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An Interplay of Wigner–Ville Distribution and 2D Hyper-Complex Quadratic-Phase Fourier Transform

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Abstract: Two-dimensional hyper-complex (Quaternion) quadratic-phase Fourier transforms (Q-QPFT) have gained much popularity in recent years because of their applications in many areas, including color image and signal processing. At the same time, the applications of Wigner–Ville distribution (WVD) in signal analysis and image processing cannot be ruled out. In this paper, we study the two-dimensional hyper-complex (Quaternion) Wigner–Ville distribution associated with the quadratic-phase Fourier transform (WVD-QPFT) by employing the advantages of quaternion quadratic-phase Fourier transforms (Q-QPFT) and Wigner–Ville distribution (WVD). First, we propose the definition of the WVD-QPFT and its relationship with the classical Wigner–Ville distribution in the quaternion setting. Next, we investigate the general properties of the newly defined WVD-QPFT, including complex conjugate, symmetry-conjugation, nonlinearity, boundedness, reconstruction formula, Moyal's formula, and Plancherel formula. Finally, we propose the convolution and correlation theorems associated with WVD-QPFT.

Keywords: quaternion quadratic-phase Fourier transform; Wigner–Ville distribution; boundedness; Moyal's formula; convolution; correlation



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1. Introduction

The quadratic-phase Fourier transform (QPFT), which was introduced by Castro et al. [1,2], is the generalized version of the classical Fourier transform. This novel transform has overtaken all the existing signal processing tools as it develops a unique analysis of both transient and non-transient signals in a fashion that is easy and insightful. To be specific, QPFT is a generalization of already known integral transforms with kernels in the exponential form, like Fourier, fractional Fourier, and linear canonical transforms. The importance of QPFT lies in the treatment of problems requiring several controllable parameters that arise in various branches of science and engineering, such as harmonic analysis, sampling theory, image processing, and many others ([2–7]). The authors in [8–11] successfully studied the generalized Wigner–Ville distribution (WVD) and ambiguity function (AF) associated with the linear canonical transform, offset linear canonical transform, and QPFT, which are very useful signal processing tools for the non-stationary signals. They also studied the properties of the generalized Wigner–Ville distribution and ambiguity function by employing a new integral transform. The ambiguity function (AF) also plays a major role in the non-stationary signal analysis and has been used in various fields such as radar signal and image processing, sonar technology, optical information processing, and many others ([12–15]).

One of the interests of the researchers of present era is the quaternion (2D-Hyper-complex) Fourier transform (QFT), which is actually the generalization of the real and complex Fourier transform (FT). The research community has successfully studied the

useful properties of the QFT such as modulation, shift, differentiation, energy conservation, convolution, correlation, the uncertainty principle, and so on. The QFT plays a vital part in the representation of (multidimensional) signals [16–18]. It transforms a signal from the real (or quaternionic) 2D spectrum to a quaternion-valued frequency spectrum. The QFT can be better used for the color image analysis [19,20]. In [21], the authors were successful in applying the QFT to image preprocessing and speech recognition. In the quaternion setting, many transformations have a close relationship with QFT, for example, the quaternion wavelet transform, the quaternion windowed Fourier transform, the quaternion fractional Fourier transform, the quaternion linear canonical transform, and the quaternion offset linear canonical transform. The QFT has a wide range of applications see [22]. The quaternion linear canonical transform, the quaternion offset linear canonical transform, and the quaternion fractional Fourier transform are the generalized versions of the QFT, which are effective signal processing tools. These integral transforms play a vital role in the efficient representation of quaternion-valued signals and have found noteworthy applications in diverse areas of signal and image processing, such as color image processing, speech recognition, edge detection, and data compression [23–25]. Recently, in [26], the authors introduced the quaternion quadratic-phase Fourier transform (Q-QPFT) as a generalization of QFT. The authors studied its main properties and associated theorems. Q-QPFT transforms a quaternionic 2D signal into a quaternion-valued frequency-domain signal. Nevertheless, due to its global kernel, the Q-QPFT, as in the case of the classic FT, is not able to indicate the time localization of the spectral components of the Q-QPFT; therefore, the Q-QPFT is not suitable for processing the non-stationary quaternion signal, whose Q-QPFT spectral characteristics change with time. In this regard, different authors generalize the Q-QPFT to new integral transforms such as the short-time quadratic-phase Fourier transform in the quaternion setting [27,28]. The generalization of Q-QPFT to other time-frequency tools is still in its infancy.

The Wigner–Ville distribution (WVD), which is also called the Wigner–Ville transform (WVT) is an important tool in the time-frequency signal analysis. In recent years, WVD has evolved as a useful tool in diverse fields such as biomedical engineering; optics; and signal, image, and video processing. Various researchers have generalized WVD to quaternion algebra by means of different quaternionic integral transformations such as quaternion linear canonical transform and quaternion offset linear canonical transform (see [29–33]).

Motivated by the generalizations of QFTs, the Q-QPFT, and the Wigner–Ville distribution, we, in this paper, were successful in proposing a novel integral transform coined the Wigner–Ville distribution, which is associated with quaternion quadratic-phase Fourier transform (WVD-QQPFT). The WVD-QQPFT addresses the limitation of Q-QPFT by combining both the results and flexibility of the two transforms WVD and Q-QPFT. It is hoped that the proposed transform can provide better representations of non-stationary quaternion-valued signals. Besides studying all of the fundamental properties of the newly defined WVD-QQPFT, we derive some properties, including complex conjugate, symmetry-conjugation, nonlinearity, boundedness, reconstruction formula, Moyal’s formula, and Plancherel formula. Moreover, we propose the convolution and correlation theorems associated with WVD-QQPFT.

The rest of the paper is organised as follows. In Section 2, we discuss some preliminary results required in subsequent sections. In Section 3, we formally introduce the definition of the Wigner–Ville distribution associated with the quaternionic quadratic-phase Fourier transform (WVD-QQPFT). Then, we investigate several basic properties of the WVD-QQPFT, which are important for signal representation in signal processing. In Section 4, we first define the convolution and correlation for the QQPFT. We then establish the new convolution and correlation for the WVD-QQPFT. Finally a conclusion is drawn in Section 5.

2. Preliminary

2.1. Quaternion Algebra

The quaternion algebra can be viewed as an extension of the complex numbers to the 4D algebra. W. R. Hamilton invented it in 1843, and it was denoted by the symbol \mathbb{H} in his honour. Each element of \mathbb{H} has a Cartesian representation as follows:

$$\mathbb{H} = \{q|q := q_0 + iq_1 + jq_2 + kq_3, q_i \in \mathbb{R}, i = 0, 1, 2, 3\}$$

where i, j, k are imaginary units and they obey Hamilton's multiplication rules (see [34])

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \\ ij &= -ji = k, jk = -kj = i, ki = -ik = j. \end{aligned}$$

Let $[q]_0$ and $\underline{q} = i[q]_1 + j[q]_2 + k[q]_3$ represent the real scalar part and the vector part of quaternion number $q = [q]_0 + i[q]_1 + j[q]_2 + k[q]_3$, respectively. Then, from [35,36] the real scalar part has a cyclic multiplication symmetry

$$[pql]_0 = [qlp]_0 = [lpq]_0, \quad \forall q, p, l \in \mathbb{H}. \quad (1)$$

The conjugation is defined by $\bar{q} = [q]_0 - i[q]_1 - j[q]_2 - k[q]_3$, and the norm of $q \in \mathbb{H}$ by

$$|q| = \sqrt{q\bar{q}} = \sqrt{[q]_0^2 + [q]_1^2 + [q]_2^2 + [q]_3^2}.$$

One can verify that

$$\overline{pq} = \bar{q}\bar{p} \quad |qp| = |q||p|, \quad \forall q, p \in \mathbb{H}.$$

The quaternion modules $L^2(\mathbb{R}^2, \mathbb{H})$ are defined as

$$L^2(\mathbb{R}^2, \mathbb{H}) := \{f|f : \mathbb{R}^2 \rightarrow \mathbb{H}, \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} |f(x_1, x_2)|^2 dx_1 dx_2 < \infty\}.$$

The inner product of quaternion functions f, g defined on $L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} = \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2,$$

with the symmetric real scalar part

$$\langle f, g \rangle = \frac{1}{2} \{ \langle f, g \rangle + \langle g, f \rangle \} = \int_{\mathbb{R}^2} [f(\mathbf{x}) \overline{g(\mathbf{x})}]_0 d\mathbf{x}.$$

The associated scalar norm of $f(\mathbf{x}) \in L^2(\mathbb{R}^2, \mathbb{H})$ is given by

$$\|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 = \langle f, f \rangle = \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} < \infty.$$

Lemma 1. If $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then the Cauchy–Schwarz inequality holds [36]

$$\left| \langle f, g \rangle_{L^2(\mathbb{R}^2, \mathbb{H})} \right|^2 \leq \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|g\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.$$

If and only if $f = -\lambda g$ for some quaternionic parameter $\lambda \in \mathbb{H}$, the equality holds.

2.2. Quaternion Quadratic-Phase Fourier Transform

Definition 1 ((Q-QPFT)[26–28]). Let $\mathbf{m}_s = (A_s, B_s, C_s, D_s, E_s)$ for $s = 1, 2$; then, the two-sided Q-QPFT of any signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[f](\mathbf{w}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(t_1, w_1) f(\mathbf{t}) \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(t_2, w_2) d\mathbf{t} \quad (2)$$

where $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(t_1, w_1)$ and $\Lambda_{\mathbf{m}_2}^{\mathbf{j}}(t_2, w_2)$ are kernel signals given by

$$\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(t_1, w_1) = \exp\{i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1^2 + D_1 t_1 + E_1 w_1)\}, \quad (3)$$

$$\Lambda_{\mathbf{m}_2}^{\mathbf{j}}(t_2, w_2) = \exp\{j(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2^2 + D_2 t_2 + E_2 w_2)\} \quad (4)$$

where $A_s, B_s, C_s, D_s, E_s \in \mathbb{R}$, $B_s \neq 0$ and $s = 1, 2$.

Lemma 2 ((Reconstruction formula for Q-QPFT) [26–28]). Every signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$, can be reconstructed back by the following formula:

$$\begin{aligned} f(\mathbf{t}) &= \mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{-1} \{ \mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[f] \}(\mathbf{t}) \\ &= \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(t_1, w_1)} \mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[f](\mathbf{w}) \overline{\Lambda_{\mathbf{m}_2}^{\mathbf{j}}(t_2, w_2)} d\mathbf{w}. \end{aligned} \quad (2.6)$$

From Lemma 2, we have the following result:

Proposition 1 (Plancherel formula for Q-QPFT). Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be two quaternion signals; then, we have

$$\langle \mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[f], \mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[g] \rangle = \frac{1}{|B_1 B_2|} \langle f, g \rangle. \quad (5)$$

For $f = g$, we have

$$\|f\|^2 = |B_1 B_2| \|\mathbb{Q}_{\mathbf{m}_1, \mathbf{m}_2}^{\mathbb{H}}[f]\|^2. \quad (6)$$

3. Wigner–Ville Distribution Associated with the Quaternion Quadratic-Phase Fourier Transform (WVD-QQPFT)

Definition 2. Let $\mathbf{m}_s = (A_s, B_s, C_s, D_s, E_s)$, $s = 1, 2$ be a given set of real parameters with $B_s \neq 0$; then, the cross Wigner–Ville distribution associated with the two-sided quaternion quadratic-phase Fourier transform (WVD-QQPFT) of any signals $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, is defined by

$$\mathcal{W}_{f,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau \quad (7)$$

where $\mathbf{w} = (w_1, w_2)$, $\mathbf{t} = (t_1, t_2)$, $\tau = (\tau_1, \tau_2)$ and $\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1)$ and $\Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2)$ are kernel signals given by (3) and (4), respectively.

Remark 1. If we take $f = g$ in (7), then we call it auto-WVD-QQPFT, and it is denoted as $\mathcal{W}_f^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w})$ and defined by

$$\mathcal{W}_f^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau. \quad (8)$$

Let us define the quaternion correlation product

$$R_{f,g}(\mathbf{t}, \tau) = f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)}.$$

Then, (7) can be reshaped as

$$\begin{aligned}\mathcal{W}_{f,g}^{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{t},\mathbf{w}) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) R_{f,g}(\mathbf{t}, \tau) \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau \\ &= \mathbb{Q}_{\mathbf{m}_1,\mathbf{m}_2}^{\mathbb{H}}[R_{f,g}(\mathbf{t}, \tau)](\mathbf{w}).\end{aligned}\quad (9)$$

Applying the inverse quaternion QPFT on both sides of (9), we obtain

$$R_{f,g}(\mathbf{t}, \tau) = \mathbb{Q}_{\mathbf{m}_1,\mathbf{m}_2}^{-1} \{ \mathcal{W}_{f,g}^{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \}$$

which implies

$$\begin{aligned}& f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \\ &= \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1)} \mathcal{W}_{f,g}^{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \overline{\Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2)} d\mathbf{w} \\ &= \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} e^{-i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + C_1 w_1^2 + D_1 \tau_1 + E_1 w_1)} \mathcal{W}_{f,g}^{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \\ &\quad \times e^{-j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + C_2 w_2^2 + D_2 \tau_2 + E_2 w_2)} d\mathbf{w}.\end{aligned}\quad (10)$$

Remark 2. By varying the parameter $\mathbf{m}_s, s = 1, 2$ the proposed transform (7) boils down to various well known integral transforms associated with WVD, viz: for $\mathbf{m}_1 = (0, -1, 0, 0, 0)$ and $\mathbf{m}_2 = (0, -1, 0, 0, 0)$, we obtain classical quaternion WVD and for $\mathbf{m}_1 = (\cot \theta_1, -\csc \theta_1, \cot \theta_1, 0, 0)$, and $\mathbf{m}_2 = (\cot \theta_2, -\csc \theta_2, \cot \theta_2, 0, 0)$, we obtain quaternion fractional WVD.

Prior to investigating the properties of WVD-QQPFT, we will establish its relation with the quaternion Wigner–Ville distribution.

3.1. Relationship with the Quaternion Wigner–Ville Distribution.

From the Definition 2, we obtain

$$\begin{aligned}\mathcal{W}_{f,g}^{\mathbf{m}_1,\mathbf{m}_2}(\mathbf{t},\mathbf{w}) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + C_1 w_1^2 + D_1 \tau_1 + E_1 w_1)} f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \\ &\quad \times e^{j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + C_2 w_2^2 + D_2 \tau_2 + E_2 w_2)} d\tau \\ &= \frac{1}{2\pi} e^{iw_1(C_1 w_1 + E_1)} \int_{\mathbb{R}^2} e^{i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + D_1 \tau_1)} f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \\ &\quad \times e^{j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + D_2 \tau_2)} d\tau \times e^{jw_2(C_2 w_2 + E_2 w_2)} \\ &= \frac{1}{2\pi} e^{iw_1(C_1 w_1 + E_1)} \int_{\mathbb{R}^2} e^{i\tau_1(B_1 w_1 + D_1)} f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} e^{j\tau_2(B_2 w_2 + D_2)} d\tau \\ &\quad \times e^{jw_2(C_2 w_2 + E_2 w_2)}, \quad \text{if } A_s = 0, s = 1, 2 \\ &= \frac{1}{2\pi} e^{iw_1(C_1 w_1 + E_1)} \mathcal{W}_{f,g}(\mathbf{t}, -\mathbf{B}\mathbf{w} - \mathbf{D}) e^{jw_2(C_2 w_2 + E_2 w_2)}, \quad \text{if } A_s = 0, s = 1, 2\end{aligned}$$

where $\mathcal{W}_{f,g}(t, w)$ represents the quaternion Wigner–Ville distribution [37].

3.2. General Properties of WVD-QQPFT

In this subsection, we present some general properties of the novel WVD-QQPFT and their detailed proofs.

Theorem 1 (Complex conjugate). For $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then, we have

$$\overline{\mathcal{W}_{f,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w})} = \mathcal{W}_{g,f}^{\mathbf{m}'_1, \mathbf{m}'_2}(\mathbf{t}, \mathbf{w}) \quad (11)$$

where $\mathbf{m}'_s = (-A_s, -B_s, -C_s, -D_s, -E_s)$, $s = 1, 2$.

Proof. From Definition 2, we have

$$\begin{aligned} & \overline{\mathcal{W}_{f,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w})} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) g\left(\mathbf{t} - \frac{\tau}{2}\right) \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \overline{e^{i(A_1\tau_1^2 + B_1\tau_1 w_1 + C_1 w_1^2 + D_1\tau_1 + E_1 w_1)} f\left(\mathbf{t} + \frac{\tau}{2}\right) g\left(\mathbf{t} - \frac{\tau}{2}\right)} \\ & \quad \times \overline{e^{j(A_2\tau_2^2 + B_2\tau_2 w_2 + C_2 w_2^2 + D_2\tau_2 + E_2 w_2)} d\tau} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(A_1\tau_1^2 + B_1\tau_1 w_1 + C_1 w_1^2 + D_1\tau_1 + E_1 w_1)} g\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} + \frac{\tau}{2}\right)} \\ & \quad \times e^{-j(A_2\tau_2^2 + B_2\tau_2 w_2 + C_2 w_2^2 + D_2\tau_2 + E_2 w_2)} d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(A_1\tau_1^2 + B_1\tau_1 w_1 + C_1 w_1^2 + D_1\tau_1 + E_1 w_1)} g\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} + \frac{\tau}{2}\right)} \\ & \quad \times e^{-j(A_2\tau_2^2 + B_2\tau_2 w_2 + C_2 w_2^2 + D_2\tau_2 + E_2 w_2)} d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i((-A_1)\tau_1^2 + (-B_1)\tau_1 w_1 + (-C_1)w_1^2 + (-D_1)\tau_1 + (-E_1)w_1)} g\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} + \frac{\tau}{2}\right)} \\ & \quad \times e^{j((-A_2)\tau_2^2 + (-B_2)\tau_2 w_2 + (-C_2)w_2^2 + (-D_2)\tau_2 + (-E_2)w_2)} d\tau \\ &= \mathcal{W}_{g,f}^{\mathbf{m}'_1, \mathbf{m}'_2}(\mathbf{t}, \mathbf{w}), \quad \mathbf{m}'_s = (-A_s, -B_s, -C_s, -D_s, -E_s), \quad s = 1, 2. \end{aligned}$$

This completes the proof. \square

Theorem 2 (Symmetry-conjugation). Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ be a quaternion signal. Then,

$$\mathcal{W}_{f(-t)}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) = \mathcal{W}_f^{\mathbf{m}'_1, \mathbf{m}'_2}(-\mathbf{t}, -\mathbf{w}) \quad (12)$$

where $\mathbf{m}'_s = (A_s, B_s, C_s, -D_s, -E_s)$, $s = 1, 2$.

Proof. From (1), we obtain

$$\begin{aligned} & \mathcal{W}_{f(-t)}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{\mathbf{m}_1}^{\mathbf{i}}(\tau_1, w_1) f\left(-\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(-\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{\mathbf{m}_2}^{\mathbf{j}}(\tau_2, w_2) d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(A_1\tau_1^2 + B_1\tau_1 w_1 + C_1 w_1^2 + D_1\tau_1 + E_1 w_1)} f\left(-\mathbf{t} - \frac{\tau}{2}\right) \overline{f\left(-\mathbf{t} + \frac{\tau}{2}\right)} \\ & \quad \times e^{j(A_2\tau_2^2 + B_2\tau_2 w_2 + C_2 w_2^2 + D_2\tau_2 + E_2 w_2)} d\tau. \end{aligned}$$

On setting $-\tau = \tau'$ in above equation, we have

$$\begin{aligned}
 & \mathcal{W}_{f(-t)}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(A_1(-\tau'_1)^2 + B_1(-\tau'_1)w_1 + C_1w_1^2 + D_1(-\tau'_1) + E_1w_1)} f\left(-\mathbf{t} + \frac{\tau'}{2}\right) \\
 & \quad \times \overline{f\left(-\mathbf{t} - \frac{\tau'}{2}\right)} e^{j(A_2(-\tau'_2)^2 + B_2(-\tau'_2)w_2 + C_2w_2^2 + D_2(-\tau'_2) + E_2w_2)} d\tau' \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{j(A_1\tau_1'^2 + B_1\tau_1'(-w_1) + C_1(-w_1)^2 + (-D_1)\tau_1' + (-E_1)(-w_1))} f\left(-\mathbf{t} + \frac{\tau'}{2}\right) \\
 & \quad \times \overline{f\left(-\mathbf{t} - \frac{\tau'}{2}\right)} e^{j(A_2\tau_2'^2 + B_2\tau_2'(-w_2) + C_2(-w_2)^2 + (-D_2)\tau_2' + (-E_2)(-w_2))} d\tau' \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1', -w_1) f\left(-\mathbf{t} + \frac{\tau'}{2}\right) \overline{f\left(-\mathbf{t} - \frac{\tau'}{2}\right)} \Lambda_{m_2}^j(\tau_2', -w_2) d\tau' \\
 &= \mathcal{W}_f^{m_1', m_2'}(-\mathbf{t}, -\mathbf{w})
 \end{aligned}$$

where $\mathbf{m}'_s = (A_s, B_s, C_s, -D_s, -E_s)$, $s = 1, 2$.

This completes the proof. \square

Next, we establish the nonlinearity of WVD-QQPFT, which states that WVD-QQPFT does not satisfy the superposition principle, which is not suitable to the analysis of multi-component signals.

Theorem 3 (Nonlinearity). *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, then,*

$$\mathcal{W}_{f+g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) = \mathcal{W}_{f,f}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{g,f}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{g,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}). \quad (13)$$

Proof. By applying Definition 2, we have

$$\begin{aligned}
 & \mathcal{W}_{f+g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) \left[f\left(\mathbf{t} + \frac{\tau}{2}\right) + g\left(\mathbf{t} + \frac{\tau}{2}\right) \right] \\
 & \quad \times \overline{\left[f\left(\mathbf{t} - \frac{\tau}{2}\right) + g\left(\mathbf{t} - \frac{\tau}{2}\right) \right]} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) \left[f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} - \frac{\tau}{2}\right)} + f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \right. \\
 & \quad \left. + g\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} - \frac{\tau}{2}\right)} + g\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \right] \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 & \quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 & \quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) g\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{f\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 & \quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) g\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\
 &= \mathcal{W}_{f,f}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{g,f}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) + \mathcal{W}_{g,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}).
 \end{aligned}$$

This completes the proof. \square

Theorem 4 (Boundedness). *For $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$\left| \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \right| \leq \frac{2}{\pi} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|g\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \quad (14)$$

Proof. By the virtue of Cauchy–Schwarz inequality in quaternion domain, we have

$$\begin{aligned}
 & \left| \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \right|^2 \\
 &= \left| \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \right|^2 \\
 &\leq \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \left| \Lambda_{m_1}^i(\tau_1, w_1) f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \Lambda_{m_2}^j(\tau_2, w_2) \right| d\tau \right)^2 \\
 &= \frac{1}{4\pi^2} \left(\int_{\mathbb{R}^2} \left| f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \right| d\tau \right)^2 \\
 &\leq \frac{1}{4\pi^2} \left(\int_{\mathbb{R}^2} \left| f\left(\mathbf{t} + \frac{\tau}{2}\right) \right|^2 d\tau \right) \left(\int_{\mathbb{R}^2} \left| \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \right|^2 d\tau \right) \\
 &= \frac{1}{4\pi^2} \left(4 \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \right) \left(4 \int_{\mathbb{R}^2} |f(\mathbf{z})|^2 d\mathbf{z} \right) \\
 &= \frac{4}{\pi^2} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|g\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.
 \end{aligned}$$

Here, we have applied the change of variables $\mathbf{x} = \mathbf{t} + \frac{\tau}{2}$ and $\mathbf{z} = \mathbf{t} - \frac{\tau}{2}$ in the last second step.

On further simplifying above equation, we have

$$\left| \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \right| \leq \frac{2}{\pi} \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})} \|g\|_{L^2(\mathbb{R}^2, \mathbb{H})}. \quad (15)$$

This completes the proof. \square

The following theorem guarantees the reconstruction of the input quaternion signal from the corresponding WVD-QQPFT within a constant factor.

Theorem 5 (Reconstruction formula). *Let f and g be two quaternion signals in $L^2(\mathbb{R}^2, \mathbb{H})$, with $g(\mathbf{0}) \neq \mathbf{0}$, then, we have the following inversion formula of the WVD-QQPFT:*

$$f(\mathbf{t}) = \frac{1}{g(\mathbf{0})} \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{m_1}^i(\tau_1, w_1)} \mathcal{W}_{f,g}^{m_1, m_2}\left(\frac{\mathbf{t}}{2}, \mathbf{w}\right) \overline{\Lambda_{m_2}^j(\tau_2, w_2)} d\mathbf{w}. \quad (16)$$

Proof. From (10), we have

$$\begin{aligned}
 & f\left(\mathbf{t} + \frac{\tau}{2}\right) \overline{g\left(\mathbf{t} - \frac{\tau}{2}\right)} \\
 &= \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{m_1}^i(\tau_1, w_1)} \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \overline{\Lambda_{m_2}^j(\tau_2, w_2)} d\mathbf{w} \\
 &= \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} e^{-i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + C_1 w_1^2 + D_1 \tau_1 + E_1 w_1)} \mathcal{W}_{f,g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\
 &\quad \times e^{-j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + C_2 w_2^2 + D_2 \tau_2 + E_2 w_2)} d\mathbf{w}.
 \end{aligned}$$

On setting $\mathbf{t} = \frac{\tau}{2}$ and applying change of variable $\mathbf{s} = 2\mathbf{t}$, we obtain

$$f(\mathbf{s}) \overline{g(\mathbf{0})} = \frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \overline{\Lambda_{m_1}^i(s_1, w_1)} \mathcal{W}_{f,g}^{m_1, m_2}\left(\frac{\mathbf{s}}{2}, \mathbf{w}\right) \overline{\Lambda_{m_2}^j(s_2, w_2)} d\mathbf{w}.$$

This completes the proof. \square

Theorem 6 (Moyal's Formula). Let $\mathcal{W}_{f_1, g_1}^{m_1, m_2}$ and $\mathcal{W}_{f_2, g_2}^{m_1, m_2}$ be the WVD-QQPFT of the quaternion signals f_1, g_1 and f_2, g_2 , respectively. Then,

$$\langle \mathcal{W}_{f_1, f_2}^{m_1, m_2}, \mathcal{W}_{g_1, g_2}^{m_1, m_2} \rangle = \frac{4}{|B_1 B_2|} [\langle f_1, f_2 \rangle \langle g_1, g_2 \rangle]_0. \quad (17)$$

Proof. By the definition of Wigner–Ville distribution associated with QQPFT and inner product relation, we have

$$\begin{aligned} & \langle \mathcal{W}_{f_1, f_2}^{m_1, m_2}, \mathcal{W}_{g_1, g_2}^{m_1, m_2} \rangle \\ &= \int_{\mathbb{R}^4} \left[\mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \overline{\mathcal{W}_{f_2, g_2}^{m_1, m_2}(\mathbf{t}, \mathbf{w})} \right]_0 d\mathbf{w} d\mathbf{t} \\ &= \int_{\mathbb{R}^4} \left[\mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) f_2\left(\mathbf{t} + \frac{\tau}{2}\right) g_2\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{\Lambda_{m_2}^j(\tau_2, w_2)} d\tau \right]_0 d\mathbf{w} d\mathbf{t} \\ &= \int_{\mathbb{R}^6} \left[\mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \frac{1}{2\pi} \Lambda_{m_2}^{-j}(\tau_2, w_2) g_2\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f_2\left(\mathbf{t} + \frac{\tau}{2}\right)} \Lambda_{m_1}^{-i}(\tau_1, w_1) \right]_0 d\tau d\mathbf{w} d\mathbf{t} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^6} \left[\Lambda_{m_1}^{-i}(\tau_1, w_1) \mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \Lambda_{m_2}^{-j}(\tau_2, w_2) g_2\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f_2\left(\mathbf{t} + \frac{\tau}{2}\right)} \right]_0 d\tau d\mathbf{w} d\mathbf{t} \\ &= \frac{1}{|B_1 B_2|} \int_{\mathbb{R}^4} \left[\frac{|B_1 B_2|}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^{-i}(\tau_1, w_1) \mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \Lambda_{m_2}^{-j}(\tau_2, w_2) d\mathbf{w} \right. \\ &\quad \left. \times g_2\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f_2\left(\mathbf{t} + \frac{\tau}{2}\right)} \right]_0 d\tau d\mathbf{t} \\ &= \frac{1}{|B_1 B_2|} \int_{\mathbb{R}^4} \left[f_1\left(\mathbf{t} + \frac{\tau}{2}\right) g_1\left(\mathbf{t} - \frac{\tau}{2}\right) g_2\left(\mathbf{t} - \frac{\tau}{2}\right) \overline{f_2\left(\mathbf{t} + \frac{\tau}{2}\right)} \right]_0 d\tau d\mathbf{t}. \end{aligned}$$

On setting $\mathbf{t} + \frac{\tau}{2} = \mathbf{x}$ and $\mathbf{t} - \frac{\tau}{2} = \mathbf{y}$, the above equation becomes

$$\begin{aligned} \langle \mathcal{W}_{f_1, g_1}^{m_1, m_2}, \mathcal{W}_{f_2, g_2}^{m_1, m_2} \rangle &= \frac{4}{|B_1 B_2|} \int_{\mathbb{R}^4} \left[f_1(\mathbf{x}) \overline{g_1(\mathbf{y})} g_2(\mathbf{y}) \overline{f_2(\mathbf{x})} \right]_0 d\mathbf{x} d\mathbf{y} \\ &= \frac{4}{|B_1 B_2|} \left[\int_{\mathbb{R}^2} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} d\mathbf{x} \int_{\mathbb{R}^2} g_2(\mathbf{y}) \overline{g_1(\mathbf{y})} d\mathbf{y} \right]_0 \\ &= \frac{4}{|B_1 B_2|} [\langle f_1, f_2 \rangle \langle g_1, g_2 \rangle]_0. \end{aligned}$$

This completes the proof. \square

Remark 3 (Plancherel's theorem). For $f_1 = f_2 = f$ and $g_1 = g_2 = g$, then Moyal's formula yields that the signal energy is preserved by the WVD-QQPFT.

$$\frac{|B_1 B_2|}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left| \mathcal{W}_{f_1, g_1}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \right|^2 d\mathbf{w} d\mathbf{t} = \|f\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2 \|g\|_{L^2(\mathbb{R}^2, \mathbb{H})}^2.$$

In the next section, we establish our main results, i.e., convolution and correlation theorems for the WVD associated with quaternion quadratic-phase Fourier transform.

4. Convolution and Correlation Theorems for the WVD-QQPFT

Definition 3. For a pair of real-valued functions $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we define the convolution operator of the QQPFT as

$$(f \star g)(t) = \int_{\mathbb{R}^2} \Xi(z_1, t_1) f(\mathbf{z}) g(t - z) \Xi(z_2, t_2) d\mathbf{z} \quad (18)$$

where $\Xi_1(z_1, t_1)$ and $\Xi_2(z_2, t_2)$ are known as weight functions.

We assume

$$\Xi_1(z_1, t_1) = e^{-4iA_1z_1(t_1-z_1)} \quad \text{and} \quad \Xi_2(z_2, t_2) = e^{-4jA_2z_2(t_2-z_2)}. \quad (19)$$

As a consequence of the above definition, we obtain the following important theorem, which invokes the WVD-QQPFT to separate the unwanted components appearing in the input signal and see its effect on the convolution given in (18).

Theorem 7. (Convolution for WVD-QOLCT). Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be two real-valued signals. If we assume that $\mathcal{W}_{f,g}^{m_1, m_2}$ is a real-valued function, then the following result holds:

$$\begin{aligned} \mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ = 2\pi e^{-iw_1(C_1w_1+E_1)} \left\{ \int_{\mathbb{R}^2} e^{-8iA_1u_1(t_1-u_1)} \mathcal{W}_{f,f}^{m_1, m_2}(\mathbf{u}, \mathbf{w}) \mathcal{W}_{g,g}^{m_1, m_2}(\mathbf{t} - \mathbf{u}, \mathbf{w}) \right. \\ \left. \times e^{-8jA_2u_2(t_2-u_2)} d\mathbf{u} \right\} \times e^{-jw_2(C_2w_2+E_2)}. \end{aligned}$$

Proof. Applying the definition of the WVD-QQPFT, we have

$$\mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) \left[(f \star g)\left(\mathbf{t} + \frac{\tau}{2}\right) \right] \left[\overline{(f \star g)\left(\mathbf{t} - \frac{\tau}{2}\right)} \right] \Lambda_{m_2}^j(\tau_2, w_2) d\tau. \quad (20)$$

Now, using Definition 18 in (20), we have

$$\begin{aligned} \mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) \left\{ \int_{\mathbb{R}^2} \Xi_1\left(z_1, t_1 + \frac{\tau_1}{2}\right) f(\mathbf{z}) g\left(\mathbf{t} + \frac{\tau}{2} - \mathbf{z}\right) \Xi_2\left(z_2, t_2 + \frac{\tau_2}{2}\right) d\mathbf{z} \right. \\ \left. \times \int_{\mathbb{R}^2} \Xi_1\left(y_1, t_1 - \frac{\tau_1}{2}\right) \overline{f(\mathbf{y}) g\left(\mathbf{t} - \frac{\tau}{2} - \mathbf{y}\right)} \Xi_2\left(y_2, t_2 - \frac{\tau_2}{2}\right) d\mathbf{y} \right\} \Lambda_{m_2}^j(\tau_2, w_2) d\tau \\ = \frac{1}{2\pi} \int_{\mathbb{R}^6} \Lambda_{m_1}^i(\tau_1, w_1) e^{-4iA_1z_1((t_1+\frac{\tau_1}{2})-z_1)} f(\mathbf{z}) g\left(\mathbf{t} + \frac{\tau}{2} - \mathbf{z}\right) e^{-4jA_2z_2((t_2+\frac{\tau_2}{2})-z_2)} \\ \times e^{-4iA_1y_1((t_1-\frac{\tau_1}{2})-y_1)} \overline{f(\mathbf{y}) g\left(\mathbf{t} - \frac{\tau}{2} - \mathbf{y}\right)} e^{-4jA_2y_2((t_2-\frac{\tau_2}{2})-y_2)} \Lambda_{m_2}^j(\tau_2, w_2) d\mathbf{z} d\mathbf{y} d\tau \\ = \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1\tau_1^2+B_1\tau_1w_1+C_1w_1^2+D_1\tau_1+E_1w_1)} e^{-4iA_1z_1((t_1+\frac{\tau_1}{2})-z_1)} f(\mathbf{z}) g\left(\mathbf{t} + \frac{\tau}{2} - \mathbf{z}\right) \\ \times e^{-4jA_2z_2((t_2+\frac{\tau_2}{2})-z_2)} e^{-4iA_1y_1((t_1-\frac{\tau_1}{2})-y_1)} \overline{f(\mathbf{y}) g\left(\mathbf{t} - \frac{\tau}{2} - \mathbf{y}\right)} e^{-4jA_2y_2((t_2-\frac{\tau_2}{2})-y_2)} \\ \times e^{j(A_2\tau_2^2+B_2\tau_2w_2+C_2w_2^2+D_2\tau_2+E_2w_2)} d\mathbf{z} d\mathbf{y} d\tau. \end{aligned} \quad (21)$$

On setting $\mathbf{z} = \mathbf{u} + \frac{\mathbf{p}}{2}$ and $\mathbf{y} = \mathbf{u} - \frac{\mathbf{p}}{2}$ in above equation, we obtain

$$\begin{aligned} \mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ = \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1\tau_1^2+B_1\tau_1w_1+C_1w_1^2+D_1\tau_1+E_1w_1)} e^{-4iA_1(u_1+\frac{p_1}{2})((t_1+\frac{\tau_1}{2})-(u_1+\frac{p_1}{2}))} f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right) \\ \times g\left(\mathbf{t} + \frac{\tau}{2} - \left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)\right) e^{-4jA_2(u_2+\frac{p_2}{2})((t_2+\frac{\tau_2}{2})-(u_2+\frac{p_2}{2}))} \\ \times e^{-4iA_1(u_1-\frac{p_1}{2})((t_1-\frac{\tau_1}{2})-(u_1-\frac{p_1}{2}))} \overline{f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right) g\left(\mathbf{t} - \frac{\tau}{2} - \left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)\right)} \\ \times e^{-4jA_2(u_2-\frac{p_2}{2})((t_2-\frac{\tau_2}{2})-(u_2-\frac{p_2}{2}))} e^{j(A_2\tau_2^2+B_2\tau_2w_2+C_2w_2^2+D_2\tau_2+E_2w_2)} d\mathbf{p} d\mathbf{q} d\mathbf{u}. \end{aligned}$$

Taking $\tau = \mathbf{p} + \mathbf{q}$ and noting that f, g are real-valued, we obtain

$$\begin{aligned}
& \mathcal{W}_{f \star g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1(p_1+q_1)^2 + B_1(p_1+q_1)w_1 + C_1w_1^2 + D_1(p_1+q_1) + E_1w_1)} \\
&\quad \times e^{-4iA_1(u_1 + \frac{p_1}{2})((t_1 + \frac{p_1+q_1}{2}) - (u_1 + \frac{p_1}{2}))} f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right) \\
&\quad \times g\left(\mathbf{t} + \frac{\mathbf{p} + \mathbf{q}}{2} - \left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)\right) e^{-4jA_2(u_2 + \frac{p_2}{2})((t_2 + \frac{p_2+q_2}{2}) - (u_2 + \frac{p_2}{2}))} \\
&\quad \times e^{-4iA_1(u_1 - \frac{p_1}{2})((t_1 - \frac{p_1+q_1}{2}) - (u_1 - \frac{p_1}{2}))} f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right) g\left(\mathbf{t} - \frac{\mathbf{p} + \mathbf{q}}{2} - \left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)\right) \\
&\quad \times e^{-4jA_2(u_2 - \frac{p_2}{2})((t_2 - \frac{p_2+q_2}{2}) - (u_2 - \frac{p_2}{2}))} \\
&\quad \times e^{j(A_2(p_2+q_2)^2 + B_2(p_2+q_2)w_2 + C_2w_2^2 + D_2(p_2+q_2) + E_2w_2)} d\mathbf{p}d\mathbf{q}d\mathbf{u} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{-8iA_1u_1(t_1-u_1) - 2iA_1p_1q_1} e^{i(A_1p_1^2 + B_1p_1w_1 + C_1w_1^2 + D_1p_1 + E_1w_1)} f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right) \overline{f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)} \\
&\quad \times e^{i(A_1q_1^2 + B_1q_1w_1 + C_1w_1^2 + D_1q_1 + E_1w_1 + 2A_1p_1q_1 - C_1w_1^2 - E_1w_1)} \\
&\quad \times e^{j(A_2p_2^2 + B_2p_2w_2 + C_2w_2^2 + D_2p_2 + E_2w_2)} g\left(\mathbf{t} - \mathbf{u} + \frac{\mathbf{q}}{2}\right) \overline{g\left(\mathbf{t} - \mathbf{u} + \frac{\mathbf{q}}{2}\right)} \\
&\quad \times e^{j(A_2q_2^2 + B_2q_2w_2 + C_2w_2^2 + D_2q_2 + E_2w_2 + 2A_2p_2q_2 - C_2w_2^2 - E_2w_2)} e^{-8jA_2u_2(t_2-u_2) - 2jA_2p_2q_2} d\mathbf{p}d\mathbf{q}d\mathbf{u} \\
&= 2\pi \int_{\mathbb{R}^2} e^{-8iA_1u_1(t_1-u_1)} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(A_1p_1^2 + B_1p_1w_1 + C_1w_1^2 + D_1p_1 + E_1w_1)} f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right) \overline{f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)} \right. \\
&\quad \times e^{j(A_2p_2^2 + B_2p_2w_2 + C_2w_2^2 + D_2p_2 + E_2w_2)} d\mathbf{p} \left. \right\} e^{-iw_1(C_1w_1 + E_1)} \\
&\quad \times \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(A_1q_1^2 + B_1q_1w_1 + C_1w_1^2 + D_1q_1 + E_1w_1)} g\left(\mathbf{t} - \mathbf{u} + \frac{\mathbf{q}}{2}\right) \overline{g\left(\mathbf{t} - \mathbf{u} + \frac{\mathbf{q}}{2}\right)} \\
&\quad \times e^{j(A_2q_2^2 + B_2q_2w_2 + C_2w_2^2 + D_2q_2 + E_2w_2)} d\mathbf{q} \left. \right\} e^{-8jA_2u_2(t_2-u_2)} d\mathbf{u} e^{-jw_2(C_2w_2 + E_2)} \\
&= 2\pi \int_{\mathbb{R}^2} e^{-8iA_1u_1(t_1-u_1)} \mathcal{W}_{f,f}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{u}, \mathbf{w}) e^{-iw_1(C_1w_1 + E_1)} \mathcal{W}_{g,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t} - \mathbf{u}, \mathbf{w}) \\
&\quad \times e^{-8jA_2u_2(t_2-u_2)} d\mathbf{u} e^{-jw_2(C_2w_2 + E_2)} \\
&= 2\pi e^{-iw_1(C_1w_1 + E_1)} \left\{ \int_{\mathbb{R}^2} e^{-8iA_1u_1(t_1-u_1)} \mathcal{W}_{f,f}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{u}, \mathbf{w}) \mathcal{W}_{g,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t} - \mathbf{u}, \mathbf{w}) \right. \\
&\quad \times e^{-8jA_2u_2(t_2-u_2)} d\mathbf{u} \left. \right\} \times e^{-jw_2(C_2w_2 + E_2)}.
\end{aligned}$$

This completes the proof. \square

Next, we will derive the correlation theorem in the WVD-QQPFT. Let us define the correlation for the QQPFT.

Definition 4. For any two real-valued functions $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, we define the correlation operator of the QQPFT as

$$(f \circ g)(t) = \int_{\mathbb{R}^2} e^{4iA_1z_1(t_1+z_1)} \overline{f(\mathbf{z})} g(\mathbf{t} + \mathbf{z}) e^{4jA_2z_2(t_2+z_2)} d\mathbf{z}. \quad (22)$$

Theorem 8. (Correlation for WVD-QOLCT). Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be two real-valued signals. If we assume that $\mathcal{W}_{f,g}^{\mathbf{m}_1, \mathbf{m}_2}$ is a real-valued function, then the following result holds:

$$\begin{aligned}
& \mathcal{W}_{f \circ g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t}, \mathbf{w}) \\
&= 2\pi e^{-iw_1(C_1w_1 + E_1)} \left\{ \int_{\mathbb{R}^2} e^{4iA_1(2u_1(t_1+u_1))} \mathcal{W}_{f,f}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{u}, -\mathbf{w}) \mathcal{W}_{g,g}^{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{t} + \mathbf{u}, \mathbf{w}) \right. \\
&\quad \times e^{4jA_2(2u_2(t_2+u_2))} d\mathbf{u} \left. \right\} \times e^{-jw_2(C_2w_2 + E_2)}. \quad (23)
\end{aligned}$$

Proof. Applying the definition of the WVD-QOLCT, we have

$$\mathcal{W}_{f \circ g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda_{m_1}^i(\tau_1, w_1) \left[(f \circ g)\left(\mathbf{t} + \frac{\tau}{2}\right) \right] \left[\overline{(f \circ g)\left(\mathbf{t} - \frac{\tau}{2}\right)} \right] \Lambda_{m_2}^j(\tau_2, w_2) d\tau. \quad (24)$$

Now using Definition 4 in (24), we have

$$\begin{aligned} & \mathcal{W}_{f \circ g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^6} \Lambda_{m_1}^i(\tau_1, w_1) e^{4iA_1 z_1((t_1 + \frac{\tau_1}{2}) + z_1)} \overline{f(\mathbf{z})} g\left(\mathbf{z} + \mathbf{t} + \frac{\tau}{2}\right) e^{4jA_2 z_2((t_2 + \frac{\tau_2}{2}) + z_2)} \\ & \quad \times e^{4iA_1 y_1(y_1 + (t_1 - \frac{\tau_1}{2}))} \overline{f(\mathbf{y})} g\left(\mathbf{y} + \mathbf{t} - \frac{\tau}{2}\right) e^{4jA_2 y_2(y_2 + (t_2 - \frac{\tau_2}{2}))} \Lambda_{m_2}^j(\tau_2, w_2) d\mathbf{z} dy d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + C_1 w_1^2 + D_1 \tau_1 + E_1 w_1)} e^{4iA_1 z_1((t_1 + \frac{\tau_1}{2}) + z_1)} \overline{f(\mathbf{z})} g\left(\mathbf{z} + \mathbf{t} + \frac{\tau}{2}\right) \\ & \quad \times e^{4jA_2 z_2((t_2 + \frac{\tau_2}{2}) + z_2)} e^{4iA_1 y_1(y_1 + (t_1 - \frac{\tau_1}{2}))} \overline{f(\mathbf{y})} g\left(\mathbf{y} + \mathbf{t} - \frac{\tau}{2}\right) e^{4jA_2 y_2(y_2 + (t_2 - \frac{\tau_2}{2}))} \\ & \quad \times e^{j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + C_2 w_2^2 + D_2 \tau_2 + E_2 w_2)} d\mathbf{z} dy d\tau. \end{aligned}$$

On setting $\mathbf{z} = \mathbf{u} + \frac{\mathbf{p}}{2}$ and $\mathbf{y} = \mathbf{u} - \frac{\mathbf{p}}{2}$ in above equation, we obtain

$$\begin{aligned} & \mathcal{W}_{f \circ g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1 \tau_1^2 + B_1 \tau_1 w_1 + C_1 w_1^2 + D_1 \tau_1 + E_1 w_1)} e^{4iA_1(u_1 + \frac{p_1}{2})((u_1 + \frac{p_1}{2}) + (t_1 + \frac{\tau_1}{2}))} \overline{f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)} \\ & \quad \times g\left(\mathbf{t} + \frac{\tau}{2} + \left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)\right) e^{4jA_2(u_2 + \frac{p_2}{2})((t_2 + \frac{\tau_2}{2}) + (u_2 + \frac{p_2}{2}))} \\ & \quad \times e^{4iA_1(u_1 - \frac{p_1}{2})((t_1 - \frac{\tau_1}{2}) + (u_1 - \frac{p_1}{2}))} \overline{f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)} g\left(\mathbf{t} - \frac{\tau}{2} + \left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)\right) \\ & \quad \times e^{4jA_2(u_2 - \frac{p_2}{2})((t_2 - \frac{\tau_2}{2}) + (u_2 - \frac{p_2}{2}))} e^{j(A_2 \tau_2^2 + B_2 \tau_2 w_2 + C_2 w_2^2 + D_2 \tau_2 + E_2 w_2)} dp dq du. \end{aligned}$$

Taking $\tau = \mathbf{q} - \mathbf{p}$ and noting that f, g are real-valued, we obtain

$$\begin{aligned} & \mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^6} e^{i(A_1(q_1 - p_1)^2 + B_1(q_1 - p_1)w_1 + C_1 w_1^2 + D_1(q_1 - p_1) + E_1 w_1)} \\ & \quad \times e^{4iA_1(u_1 + \frac{p_1}{2})((u_1 + \frac{p_1}{2}) + (t_1 + \frac{(q_1 - p_1)}{2}))} \overline{f\left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)} \\ & \quad \times g\left(\mathbf{t} + \frac{\mathbf{q} - \mathbf{p}}{2} + \left(\mathbf{u} + \frac{\mathbf{p}}{2}\right)\right) e^{4jA_2(u_2 + \frac{p_2}{2})((t_2 + \frac{(q_2 - p_2)}{2}) + (u_2 + \frac{p_2}{2}))} \\ & \quad \times e^{4iA_1(u_1 - \frac{p_1}{2})((t_1 - \frac{(q_1 - p_1)}{2}) + (u_1 - \frac{p_1}{2}))} \overline{f\left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)} g\left(\mathbf{t} - \frac{\mathbf{q} - \mathbf{p}}{2} + \left(\mathbf{u} - \frac{\mathbf{p}}{2}\right)\right) \\ & \quad \times e^{4jA_2(u_2 - \frac{p_2}{2})((t_2 - \frac{(q_2 - p_2)}{2}) + (u_2 - \frac{p_2}{2}))} \\ & \quad \times e^{j(A_2(q_2 - p_2)^2 + B_2(q_2 - p_2)w_2 + C_2 w_2^2 + D_2(q_2 - p_2) + E_2 w_2)} dp dq du. \end{aligned}$$

Following the procedure of the Theorem 7, we have

$$\begin{aligned} & \mathcal{W}_{f \star g}^{m_1, m_2}(\mathbf{t}, \mathbf{w}) \\ &= 2\pi e^{-iw_1(C_1 w_1 + E_1)} \left\{ \int_{\mathbb{R}^2} e^{4iA_1(2u_1(t_1 + u_1))} \mathcal{W}_{f, f}^{m_1, m_2}(\mathbf{u}, -\mathbf{w}) \mathcal{W}_{g, g}^{m_1, m_2}(\mathbf{t} + \mathbf{u}, \mathbf{w}) \right. \\ & \quad \left. \times e^{4jA_2(2u_2(t_2 + u_2))} d\mathbf{u} \right\} \times e^{-jw_2(C_2 w_2 + E_2)}. \end{aligned}$$

This completes the proof. \square

5. Conclusions

In this paper, we intertwined the advantages of the classical Wigner–Ville distribution and quaternion quadratic-phase Fourier transforms and introduced the notion of the novel WVD-QQPFT. Based on the properties of Q-QPFT and classical quaternion Wigner–Ville distribution (QWVD), the relationship between these two transforms is presented. Vital properties such as complex conjugate, symmetry-conjugation, nonlinearity, boundedness, reconstruction formula, the Moyals formula, and the Plancherel formula are derived. Finally, the convolution and correlation theorems associated with WVD-QQPFT are proposed. In our future works, we shall study the wavelet transform in the quaternion quadratic-phase domain and its relationship with the proposed WVD-QQPFT.

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Abbreviations

The following abbreviations are used in this manuscript:

QWQPFT	Quaternion windowed quadratic-phase Fourier transform
QPFT	Quadratic-phase Fourier transform
Q-QPFT	Quaternion quadratic-phase Fourier transform
WVD	Wigner–Ville distribution

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