa_1], \\ {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi} \varpi(\varkappa), & if \ \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{8}$$

where $\mathbb{I}^{1;\psi}\varpi(t) = \int_0^{\varkappa_1} \psi'(t)\varpi(t)dt$ represents classical integral with respect to ψ on $[0, \varkappa_1]$ and ${}^{PRL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi}\varpi(\varkappa) = \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \frac{\psi'(t)}{(\psi(\varkappa)-\psi(t))^{1-\vartheta}} \varpi(t)dt$ is a ψ-RL fractional integral (see [2]).

We recall the definitions of Ulam–Hyers (UH) stability and generalized Ulam–Hyers (GUH) stability.

Definition 4 ([43])**.** *ψ-PCPP (6) is UH stable if there exists a $\chi_{\varphi} > 0$ such that $\forall\ \varepsilon > 0$, and for each solution $\omega \in \mathcal{C}$ of the inequality*

$$\left| {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} \omega(\varkappa) - \varphi(\varkappa, \omega(\varkappa), \omega(\lambda_1\varkappa), \ldots, \omega(\lambda_m\varkappa)) \right| \leq \varepsilon, \quad \varkappa \in \mathbb{J}, \tag{9}$$

there exists a solution $\upsilon \in \mathcal{C}$ of ψ-PCPP (6) satisfies

$$|\omega(\varkappa) - \upsilon(\varkappa)| \leq \chi_{\varphi}\varepsilon. \tag{10}$$

Additionally, if there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$|\omega(\varkappa) - \upsilon(\varkappa)| \leq \chi_{\varphi}\phi(\varepsilon), \quad \varkappa \in \mathbb{J},$$

then the concerned solution is GUH stable.

For our forthcoming analysis, we need Banach's contraction map [44] and Krasnoselskii's fixed-point theorem [45].

3. Main Results

In this section, we provide some qualitative analyses of the ψ-piecewise Caputo pantograph FDE (6). First, certain crucial results are provided for the benefit of the upcoming analysis.

Remark 2.
(i) If $\psi(\varkappa) = \varkappa$, then (7) reduces to a piecewise Caputo FD [41].
(ii) If $\psi(\varkappa) = \varkappa$, then (8) reduces to a piecewise RL fractional integral [41].

Lemma 2. *Let $\vartheta \in (0,1]$ and for a given function $\varpi \in \mathcal{C}$. Then,*

$$ {}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi}\, {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}\varpi(\varkappa) = \begin{cases} \mathbb{I}^{1;\psi}\mathbb{D}^{1;\psi}\varpi(\varkappa) = \varpi(\varkappa_1) - \varpi(0),\ if\ \ \varkappa \in [0,\varkappa_1], \\ {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi}\, {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi}\varpi(\varkappa) = \varpi(\varkappa) - \varpi(\varkappa_1), if\ \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{11}$$

and

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}\, {}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi}\varpi(\varkappa) = \begin{cases} \mathbb{D}^{1;\psi}\,\mathbb{I}^{1;\psi}\varpi(\varkappa) = \varpi(\varkappa),\ if\ \ \varkappa \in [0,\varkappa_1], \\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta}\, {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta}\varpi(\varkappa) = \varpi(\varkappa),\ if\ \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{12}$$

Proof. For $\varkappa \in [\varkappa_1, b]$, the proof can be accomplished following similar kinds of steps as in the proof for Theorem 4 presented by Almeida [24].

For $\varkappa \in [0, \varkappa_1]$ in (11), we have from (8) and (7) that

$$\mathbb{I}^{1;\psi}\mathbb{D}^{1;\psi}\varpi(\varkappa) = \int_0^{\varkappa_1} \psi'(t)\mathbb{D}^{1;\psi}\varpi(t)dt = \int_0^{\varkappa_1} \psi'(t)\,\frac{\varpi'(t)}{\psi'(t)}dt = \int_0^{\varkappa_1} \varpi'(t)dt = \varpi(\varkappa_1) - \varpi(0).$$

For $\varkappa \in [0, \varkappa_1]$ in (12), using the Lemma 2.4 [2], for $n-1 < \vartheta \le n \in \mathbb{N}$ and $\varpi \in \mathcal{C}$, then $\mathbb{D}^{\vartheta;\psi}\,\mathbb{I}^{\vartheta;\psi}\varpi(\varkappa) = \varpi(\varkappa)$.

As a special case for $\vartheta = 1$, then $\mathbb{D}^{1;\psi}\mathbb{I}^{1;\psi}\varpi(\varkappa) = \varpi(\varkappa)$. □

Lemma 3. *Let $0 < \vartheta \le 1,\ 0 < \lambda_1 < \cdots < \lambda_m < 1$, and $\varphi : \mathbb{J} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $g : \mathcal{C} \to \mathbb{R}$ be continuous. Then, the ψ-PCPP (6) is equivalent to*

$$v(\varkappa) = \begin{cases} v_0 + g(v) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt,\ \ if\ \ \varkappa \in [0,\varkappa_1], \\ v_{\varkappa_1} + g(v) + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \\ \quad \times\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt,\ \ if\ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{13}$$

Proof. Let us assume we have the ψ-PCPP (6) and show that $v \in \mathcal{C}$ satisfies (13). By Definition 3, we have

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}v(\varkappa) = \begin{cases} \frac{1}{\psi'(\varkappa)}v'(\varkappa) : if\ \ \varkappa \in [0,\varkappa_1], \\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi}v(\varkappa) : if\ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

In view of Lemma 2, we have

$$ {}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi}\, {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}v(\varkappa) = \begin{cases} \mathbb{I}^{1;\psi}\mathbb{D}^{1;\psi}v(\varkappa) = v(\varkappa_1) - v(0),\ if\ \ \varkappa \in [0,\varkappa_1], \\ {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi}\, {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi}v(\varkappa) = v(\varkappa) - v(\varkappa_1), if\ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{14}$$

Applying ${}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi}$ on (6), we have

$$ {}^{PRL}\mathbb{I}_{0^+}^{\vartheta;\psi}\, {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}v(\varkappa) = \begin{cases} \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt,\ if\ \ \varkappa \in [0,\varkappa_1], \\ {}^{PRL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi}\,\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t)),\ if\ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{15}$$

Comparing (14) and (15), we obtain

Case 1: For $\varkappa \in [0, \varkappa_1]$,

$$v(\varkappa) = v(0) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt.$$

Case 2: For $\varkappa \in [\varkappa_1, b]$,

$$v(\varkappa) = v(\varkappa_1) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt.$$

Using the condition in both cases, we obtain

$$v(\varkappa) = \begin{cases} v_0 + g(v) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt, & if \ \ \varkappa \in [0, \varkappa_1], \\ v_{\varkappa_1} + g(v) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} & \\ \times \varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt, & if \ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

which is (13).

Let $v \in \mathcal{C}$ satisfy (13), and we prove that (6) holds. Applying ${}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}$ on (13), we have

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} v(\varkappa) = \begin{cases} \frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa}\left(v_0 + g(v) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt\right) \ if \ \ \varkappa \in [0, \varkappa_1], \\ {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi}\left(v_{\varkappa_1} + g(v) + {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi} \varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))\right), \ if \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{16}$$

From the fact that classical derivative and Caputo derivative of any constant function are zero, Lemma 2 shows that

$$\begin{aligned} & \frac{1}{\psi'(\varkappa)} \frac{d}{d\varkappa} \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))dt \\ = \ & \varphi(\varkappa, v(\varkappa), v(\lambda_1 \varkappa), \ldots, v(\lambda_m \varkappa)), \text{ on } 0 \leq \varkappa \leq \varkappa_1, \end{aligned}$$

and

$$\begin{aligned} & {}^{C}\mathbb{D}_{\varkappa_1}^{\vartheta;\psi} \, {}^{RL}\mathbb{I}_{\varkappa_1}^{\vartheta;\psi} \varphi(\varkappa, v(\varkappa), v(\lambda \varkappa)) \\ = \ & \varphi(\varkappa, v(\varkappa), v(\lambda_1 \varkappa), \ldots, v(\lambda_m \varkappa)), \text{ on } \ \varkappa_1 \leq \varkappa \leq b. \end{aligned}$$

Hence

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1 \varkappa), \ldots, v(\lambda_m \varkappa)), \text{ for each } \varkappa \in \mathbb{J}.$$

Moreover, $v(0) = v_0 + g(v)$ on $[0, \varkappa_1]$, and $v(\varkappa_1) = v_{\varkappa_1} + g(v)$, on $[\varkappa_1, b]$. □

Remark 3. *Let $m = 1$ in Lemma 3. Then, we have the following ψ-piecewise Caputo pantograph problem (ψ-PCPP):*

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda \varkappa)), \ \ \varkappa \in \mathbb{J}, \\ v(0) = v_0 + g(v). \end{cases} \tag{17}$$

In particular, if we replace ${}^{C}\mathbb{D}_{0^+}^{\vartheta}$ instead of ${}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi}$ with $\psi(\varkappa) = \varkappa$, then (17) reduces to (4), which was considered by Balachandran et al. [40].

Corollary 1. *Let $0 < \vartheta \leq 1, \ \ 0 < \lambda < 1$, and $\varphi : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathcal{C} \to \mathbb{R}$ be continuous. Then, the ψ-PCPP (17) is equivalent to*

$$v(\varkappa) = \begin{cases} v_0 + g(v) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda t))dt, & if \ \ \varkappa \in [0, \varkappa_1], \\ v_{\varkappa_1} + g(v) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} & \\ \times \varphi(t, v(t), v(\lambda t))dt, & if \ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{18}$$

As per Lemma 3, we define an operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{K}v)(\varkappa) = \begin{cases} (\mathcal{K}_1 v)(\varkappa), & if \ \ \varkappa \in [0, \varkappa_1], \\ (\mathcal{K}_2 v)(\varkappa) & if \ \ \varkappa \in [\varkappa_1, b], \end{cases}$$

where

$$(\mathcal{K}_1 v)(\varkappa) = v_0 + g(v) + \int_0^{\varkappa} \psi'(t)\varphi(t, v(t), v(\lambda_1 t), \dots, v(\lambda_m t))dt, \ \ if \ \ \varkappa \in [0, \varkappa_1],$$

$$\begin{array}{c}(\mathcal{K}_2 v)(\varkappa) = v_{\varkappa_1} + g(v) + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta - 1} \\ \times \varphi(t, v(t), v(\lambda_1 t), \dots, v(\lambda_m t))dt, \quad if \ \ \varkappa \in [\varkappa_1, b].\end{array}$$

Note that problem (6) has solutions if and only if the operator $\mathcal{K}$ has fixed points, i.e., $(\mathcal{K}v)(\varkappa) = v(\varkappa)$.

The uniqueness result is based on the Banach contraction map [44].

Theorem 1. *Assume that:*

(H1). *There exists $L_0, L_i > 0$ $(i = 1, 2, \dots, m)$ such that*

$$\left|\varphi(\varkappa, v, v_{\lambda_1}, \dots, v_{\lambda_m}) - \varphi(\varkappa, \omega, \omega_{\lambda_1}, \dots, \omega_{\lambda_m})\right| \leq L_0|v - \omega| + \sum_{i=1}^{m} L_i\left|v_{\lambda_i} - \omega_{\lambda_i}\right|,$$

for each $\varkappa \in \mathbb{J}$, $v, v_{\lambda_i}, \omega_{\lambda_i} \in \mathbb{R}$;

(H2). *There exists $L_g > 0$ such that $0 < L_g < 1$ and $|g(v) - g(\omega)| \leq L_g|v - \omega|$, for $v, \omega \in \mathcal{C}$. If*

$$\begin{aligned} \aleph_1 &= L_g + \zeta\varkappa_1\left(L_0 + \sum_{i=1}^{m} L_i\right) < 1, \\ \aleph_2 &= L_g + \frac{(\psi(b) - \psi(\varkappa_1))^{\vartheta}}{\Gamma(\vartheta + 1)}\left(L_0 + \sum_{i=1}^{m} L_i\right) < 1, \end{aligned} \tag{19}$$

then the ψ-PCPP (6) has a unique solution on $\mathbb{J}$.

Proof. Let $\sup_{\varkappa \in \mathbb{J}}|\varphi(\varkappa, 0, 0, \dots, 0)| = M_\varphi < \infty$, and $\sup_{\varkappa \in \mathbb{J}}|g(0)| = M_g < \infty$. Becaue $\psi \in \mathcal{C}^1$, there exists a ζ such that $\sup_{t \in \mathbb{J}}|\psi'(t)| \leq \zeta$. Choose $r > 0$ such that

$$r \geq \begin{cases} \frac{|v_0| + M_g + M_\varphi\zeta\varkappa_1}{1 - \aleph_1}, \text{ on } [0, \varkappa_1], \\ \frac{|v_{\varkappa_1}| + M_g + M_\varphi\frac{(\psi(b) - \psi(\varkappa_1))^{\vartheta}}{\Gamma(\vartheta + 1)}}{1 - \aleph_2}, \text{ on } [\varkappa_1, b]. \end{cases} \tag{20}$$

Now, we show that $\mathcal{K}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{v \in \mathcal{C} : \|v\| \leq r\}$. For any $v \in \mathcal{B}_r$, and $\varkappa \in [0, \varkappa_1]$, we have

$$\begin{aligned} |(\mathcal{K}_1 v)(\varkappa)| &\leq \sup_{\varkappa \in [0, \varkappa_1]}\left\{|v_0| + |g(v)| + \int_0^{\varkappa} \psi'(t)|\varphi(t, v(t), v(\lambda_1 t), \dots, v(\lambda_m t))|dt\right\} \\ &\leq |v_0| + |g(v)| + \int_0^{\varkappa_1} \psi'(t)|\varphi(t, v(t), v(\lambda_1 t), \dots, v(\lambda_m t))|dt \\ &\leq |v_0| + |g(v) - g(0)| + M_g + \int_0^{\varkappa_1} \psi'(t)M_\varphi dt + \int_0^{\varkappa_1} \psi'(t) \\ &\quad \times|\varphi(t, v(t), v(\lambda_1 t), \dots, v(\lambda_m t)) - \varphi(t, 0, 0, \dots, 0)|dt \\ &\leq |v_0| + L_g|v| + M_g + M_\varphi\zeta\varkappa_1 + \int_0^{\varkappa_1} \psi'(t)\left(L_0|v| + \sum_{i=1}^{m} L_i|v(\lambda_i t)|\right)dt \\ &\leq |v_0| + L_g\|v\| + M_g + M_\varphi\zeta\varkappa_1 + \zeta\varkappa_1\left(L_0\|v\| + \sum_{i=1}^{m} L_i\|v\|\right) \\ &\leq |v_0| + M_g + M_\varphi\zeta\varkappa_1 + \left(L_g + \zeta\varkappa_1(L_0 + \sum_{i=1}^{m} L_i)\right)r \\ &\leq (1 - \aleph_1)r + \aleph_1 r = r. \end{aligned} \tag{21}$$

For any $v \in \mathcal{B}_r$, and $\varkappa \in [\varkappa_1, b]$, we have

$$\begin{aligned}
|(\mathcal{K}_2 v)(\varkappa)| &\leq \sup_{\varkappa\in[\varkappa_1,b]}\Big\{|v_{\varkappa_1}| + |g(v)| + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}\psi'(t)(\psi(\varkappa)-\psi(t))^{\vartheta-1} \\
&\quad \times|\varphi(t,v(t),v(\lambda_1 t),\ldots,v(\lambda_m t))|dt\Big\} \\
&\leq |v_{\varkappa_1}| + |g(v)| + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{b}\psi'(t)(\psi(b)-\psi(t))^{\vartheta-1} \\
&\quad \times|\varphi(t,v(t),v(\lambda_1 t),\ldots,v(\lambda_m t))|dt \\
&\leq |v_{\varkappa_1}| + |g(v)-g(0)| + M_g + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{b}\psi'(t)(\psi(b)-\psi(t))^{\vartheta-1}M_\varphi dt \\
&\quad + \int_{\varkappa_1}^{b}\psi'(t)(\psi(b)-\psi(t))^{\vartheta-1}|\varphi(t,v(t),v(\lambda_1 t),\ldots,v(\lambda_m t)) - \varphi(t,0,0,\ldots,0)|dt \\
&\leq |v_{\varkappa_1}| + L_g\|v\| + M_g + M_\varphi\frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \\
&\quad + \frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}\left(L_0\|v\| + \sum_{i=1}^m L_i\|v\|\right) \\
&\leq |v_{\varkappa_1}| + M_g + M_\varphi\frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} + \left(L_g + \frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}(L_0 + \sum_{i=1}^m L_i)\right)r \\
&\leq (1-\aleph_1)r + \aleph_2 r = r.
\end{aligned} \tag{22}$$

Equations (21) and (22) show that $\mathcal{K}\mathcal{B}_r \subset \mathcal{B}_r$.

Next, let $v,\overline{v} \in \mathcal{C}$. Then

Case 1: For $\varkappa \in [0, \varkappa_1]$,

$$\begin{aligned}
&|(\mathcal{K}_1 v)(\varkappa) - (\mathcal{K}_1\overline{v})(\varkappa)| \\
&\leq |g(v) - g(\overline{v})| + \int_0^{\varkappa_1}\psi'(t) \\
&\times|\varphi(t,v(t),v(\lambda_1 t),\ldots,v(\lambda_m t)) - \varphi(t,\overline{v}(t),\overline{v}(\lambda_1 t),\ldots,\overline{v}(\lambda_m t))|dt \\
&\leq L_g|v-\overline{v}| + \int_0^{\varkappa_1}\psi'(t)\left(L_0|v(t)-\overline{v}(t)| + \sum_{i=1}^m L_i|v(\lambda_i t) - \overline{v}(\lambda_i t)|\right)dt \\
&\leq \left[L_g + \zeta\varkappa_1\left(L_0 + \sum_{i=1}^m L_i\right)\right]\|v-\overline{v}\|_{\mathcal{C}} \\
&= \aleph_1\|v-\overline{v}\|_{\mathcal{C}}.
\end{aligned} \tag{23}$$

Case 2: For $\varkappa \in [\varkappa_1, b]$,

$$\begin{aligned}
&|(\mathcal{K}_2 v)(\varkappa) - (\mathcal{K}_2\overline{v})(\varkappa)| \\
&\leq |g(v) - g(\overline{v})| + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}\psi'(t)(\psi(\varkappa)-\psi(t))^{\vartheta-1} \\
&\times|\varphi(t,v(t),v(\lambda_1 t),\ldots,v(\lambda_m t)) - \varphi(t,\overline{v}(t),\overline{v}(\lambda_1 t),\ldots,\overline{v}(\lambda_m t))|dt \\
&\leq L_g|v-\overline{v}| + \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^{\varkappa}\psi'(t)(\psi(\varkappa)-\psi(t))^{\vartheta-1} \\
&\times\left(L_0|v(t)-\overline{v}(t)| + \sum_{i=1}^m L_i|v(\lambda_i t) - \overline{v}(\lambda_i t)|\right)dt \\
&\leq L_g\|v-\overline{v}\|_{\mathcal{C}} + \frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}\left(L_0 + \sum_{i=1}^m L_i\right)\|v-\overline{v}\|_{\mathcal{C}} \\
&\leq \left(L_g + \frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}\left(L_0 + \sum_{i=1}^m L_i\right)\right)\|v-\overline{v}\|_{\mathcal{C}} \\
&= \aleph_2\|v-\overline{v}\|_{\mathcal{C}}.
\end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$\|\mathcal{K}v - \mathcal{K}\overline{v}\|_{\mathcal{C}} \leq \begin{cases} \aleph_1\|v-\overline{v}\|_{\mathcal{C}}, & on \in [0,\varkappa_1], \\ \aleph_2\|v-\overline{v}\|_{\mathcal{C}} & on \in [\varkappa_1,b]. \end{cases}$$

Because $\aleph_1, \aleph_2 < 1$, $\mathcal{K}$ is a contraction map. Therefore, we deduce from the Banach contraction map that ψ-PCPP (6) has a unique solution existing on $\mathbb{J}$. □

Next, we provide existence results based on Krasnoselskii's fixed-point theorem [45].

Theorem 2. *Let $\varphi : \mathbb{J} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, and $g : \mathcal{C} \to \mathbb{R}$ be continuous, satisfying (H1) and (H2). In addition, we assume that:*

(H3) $\left|\varphi(t, v, v_{\lambda_1}, \dots, v_{\lambda_m})\right| \leq \mu_\varphi(\varkappa)$ *for each* $\varkappa \in \mathbb{J}$, $v, v_{\lambda_i} \in \mathbb{R}$, *and* $\mu_\varphi \in \mathcal{C}$.

(H4) $|g(v)| \leq \mu_g|v|$, *for* $v \in \mathcal{C}$, $\mu_g > 0$.

If $0 \neq \mu_g < 1$, *then* ψ*-PCPP (6) has a least one solution on* $\mathbb{J}$.

Proof. Choose

$$r \geq \begin{cases} \frac{1}{1-\mu_g}\left(|v_0| + \mu_\varphi^*\zeta\varkappa_1\right), \text{ on } \varkappa \in [0,\varkappa_1], \\ \frac{1}{1-\mu_g}\left(|v_{\varkappa_1}| + \frac{\mu_\varphi^*(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}\right), \text{ on } \varkappa \in [\varkappa_1,b], \end{cases}$$

where $\mu_\varphi^* = \sup_{\varkappa\in\mathbb{J}}\left|\mu_\varphi(\varkappa)\right|$, and $\sup_{t\in\mathbb{J}}|\psi'(t)| \leq \zeta$.

Consider the operators $\mathcal{P}, \mathcal{O} : \mathcal{C}(\mathbb{J}, \mathcal{B}_r) \to \mathcal{C}(\mathbb{J}, \mathcal{B}_r)$ defined by

$$(\mathcal{P}v)(\varkappa) = \begin{cases} v_0 + g(v), \text{ if } \varkappa \in [0,\varkappa_1], \\ v_{\varkappa_1} + g(v), \text{ if } \varkappa \in [\varkappa_1,b], \end{cases}$$

and

$$(\mathcal{O}v)(\varkappa) = \begin{cases} \int_0^\varkappa \psi'(t)\varphi(t,v(t),v(\lambda_1 t),\dots,v(\lambda_m t))dt, & if \ \varkappa \in [0,\varkappa_1], \\ \frac{1}{\Gamma(\vartheta)}\int_{\varkappa_1}^\varkappa \psi'(t)(\psi(\varkappa)-\psi(t))^{\vartheta-1}\varphi(t,v(t),v(\lambda_1 t),\dots,v(\lambda_m t))dt, & if \ \varkappa \in [\varkappa_1,b], \end{cases}.$$

where $(\mathcal{P}v + \mathcal{O}v)(\varkappa) = (\mathcal{K}v)(\varkappa)$.

For any $v, \omega \in \mathcal{B}_r$, we have

Case 1: For $\varkappa \in [0,\varkappa_1]$,

$$\begin{aligned} |(\mathcal{P}v + \mathcal{O}v)(\varkappa)| &\leq |(\mathcal{P}v)(\varkappa)| + |(\mathcal{O}v)(\varkappa)| \\ &\leq |v_0| + |g(v)| + \int_0^\varkappa |\psi'(t)||\varphi(t,v(t),v(\lambda_1 t),\dots,v(\lambda_m t))|dt \\ &\leq |v_0| + \mu_g|v| + \int_0^\varkappa |\psi'(t)|\left|\mu_\varphi(t)\right|dt \\ &\leq |v_0| + \mu_g\|v\|_{\mathcal{C}} + \mu_\varphi^*\zeta\varkappa_1 \\ &\leq |v_0| + \mu_\varphi^*\zeta\varkappa_1 + \mu_g r \\ &\leq r. \end{aligned}$$

Case 2: For $\varkappa \in [\varkappa_1, b]$,

$$\begin{aligned}
|(\mathcal{P}v + \mathcal{O}v)(\varkappa)| &\leq |(\mathcal{P}v)(\varkappa)| + |(\mathcal{O}v)(\varkappa)| \\
&\leq |v_{\varkappa_1}| + |g(v)| \\
&\quad + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} |\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))| dt \\
&\leq |v_{\varkappa_1}| + \mu_g |v| + \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} |\mu_\varphi(t)| dt \\
&\leq |v_{\varkappa_1}| + \mu_g \|v\|_{\mathcal{C}} + \mu_\varphi^* \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \\
&\leq |v_{\varkappa_1}| + \mu_g r + \mu_\varphi^* \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \\
&\leq r.
\end{aligned}$$

Hence, we deduce that $\|\mathcal{P}v + \mathcal{O}v\|_{\mathcal{C}} \leq r$.
Next, for any $\varkappa \in \mathbb{J}$ and $v, \omega \in \mathcal{B}_r$, we have

$$\begin{aligned}
|\mathcal{P}v(\varkappa) - \mathcal{P}\omega(\varkappa)| &\leq \begin{cases} |g(v) - g(\omega)|, \text{ if } \varkappa \in [0, \varkappa_1], \\ |g(v) - g(\omega)|, \text{ if } \varkappa \in [\varkappa_1, b], \end{cases} \\
&\leq \begin{cases} L_g |v - \omega|, \text{ if } \varkappa \in [0, \varkappa_1], \\ L_g |v - \omega|, \text{ if } \varkappa \in [\varkappa_1, b]. \end{cases}
\end{aligned}$$

Thus, $\|\mathcal{P}v - \mathcal{P}\omega\|_{\mathcal{C}} \leq L_g \|v - \omega\|_{\mathcal{C}}$. As $L_g < 1$, $\mathcal{P}$ is a contraction map.
Finally, we show that $\mathcal{O}$ is continuous and compact.
Initially, we show that $\mathcal{O}$ is continuous.
Let $\{v_n\}_{n \geq 1}$ in $\mathcal{B}_r$ such that $v_n \to v$ in $\mathcal{B}_r$. Then, $\Phi_{v_n}(\varkappa) \to \Phi_v(\varkappa)$ $n \to \infty$, where

$$\Phi_{v_n}(\varkappa) := \varphi(\varkappa, v_n(\varkappa), v_n(\lambda_1 \varkappa), \ldots, v_n(\lambda_m \varkappa)) \text{ and } \Phi_v(\varkappa) := \varphi(\varkappa, v(\varkappa), v(\lambda_1 \varkappa), \ldots, v(\lambda_m \varkappa)).$$

For each $\varkappa$, we have

$$\begin{aligned}
|\mathcal{O}v_n(\varkappa) - \mathcal{O}v(\varkappa)| &= \begin{cases} \int_0^{\varkappa} \psi'(t) |\Phi_{v_n}(t) - \Phi_v(t)| dt, \ \ if \ \ \varkappa \in [0, \varkappa_1], \\ \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} |\Phi_{v_n}(t) - \Phi_v(t)| dt, \ \ if \ \ \varkappa \in [0, \varkappa_1], \end{cases} \\
&\leq \begin{cases} \int_0^{\varkappa} \psi'(t) \|\Phi_{v_n}(\cdot) - \Phi_v(\cdot)\| dt, \ \ if \ \ \varkappa \in [0, \varkappa_1], \\ \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \|\Phi_{v_n}(\cdot) - \Phi_v(\cdot)\| dt, \ \ if \ \ \varkappa \in [0, \varkappa_1]. \end{cases}
\end{aligned}$$

The continuity of φ and ψ implies that $|\mathcal{O}v_n(\varkappa) - \mathcal{O}v(\varkappa)| \to 0$, as $n \to \infty$.
Next, $\mathcal{O}$ is uniformly bounded on $\mathcal{B}_r$ as

$$|(\mathcal{O}v)(\varkappa)| \leq \begin{cases} \mu_\varphi^* \zeta \varkappa_1, \ \ if \ \ \varkappa \in [0, \varkappa_1], \\ \mu_\varphi^* \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}, \ \ if \ \ \varkappa \in [\varkappa_1, b]. \end{cases}$$

At last, we show the compactness of $\mathcal{O}$. Let $\varkappa \in [0, \varkappa_1]$ with $\varkappa_\epsilon < \varkappa_\delta \in [0, \varkappa_1]$. Then,

$$\begin{aligned}
|(\mathcal{O}v)(\varkappa_\epsilon) - (\mathcal{O}v)(\varkappa_\delta)| &= \left| \int_{\varkappa_\epsilon}^{\varkappa_\delta} \psi'(t) \varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t)) dt \right| \\
&\leq \int_{\varkappa_\epsilon}^{\varkappa_\delta} |\psi'(t)| |\varphi(t, v(t), v(\lambda_1 t), \ldots, v(\lambda_m t))| dt \\
&\leq (\varkappa_\delta - \varkappa_\epsilon) \zeta \mu_\varphi^*. \qquad (25)
\end{aligned}$$

Let $\varkappa \in [\varkappa_1, b]$ with $\varkappa_\epsilon < \varkappa_\delta \in [\varkappa_1, b]$. Then,

$$
\begin{aligned}
&|(\mathcal{O}v)(\varkappa_\epsilon) - (\mathcal{O}v)(\varkappa_\delta)| \\
&\leq \left| \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_\epsilon} \psi'(t)(\psi(\varkappa_\epsilon) - \psi(t))^{\vartheta-1} \Phi_v(t) dt \right. \\
&\left. - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_\delta} \psi'(t)(\psi(\varkappa_\delta) - \psi(t))^{\vartheta-1} \Phi_v(t) dt \right| \\
&\leq \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa_\epsilon} \psi'(t) \Big[(\psi(\varkappa_\epsilon) - \psi(t))^{\vartheta-1} - (\psi(\varkappa_\delta) - \psi(t))^{\vartheta-1} \Big] |\Phi_v(t)| dt \\
&+ \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_\epsilon}^{\varkappa_\delta} \psi'(t)(\psi(\varkappa_\delta) - \psi(t))^{\vartheta-1} |\Phi_v(t)| dt \\
&\leq \frac{\mu_\varphi^\star}{\Gamma(\vartheta+1)} \Big[(\psi(\varkappa_\delta) - \psi(\varkappa_1))^\vartheta - (\psi(\varkappa_\epsilon) - \psi(\varkappa_1))^\vartheta + 2(\psi(\varkappa_\delta) - \psi(\varkappa_\epsilon))^\vartheta \Big]. \qquad (26)
\end{aligned}
$$

From (25) and (26), we obtain

$$|(\mathcal{O}v)(\varkappa_\epsilon) - (\mathcal{O}v)(\varkappa_\delta)| \to 0, \text{ as } \varkappa_\delta - \varkappa_\epsilon \to 0.$$

Thus, $\mathcal{O}$ is equicontinuous. As per the previous steps, $\mathcal{O}$ is relatively compact on $\mathcal{B}_r$. Consequently, the Arzela–Ascoli lemma shows that $\mathcal{O}$ is compact on $\mathcal{B}_r$. From to Krasnoselskii's theorem [45], the ψ-PCPP (6) has a least one solution on $\mathbb{J}$. □

UH Stability Analysis

In this part, we provide the UH and GUH stability of ψ-PCPP (6).

Remark 4. *$\omega \in \mathcal{C}$ satisfies (9) if there exists $\varsigma \in \mathcal{C}$ with*
(i) $|\varsigma(\varkappa)| \leq \varepsilon$, $\varkappa \in \mathbb{J}$;
(ii) For all $\varkappa \in \mathbb{J}$,

$$ {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} \omega(\varkappa) = \varphi(\varkappa, \omega(\varkappa), \omega(\lambda_1 \varkappa), \ldots, \omega(\lambda_m \varkappa)) + \varsigma(\varkappa). \qquad (27)$$

Lemma 4. *Let $0 < \vartheta \leq 1$, and $\omega \in \mathcal{C}$ is a solution of (9). Then, ω satisfies*

$$
\begin{cases}
\left|\omega(\varkappa) - \mathcal{W}_0 - \int_0^\varkappa \psi'(t)\Phi_\omega(t)dt\right| \leq \zeta\varkappa_1\varepsilon, \; if \;\; \varkappa \in [0, \varkappa_1], \\
\left|\omega(\varkappa) - \mathcal{W}_1 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^\varkappa \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}\Phi_\omega(t)dt\right| \leq \frac{(b-\varkappa_1)^\vartheta}{\Gamma(\vartheta+1)}\varepsilon, \; if \;\; \varkappa \in [\varkappa_1, b],
\end{cases}
$$

where $\Phi_\omega(t) := \varphi(t, \omega(t), \omega(\lambda_1 t), \ldots, \omega(\lambda_m t))$, $\mathcal{W}_0 = \omega_0 + g(\omega)$, ζ is a constant with $\sup_{t\in\mathbb{J}}|\psi'(t)| \leq \zeta$, and $\mathcal{W}_1 = \omega_{\varkappa_1} + g(\omega)$.

Proof. Let ω be a solution of (9). It follows from (ii) of Remark 4 that

$$
\begin{cases}
{}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} \omega(\varkappa) = \varphi(\varkappa, \omega(\varkappa), \omega(\lambda_1\varkappa), \ldots, \omega(\lambda_m\varkappa)) + \varsigma(\varkappa) \\
\omega(0) = \omega_0 + g(\omega).
\end{cases} \qquad (28)
$$

Then, the solution of problem (28) is

$$
\omega(\varkappa) = \begin{cases}
\mathcal{W}_0 + \int_0^\varkappa \psi'(t)[\Phi_\omega(t) + \varsigma(t)]dt, \; if \;\; \varkappa \in [0, \varkappa_1], \\
\mathcal{W}_1 + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^\varkappa \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}[\Phi_\omega(t) + \varsigma(t)]dt, \quad if \;\; \varkappa \in [\varkappa_1, b].
\end{cases}
$$

Once more, by (i) of Remark 4, we obtain

$$
\begin{cases}
\left|\omega(\varkappa) - \mathcal{W}_0 - \int_0^\varkappa \psi'(t)\Phi_\omega(t)dt\right| \leq \int_0^\varkappa \psi'(t)|\varsigma(t)|dt \leq \varepsilon\zeta\varkappa_1, \; \text{for } \varkappa \in [0, \varkappa_1], \\
\left|\omega(\varkappa) - \mathcal{W}_1 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^\varkappa \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}\Phi_\omega(t)dt\right| \\
\leq \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^\varkappa \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}|\varsigma(t)|dt \leq \frac{(\psi(b)-\psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)}\varepsilon, \; \text{for } \varkappa \in [\varkappa_1, b].
\end{cases}
$$

□

Theorem 3. *Under the hypotheses of Theorem 1. Then, the solution of the ψ-PCPP (6) is UH and GUH stable.*

Proof. Let $\omega \in \mathcal{C}$ be a solution of (9), and $v \in \mathcal{C}$ be a unique solution of the following problem

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\vartheta;\psi} v(\varkappa) = \Phi_v(\varkappa), \\ v(0) = \omega(0). \end{cases}$$

From Lemma 1, we have

$$v(\varkappa) = \begin{cases} \mathcal{V}_0 + \int_0^{\varkappa} \psi'(t)\Phi_v(t)dt, & if \ \ \varkappa \in [0, \varkappa_1], \\ \mathcal{V}_1 + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}\Phi_v(t)dt, & if \ \ \varkappa \in [\varkappa_1, b], \end{cases} \tag{29}$$

where $\mathcal{V}_0 = v_0 + g(v)$ and $\mathcal{V}_1 = v_{\varkappa_1} + g(v)$. Clearly, if $v(0) = \omega(0)$, then $v_0 = v_{\varkappa_1}$, $\mathcal{V}_0 = \mathcal{W}_0$, and $\mathcal{V}_1 = \mathcal{W}_1$. Hence, (29) becomes

$$v(\varkappa) = \begin{cases} \mathcal{W}_0 + \int_0^{\varkappa} \psi'(t)\Phi_v(t)dt, & if \ \ \varkappa \in [0, \varkappa_1], \\ \mathcal{W}_1 + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1}\Phi_v(t)dt, & if \ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{30}$$

Using Lemma 4 and (H1), for $\varkappa \in [0, \varkappa_1]$, we have

$$\begin{aligned} &|\omega(\varkappa) - v(\varkappa)| \\ &= \left|\omega(\varkappa) - \mathcal{W}_0 - \int_0^{\varkappa} \psi'(t)\Phi_v(t)dt\right| \\ &\leq \left|\omega(\varkappa) - \mathcal{W}_0 - \int_0^{\varkappa} \psi'(t)\Phi_\omega(t)dt\right| + \int_0^{\varkappa} \psi'(t)|\Phi_\omega(t) - \Phi_v(t)|dt \\ &\leq \zeta\varkappa_1\varepsilon + \zeta \int_0^{\varkappa} \left(L_0|v(t) - \omega(t)| + \sum_{i=1}^{m} L_i|v(\lambda_i t) - \omega(\lambda_i t)| \right) dt \\ &\leq \zeta\varkappa_1\varepsilon + \zeta \int_0^{\varkappa} (L_0|v(t) - \omega(t)| + L_1|v(\lambda_1 t) - \omega(\lambda_1 t)| + \cdots + L_m|v(\lambda_m t) - \omega(\lambda_m t)|)dt \\ &\leq \zeta\varkappa_1\varepsilon + \zeta(L_0 + L_1 + \cdots + L_m) \int_0^{\varkappa} |v(t) - \omega(t)|dt. \end{aligned}$$

Using classical Gronwall's Lemma [46], we obtain

$$\begin{aligned} |\omega(\varkappa) - v(\varkappa)| &\leq \zeta\varkappa_1\varepsilon \exp\left(\int_0^{\varkappa} \zeta(L_0 + L_1 + \cdots + L_m) \right) dt \\ &= \zeta\varkappa_1\varepsilon \exp(\zeta(L_0 + L_1 + \cdots + L_m)\varkappa_1) := \varepsilon\chi_{\varphi}^0. \end{aligned} \tag{31}$$

For $\varkappa \in [\varkappa_1, b]$, we have

$$
\begin{aligned}
&|\omega(\varkappa) - v(\varkappa)| \\
&= \left| \omega(\varkappa) - \mathcal{W}_0 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \Phi_v(t) dt \right| \\
&\leq \left| \omega(\varkappa) - \mathcal{W}_0 - \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \Phi_\omega(t) dt \right| \\
&+ \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} |\Phi_\omega(t) - \Phi_v(t)| dt \\
&\leq \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \varepsilon + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \\
&\times \left(L_0 |v(t) - \omega(t)| + \sum_{i=1}^{m} L_i |v(\lambda_i t) - \omega(\lambda_i t)| \right) dt \\
&\leq \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \varepsilon + \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} \\
&\times (L_0 |v(t) - \omega(t)| + L_1 |v(\lambda_1 t) - \omega(\lambda_1 t)| + \cdots + L_m |v(\lambda_m t) - \omega(\lambda_m t)|) dt \\
&\leq \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \varepsilon + \frac{(L_0 + L_1 + \cdots + L_m)}{\Gamma(\vartheta)} \\
&\times \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} |v(t) - \omega(t)| dt.
\end{aligned}
$$

Using the generalized fractional Gronwall's Lemma [47], we obtain

$$
\begin{aligned}
|\omega(\varkappa) - v(\varkappa)| &\leq \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \varepsilon + \frac{(L_0 + L_1 + \cdots + L_m)}{\Gamma(\vartheta+1)} \varepsilon \\
&\times \frac{1}{\Gamma(\vartheta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\vartheta-1} (\psi(b) - \psi(\varkappa_1))^\vartheta dt \\
&\leq \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \varepsilon + \frac{(L_0 + L_1 + \cdots + L_m)}{\Gamma(\vartheta+1)} \varepsilon \frac{(\psi(b) - \psi(\varkappa_1))^{2\vartheta}}{\Gamma(\vartheta+1)} \\
&= \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \left(1 + \frac{(L_0 + L_1 + \cdots + L_m)}{\Gamma(\vartheta+1)} (\psi(b) - \psi(\varkappa_1))^\vartheta \right) \varepsilon \\
&: = \varepsilon \chi_\varphi^1.
\end{aligned} \tag{32}
$$

It follows from (31) and (32) that

$$
|\omega(\varkappa) - v(\varkappa)| \leq \left\{ \begin{array}{l} \varepsilon \chi_\varphi^0, for\ \varkappa \in [0, \varkappa_1] \\ \varepsilon \chi_\varphi^1, for\ \varkappa \in [\varkappa_1, b] \end{array} \right. , \tag{33}
$$

where

$$
\chi_\varphi^0 = \zeta \varkappa_1 \exp(\zeta(L_0 + L_1 + \cdots + L_m)\varkappa_1), \text{ and}
$$

$$
\chi_\varphi^1 = \frac{(\psi(b) - \psi(\varkappa_1))^\vartheta}{\Gamma(\vartheta+1)} \left(1 + \frac{(L_0 + L_1 + \cdots + L_m)}{\Gamma(\vartheta+1)} (\psi(b) - \psi(\varkappa_1))^\vartheta \right).
$$

Hence, the ψ-PCPP (6) is UH stable in $\mathcal{C}$.

Moreover, from Definition 4, there exists a nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(\epsilon) = \epsilon$. Then, from (33), we have

$$
|\omega(\varkappa) - v(\varkappa)| \leq \left\{ \begin{array}{l} \chi_\varphi^0 \phi(\epsilon), for\ \varkappa \in [0, \varkappa_1], \\ \chi_\varphi^1 \phi(\epsilon), for\ \varkappa \in [\varkappa_1, b], \end{array} \right.
$$

with $\phi(0) = 0$, which proves that ψ-PCPP (6) is GUH stable in $\mathcal{C}$. □

Remark 5. *Consider a more general problem as a system that contains a number of problems similar to our current problem (6) as follows:*

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\vartheta_i;\psi} v_i(\varkappa) = \varphi_i\Big(\varkappa, v_{1,\lambda_j}(\varkappa), v_{2,\lambda_j}(\varkappa), \ldots, v_{n,\lambda_j}(\varkappa)\Big), \ \ \varkappa \in [0,b], \\ v_i(0) = v_{0i} + g(v_i), \ \ i = 1,\ldots,n, \ j = 1,\ldots,m, \end{cases} \tag{34}$$

where $0 < \vartheta_i \leq 1$, $0 < \lambda_j < 1$, ${}^{PC}\mathbb{D}_{0^+}^{\vartheta_i;\psi}$ *is the generalized piecewise FD of order* ϑ_i*, and*

$$\begin{aligned} v_{1,\lambda_j}(\varkappa) &= v_1(\varkappa), v_1(\lambda_1\varkappa), \ldots, v_1(\lambda_m\varkappa), \\ v_{2,\lambda_j}(\varkappa) &= v_2(\varkappa), v_2(\lambda_1\varkappa), \ldots, v_2(\lambda_m\varkappa), \\ &\vdots \\ v_{n,\lambda_j}(\varkappa) &= \varkappa, v_n(\varkappa), v_n(\lambda_1\varkappa), \ldots, v_n(\lambda_m\varkappa). \end{aligned}$$

The pantograph system (34) can be written as:

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{\Theta;\psi}\mathcal{V}(\varkappa) = \Phi(\varkappa, \mathcal{V}_\lambda(\varkappa)), \ \ \varkappa \in [0,b], \\ \mathcal{V}(0) = \mathcal{V}_0 + g(\mathcal{V}), \end{cases} \tag{35}$$

where

$$\mathcal{V}(\varkappa) = \begin{bmatrix} v_1(\varkappa) \\ v_2(\varkappa) \\ \vdots \\ v_n(\varkappa) \end{bmatrix}, \ \ \Phi(\varkappa, \mathcal{V}_\lambda(\varkappa)) = \begin{bmatrix} \varphi_1(\varkappa, v_{1,\lambda_j}(\varkappa)) \\ \varphi_2(\varkappa, v_{2,\lambda_j}(\varkappa)) \\ \vdots \\ \varphi_n(\varkappa, v_{n,\lambda_j}(\varkappa)) \end{bmatrix}, \ \ \text{and}$$

$$\mathcal{V}(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \\ \vdots \\ v_n(0) \end{bmatrix}, \ \ \mathcal{V}_0 = \begin{bmatrix} v_{01} \\ v_{02} \\ \vdots \\ v_{0n} \end{bmatrix}, \ \ \Theta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_n \end{bmatrix}.$$

Using Lemma 1, system (35) has the following solution:

$$\mathcal{V}(\varkappa) = \begin{cases} \mathcal{V}_0 + g(\mathcal{V}) + \int_0^{\varkappa} \psi'(t)\Phi(\mathcal{V}_\lambda(t))dt, \ \ if \ \ \varkappa \in [0, \varkappa_1], \\ \mathcal{V}_{\varkappa_1} + g(\mathcal{V}) + \frac{1}{\Gamma(\Theta)} \int_{\varkappa_1}^{\varkappa} \psi'(t)(\psi(\varkappa) - \psi(t))^{\Theta-1}\Phi(t, \mathcal{V}_\lambda(t))dt, \ \ \ if \ \ \varkappa \in [\varkappa_1, b]. \end{cases} \tag{36}$$

Remark 6. *Following the approaches of proof used in the preceding, we can obtain the same results (Theorems 1–3) for the nonlinear pantograph system (35) in view of formula (36).*

4. Example

In this portion, we present an exhaustive example to illustrate the reported results. Consider the following ψ-PCPP

$$\begin{cases} {}^{PC}\mathbb{D}_{0^+}^{0.25;e^{\frac{\varkappa}{2}}} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa)), \ \ \varkappa \in [0,1], \\ v(0) = 0.7 + \sum_{i=1}^{n} c_i v(\varkappa_i), \end{cases} \tag{37}$$

or

$$\begin{cases} \mathbb{D}^{1;e^{\frac{\varkappa}{2}}} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa)), \text{ if } \varkappa \in [0, 0.5], \\ v(0) = 0.7 + \sum_{i=1}^{n} c_i v(\varkappa_i), \\ \\ {}^{C}\mathbb{D}_{0.5^+}^{0.25;e^{\frac{\varkappa}{2}}} v(\varkappa) = \varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa)), \ if \ \ \varkappa \in [0.5, 1], \\ v(0.5) = 0.8 + \sum_{i=1}^{n} c_i v(\varkappa_i), \end{cases}$$

where $\vartheta = 0.25$, $\psi(\varkappa) = e^{\frac{\varkappa}{2}}$, $v_0 = 0.7$, $v_{\varkappa_1} = 0.8$, $0 < \varkappa_1 = 0.5 < \cdots < \varkappa_n < 1 = b$, and c_i are positive constants with $\sum_{i=1}^{n} c_i < \frac{3}{5}$. Set $m = 2$ $\left(\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{4}\right)$,

$$\varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa)) = \frac{\sin\left|v(\varkappa) + v(\frac{\varkappa}{3})\right|}{9 + 2\varkappa} + \frac{\cos\left|v(\frac{\varkappa}{4})\right|}{4 + 2\varkappa},$$

for $\varkappa \in [0,1]$, $v \in [0,\infty)$, and

$$g(v) = \sum_{i=1}^{n} c_i v(\varkappa_i), \ v \in [0,\infty).$$

(I) Let $v, \omega \in [0,\infty)$, $\varkappa \in [0,1]$. Then

$$\begin{aligned}
&|\varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa)) - \varphi(\varkappa, \omega(\varkappa), \omega(\lambda_1\varkappa), \omega(\lambda_2\varkappa))| \\
&= \left| \frac{\sin\left|v(\varkappa) + v(\frac{\varkappa}{3})\right|}{9 + 2\varkappa} + \frac{\cos\left|v(\frac{\varkappa}{4})\right|}{4 + 2\varkappa} - \frac{\sin\left|\omega(\varkappa) + \omega(\frac{\varkappa}{3})\right|}{9 + 2\varkappa} - \frac{\cos\left|\omega(\frac{\varkappa}{4})\right|}{4 + 2\varkappa} \right| \\
&\leq \frac{1}{9 + 2\varkappa} \left| \left(\left|v(\varkappa) + v(\frac{\varkappa}{3})\right| - \left|\omega(\varkappa) + \omega(\frac{\varkappa}{3})\right| \right) + \frac{1}{4 + 2\varkappa} \left(\left|v(\frac{\varkappa}{4})\right| - \left|\omega(\frac{\varkappa}{4})\right| \right) \right| \\
&\leq \frac{1}{9 + 2\varkappa} \left(|v(\varkappa) - \omega(\varkappa)| + \left|v(\frac{\varkappa}{3}) - \omega(\frac{\varkappa}{3})\right| \right) + \frac{1}{4 + 2\varkappa} \left|v(\frac{\varkappa}{4}) - \omega(\frac{\varkappa}{4})\right| \\
&\leq \frac{1}{9}|v(\varkappa) - \omega(\varkappa)| + \frac{1}{9}\left|v(\frac{\varkappa}{3}) - \omega(\frac{\varkappa}{3})\right| + \frac{1}{4}\left|v(\frac{\varkappa}{4}) - \omega(\frac{\varkappa}{4})\right|.
\end{aligned}$$

Thus, (Hy_1) holds with $L_0 = L_1 = \frac{1}{9}$ and $L_2 = \frac{1}{4}$. Additionally, for $v, \omega \in [0,\infty)$, we have

$$|g(v) - g(\omega)| = \left| \sum_{i=1}^{n} c_i v(\varkappa_i) - \sum_{i=1}^{n} c_i \omega(\varkappa_i) \right| \leq \sum_{i=1}^{n} c_i |v - \omega| \leq \frac{3}{5}|v - \omega|.$$

Therefore, (Hy_2) holds with $L_g = \frac{3}{5}$. To fulfill condition (19), we have $\sup_{\varkappa\in[0,1]} |\psi'(\varkappa)| = \sup_{\varkappa\in[0,1]} \left|\frac{1}{2}e^{\frac{\varkappa}{2}}\right| = \frac{\sqrt{e}}{2} < 1 := \zeta$. Hence, $\aleph_1 = \frac{301}{360} < 1$, and $\aleph_2 \approx 0.64 < 1$. Thus, Theorem 1 shows that ψ-PCPP (37) has a unique solution $[0,1]$.

(II) For $\varepsilon > 0$ with $\chi_{\varphi}^{0} = \frac{1}{2}e^{\frac{17}{72}} > 0$, and $\chi_{\varphi}^{1} \approx 1.20 > 0$. It follows from Theorem 3 that the ψ-PCPP (37) is HU and GUH stable.

(III) For $\varkappa \in [0,1]$ and $v \in [0,\infty)$, we obtain

$$|\varphi(\varkappa, v(\varkappa), v(\lambda_1\varkappa), v(\lambda_2\varkappa))| \leq \frac{1}{9 + 2\varkappa} + \frac{1}{4 + 2\varkappa}$$

and

$$|g(v)| = \left| \sum_{i=1}^{n} c_i v(\varkappa_i) \right| \leq \sum_{i=1}^{n} c_i |v| \leq \frac{3}{5}|v|.$$

Consequently, (Hy_3) and ((Hy_4) hold with $\mu_{\varphi}(\varkappa) = \frac{1}{9+2\varkappa} + \frac{1}{4+2\varkappa}$, $\mu_{\varphi}^{*} = \frac{49}{36}$, and $\mu_g = \frac{3}{5} < 1$. Thus, all the assumptions of Theorem 2 are satisfied. Hence, ψ-PCPP (37) has a solution on $[0,1]$.

5. Conclusions

Atangana and Araz [41] suggested the idea of piecewise derivatives. In this regard, we created and expanded the existence, uniqueness, and UH–GUH stability results for nonlocal pantograph equations under ψ-piecewise Caputo FDs as an additional contribution to this subject. Based on the fixed-point theorems of Banach and Krasnoselskii, we offered numerous new results of existence and uniqueness. Moreover, results pertaining to UH/GUH stability were obtained utilizing traditional methodologies of nonlinear functional analysis. An example to validate the theoretical findings was provided. In light of our recent discoveries, a more general problem for the pantograph system that includes problems related to the study's subject was presented. In the future, it will be interesting to study the current pantograph systems under piecewise FDs in the Caputo Fabrizio, and Atangana–Baleanu sense [41,48,49].

Author Contributions: Conceptualization, M.S.A., W.S. and H.Z.A.; formal analysis, M.S.A., W.S, H.Z.A., N.A. and M.D.A.; methodology, M.S.A., W.S., H.Z.A. and N.A., investigation, M.S.A., W.S, H.Z.A., N.A. and M.D.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No real data were used to support this study. The data used in this study are hypothetical.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Osler, J. Leibniz rule for fractional derivatives generalized and an application to infinite series. *SIAM J. Appl. Math.* **1970**, *18*, 658–674. [CrossRef]
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier B.V.: Amsterdam, The Netherlands, 2006.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. Fractional Integrals and Derivatives. In *Theory and Applications*; Gordon and Breach: Amsterdam, The Netherlands, 1993.
4. Diethelm, K.; Ford, N.J. Analysis of fractional differential equations. *J. Math. Anal. Appl.* **2022**, *265*, 229–248. [CrossRef]
5. Abdo, M.S.; Ibrahim, A.G.; Panchal, S.K. Nonlinear implicit fractional differential equation involving ψ-Caputo fractional derivative. *Proc. Jangjeon Math. Soc.* **2019**, *22*, 387–400.
6. Abdo, M.S.; Abdeljawad, T.; Shah, K.; Ali, S.M. On nonlinear coupled evolution system with nonlocal subsidiary conditions under fractal-fractional order derivative. *Math. Meth. Appl. Sci.* **2021**, *44*, 6581–6600. [CrossRef]
7. Abdo, M.S.; Abdeljawad, T.; Ali, S.M.; Shah, K.; Jarad, F. Existence of positive solutions for weighted fractional order differential equations. *Chaos Solitons Fractals* **2020**, *141*, 110341. [CrossRef]
8. Jarad, F.; Abdeljawad, T.; Hammouch, Z. On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative. *Chaos Solitons Fractals* **2018**, *117*, 16–20. [CrossRef]
9. Abbas, S.; Benchohra, M.; N'Guérékata, G.M. *Topics in Fractional Differential Equations*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012; Volume 27.
10. Kucche, K.D.; Mali, A.D. On the nonlinear (k,ψ)-Hilfer fractional differential equations. *Chaos Solitons Fractals* **2021**, *152*, 111335. [CrossRef]
11. Alaoui, M.K.; Fayyaz, R.; Khan, A.; Shah, R.; Abdo, M.S. Analytical Investigation of Noyes–Field Model for Time-Fractional Belousov–Zhabotinsky Reaction. *Complexity* **2021**, *2021*, 3248376. [CrossRef]
12. Abood, B.N.; Redhwan, S.S.; Abdo, M.S. Analytical and approximate solutions for generalized fractional quadratic integral equation. *Nonlinear Funct. Anal. Appl.* **2021**, *26*, 497–512.
13. Atangana, A.; G ómez-Aguilar, G.F. Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu. *Numer. Meth. Partial Diff. Equ.* **2018**, *34*, 1502–1523. [CrossRef]
14. Al-Mdallal, Q.M. On the numerical solution of fractional Sturm- Liouville problems. *Int. J. Comput.* **2010**, *87*, 2837–2845. [CrossRef]
15. Ahmad, B.; Ntouyas, S.K. Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. *Appl. Math. Comput.* **2015**, *266*, 615–622. [CrossRef]
16. Ntouyas, S.K. Global existence results for certain second order delay integrodifferential equations with nonlocal conditions. *Dyn. Syst. Appl.* **1998**, *7*, 415–426.
17. Kumar, A.; Chauhan, H.V.S.; Ravichandran, C.; Nisar, K.S.; Baleanu, D. Existence of solutions of non-autonomous fractional differential equations with integral impulse condition. *Adv. Differ. Equ.* **2020**, *2020*, 434. [CrossRef]
18. Abbas, S.; Benchohra, M.; Lazreg, J.E.; Zhou, Y. A survey on Hadamard and Hilfer fractional differential equations: Analysis and stability. *Chaos Solitons Fractals* **2017**, *102*, 47–71. [CrossRef]
19. Lakshmikantham, V.; Vatsala, A.S. Basic theory of fractional differential equations. *Nonlinear Anal.* **2008**, *69*, 2677–2682. [CrossRef]
20. Furati, K.M.; Kassim, M.D. Existence and uniqueness for a problem involving Hilfer fractional derivative. *Comput. Math. Appl.* **2012**, *64*, 1616–1626. [CrossRef]
21. Wang, J.; Zhang, Y. Nonlocal initial value problems for differential equations with Hilfer fractional derivative. *Applied Mathematics Comput.* **2015**, *266*, 850–859. [CrossRef]

22. Tunç, C.; Golmankhaneh, A.K.; Branch, U. On stability of a class of second alpha-order fractal differential equations. *AIMS Math.* **2020**, *5*, 2126–2142. [CrossRef]
23. Bohner, M.; Tunç, O.; Tunç, C. Qualitative analysis of Caputo fractional integro-differential equations with constant delays. *Comput. Appl. Math.* **2021**, *40*, 214. [CrossRef]
24. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
25. Sousa, J.V.C.; Oliveira, E.C. On the ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
26. Hale, J.K.; Lunel, S.M. *Introduction to Functional Differential Equations*; Springer Science and Business Media: New York, NY, USA, 2013.
27. Brunt, B.; Zaidi, A.A.; Lynch, T. Cell division and the pantograph equation. *ESAIM Proc. Surv.* **2018**, *62*, 158–167. [CrossRef]
28. Sedaghat, S.; Ordokhani, Y.; Dehghan, M. Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 4815–4830. [CrossRef]
29. Bahsi, M.; Cevik, M.; Sezer, M. Orthoexponential polynomial solutions of delay pantograph differential equations with residual error estimation. *Appl. Math. Comput.* **2015**, *271*, 11–21.
30. Ockendon, J.R.; Taylor, A.B. The dynamics of a current collection system for an electric locomotive. *Proc. R Soc. Lond. Ser. A* **1971**, *322*, 447–468.
31. Ajello, W.G.; Freedman, H.I.; Wu, J. A model of stage structured population growth with density depended time delay. *SIAM J. Appl. Math.* **1992**, *52*, 855–869.
32. Weiner, G.J. Activation of NK Cell Responses and Immunotherapy of Cancer. *Curr. Cancer Res.* **2014**, *12*, 57–66.
33. Langley, J.K. A certain functional-differential equation. *J. Math. Anal. Appl.* **2000**, *244*, 564–567. [CrossRef]
34. Liu, M.Z. Asymptotic behavior of functional-differential equations with proportional time delays. *Eur. J. Appl. Math.* **1996**, *7*, 11–30. [CrossRef]
35. Li, D.; Liu, M.Z. Runge-Kutta methods for the multi-pantograph delay equation. *Appl. Math. Comput.* **2005**, *163*, 383–395. [CrossRef]
36. Derfel, G.A.; Iserles, A. The pantograph equation in the complex plane. *J. Math. Anal. Appl.* **1997**, *213*, 117–132 [CrossRef]
37. Iserles, A. Exact and discretized stability of the pantograph equation. *Appl. Numer. Math.* **1997**, *24*, 295–308. [CrossRef]
38. Liu, M.Z.; Li, D. Properties of analytic solution and numerical solution of multi-pantograph equation. *Appl. Math. Comput.* **2004**, *155*, 853–871. [CrossRef]
39. Sezer, M.; Yalcinbas, S.; Sahin, N. Approximate solution of multi-pantograph equation with variable coefficients. *J. Comput. Appl. Math.* **2008**, *214*, 406–416. [CrossRef]
40. Balachandran, K.; Kiruthika, S.; Trujillo, J. Existence of solutions of nonlinear fractional pantograph equations. *Acta Math. Sci.* **2013**, *33*, 712–720. [CrossRef]
41. Atangana, A.; Araz, S.I. New concept in calculus: Piecewise differential and integral operators. *Chaos Solitons Fractals* **2021**, *145*, 110638. [CrossRef]
42. Bashir, A.; Sivasundaram, S. Some existence results for fractional integro-differential equations with nonlocal conditions. *Commun. Appl. Anal.* **2008**, *12*, 107–112.
43. Shah, K.; Abdeljawad, T.; Abdalla, B.; Abualrub, M.S. Utilizing fixed point approach to investigate piecewise equations with non-singular type derivative. *AIMS Math.* **2022**, *7*, 14614–14630. [CrossRef]
44. Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
45. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Usp. Mat. Nauk.* **1955**, *10*, 123–127.
46. Gronwall, T.H. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. Math.* **2019**, *2*, 292–296. [CrossRef]
47. Sousa, J.V.C.; Oliveira, E.C. A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator. *Diff. Equ. Appl.* **2019**, *11*, 87–106.
48. Caputo, M.; Fabrizio, M. New definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
49. Atangana, A.; Baleanu, D. New fractional derivative with non-local and non-singular kernel. *Therm. Sci.* **2016**, *20*, 757–763. [CrossRef]