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Characteristics of Sasakian Manifolds Admitting Almost $*$-Ricci Solitons

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Abstract: This article presents some results of a geometric classification of Sasakian manifolds (SM) that admit an almost $*$-Ricci soliton (RS) structure (g, ω, X). First, we show that a complete SM equipped with an almost $*$-RS with $\omega \neq$ const is a unit sphere. Then we prove that if an SM has an almost $*$-RS structure, whose potential vector is a Jacobi vector field on the integral curves of the characteristic vector field, then the manifold is a null or positive SM. Finally, we characterize those SM represented as almost $*$-RS, which are $*$-RS, $*$-Einstein or $*$-Ricci flat.

Keywords: sasakian manifold; (almost) $*$-Ricci soliton; $*$-Einstein manifold; infinitesimal contact transformation; conformal vector field

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1. Introduction

Nonlinear systems that support solitons can generate self-similarity and fractals on successively smaller scales. Very often, the solitons of a given system are all self-similar to each other, see [1].

Several authors study Ricci solitons (RS) in contact metric geometry, where Sasakian manifolds (SM) are especially important. SM, which are odd-dimensional analogues of K'ahler manifolds, are used to build geometrical examples, e.g., manifolds of special holonomy, Einstein manifolds (EM), K'ahler manifolds and orbifolds. The geometry of SM (in particular, η-EM and Sasaki–Einstein manifolds) attracts much attention from mathematicians. We refer the reader to [2–4] for the latest developments in the theory of these manifolds.

Many tools for K'ahler manifolds have analogues for contact manifolds, among them the $*$-Ricci tensor, introduced in [5] for almost Hermitian manifolds. In [6], the $*$-Ricci tensor was applied to a real hypersurface in a non-flat complex space form. Namely, for an almost contact manifold $M^{2n+1}(\varphi, \zeta, \eta, g)$ and any vector fields Y_1, Y_2, Y_3 on M^{2n+1}, we get

$$\mathrm{Ric}^*_g(Y_1, Y_2) = \frac{1}{2}\,\mathrm{trace_g}\left\{Y_3 \to R(Y_1, \varphi\, Y_2)\varphi Y_3\right\}. \tag{1}$$

If Ric^*_g vanishes identically then such M^{2n+1} is called $*$-*Ricci flat*. A $*$-*RS* structure has been introduced in [7], using Ric^*_g instead of the Ricci tensor in the equation of RS,

$$\pounds_X\, g + 2\,\mathrm{Ric}^*_g = 2\,\omega\, g. \tag{2}$$

Here, $\pounds$ is the Lie derivative. If ω in (2) belongs to $C^\infty(M)$ and is not a constant, then we get an *almost* $*$-*RS*, the triple (g, ω, X). An almost $*$-RS is *shrinking* for $\omega > 0$, *steady* for $\omega = 0$ and *expanding* for $\omega < 0$. Observe that a $*$-RS is $*$-*EM* (i.e., the $*$-Ricci tensor and the metric tensor are homothetic) if X is a Killing vector field (KVF), see [8]. Thus, it is a generalization of a $*$-EM. In particular, if $X = \nabla f$ (the gradient of $f \in C^\infty(M)$) in (2), then we obtain a *gradient almost* $*$*-RS*. Some generalizations of RS were studied by several authors, see [9–15].

Ghosh-Patra [16] first studied $*$-RS and gradient almost $*$-RS on a contact metric manifold, in particular, on an SM and (k, μ)-contact manifolds. Later on, $*$-RS were studied on almost contact manifolds in [15,17–20]. Wang [20] proved that "if the metric of a Kenmotsu 3-manifold represents a $*$-RS, then the manifold is locally the hyperbolic space $\mathbb{H}^3(-1)$", and it was proved in [18] that "if a non-Kenmotsu $(k, \mu)'$-almost Kenmotsu manifold has a $*$-RS structure, then it is locally isometric to $\mathbb{H}^{n+1}(-4)\times\mathbb{R}$ under some restrictions".

There is a natural question: "*under what conditions a (gradient) almost RS is a unit sphere*"? Several affirmative answers to this question on Riemannian and contact metric manifolds, are given in [9,10,12–14,21,22]. An almost $*$-RS is a generalization of $*$-RS and $*$-EM; thus we ask a question: "under what conditions a (gradient) almost $*$-RS is a unit sphere?"

In [16], they answered this question by showing that *"if a complete SM admits a gradient almost* $*$*-RS, then it is a unit sphere"*. Here, we consider non-gradient almost $*$-RS in the case of SM, and we are looking for conditions under which SM having an almost $*$-RS structure is a unit sphere. Our main achievement is the following.

Theorem 1. *If a complete SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ of dimension greater than three has an almost $*$-RS structure (g, ω, X) with $\omega \neq const$, then it is the unit sphere $\mathbb{S}^{2n+1}$.*

The following question arises: "under what conditions an almost RS is an RS, or, EM"? Several answers to this question can be found in [14,22–24]. An almost $*$-RS generalizes $*$-RS and $*$-EM; so the following question arises:

"when an almost $*$-RS is a $*$-RS, for example, an $*$-EM ?"

In [16], they found such a condition on an SM, and this question on a Kenmotsu manifold was studied in [15]. In this regard, we are interested in characterizing those SM represented as almost $*$-RS, which are $*$-RS or are $*$-EM. In the following theorem, we assume that the potential vector field is a Jacobi vector field on the integral curves of the Reeb vector field.

Theorem 2. *If an SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ has an almost $*$-RS structure (g, ω, X) such that X is a Jacobi field on the ζ-integral curves, then (g, ω, X) is a $*$-RS on M.*

Now, we recall some definitions, e.g., [3]. A SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ is an *η-EM* if

$$\mathrm{Ric}_g = \alpha_1\, g + \alpha_2\, \eta \otimes \eta,$$

where functions α_1 and α_2 belong to $C^\infty(M)$. For K-contact manifolds (for example, SM) of dimension ≥ 4, α_1, α_2 are real constants, see [25], thus, the scalar curvature is constant. An η-Einstein SM is a *null-SM* when $\alpha_1 = -2$ and $\alpha_2 = 2n + 2$, and is a *positive-SM* when $\alpha_1 > -2$, see [3]. Using ([16], Theorem 8), we get the following consequence of Theorem 2.

Corollary 1. *If an SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ has an almost $*$-RS structure (g, ω, X) such that X is a Jacobi vector field on the ζ-integral curves, then M is positive SM and X is KVF (and g is $*$-EM), or M is null-SM and φ is invariant under X.*

Remark 1. *Observe from [24] that if an SM has an almost RS structure, whose potential field is a Jacobi vector field on the ζ-integral curves, then the manifold is null-SM with the expanding RS.*

A vector field Y on a contact manifold (M, η) that preserves the contact form η is called an *infinitesimal contact transformation*, see [26,27], i.e.,

$$\pounds_Y \eta = \nu\, \eta, \tag{3}$$

for some $\nu \in C^{\infty}(M)$; and if $\nu = 0$, then Y is said to be *strict*. A vector field Y preserving φ, ζ, η and g on a contact metric manifold is called an *infinitesimal automorphism*. In [16], they addressed the question by showing that *"if an SM represents an almost $*$-RS such that the potential vector field is an infinitesimal contact transformation, then it is a $*$-RS"*. We improve this result in the following statement.

Theorem 3. *Let an SM represent an almost $*$-RS (g, ω, X) such that X is an infinitesimal contact transformation. Then X leaves φ invariant, and the manifold is η-EM of scalar curvature $(\omega + 4n)n$.*

Assuming compactness we get one more sufficient condition for g to be $*$-Ricci flat (e.g., $*$-EM).

Theorem 4. *Let a compact SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ admit an almost $*$-RS (g, ω, X) such that X is an infinitesimal contact transformation. Then the soliton is steady $(\omega = 0)$, X is an infinitesimal automorphism, and g is $*$-Ricci flat of scalar curvature $4n^2$.*

Finally, we show that the property "potential vector field and the characteristic vector field are parallel" provides "$*$-Ricci flat" (e.g., $*$-EM).

Theorem 5. *Let an SM $M^{2n+1}(\varphi, \zeta, \eta, g)$ admit an almost $*$-RS (g, ω, X) such that X is parallel to ζ. Then the soliton is steady $(\omega = 0)$, V is KVF, and g is $*$-Ricci flat of scalar curvature $4n^2$.*

Remark 2. *Observe from [14] that if a compact SM has an almost RS structure (g, ω, X) such that X is an infinitesimal contact transformation, then the soliton is shrinking ($\omega = 2n$), X is strict and leaves φ invariant, and the manifold is EM of scalar curvature $2n(2n+1)$. By Theorem 4, the soliton is steady $(\omega = 0)$, X is strict and leaves φ, ζ and η invariant, and the manifold is $*$-EM of scalar curvature $4n^2$. In addition, note that an SM considered as an RS or an almost RS with X parallel to ζ, is an EM of scalar curvature $2n(2n+1)$, and X is a KVF, see [28].*

Within the framework of Theorems 3–5, we not only find sufficient conditions for $*$-EM, but also characterize the structural tensor fields and the potential vector field.

2. Preliminaries

Here, we recall some properties of SM, see [2,26]. We suppose that all manifolds are smooth and connected. The curvature tensor R on a Riemannian manifold (M, g) is given by

$$R(Y_1, Y_2) = [\nabla_{Y_1}, \nabla_{Y_2}] - \nabla_{[Y_1, Y_2]}, \quad Y_1, Y_2 \in \mathcal{X}(M), \tag{4}$$

where ∇ is the Levi-Civita connection and $\mathcal{X}(M)$ is the space of vector fields on M. They define the Ricci operator Q by

$$g(QY_1, Y_2) = \mathrm{Ric}_g(Y_1, Y_2) = \mathrm{trace}_g \{Y_3 \to R(Y_3, Y_1)Y_2\}, \quad Y_1, Y_2, Y_3 \in \mathcal{X}(M)$$

is a symmetric $(1,1)$-tensor. Then $r = \mathrm{trace}_g\, Q$ is the scalar curvature. We get the following formula (follows from twice contracted second Bianchi identity, see, e.g., [29]):

$$\frac{1}{2}\, g(Y_1, \nabla r) = (\mathrm{div}\, Q)(Y_1) = \textstyle\sum_i g((\nabla_{E_i} Q)Y_1, E_i), \quad Y_1 \in \mathcal{X}(M), \tag{5}$$

where $\{E_i\}$ is any local orthonormal basis on M.

For the Lie derivative of $f \in C^{\infty}(M)$ along $Y_1 \in \mathcal{X}(M)$ we have

$$\pounds_{Y_1} f = Y_1(f) = g(Y_1, \nabla f),$$

where the Hessian of f is defined by

$$\mathrm{Hess}_f(Y_1, Y_2) = \nabla^2 f(Y_1, Y_2) = (\nabla_{Y_1} \nabla f)Y_2 = Y_1 Y_2(f) - (\nabla_{Y_1} Y_2)(f), \quad Y_1, Y_2 \in \mathcal{X}(M).$$

The hessian Hess_f is symmetric and $g(\nabla_{Y_1} \nabla f, Y_2) = g(\nabla_{Y_2} \nabla f, Y_1)$ is true, so it is a section of $T^*M \otimes T^*M$. A vector field Y on (M, g) is conformal if

$$\pounds_Y\, g = f\, g$$

for some $f \in C^\infty(M)$. Such Y is called homothetic if $f = constant$, and Y is said to be KVF if $f = 0$.

A manifold M^{2n+1} is a contact manifold if $\eta \wedge (d\eta)^n \neq 0$ for some global 1-form η. In this case, ζ satisfying $d\eta(\zeta, \cdot) = 0$ and $\eta(\zeta) = 1$ is called a characteristic vector field. Polarization of $d\eta$ on $\mathcal{D}_\equiv = \ker \eta$ allows to find a $(1,1)$-tensor φ and a Riemannian structure g satisfying

$$\varphi^2 = -id_{TM} + \eta \otimes \zeta, \quad \eta = g(\zeta, \cdot), \tag{6}$$
$$d\eta(\cdot\,, \cdot) = g(\cdot\,, \varphi\, \cdot). \tag{7}$$

They call such $M^{2n+1}(\varphi, \zeta, \eta, g)$ a contact metric manifold with associated metric g. By (6), we get

$$\varphi(\zeta) = 0, \quad \eta \circ \varphi = 0, \quad \text{and} \quad \mathrm{rank}(\varphi) = 2n.$$

Further, a contact metric manifold is called SM if the following condition is valid, see [3,26]:

$$[\varphi Y_1, \varphi Y_2] = -2\, d\eta(Y_1, Y_2)\zeta, \quad Y_1, Y_2 \in \mathcal{X}(M).$$

The curvature tensor of an SM satisfies

$$R(Y_1, Y_2)\zeta = \eta(Y_2)Y_1 - \eta(Y_1)Y_2, \quad Y_1, Y_2 \in \mathcal{X}(M). \tag{8}$$

Recall that if ζ is KVF, then M is called a K-contact manifold. A SM is K-contact, the converse is valid in dimension 3 only, see ([26], p. 87). SM satisfy the following, see ([26], p. 113):

$$\nabla_{Y_1} \zeta = -\varphi\, Y_1, \quad Y_1 \in \mathcal{X}(M), \tag{9}$$
$$Q\, \zeta = 2\, n\, \zeta. \tag{10}$$

According to ([16], Lemma 2.1):

$$\nabla_\zeta\, Q = Q\, \varphi - \varphi\, Q;$$

therefore, $\nabla_\zeta\, Q = 0$, as φ and the Ricci operator commute on an SM, see [26]. Using (9) we find the derivative of (10) in the direction of $Y_1 \in \mathcal{X}(M)$,

$$(\nabla_{Y_1} Q)\zeta = Q\varphi Y_1 - 2n\, \varphi Y_1. \tag{11}$$

The $*$-Ricci tensor Ric^*_g on an SM has the view (see [16], Lemma 5):

$$\mathrm{Ric}^*_g(Y_1, Y_2) = \mathrm{Ric}_g(Y_1, Y_2) - (2n-1)\, g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2), \quad Y_1, Y_2 \in \mathcal{X}(M). \tag{12}$$

Thus, we can write the $*$-RS structure (2) on an SM as

$$(\pounds_V\, g)(Y_1, Y_2) + 2\, \mathrm{Ric}_g(Y_1, Y_2) = 2\,(\omega + 2n - 1)\, g(Y_1, Y_2) + 2\, \eta(Y_1)\eta(Y_2) \tag{13}$$

for all $Y_1, Y_2 \in \mathcal{X}(M)$.

Remark 3. *Contracting the derivative of* (8) *using a local orthonormal basis* $\{E_i\}_{1\le i\le 2n+1}$ *on* M *and using* (9), *we get the following:*

$$(\operatorname{div} R)(Y_1,Y_2)\zeta - g(R(Y_1,Y_2)\varphi E_i,E_i) = -2g(\varphi Y_1,Y_2).$$

Contracting the second Bianchi identity, they reduce the above equation to the following:

$$g((\nabla_{Y_1}Q)Y_2 - (\nabla_{Y_2}Q)Y_1,\zeta) - g(R(Y_1,Y_2)\varphi E_i,E_i) = -2g(\varphi Y_1,Y_2). \tag{14}$$

Using (11), *from (14) we obtain*

$$g(R(Y_1,Y_2)\varphi E_i,E_i) = g(Q\varphi Y_1,Y_2) + g(\varphi Q Y_1,Y_2) - 2(2n-1)g(\varphi Y_1,Y_2).$$

Replacing Y_2 *by* φY_2 *in the preceding equation and using the definition* (1), *we represent the* $*$*-Ricci tensor of an SM in the form* (12).

To prove Theorems 1 and 2, we need the following three results.

Proposition 1. *If an SM admits an almost* $*$*-RS structure, then we get the following:*

$$(\pounds_X\nabla)(Y_1,\zeta) + 2\,Q\varphi Y_1 = 2(2n-1)\,\varphi Y_1 + Y_1(\omega)\zeta + \zeta(\omega)Y_1 - \eta(Y_1)\,\nabla\omega, \quad Y_1\in\mathcal{X}(M). \tag{15}$$

Proof. Using the derivative of (13) for an arbitrary $Y_3\in\mathcal{X}(M)$ and (9), gives

$$\begin{aligned}(\nabla_{Y_3}\pounds_X g)(Y_1,Y_2) + 2\,(\nabla_{Y_3}\operatorname{Ric}_g)(Y_1,Y_2) &= 2\,Y_3(\omega)\,g(Y_1,Y_2)\\ -\,2\,\eta(Y_1)\,g(Y_2,\varphi Y_3) - 2\,\eta(Y_2)\,g(Y_1,\varphi Y_3)&\end{aligned} \tag{16}$$

for $Y_1,Y_2\in\mathcal{X}(M)$. Following Yano ([30] p. 23), for $Y_1,Y_2,Y_3\in\mathcal{X}(M)$ we write

$$(\pounds_X\nabla_{Y_3}g - \nabla_{Y_3}\pounds_X g - \nabla_{[X,Y_3]}g)(Y_1,Y_2) = -g((\pounds_X\nabla)(Y_3,Y_1),Y_2) - g((\pounds_X\nabla)(Y_3,Y_2),Y_1).$$

Inserting (16) in the above relation and using $\nabla g = 0$ yields the following:

$$\begin{aligned}&g((\pounds_X\nabla)(Y_3,Y_1),Y_2) + g((\pounds_X\nabla)(Y_3,Y_2),Y_1) + 2\,(\nabla_{Y_3}\operatorname{Ric}_g)(Y_1,Y_2)\\ &= 2\,\{Y_3(\omega)\,g(Y_1,Y_2) - \eta(Y_1)\,g(Y_2,\varphi Y_3) - \eta(Y_2)\,g(Y_1,\varphi Y_3)\}.\end{aligned}$$

Due to the symmetry $(\pounds_X\nabla)(Y_1,Y_2) = (\pounds_X\nabla)(Y_2,Y_1)$, using cyclical permutations of Y_1, Y_2, Y_3 in the preceding equation, we get

$$\begin{aligned}g((\pounds_X\nabla)(Y_1,Y_2),Y_3) = {}&(\nabla_{Y_3}\operatorname{Ric}_g)(Y_1,Y_2) - (\nabla_{Y_1}\operatorname{Ric}_g)(Y_2,Y_3) - (\nabla_{Y_2}\operatorname{Ric}_g)(Y_3,Y_1)\\ &+ Y_1(\omega)\,g(Y_2,Y_3) + Y_2(\omega)\,g(Y_3,Y_1) - Y_3(\omega)\,g(Y_1,Y_2)\\ &- 2\,\eta(Y_2)\,g(\varphi Y_1,Y_3) - 2\,\eta(Y_1)\,g(\varphi Y_2,Y_3).\end{aligned} \tag{17}$$

Since the Ricci operator Q is self-adjoint, using $\nabla_\zeta Q = 0$ and (11), and replacing Y_2 by ζ in (17), we get (15). □

Proposition 2. *Let an SM represent an almost* $*$*-RS and* ζ *leave* ω *invariant. Then we get*

$$\begin{aligned}R(Y_1,Y_2)\nabla\omega - g(R(Y_1,Y_2)\nabla\omega,\zeta)\zeta = {}&4\{(\nabla_{Y_1}Q)Y_2 - (\nabla_{Y_2}Q)Y_1\} - 2\{Y_1(\omega)Y_2 - Y_2(\omega)Y_1\}\\ &+ 2\,\{Y_1(\omega)\,\eta(Y_2)\zeta - Y_2(\omega)\,\eta(Y_1)\zeta\} + 2(\omega-2)\,\{2\,g(Y_1,\varphi Y_2)\zeta\\ &+ \eta(Y_1)\,\varphi Y_2 - \eta(Y_2)\,\varphi Y_1\} + \eta(Y_2)\,\nabla_{Y_1}\nabla_\zeta\nabla\omega - \eta(Y_1)\,\nabla_{Y_2}\nabla_\zeta\nabla\omega\\ &+ 2\,g(\varphi Y_2,Y_1)\nabla_\zeta\nabla\omega + g(\varphi Y_2,\nabla_{Y_1}\nabla\omega)\zeta - g(\varphi Y_1,\nabla_{Y_2}\nabla\omega)\zeta\\ &+ g(\zeta,\nabla_{Y_1}\nabla\omega)\,\varphi Y_2 - g(\zeta,\nabla_{Y_2}\nabla\omega)\,\varphi Y_1, \quad Y_1,Y_2\in\mathcal{X}(M).\end{aligned} \tag{18}$$

Proof. Applying the Lie derivative for $R(Y_1,\zeta)\zeta = Y_1 - \eta(Y_1)\zeta$ for all $Y_1 \in \mathcal{X}(M)$ (follows from (8)) along X and using (8), yields

$$(\pounds_X R)(Y_1,\zeta)\zeta + R(Y_1,\zeta)\pounds_X\zeta + \eta(\pounds_X\zeta)Y_1 + (\pounds_X g)(Y_1,\zeta)\zeta + g(Y_1,\pounds_X\zeta)\zeta = 0. \tag{19}$$

Replacing (Y_1,Y_2) by (Y_1,ζ) in (13) and using (10), we acquire the following:

$$(\pounds_X g)(Y_1,\zeta) = 2\omega\eta(Y_1).$$

Plugging it into the Lie derivative of $\eta(Y_1) = g(Y_1,\zeta)$ and $\eta(\zeta) = 1$, we find $\eta(\pounds_X\zeta) = -\omega$ and $(\pounds_X\eta)(\zeta) = \omega$. Thus, in view of (8), Equation (19) gives us

$$(\pounds_X R)(Y_1,\zeta)\zeta = 2\omega\{Y_1 - \eta(Y_1)\zeta\}, \quad Y_1 \in \mathcal{X}(M). \tag{20}$$

By conditions, $\zeta(\omega) = 0$ is valid, thus (15) reduces to the following:

$$(\pounds_X \nabla)(Y_1,\zeta) + 2\,Q\varphi Y_1 = 2(2n-1)\,\varphi Y_1 + Y_1(\omega)\zeta - \eta(Y_1)\,\nabla\omega, \quad Y_1 \in \mathcal{X}(M). \tag{21}$$

Using ζ-derivative of (21), gives

$$(\nabla_\zeta \pounds_X \nabla)(Y_1,\zeta) = g(Y_1,\nabla_\zeta\nabla\omega)\zeta - \eta(Y_1)\,\nabla_\zeta\nabla\omega, \tag{22}$$

where equalities $\nabla_\zeta\, Q = \nabla_\zeta\,\zeta = \nabla_\zeta\,\varphi = 0$ for an SM were used. On the other hand, using $Y_1 = \zeta$ in (21), then differentiating along Y_1 and using (9) and the symmetry of $\pounds_X\nabla$, we get

$$(\nabla_{Y_1}(\pounds_X\nabla))(\zeta,\zeta) = 2\,(\pounds_X\nabla)(\varphi Y_1,\zeta) - \nabla_{Y_1}\nabla\omega. \tag{23}$$

Next, differentiating $g(\zeta,\nabla\omega) = 0$ along $Y_1 \in \mathcal{X}(M)$ and using (6), gives $(\varphi Y_1)(\omega) = g(\zeta,\nabla_{Y_1}\nabla\omega)$; therefore, it suffices to combine (6), (10), (21) and (23) to arrive at the result

$$(\nabla_{Y_1}\pounds_X\nabla)(\zeta,\zeta) = 4\,QY_1 - 4(2n-1)Y_1 - 4\,\eta(Y_1)\zeta + 2g(\zeta,\nabla_{Y_1}\nabla\omega)\zeta - \nabla_{Y_1}\nabla\omega. \tag{24}$$

We need the following commutation result, see ([30], p. 23):

$$(\pounds_X R)(Y_1,Y_2)Y_3 = (\nabla_{Y_1}\pounds_X\nabla)(Y_2,Y_3) - (\nabla_{Y_2}\pounds_X\nabla)(Y_1,Y_3). \tag{25}$$

Next, replacing both Y_2 and Y_3 by ζ in (25) and then plugging the values of $(\pounds_X R)(Y_1,\zeta)\zeta$, $(\nabla_\zeta\pounds_X\nabla)(Y_1,\zeta)$ and $(\nabla_{Y_1}\pounds_X\nabla)(\zeta,\zeta)$ from (20), (22) and (24), respectively, we get

$$\nabla_{Y_1}\nabla\omega = 4\,QY_1 - 2\{\omega + 2(2n-1)\}Y_1 + 2(\omega-2)\,\eta(Y_1)\zeta + g(\zeta,\nabla_X\nabla\omega)\zeta + \eta(Y_1)\,\nabla_\zeta\nabla\omega \tag{26}$$

for $Y_1 \in \mathcal{X}(M)$. Using (9) in the Y_2-derivative of (26), we acquire

$$\begin{aligned}\nabla_{Y_2}\nabla_{Y_1}\nabla\omega = {}& 4\{(\nabla_{Y_2}Q)Y_1 + Q(\nabla_{Y_2}Y_1)\} - 2Y_2(\omega)\{Y_1 - \eta(Y_1)\zeta\} - 2\{\omega + 2(2n-1)\}\nabla_{Y_2}Y_1 \\ &+ 2(\omega-2)\{\eta(\nabla_{Y_2}Y_1)\zeta - g(Y_1,\varphi Y_2)\zeta - \eta(Y_1)\varphi Y_2\} + \{\eta(\nabla_{Y_2}Y_1) \\ &- g(Y_1,\varphi Y_2)\}\nabla_\zeta\nabla\omega + \eta(Y_1)\nabla_{Y_2}\nabla_\zeta\nabla\omega - g(\varphi Y_2,\nabla_{Y_1}\nabla\omega)\zeta \\ &+ g(\zeta,\nabla_{Y_2}\nabla_{Y_1}\nabla\omega)\zeta - g(\zeta,\nabla_{Y_1}\nabla\omega)\varphi Y_2.\end{aligned}$$

Since Hess_ω is symmetric and φ is skew-symmetric, using (26) and the above equation in (4) completes the proof of (18). $\square$

Recall that the contact metric structure commutes with the Ricci operator, i.e., $Q\varphi = \varphi Q$, e.g., [26]. Its covariant derivative and (5) provide the following.

Lemma 1 (see [12])**.** *For an SM M and all $Y_1 \in \mathcal{X}(M)$, we have*

$$(i)\ \sum_{i=1}^{2n+1} g((\nabla_{\varphi Y_1}Q)\,\varphi E_i, E_i) = 0, \quad (ii)\ \sum_{i=1}^{2n+1} g((\nabla_{\varphi E_i}Q)\,\varphi Y_1, E_i) = -\frac{1}{2}\,Y_1(r),$$

where $\{E_i\}_{1\leq i\leq 2n+1}$ *is a local orthonormal basis on* M.

3. Proof of Results

Proof of Theorem 1. Since the characteristic vector field is KVF, $\pounds_\zeta\, g = 0 = \pounds_\zeta\, \mathrm{Ric}_g$ is valid. Applying this to the Lie derivative of (13) along ζ, and using $\pounds_{Y_1}\pounds_{Y_2}\, g - \pounds_{Y_2}\pounds_{Y_1}\, g = \pounds_{[Y_1,Y_2]}\, g$, e.g., [30], we get

$$\pounds_{[X,\zeta]}\, g = -2\,\zeta(\omega)\, g. \tag{27}$$

By (27), $[X,\zeta]$ is a conformal vector field. This gives us the following alternatives:

(I) $[X,\zeta]$ is non-homothetic, (II) $[X,\zeta]$ is homothetic.

Okumura [31] proved that "*if a complete SM of dimension* > 3 *has a non-Killing conformal vector field, then it is a unit sphere*". Applying this theorem for (I), we conclude that (M, g) is a unit sphere.

We will finish the proof by showing a contradiction for case (II). Sharma [32] proved the following: "*a homothetic vector field on an SM (more generally, K-contact manifold) is necessarily a KVF*". So, (27) implies that ζ leaves ω invariant. Thus, (18) holds. Contracting it over Y_1 and then using (5), (8), we obtain

$$\begin{aligned}\mathrm{Ric}_g(Y_2,\nabla\omega) &= 4\,g(\varphi\,Y_2,\nabla_\zeta\nabla\omega) + \eta(Y_2)\ \mathrm{div}(\nabla_\zeta\nabla\omega)\\ &\quad - g(\zeta,\nabla_{Y_2}\nabla_\zeta\nabla\omega) - 2\,g(Y_2,\nabla r) + 2(2n-1)\,g(Y_2,\nabla\omega),\end{aligned} \tag{28}$$

where we used $\mathrm{trace}_g\,\varphi = 0 = \varphi\zeta$, the skew-symmetry of φ and symmetry of Hess_ω. Differentiating the equality $g(\zeta,\nabla_\zeta\nabla\omega) = 0$ along $Y_2 \in \mathcal{X}(M)$ and using (9), gives

$$g(\zeta,\nabla_{Y_2}\nabla_\zeta\nabla\omega) = g(\varphi\,Y_2,\nabla_\zeta\nabla\omega).$$

Thus, (28) can be rewritten as

$$\mathrm{Ric}_g(Y_2,\nabla\omega) = 3\,g(\varphi\,Y_2,\nabla_\zeta\nabla\omega) + \eta(Y_2)\ \mathrm{div}(\nabla_\zeta\nabla\omega) - 2\,Y_2(r) + 2(2n-1)\,Y_2(\omega) \tag{29}$$

for all $Y_2 \in \mathcal{X}(M)$. Next, recall the following result for SM, see [26]:

$$R(\varphi Y_2,\varphi Y_1)Y_3 = R(Y_2,Y_1)Y_3 + g(Y_2,Y_3)Y_1 - g(Y_1,Y_3)Y_2 - g(Y_2,Y_3)Y_1 + g(Y_1,Y_3)Y_2$$

for all $Y_1, Y_2, Y_3 \in \mathcal{X}(M)$. Substituting $Y_1 = \varphi\,Y_1$ and $Y_2 = \varphi\,Y_2$ in (18) and using the last formula, in view of (6), $\varphi\,\zeta = 0$ and the skew-symmetry of φ, we obtain

$$\begin{aligned}R(Y_1,Y_2)\nabla\omega - g(R(Y_1,Y_2)\nabla\omega,\,\zeta)\zeta &= 4\,\{(\nabla_{\varphi Y_1}Q)\varphi Y_2 - (\nabla_{\varphi Y_2}Q)\varphi Y_1\} + Y_2(\omega)\,\{2Y_1 - \eta(Y_1)\zeta\}\\ &\quad - Y_1(\omega)\,\{2Y_2 - \eta(Y_2)\zeta\} + 4(\omega-2)\,g(Y_1,Y_2)\zeta + 2\,g(Y_1,Y_2)\nabla_\zeta\nabla\omega\\ &\quad + 2\,\eta(Y_2)\,g(\zeta,\nabla_{Y_1}\nabla\omega)\zeta - 2\,\eta(Y_1)\,g(\zeta,\nabla_{Y_2}\nabla\omega)\zeta - g(Y_2,\nabla_{Y_1}\nabla\omega)\zeta\\ &\quad - g(Y_1,\nabla_{Y_2}\nabla\omega)\zeta + g(\zeta,\nabla_{Y_2}\nabla\omega)Y_1 - g(\zeta,\nabla_{Y_1}\nabla\omega)Y_2.\end{aligned} \tag{30}$$

On the other hand, contracting (30) over Y_1 and applying (8), (5), (29) and Lemma 1, we find

$$\eta(Y_2)\,\mathrm{div}(\nabla_\zeta\nabla\omega) + 2(n-1)\big\{Y_2(\omega) - g(\varphi Y_2,\nabla_\zeta\nabla\omega)\big\} = 0, \tag{31}$$

where we have used the symmetry of Hess_ω and that ζ leaves ω invariant. Next, replacing Y_2 by φY_2 in (31), noting that (6) and using $g(\zeta,\nabla_\zeta\nabla\omega) = 0$, we acquire

$$2(n-1)\big\{g(\varphi Y_2,\nabla\omega) + g(Y_2,\nabla_\zeta\nabla\omega)\big\} = 0, \quad Y_2 \in \mathcal{X}(M). \tag{32}$$

Furthermore, differentiating $g(\zeta,\nabla\omega) = 0$ along $Y_2 \in \mathcal{X}(M)$ and using (9), we achieve

$$g(\zeta,\nabla_{Y_2}\nabla\omega) = g(\varphi Y_2,\nabla\omega).$$

Thus, (32) for $n > 1$ gives us $g(\varphi Y_2, \nabla\omega) = 0$; consequently, $\nabla\omega = 0$. Hence, ω is constant – a contradiction with the conditions of the theorem. □

Proof of Theorem 2. Applying Proposition 1 to the well-known formula:

$$\nabla_{Y_1}\nabla_{Y_2}X - \nabla_{\nabla_{Y_1}Y_2}X - R(Y_1, X)Y_2 = (\pounds_X\nabla)(Y_1, Y_2),$$

see ([30], p. 23), we acquire

$$\nabla_{Y_1}\nabla_{\zeta}X - \nabla_{\nabla_{Y_1}\zeta}X - R(Y_1, X)\zeta = -2Q\varphi Y_1 + 2(2n-1)\varphi Y_1 + Y_1(\omega)\zeta + \zeta(\omega)Y_1 - \eta(Y_1)\nabla\omega. \tag{33}$$

By conditions, X is a Jacobi vector field on the ζ-integral curves, see [33], i.e.,

$$\nabla_\zeta\nabla_\zeta X + R(X, \zeta)\zeta = 0.$$

Using $Y_1 = \zeta$ in (33) and $\nabla_\zeta\,\zeta = 0$ (that is a consequence of (9)), we achieve the equality $\nabla\omega = 2\,\zeta(\omega)\zeta$, or, using the exterior derivative,

$$d\omega = 2\,\zeta(\omega)\,\eta.$$

Applying exterior derivative, the Poincaré lemma ($d^2 = 0$), and the wedge product with η, we acquire $\zeta(\omega)\,\eta \wedge d\eta = 0$; thus, $\zeta(\omega) = 0$, as $\eta \wedge d\eta$ is nowhere zero on a contact manifold. Thus, $d\omega = 0$, i.e., $\omega = constant$. □

Corollary 4 follows from ([16], Theorem 8) and our Theorem 2.

Proof of Theorem 3. Using $d \circ \pounds_X = \pounds_X \circ d$ (d commutes with the Lie derivative) and applying the operator d to (3), gives

$$(\pounds_X d\eta)(Y_1, Y_2) = \frac{1}{2}\Big\{Y_1(\nu)\,\eta(Y_2) - Y_2(\nu)\,\eta(Y_1)\Big\} + \nu\, d\eta(Y_1, Y_2) \tag{34}$$

for a function $\nu \in C^\infty(M)$ and any $Y_1, Y_2 \in \mathcal{X}(M)$. Applying the Lie derivative of (7) in the X-direction and using (2), (3) and (34), we obtain

$$2(\pounds_X\varphi)(Y_1) + 2(2\omega - \nu + 2(2n-1))\,\varphi Y_1 = 4Q\varphi Y_1 + \eta(Y_1)\,\nabla\nu - Y_1(\nu)\,\zeta. \tag{35}$$

From the first equality of (6), for $Y_1 \in \mathcal{X}(M)$ we get

$$(\pounds_X\varphi)(\varphi Y_1) + \varphi(\pounds_X\varphi)(Y_1) = (\pounds_X\eta)(Y_1)\zeta + \eta(Y_1)\pounds_X\zeta. \tag{36}$$

As a result of (10), $*$-RS Equation (2) gives us

$$(\pounds_X g)(Y_1, \zeta) = 2\omega\,\eta(Y_1).$$

Taking into account this, as well as (3), it suffices to show that

$$g(\pounds_X\zeta, Y_1) = (\nu - 2\omega)\,\eta(Y_1). \tag{37}$$

By direct calculation using $\varphi\zeta = 0$ we get $(\pounds_X\varphi)(\zeta) = 0$; hence, (35) gives $\nabla\nu = \zeta(\nu)\,\zeta$. Thus, by (9) we get

$$\mathrm{Hess}_\nu(Y_1, Y_2) = Y_1(\zeta(\nu))\,\eta(Y_2) - \zeta(\nu)\,g(\varphi Y_1, Y_2). \tag{38}$$

Since φ is skew-symmetric and Hess_ν is symmetric, by (7) and (38), for Y_1, Y_2 orthogonal to ζ we achieve

$$\zeta(\nu)\,d\eta(Y_1, Y_2) = 0.$$

Thus, $\zeta(\nu) = 0$, as $d\eta$ is nonzero; hence, $\nabla\nu = 0$. Thus, ν is constant. Combining (2), (37) and $\eta(\zeta) = 1$, we get $\nu = \omega$. Substituting this and (10) in (35), gives

$$(\pounds_X\varphi)(\varphi Y_1) = \varphi(\pounds_X\varphi)(Y_1) = -2\,QY_1 + (\omega + 2(2n-1))Y_1 - (\omega - 2)\,\eta(Y_1)\zeta, \tag{39}$$

where the equality $Q\varphi = \varphi Q$ (for an SM) has been used, see ([26], p. 116). Now, by (3), (36), (37), (39) and $\nu = \omega$, we obtain the relation

$$2\,\mathrm{Ric}_g = (\omega + 2(2n-1))\,g - (\omega - 2)\,\eta\otimes\eta. \tag{40}$$

Hence, (M, g) is an η-EM of scalar curvature $n(\omega + 4n)$. Substituting (40) into (35), we find $\pounds_X\varphi = 0$; thus, X leaves φ invariant. □

Proof of Theorem 4. The volume form Ω on a contact metric manifold satisfies $\Omega = \eta\wedge(d\eta)^n \neq 0$; therefore, its Lie derivative in the X-direction and (3) give

$$\pounds_X\Omega = (n+1)\nu\,\Omega.$$

Proceeding, we obtain the equality $\pounds_X\Omega = (\operatorname{div} X)\Omega$, from which we deduce $\operatorname{div} X = (n+1)\nu$. Applying the Divergence theorem (for compact M), this gives $\nu = 0$; consequently, $\omega = 0$ by the proof of Theorem 3. Thus, (3) and (37), respectively, follows from the condition that X leaves η and ζ invariant. Moreover, Equation (40) becomes

$$\mathrm{Ric}_g = (2n-1)\,g + \eta\otimes\eta. \tag{41}$$

Thus, from (12) we conclude that (M, g) is $*$-Ricci flat of zero $*$-scalar curvature r^*. Using (2), we find that X is KVF. Applying Theorem 3, completes the proof. □

Proof of Theorem 5. By conditions, $X = \sigma\,\zeta$, where σ is a non-zero smooth function. By the skew-symmetry of φ and (9), we obtain

$$(\pounds_X g)(Y_1, Y_2) = g(\nabla_{Y_1}X, Y_2) + g(\nabla_X Y_2, Y_1) = Y_1(\sigma)\eta(Y_2) + Y_2(\sigma)\eta(Y_1). \tag{42}$$

Thus, (13) becomes

$$Y_1(\sigma)\eta(Y_2) + Y_2(\sigma)\eta(Y_1) + 2\,\mathrm{Ric}_g(Y_1, Y_2) = 2(\omega + 2n - 1)\,g(Y_1, Y_2) + 2\,\eta(Y_1)\eta(Y_2). \tag{43}$$

Using $Y_2 = \zeta$ in (43) and (10), yields

$$Y_1(\sigma) = (2\omega - \zeta(\sigma))\eta(Y_1), \quad Y_1 \in \mathcal{X}(M). \tag{44}$$

Again, using $Y_1 = \zeta$ and $Y_2 = \zeta$ in (43) and applying (10), we get $\zeta(\sigma) = \omega$; hence, from (44) it follows that $Y_1(\sigma) = \zeta(\sigma)\,\eta(Y_1)$, for $Y_1 \in \mathcal{X}(M)$. By the above argument, we get that σ is constant. By (42), X is KV, and using (44) we get $\omega = 0$. The above reduces (43) to (41), and therefore, g is $*$-Ricci flat (follows from (12)) and of constant scalar curvature $4n^2$. □

4. Conclusions

A modern geometrical concept of an almost $*$-RS can be important in differential geometry and theoretical physics. We study the interaction of this structure on a smooth manifold with the well-known Sasakian structure and prove some results of geometric classification. Theorem 1 contains a condition for a complete SM equipped with an almost $*$-RS to be a unit sphere. Theorems 2–5 having local character, contain conditions, ensuring that an SM equipped with an almost $*$-RS structure is a $*$-RS, e.g., a $*$-EM. Using the fact that an almost $*$-Ricci tensor on an SM can be written as (12), an almost $*$-RS reduces to the form (13), which is less general than the almost η-RS equation for an SM:

$$\frac{1}{2}\pounds_X\,g + \mathrm{Ric}_g + \omega\,g + \delta\,\eta\otimes\eta = 0. \tag{45}$$

In connection with the above, the following question arises: "are our Theorems 1–5 true under the condition (45) instead of (13), where ω and δ are arbitrary smooth functions on M?"

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