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Determination of Three-Dimensional Brinkman—Forchheimer-Extended Darcy Flow

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Abstract: The aim of this study is to determine a 3D incompressible Brinkman–Forchheimer-extended Darcy fluid flow. Based on global well-posedness and regularity of solutions with a periodic boundary condition, the determining modes for weak and regular solutions is achieved via the generalized Grashof number for a 3D non-autonomous Brinkman–Forchheimer-Darcy fluid flow in porous medium. Furthermore, the asymptotic determination of the complete trajectories inside an attractor via Fourier functionals is shown for a 3D autonomous Brinkman–Forchheimer-extended Darcy model.

Keywords: Brinkman–Forchheimer-extended Darcy model; determination; global attractor

MSC: 35B40; 35B41; 35Q30; 76D03; 76D05

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1. Introduction

The celebrated Darcy equation is believed to originate from work in [1]. Darcy's law basically shows that the flow rate of fluid in porous materials is proportional to the pressure gradient. In current terminology, if the flow is in the x-direction and the velocity in that direction is u, then it can be expressed in the following form

$$\mu u = -k\frac{dp}{dx}, \tag{1}$$

where μ, k are viscosity and permeability, and p is the pressure of fluid flow. The constitute law (1) provides a theoretical description of porous media flow. If the velocity field can be defined by $v = (v_1, v_2, v_3)$, ρ is the density of fluid and f is the external force, then (1) can be rewritten as

$$0 = -\frac{\partial p}{\partial x_i} - \frac{\mu}{k}v_i + \rho f_i, \tag{2}$$

which describes the flow of a fluid in a saturated porous medium and provides a sufficiently low flow rate.

If the flow exceeds a certain value, it is considered that the linear relationship of (1) or (2) is insufficient to accurately describe the velocity field. Forchheimer proposed modifying the linear velocity/pressure gradient law and replacing it with the nonlinear one. According to [2,3], the left-hand side of Equation (1) can be replaced by one of three formulae

$$\alpha = au + bu^2, \;\; \alpha = mu^n, \;\; \alpha = au + bu^2 + cu^3, \tag{3}$$

where α denotes the left-hand side of Equation (1).

The porous medium is anisotropic ,and the permeability varies along all the directions. This leads to it appearing as a positive semidefinite matrix in the momentum equation, which can deduce the Brinkman–Forchheimer flow from (1)–(3) as

$$\begin{cases} \partial_t u - \nu\Delta u + au + b|u|u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \tag{4}$$

The Brinkman–Forchheimer system comes from the conservation law of the fluid in porous medium as continuous and momentum equations (see [4,5]), which is studied in [6–8] and the literature therein for the global existence of solutions and dynamic systems. Furthermore, the Brinkman–Forchheimer-Darcy model is always exported by classical Brinkman–Forchheimer flow and fully developed using the Browder–Minty theorem, which is also denoted as the Darcy-Forchheimer–Brinkman systems and extensively used in frontier areas such as petroleum engineering, hydrogeology, reactor engineering, biology, medicine and so on, see [9–21] and related literature [22,23]. The 3D Brinkman–Forchheimer-extended Darcy model can be described as

$$\begin{cases} \dfrac{\rho}{\phi}\dfrac{\partial u}{\partial t} + \dfrac{\mu}{\kappa}u + \dfrac{\rho F}{\sqrt{\kappa}}|u|u + \dfrac{\rho}{\phi^2}\nabla\cdot(u\otimes u) - \dfrac{\mu}{\phi}\nabla^2 u = -\nabla p + \rho g, \\ \gamma\dfrac{\partial p}{\partial t} + \dfrac{\partial \phi}{\partial t} + \nabla\cdot u = f, \end{cases} \tag{5}$$

which constitutes both porous media and clear fluid flow such as oil and gas reservoir with the unknown velocity field and pressure u and p. Here, $F = \frac{1.75}{\sqrt{150\phi^3}}$ denotes the Forchheimer number, $\frac{\mu}{\phi}\nabla^2 u$ is the Brinkman term, μ and κ represent the fluid viscosity and second order flows permeability, ρ and g are the mass density and gravity vector, respectively, and ϕ is the porosity. Although there are fruitful results on the mathematical analysis and numerical simulation for (5) and its extended models in [9,12,13,15–17,20–22] and literature therein, to the best of our best knowledge, there are fewer results dealing with the determination of the 3D Brinkman–Forchheimer-extended Darcy flow model, which is a special case of (5) with dimensionless representation.

The objective of this paper is to determine a 3D dimensionless Brinkman–Forchheimer-extended Darcy model defined in the periodic domain $\Omega = \prod_{i=1}^{3}(-\frac{L_i}{2}, \frac{L_i}{2})$ for some $L_i > 0$, which can be read as

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + a|u|^{2\alpha}u + b|u|^{2\beta}u + \nabla p = f, \\ \nabla\cdot u = 0, \\ u(t=0) = u_0, \end{cases} \tag{6}$$

equipped with periodic boundary conditions

$$u(x + Le_i, t) = u(x,t), \ \ i = 1,2,3,$$

where $u(x,t) : \Omega\times(0,\infty) \to \mathbb{R}^3$ denotes the velocity field, $p(x,t) : \Omega\times(0,\infty) \to \mathbb{R}$ represents the pressure as scalar function, f is the external force, the parameters $\alpha > \beta \geq 0$ are constants, and a, b are real numbers.

When $a = b = 0$, (6) reduces to the classical Navier–Stokes equations

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \nabla\cdot u = 0, \\ u(t=0) = u_0. \end{cases} \tag{7}$$

The study of finite dimensional attractors for (7) can be seen in Ladyzhenskaya [24,25] from the 1960s. Then, the research on the fractal dimension of attractors and inertial manifolds for systems (7) appeared in [26–29] and the literature therein. Moreover, information on turbulence, the determining modes, nodes and finite volume elements for 2D Navier–Stokes equations defined in the smooth domain were investigated in [27,30–35] and for non-smooth domain in [36].

When the convection term can be neglected and $\alpha = 0$, $\beta = \frac{\theta}{2}(\theta > 0)$, $a > 0$, and $b \in \mathbb{R}$, then Equation (6) reduces to generalized Brinkman–Forchheimer equations

$$\begin{cases} \partial_t u - \nu\Delta u + au + b|u|^{\theta}u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u(t = 0) = u_0. \end{cases} \tag{8}$$

The research of well-posedness and regularity of (8) can be seen in [6]. In particular, when $\theta \in (0, \infty)$, the localization technique is used in [6] for a regularity argument. The investigation on attractors and stability of a dynamical system generated by (8) can be seen in [37–40] and the literature therein. When $\theta = 2$, Equation (8) becomes the classical Brinkman–Forchheimer equation, and its well-posedness and reduction were studied in [41].

Inspired from [27,34,42], the determining modes of global solutions for an non-autonomous system and asymptotic determination of a global attractor for an autonomous case of (6) are obtained, which have the following features.

(I) The literature [43] presented a weak solution when $\alpha > \beta \geq 0$ and $a > 0$, $b \in \mathbb{R}$ and its uniqueness when $\alpha > 1$. Similarly, the strong solution and its uniqueness were shown when $\alpha > 1$.
Based on the well-posedness and regularity of Equation (6), for a non-autonomous system, the determining modes for weak and regular solutions have been illustrated. The key step for achieving determining modes is to find the generalized Grashof number for porous medium fluid flow. The difficulty lies in the estimation of a trilinear operator and a nonlinear term $F(u) = a|u|^{2\alpha}u + b|u|^{2\beta}u$ in the three-dimensional model, which is overcome by combining the utilized regularity estimates of bilinear operators with a monotone operator technique to deal with a nonlinear term $F(u) = a|u|^{2\alpha}u + b|u|^{2\beta}u$ from the new definition of generalized Grashof numbers. Furthermore, the determination can be proved for a weak solution when $0 \leq \beta \leq \frac{1}{2}$, $0 < \alpha \leq 2$ and for a regular solution when $1 < \alpha \leq \frac{7}{5}$.

(II) The global attractor of an autonomous system (6) can be obtained for the semigroup generated by the global solution. Consider the complete trajectories inside an attractor; the asymptotic determination can be constructed via Fourier functionals. The difficulty lies in the estimation of a non-linear term $F(u) = a|u|^{2\alpha}u + b|u|^{2\beta}u$, which can be overcome by using the generalized Gronwall inequality with an assumption on averaging integrations for all $\alpha > 1$.

(III) The regularity and determination for the system (6) with a non-slip boundary condition are still unknown because of $-P_L\Delta \neq -\Delta$. Especially for the regularity estimates, the pressure term does not disappear when one uses a localized technique to deal with convective and nonlinear terms, which need new skills to overcome this difficulty.

(IV) The well-celebrated models (5) and (6) have been chosen as a representative problem to highlight the utility and numerical simulation. Based on the well-posedness of the three dimensional Brinkman–Forchheimer-extended Darcy system and related models in [12,13,16,19–23], the numerical simulations, such as the continuous dependence on the Forchheimer and Brinkman coefficients and the finite element approximation for (5) and its extension, have been investigated in [9,14,15,17,18], which present the effect of increasing Grashof numbers, the parameters (Brinkamn, Forchheimer and Darcy numbers) and transportation in porous media.

Comparing with the computation of (6), the determining modes and asymptotic determination in this paper can give some theoretical analysis for the preparation of a numerical simulation of fluid flow models and also the description of asymptotic behaviors, which are important from the viewpoint of mathematical theory and computation. Moreover, the theoretical analysis presented shows good agreement with the asymptotic behavior of a numerical approximated solution.

This paper is organized as follows. In Section 2, some preliminaries and some useful lemmas are given. In Section 3, we give the existence of global weak and regular solutions under periodic boundary conditions. Section 4 is devoted to the determining modes for a three-dimensional Brinkman–Forchheimer-extended Darcy model for a non-autonomous system. In Section 5, the asymptotically determining for the autonomous system will be shown. The paper concluses with some further results for more general cases.

2. Preliminaries

2.1. Functional Spaces, Symbols and Notations

The functional space $L^2_{per}(\Omega)$ consists of all vector fields u with L_i-periodic (for $i = 1,2,3$), and the norm is defined by

$$\int_{\mathcal{O}} |u(x)|^2 dx = \|u\|^2_{(L^2(\mathcal{O}))^3}.$$

The space $H^1_{per}(\Omega)$ is also well defined as

$$\|u\|^2_{(H^1(\mathcal{O}))^3} = \frac{1}{L^2}\int_{\mathcal{O}} |u(x)|^2 dx + \int_{\mathcal{O}} \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2 dx.$$

In the space-periodic case, the spaces $\dot{V}_{per}$ and $\dot{H}_{per}$ are defined as

$$\dot{H}_{per} = \{u = \sum_{k\in Z^3\backslash\{0\}} \widehat{u}_k e^{2\pi i \frac{k}{L}x}; \widehat{u}_{-k} = \overline{\widehat{u}}_k\,, \frac{k}{L}\cdot\widehat{u}_k = 0, \sum_{k\in Z^3\backslash\{0\}} |\widehat{u}_k|^2 < \infty\},$$
$$\dot{V}_{per} = \{u = \sum_{k\in Z^3\backslash\{0\}} \widehat{u}_k e^{2\pi i \frac{k}{L}x}; \widehat{u}_{-k} = \overline{\widehat{u}}_k\,, \frac{k}{L}\cdot\widehat{u}_k = 0, \sum_{k\in Z^3\backslash\{0\}} |\frac{k}{L}|^2|\widehat{u}_k|^2 < \infty\}$$

with the inner products and norms by

$$(u,v) = \int_{\Omega} u(x)\cdot v(x)dx, \quad |u| = (u,u)^{\frac{1}{2}} = (\int_{\Omega} |u(x)|^2 dx)^{\frac{1}{2}}$$

and

$$((u,v)) = \int_{\Omega} \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx, \quad \|u\| = (\int_{\Omega} \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2 dx)^{\frac{1}{2}}$$

respectively. Furthermore, the Poincaré inequality in homogeneous spaces holds with positive constant λ_1 satisfying $\lambda_1 \sim \frac{1}{L^2}$.

2.2. Linear Operators

The Helmholtz–Leray projector is given by

$$P_L : (L^2_{per}(\Omega))^3 \to (\dot{H}_{per}(\Omega))^3$$

and the Stokes operator A is defined as

$$Au = -P_L \Delta u \;\text{ for all }\; u \in D(A) = \dot{V}_{per} \cap (H^2_{per}(\Omega))^3.$$

Since the Stokes operator A is positive, self-adjoint, it generates a set of eigenvectors $\{w_m\}_{m=1}^{\infty}$

$$Aw_m = \lambda_m w_m. \;\; m = 1, 2, \cdots,$$

and $\{w_m\}_{m=1}^{\infty}$ are the orthonormal basis in $\dot{H}_{per}$ and corresponding eigenvalues satisfy

$$0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_m \leq \cdots, \; \lambda_m \to +\infty, \;\; \text{as} \;\; m \to \infty.$$

In addition, due to the properties of operator A, the fractional powers A^s ($s \in \mathbb{R}$) can be defined as

$$A^s u = \sum_{j=1}^{\infty} \lambda_j^s (u, \omega_j)\omega_j, \;\; u \in D(A^s),$$

with the domain of A^s in $\dot{H}_{per}$ as $D(A^s) = V_{2s}$. The inner product of domain $D(A^s)$ is defined as

$$(u, v)_{D(A^s)} = (A^s u, A^s v) = \sum_{j=1}^{\infty} \lambda_j^{2s} (u, \omega_j)(v, \omega_j)$$

and norm

$$||u||_{D(A^s)} = |A^s u| = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} |(u, \omega_j)|^2 \right)^{\frac{1}{2}}$$

respectively.

The embeddings $D(A^s) \subset \dot{H}_{per} \subset D(A^{-s})$ hold (see [27]), where $D(A^{-s})$ is the dual space of $D(A^s)$. In particular, $D(A^{\frac{1}{2}}) = \dot{V}_{per}$ and $D(A^{-\frac{1}{2}}) = \dot{V}'_{per}$ satisfy the continuous embeddings

$$\dot{V}_{per} \subset \dot{H}_{per} \subset \dot{V}'_{per},$$

where $\dot{V}'_{per}$ is the dual space of $\dot{V}_{per}$, and the first embedding is dense.

The bilinear and trilinear operators on Ω are defined by

$$B(u, v) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} dx, \;\; u, v \in \dot{V}_{per}$$

and

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \;\; u, v, w \in \dot{V}_{per}.$$

The trilinear operator $b(\cdot, \cdot, \cdot)$ possesses the following properties

$$\begin{cases} b(u, v, v) = 0, \;\; u, v, w \in \dot{V}_{per}, \\ b(u, v, w) = -b(u, w, v), \;\; u, v, w \in \dot{V}_{per}, \\ b(u, v, Av) = 0, \;\; u \in \dot{V}_{per}, v \in D(A). \end{cases} \tag{9}$$

3. Well-Posedness and Regularity

In this section, the existence and uniqueness of the weak solution and regularity for the system (6) will be given.

Theorem 1. *(Well-posedness) Suppose that $f \in L^\infty(\mathbb{R}^+; \dot{H}_{per})$, $u_0 \in \dot{H}_{per}$ and $\alpha > \beta \geq 0$, $a > 0, b \in \mathbb{R}$. Then, problem (6) possesses a weak solution satisfying*

$$u \in C^0(\mathbb{R}^+; H_{weak}) \cap L^\infty_{loc}(\mathbb{R}^+; \dot{H}_{per}) \cap L^2_{loc}(\mathbb{R}^+; \dot{V}_{per}) \cap L^{2\alpha+2}_{loc}(\mathbb{R}^+; L^{2\alpha+2}(\Omega)),$$

and

$$\limsup_{t\to\infty} |u| \leq \rho_0,$$

where

$$\rho_0 = \left[\frac{1}{2\nu}\left(||f||^2_{L^\infty(\mathbb{R}^+;\dot{H}_{per})} + \left\{\left[\frac{2(1+\mathbf{1}_{\{b<0\}})}{a}\right]^{\frac{1}{\alpha}} + \mathbf{1}_{\{b<0\}}\left[\frac{4}{a}\right]^{\frac{\beta+1}{\alpha-\beta}} + 2\left[\frac{\nu}{a}\right]^{\frac{\alpha+1}{\alpha}}\right\}|\Omega|\right)\right]^{\frac{1}{2}},$$

where H_{weak} is a weak topology equipped with $\dot{H}_{per}$, and $\mathbf{1}_{\{b<0\}}$ is defined as the characteristic function of the set $\{b < 0\}$. Moreover, if $\alpha > 1$, then the weak solution is unique.

Proof. The existence of weak solutions can be obtained by the Galerkin method and compact argument as in [43]. Here, we omit the details. □

Theorem 2. *(Regularity) Suppose that $f \in L^\infty(\mathbb{R}^+; \dot{H}_{per})$, $u_0 \in \dot{V}_{per}$ and $\alpha > 1, \alpha > \beta \geq 0, a > 0, b \in \mathbb{R}$. Then, problem (6) has a unique global strong solution satisfying*

$$u \in C^0_b(\mathbb{R}^+; \dot{V}_{per}) \cap L^2_{loc}(\mathbb{R}^+; D(A)) \cap L^{2\alpha+2}_{loc}(\mathbb{R}^+; L^{2\alpha+2}(\Omega))$$

and

$$\limsup_{t\to\infty} |\nabla u| \leq \rho_1,$$

where

$$\rho_1 = \left[\frac{1}{\nu}\left\{\left(2 + (\mathcal{A}_1+1)\left[1+\frac{1}{\nu}\right]\right)||f||^2_{L^\infty(\mathbb{R}^+;\dot{H}_{per})} + (\mathcal{A}_1+1)\left[\eta_0 + \frac{\eta_1}{\nu}\right]\right\}\right]^{\frac{1}{2}},$$

with

$$\mathcal{A}_1 = 2\left\{\left[\frac{\nu^\alpha(1+2\alpha)}{2(1+2\cdot\mathbf{1}_{\{b<0\}})}\right]^{\frac{1}{1-\alpha}} + \mathbf{1}_{\{b<0\}}\left[\frac{4^\beta[|b|(1+2\beta)]^\alpha}{[a(1+2\alpha)]^\beta}\right]^{\frac{1}{\alpha-\beta}}\right\},$$

$$\eta_0 = \left\{\left[\frac{2(1+2\cdot\mathbf{1}_{\{b<0\}})}{a}\right]^{\frac{1}{\alpha}} + \mathbf{1}_{\{b<0\}}|b|\left[\frac{4|b|}{a}\right]^{\frac{\beta+1}{\alpha-\beta}}\right\}|\Omega|, \quad \eta_1 = \eta_0 + \frac{a}{2}\left[\frac{\nu}{a}\right]^{\frac{\alpha+1}{\alpha}}|\Omega|,$$

which is dependent continuously on initial data. Furthermore, if $u_0 \in L^{2\alpha+2}(\Omega)$, then $u \in L^\infty_{loc}(\mathbb{R}^+; L^{2\alpha+2}(\Omega))$ and $\partial_t u \in L^{2\alpha+2}_{loc}(\mathbb{R}^+; L^2(\Omega))$.

Proof. See, e.g., [43] for more details. □

4. Determining for a Non-Autonomous System

The section states the determining modes for a non-autonomous system, which needs some lemmas such as the following for preparation.

4.1. Some Lemmas

Lemma 1. *(See [34]) Let $\gamma(t)$ and $\eta(t)$ be two locally integrable real-valued functions that satisfy*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \eta(\tau)d\tau > 0, \tag{10}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \eta^{-}(\tau)d\tau < \infty,$$

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \gamma^{+}(\tau)d\tau = 0$$

for some $T > 0$, where $\eta^{-} = max\{-\eta(t), 0\}$ and $\gamma^{+} = max\{\gamma(t), 0\}$. Assume $\xi(t)$ is an absolutely continuous nonnegative function on $[0,\infty)$ that satisfies

$$\frac{d\xi}{dt} + \eta(t)\xi \leqslant \gamma(t) \tag{11}$$

almost everywhere on $[0,\infty)$. Then, $\xi(t) \to 0$ as $t \to \infty$.

Lemma 2. *Assume that P_m is the orthogonal projection from $\dot{H}_{per}$ to the space spanned by its finite basis $\{w_1, w_2, \ldots, w_m\}$. Then,*

$$\begin{aligned} |Q_m u|^2 &\le \frac{1}{\lambda_m}||Q_m u||^2, \ \ u \in \dot{V}_{per}, m \in N, \\ ||P_m u||^2 &\le \lambda_m |u|^2, \ \ u \in \dot{V}_{per}, m \in N, \end{aligned} \tag{12}$$

where $Q_m u = (I - P_m)u$ for identity operator I.

Proof. See, e.g., [27] for more detail. □

Lemma 3. *The Ladyzhenskaya–Agmon inequalities in three-dimensional space is presented as*

$$\begin{aligned} ||u||_{L^4(\Omega)} &\le C_1 |u|^{\frac{1}{4}} ||u||^{\frac{3}{4}}, \ \ u \in \dot{V}_{per}, \\ ||u||_{L^\infty(\Omega)} &\le C_1 ||u||^{\frac{1}{2}} |Au|^{\frac{1}{2}}, \ \ u \in D(A). \end{aligned} \tag{13}$$

Proof. See, e.g., [27] for more detail. □

Lemma 4. *There exist nonnegative constants $\kappa_0 = \kappa(\alpha)$ and $\widehat{\kappa}_0 = \widehat{\kappa}_0(\alpha)$ such that*

$$0 \le \kappa_0 |u - v|^2 (|u| + |v|)^{2\alpha} \le (|u|^{2\alpha} u - |v|^{2\alpha} v) \cdot (u - v),$$

and

$$||u|^{2\alpha} u - |v|^{2\alpha} v| \le \widehat{\kappa}_0 (|u| + |v|)^{2\alpha} |u - v|.$$

Proof. See, e.g., [43,44] for more detail. □

4.2. The Determining Modes for a Weak Solution

According to [27,32], the generalized Grashof number for medium porous fluid flow is denoted by

$$G = \frac{F}{\lambda_1 \nu^2},$$

where $F = ||f||_{L^\infty(t,t+T;\dot{H}_{per})}$.

Suppose u and v are the solutions of system (6) with different external force terms f and g, which satisfy the following systems

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + a|u|^{2\alpha}u + b|u|^{2\beta}u + \nabla p = f, \\ \nabla\cdot u = 0, \\ u(t=0) = u_0, \end{cases} \tag{14}$$

and

$$\begin{cases} \partial_t v - \nu\Delta v + (v\cdot\nabla)v + a|v|^{2\alpha}v + b|v|^{2\beta}v + \nabla q = g, \\ \nabla\cdot v = 0, \\ v(t=0) = v_0 \end{cases} \tag{15}$$

respectively.

According to [27], it gives the definition of the Galerkin projection P_m associated with the first m modes of the Stokes operator. Each solution of (6) can be expanded into the following form

$$u(x,t) = \sum_{k=1}^{\infty} \widehat{u}_k w_k(x) \ \text{ and } \ v(x,t) = \sum_{k=1}^{\infty} \widehat{v}_k w_k(x),$$

where w_k is the k-th eigenfunction of the Stokes operator. Then, the Galerkin projection corresponding to the first (say m) modes are

$$P_m u(x,t) = \sum_{k=1}^{m} \widehat{u}_k w_k(x) \ \text{ and } \ P_m v(x,t) = \sum_{k=1}^{m} \widehat{v}_k w_k(x).$$

Suppose that the external force terms f and g have the same asymptotic behavior as t and is large enough

$$||f(x,t) - g(x,t)||_{L^\infty(t,t+T;\dot{H}_{per})} \longrightarrow 0, \ \text{ as } t\to\infty.$$

Then, the first m modes are called *determining* associated with P_m (see [27]), provided that

$$||P_m u(x,t) - P_m v(x,t)||_{L^\infty(t,t+T;\dot{H}_{per})} \longrightarrow 0, \ \text{ as } t\to\infty \tag{16}$$

leads to

$$||u(x,t) - v(x,t)||_{L^\infty(t,t+T;\dot{H}_{per})} \longrightarrow 0, \ \text{ as } t\to\infty. \tag{17}$$

Now, the existence of determining modes for (6) can be shown as follows.

Theorem 3. *Suppose that* $2 \geq \alpha > 0$, $\frac{1}{2} \geq \beta > 0$, *and* $m \in N$ *fulfills the following inequality*

$$m \geq C'(\widetilde{L} + LG^2)^{\frac{3}{2}}, \tag{18}$$

where

$$L = (4|b|^2\widehat{\kappa}_0^2 + 2C_1')(2+2\nu T)\lambda_1,$$

and

$$\widetilde{L} = \frac{8|b|^2\widehat{\kappa}_0^2 + 4C_1'}{\nu^4\lambda_1}(\eta_1 + 2\nu\eta_0 T) + \frac{4|b|^2\widehat{\kappa}_0^2 + 2C_1'}{\nu^3\lambda_1}e^{-\nu t}\max\{|v(0)|^2, |u(0)|^2\}$$

for the generalized Grashof number G *of 3D Brinkman–Forchheimer equations,* $C' > 0$ *is a parameter which depends on the shape of* Ω *only. Then, the first* m *modes are called determining associated with* P_m *for (6) equipped with a periodic boundary condition.*

Proof. Let u and v be the two solutions of (14) and (15), respectively. The forcing terms f and g satisfy $\|f(t)-g(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0$ as $t \to \infty$. Denote $w = u - v$. Our goal is to show that if

$$\|P_m w(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0, \text{ as } t \to \infty,$$

then $\|w(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0$, as $t \to \infty$. To reach the goal, it is sufficient to show that

$$\|Q_m w(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0, \text{ as } t \to \infty.$$

Next, we mainly take advantage of Lemma 1 to achieve our goal.

Step 1: The estimate of $|Q_m \omega|$
Subtract (15) from (14), it yields the following equation

$$\begin{cases} \partial_t w + \nu A w + B(u,w) + B(w,v) + a(|u|^{2\alpha}u - |v|^{2\alpha}v) + b(|u|^{2\alpha}u - |v|^{2\alpha}v) = f - g, \\ w(t=0) = u_0 - v_0. \end{cases} \tag{19}$$

Taking the inner product of (19) with $Q_m w$ in $\dot{H}_{per}$, we derive

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|Q_m w|^2 + \nu||Q_m w||^2 + b(u,w,Q_m w) + b(w,v,Q_m w) + a(|u|^{2\alpha}u - |v|^{2\alpha}v, Q_m w) \\ + b(|u|^{2\beta}u - |v|^{2\beta}v, Q_m w) = (f-g, Q_m w). \end{aligned} \tag{20}$$

The last four terms on the left side of the above equation will be estimated as follows. For simplicity, denote $I_i (i = 1,2,3,4)$ as

$$I_1 = b(u,w,Q_m w),$$

$$I_2 = b(w,v,Q_m w),$$

$$I_3 = a(|u|^{2\alpha}u - |v|^{2\alpha}v, Q_m w),$$

$$I_4 = b(|u|^{2\beta}u - |v|^{2\beta}v, Q_m w).$$

By Ladyzhenskaya–Agmon's, Hölder's and Young's inequalities, note that $b(u, Q_m w, Q_m w) = 0$. The estimates of I_1, I_2 are presented as

$$\begin{aligned} I_1 = b(u,w,Q_m w) &= b(u, P_m w, Q_m w) + b(u, Q_m w, Q_m w) \\ &\le |b(u, P_m w, Q_m w)| \\ &\le C_1 |u|^{\frac{1}{4}} ||u||^{\frac{3}{4}} ||Q_m w|| |P_m w|^{\frac{1}{4}} ||P_m w||^{\frac{3}{4}} \end{aligned}$$

and

$$I_2 = b(P_m w, v, Q_m w) + b(Q_m w, v, Q_m w)$$

with

$$\begin{aligned} b(P_m w, v, Q_m w) &\le C_1 ||P_m w||_{L^4(\Omega)} ||v||_{L^4(\Omega)} ||Q_m w|| \\ &\le C_1 |v|^{\frac{1}{4}} ||v||^{\frac{3}{4}} ||Q_m w|| |P_m w|^{\frac{1}{4}} ||P_m w||^{\frac{3}{4}}, \end{aligned}$$

$$\begin{aligned} b(Q_m w, v, Q_m w) &\le C_1 ||v|| |Q_m w|^{\frac{1}{2}} ||Q_m w||^{\frac{3}{2}} \\ &\le \frac{C_1}{\nu} ||v||^4 |Q_m w|^2 + \frac{\nu}{4} ||Q_m w||^2. \end{aligned}$$

The estimates of I_3 can be given by

$$\begin{aligned}I_3 &= a(|u|^{2\alpha}u - |v|^{2\alpha}v, Q_m w)\\ &= a(|u|^{2\alpha}u - |v|^{2\alpha}v, w) - a(|u|^{2\alpha}u - |v|^{2\alpha}v, P_m w)\\ &\geq -a(|u|^{2\alpha}u - |v|^{2\alpha}v, P_m w)\end{aligned}$$

and

$$\begin{aligned}a(|u|^{2\alpha}u - |v|^{2\alpha}v, P_m w) &\leq a\widehat{\kappa}_0 \int_\Omega |w|(|u|^{2\alpha} + |v|^{2\alpha})|P_m w|dx\\ &\leq a\widehat{\kappa}_0||w||\,||P_m w||(||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)} + ||v||^{2\alpha}_{L^{2\alpha+2}(\Omega)})\\ &\leq aC_0||P_m w||^2(||u||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + ||v||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + 1)\\ &\quad + aC_0||Q_m w||\,||P_m w||(||u||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + ||v||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + 1),\end{aligned}$$

where $0 < \alpha \leq 2$ comes from $\frac{\alpha}{\alpha+1} \leq \frac{2}{3}$.

By a similar technique as above, I_4 can be estimated by

$$\begin{aligned}I_4 = b(|u|^{2\beta}u - |v|^{2\beta}v, Q_m w) &\leq |b|\widehat{\kappa}_0 \int_\Omega |w|(|u|^{2\beta} + |v|^{2\beta})|Q_m w|dx\\ &\leq |b|\widehat{\kappa}_0||w||(||u||^{2\beta}_{L^{2\beta+2}(\Omega)} + ||v||^{2\beta}_{L^{2\beta+2}(\Omega)})|Q_m w|\\ &\leq |b|\widehat{\kappa}_0||Q_m w||(||u||^{2\beta}_{L^{2\beta+2}(\Omega)} + ||v||^{2\beta}_{L^{2\beta+2}(\Omega)})|Q_m w|\\ &\quad + |b|\widehat{\kappa}_0||P_m w||(||u||^{2\beta}_{L^{2\beta+2}(\Omega)} + ||v||^{2\beta}_{L^{2\beta+2}(\Omega)})|Q_m w|,\end{aligned}$$

where $0 < \beta \leq \frac{1}{2}$ comes from $\frac{\beta}{\beta+1} \leq \frac{1}{3}$. The combination of $\dot{V}_{per} \hookrightarrow L^{2\beta+2}(\Omega)$ with Hölder's and Young's inequalities results in

$$\begin{aligned}I_4 \leq\; &\frac{2|b|^2\widehat{\kappa}_0^2}{\nu}\sup(||u||^{4\beta}, ||v||^{4\beta})|Q_m w|^2 + \frac{\nu}{4}||Q_m w||^2\\ &+ |b|C_0'(||u||^{2\beta+2}_{L^{2\beta+2}(\Omega)} + ||v||^{2\beta+2}_{L^{2\beta+2}(\Omega)} + 1)||P_m w||\,|Q_m w|.\end{aligned}$$

Substitute the estimates of I_i into Equation (20). This leads to

$$\frac{d}{dt}|Q_m w|^2 + (\nu\lambda_{m+1} - \frac{4|b|^2\widehat{\kappa}_0^2}{\nu}\sup(||u||^{4\beta}, ||v||^{4\beta}) - \frac{2C_1}{\nu}||v||^4)|Q_m w|^2 \leq \gamma(t), \tag{21}$$

where

$$\begin{aligned}\gamma(t) = \;&C_1|u|^{\frac{1}{4}}||u||^{\frac{3}{4}}||Q_m w|||P_m w|^{\frac{1}{4}}||P_m w||^{\frac{3}{4}}\\ &+ C_1|v|^{\frac{1}{4}}||v||^{\frac{3}{4}}||Q_m w|||P_m w|^{\frac{1}{4}}||P_m w||^{\frac{3}{4}}\\ &+ aC_0||P_m w||^2(||u||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + ||v||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + 1)\\ &+ aC_0||Q_m w||\,||P_m w||(||u||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + ||v||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} + 1)\\ &+ |b|C_0'||P_m w||(||u||^{2\beta+2}_{L^{2\beta+2}(\Omega)} + ||v||^{2\beta+2}_{L^{2\beta+2}(\Omega)} + 1)|Q_m w| + |f - g||Q_m w|.\end{aligned}$$

Let $\xi(t) = |Q_m w|^2$ and $\eta(t) = \nu\lambda_{m+1} - \frac{4|b|^2\widehat{\kappa}_0^2}{\nu}\sup(||u||^{4\beta}, ||v||^{4\beta}) - \frac{2C_1'}{\nu}||v||^2$. Then, (21) satisfies the form (11) in Lemma 1.

Step 2: The estimate of $\int_t^{t+T}||u||^2 d\tau$

According to Theorem 1, the energy estimate of a weak solution yields

$$2\nu \int_t^{t+T} ||u||^2 ds + a \int_t^{t+T} ||u||^{2\alpha+2}_{L^{2\alpha+2}(\Omega)} ds \leq 2(||f||^2_{L^\infty(t,t+T;\dot{H}_{per})} + \eta_0)T + |u_0|^2 e^{-\nu t} + \frac{2}{\nu}(||f||^2_{L^\infty(t,t+T;\dot{H}_{per})} + \eta_1),$$

which results in

$$||u||^2_{L^2(t,t+T;\dot{V}_{per})} \leq \frac{1}{\nu} e^{-\nu t} |u(0)|^2 + \frac{2+2\nu T}{\nu^2} ||f||^2_{L^\infty(t,t+T;\dot{H}_{per})} + \frac{2\eta_1 + 2\nu\eta_0 T}{\nu^2},$$

In particular, if the forcing term f belongs to $L^\infty(t, t+T; \dot{H}_{per})$, the boundedness of u in the space $L^2(t, t+T; \dot{V}_{per})$ can be deduced, which will be used in the next step.

Step 3: Determining modes
The results in **Steps 1** and **2** lead to

$$\begin{aligned}
\frac{1}{T}\int_t^{t+T} \eta(s) ds &= \frac{1}{T}\int_t^{t+T} \nu\lambda_{m+1} - \frac{4|b|^2\widehat{\kappa}_0^2}{\nu} \sup(||u||^{4\beta}, ||v||^{4\beta}) - \frac{2C_1'}{\nu}||v||^2 ds \\
&\geq \nu\lambda_{m+1} - \frac{(4|b|^2\widehat{\kappa}_0^2 + 2C_1')(2+2\nu T)}{\nu^3} F^2 \\
&\quad - \frac{4|b|^2\widehat{\kappa}_0^2 + 2C_1'}{\nu^2} e^{-\nu t} \max\{|v(0)|^2, |u(0)|^2\} \\
&\quad - \frac{8|b|^2\widehat{\kappa}_0^2 + 4C_1'}{\nu^3}(\eta_1 + 2\nu\eta_0 T) > 0,
\end{aligned}$$

where C_1' is a constant independent of t and x.

From [26], the approximation of eigenvalues fora Stokes operator with a periodic boundary satisfying averaging zero and a non-slip case can be given by

$$\lambda_m \sim C\lambda_1 m^{\frac{2}{d}}, \ d = 2 \text{ or } 3,$$

Thus, we derive

$$m \geq C'(\widetilde{L} + LG^2)^{\frac{3}{2}},$$

where

$$L = (4|b|^2\widehat{\kappa}_0^2 + 2C_1')(2+2\nu T)\lambda_1$$

and

$$\widetilde{L} = \frac{8|b|^2\widehat{\kappa}_0^2 + 4C_1'}{\nu^4\lambda_1}(\eta_1 + 2\nu\eta_0 T) + \frac{4|b|^2\widehat{\kappa}_0^2 + 2C_1'}{\nu^3\lambda_1} e^{-\nu t} \max\{|v(0)|^2, |u(0)|^2\}.$$

Therefore, combining the results in the above steps, we conclude that m-modes satisfy inequality (18). This implies that $P_m u$ is the determining mode for the weak solution of (6). The proof is finished. □

4.3. Determining Modes for a Regular Solution

The definition of determining modes for the regular solution of (6) is similar to the case for the weak solution, which only needs to prove

$$||u(x,t) - v(x,t)||_{L^\infty(t,t+T;\dot{V}_{per})} \longrightarrow 0 \ \text{ as } t \to \infty. \tag{22}$$

Now, the determining modes for the regular solution of (6) are given as follows.

Theorem 4. *Suppose that $1 < \alpha \leq \frac{7}{5}$ and $m \in N$ satisfy*

$$m \geq C'\Big[K\nu^2\lambda_1 G^2 + \Big\{2C(\nu) + 2\widetilde{\kappa}|b|(2\beta+1) + 2\kappa a(2\alpha+1) + 2C(\nu)\big[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\big]\Big\}\frac{\mathcal{A}_1\eta_0}{\nu^2\lambda_1}\Big]^{\frac{3}{2}}, \tag{23}$$

where

$$K = \Big\{C(\nu) + \widetilde{\kappa}|b|(2\beta+1) + \kappa a(2\alpha+1) + 2C(\nu)\big[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\big]\Big\}(2+\mathcal{A}_1)$$

for the generalized Grashof number G of 3D Brinkman–Forchheimer equations, $C' > 0$ is a parameter which depends on the shape of Ω only. Then, the first m modes are called determining modes associated with P_m for a regular solution of (6).

Proof. Similar to the technique in the proof of Theorem 3, we only need to show that

$$\|Q_m w(t)\|_{L^\infty(t,t+T;\dot{V}_{per})} \to 0, \ \text{ as } \ t \to \infty,$$

for $\|P_m w(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0$ and $\|f(t) - g(t)\|_{L^\infty(t,t+T;\dot{H}_{per})} \to 0$ as $t \to \infty$.

Step 1: The estimate of $\|Q_m\omega\|$
Taking the inner product of (19) with $AQ_m w$ in $\dot{H}_{per}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||Q_m w||^2 + \nu|AQ_m w|^2 + b(u,w,AQ_m w) + b(w,v,AQ_m w)\\&\quad + a(|u|^{2\alpha}u - |v|^{2\alpha}v, AQ_m w) + b(|u|^{2\beta}u - |v|^{2\beta}v, AQ_m w) = (f-g, AQ_m w).\end{aligned} \tag{24}$$

For simplicity, denote $J_i (i = 1,2,3,4)$ as

$$\begin{aligned}J_1 &= b(u,w,AQ_m w),\\ J_2 &= b(w,v,AQ_m w),\\ J_3 &= a(|u|^{2\alpha}u - |v|^{2\alpha}v, AQ_m w),\\ J_4 &= b(|u|^{2\beta}u - |v|^{2\beta}v, AQ_m w).\end{aligned}$$

The trilinear operator J_1 is estimated by

$$\begin{aligned}J_1 = b(u,w,AQ_m w) &= b(u,P_m w,AQ_m w) + b(u,Q_m w,AQ_m w)\\ &\leq |b(u,P_m w,AQ_m w)|\\ &\leq C_1||u||^{\frac{1}{2}}|Au|^{\frac{1}{2}}||P_m w||\,|AQ_m w|.\end{aligned}$$

Since $b(u,Q_m w,AQ_m w) = 0$ in the periodic case in [27], the equality

$$b(v,u,Au) + b(u,v,Au) + b(u,u,Av) = 0, \ \ u,v \in D(A)$$

leads to

$$\begin{aligned}J_2 = b(w,v,AQ_m w) &= b(P_m w,v,AQ_m w) - b(w,Q_m w,AQ_m w) - b(Q_m w,Q_m w,Av)\\ &= b(P_m w,v,AQ_m w) - b(Q_m w,Q_m w,Av)\end{aligned}$$

with estimates

$$\begin{aligned} b(P_m w, v, AQ_m w) &\le C_1 ||P_m w||_{L^\infty(\Omega)} ||v||\, |AQ_m w| \\ &\le C_1 ||P_m w||^{\frac{1}{2}} |AP_m w|^{\frac{1}{2}} ||v||\, |AQ_m w| \end{aligned}$$

and

$$\begin{aligned} b(Q_m w, Q_m w, Av) &\le C_1 ||Q_m w||^{\frac{3}{2}} |AQ_m w|^{\frac{1}{2}} |Av| \\ &\le C(\nu) |Av|^{\frac{4}{3}} ||Q_m w||^2 + \frac{\nu}{6} |AQ_m w|^2. \end{aligned}$$

Since $A = -\Delta$ in the periodic case, the Hölder and Young inequalities result in

$$\begin{aligned} J_3 &= a(|u|^{2\alpha} u - |v|^{2\alpha} v, AQ_m w) \\ &= a(2\alpha+1) \int_\Omega (|u|^{2\alpha} \nabla u - |v|^{2\alpha} \nabla v) \cdot \nabla Q_m w dx \\ &= a(2\alpha+1) \int_\Omega (|u|^{2\alpha} \nabla u - |u|^{2\alpha} \nabla v + |u|^{2\alpha} \nabla v - |v|^{2\alpha} \nabla v) \cdot \nabla Q_m w dx \\ &= a(2\alpha+1) \int_\Omega (|u|^{2\alpha} - |v|^{2\alpha}) \nabla v \cdot \nabla Q_m w dx + a(2\alpha+1) \int_\Omega |u|^{2\alpha} \nabla w \cdot \nabla Q_m w dx \\ &\le a(2\alpha+1) \int_\Omega f'(\varphi) w \nabla v \cdot \nabla Q_m w dx + a(2\alpha+1) \int_\Omega |u|^{2\alpha} \nabla w \cdot \nabla Q_m w dx \\ &\le a(2\alpha+1)\kappa |Av|\, ||w||\, |\nabla Q_m w| + a(2\alpha+1) ||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)} |Aw|\, ||Q_m w||^{\frac{1}{4}} ||AQ_m w|^{\frac{3}{4}}, \end{aligned}$$

where $1 < \alpha \le \frac{7}{5}$, $\kappa = 2\alpha ||\varphi||^{2\alpha-1}_{L^\infty(\Omega)}$, φ lies in u and v. Hence, the further estimate of J_3

$$\begin{aligned} J_3 \le\ & \kappa a(2\alpha+1)|Av|\, ||P_m w||\, ||Q_m w|| + \kappa a(2\alpha+1)|Av|\, ||Q_m w||^2 \\ &+ a(2\alpha+1)||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)} |AP_m w|\, ||Q_m w||^{\frac{1}{4}} ||AQ_m w|^{\frac{3}{4}} \\ &+ a(2\alpha+1)||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)} ||Q_m w||^{\frac{1}{4}} ||AQ_m w|^{\frac{7}{4}} \\ \le\ & \kappa a(2\alpha+1)|Av|\, ||P_m w||\, ||Q_m w|| + \kappa a(2\alpha+1)|Av|\, ||Q_m w||^2 \\ &+ a(2\alpha+1)||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)} |AP_m w|\, ||Q_m w||^{\frac{1}{4}} ||AQ_m w|^{\frac{3}{4}} \\ &+ C(\nu) a^8 (2\alpha+1)^8 ||u||^{16\alpha} ||Q_m w||^2 + \frac{\nu}{6} |AQ_m w|^2 \end{aligned}$$

holds from $V \hookrightarrow L^{2\alpha+2}(\Omega)$.

Similarly, the estimate of J_4 can be obtained

$$\begin{aligned} J_4 \le\ & \widetilde{\kappa} |b| (2\beta+1)|Av|\, ||P_m w||\, ||Q_m w|| + \widetilde{\kappa} |b| (2\beta+1)|Av|\, ||Q_m w||^2 \\ &+ |b|(2\beta+1)||u||^{2\beta}_{L^{2\beta+2}(\Omega)} |AP_m w|\, ||Q_m w||^{\frac{1}{4}} ||AQ_m w|^{\frac{3}{4}} \\ &+ C(\nu)|b|^8 (2\beta+1)^8 ||u||^{16\beta} ||Q_m w||^2 + \frac{\nu}{6} |AQ_m w|^2, \end{aligned}$$

where $\widetilde{\kappa} = 2\beta ||\varphi||^{2\beta-1}_{L^\infty(\Omega)}$.

Substitute the estimate of J_i into (24). This yields

$$\begin{aligned} \frac{d}{dt} ||Q_m w||^2 + \{ & \nu \lambda_{m+1} - C(\nu)|Av|^{\frac{4}{3}} - C(\nu) a^8 (2\alpha+1)^8 ||u||^{16\alpha} \\ & - C(\nu)|b|^8 (2\beta+1)^8 ||u||^{16\beta} - \kappa a(2\alpha+1)|Av| \\ & - \widetilde{\kappa} |b| (2\beta+1)|Av| \} ||Q_m w||^2 \le \gamma(t), \end{aligned} \tag{25}$$

where

$$\begin{aligned}\gamma(t) = &C_1||u||^{\frac{1}{2}}|Au|^{\frac{1}{2}}||P_m w||\ |AQ_m w| \\ &+ C_1||P_m w||^{\frac{1}{2}}|AP_m w|^{\frac{1}{2}}||v||\ |AQ_m w| \\ &+ \kappa a(2\alpha-1)|Av|\ ||P_m w||\ ||Q_m w|| \\ &+ a(2\alpha+1)||u||^{2\alpha}_{L^{2\alpha+2}(\Omega)}|AP_m w|\ ||Q_m w||^{\frac{1}{4}}||AQ_m w|^{\frac{3}{4}} \\ &+ |b|(2\beta+1)||u||^{2\beta}_{L^{2\beta+2}(\Omega)}|AP_m w|\ ||Q_m w||^{\frac{1}{4}}||AQ_m w|^{\frac{3}{4}} \\ &+ \widetilde{\kappa}|b|(2\beta-1)|Av|\ ||P_m w||\ ||Q_m w|| + |f-g||AQ_m w|.\end{aligned}$$

Setting $\xi(t) = ||Q_m w||^2$ and

$$\begin{aligned}\eta(t) =&\nu\lambda_{m+1} - C(\nu)|Av|^{\frac{4}{3}} - C(\nu)a^8(2\alpha+1)^8||u||^{16\alpha} \\ &- C(\nu)|b|^8(2\beta+1)^8||u||^{16\beta} - \kappa a(2\alpha+1)|Av| \\ &- \widetilde{\kappa}|b|(2\beta+1)|Av|,\end{aligned}$$

we conclude that (25) satisfies the form (11) in Lemma 1.

Step 2: The estimate of $\int_t^{t+T}|Au|^2 d\tau$
According to Theorem 2, we obtain the energy estimate of a regular solution

$$||u||^2_{L^\infty(t,t+T;\dot{V}_{per})} \le \frac{2}{\nu}\left[(2+\mathcal{A}_1)||f||_{L^\infty(t,t+1;\dot{H}_{per})} + \mathcal{A}_1\eta_0\right]T$$

and

$$\|Au\|^2_{L^2(t,t+T;\dot{H}_{per})} \le \frac{1}{\nu^2}\left[(2+\mathcal{A}_1)||f||_{L^\infty(t,t+1;\dot{H}_{per})} + 2\mathcal{A}_1\eta_0\right]T.$$

In particular, the hypothesis $f \in L^\infty(t,t+T;\dot{H}_{per})$ leads to the boundedness of u in the spaces $L^2(t,t+T;D(A))$, $L^\infty(t,t+T;\dot{V}_{per})$, which will be used in the next step.

Step 3: Determining modes
It remains to check the similar condition (10) in Lemma 1.
From the results in the above steps, we can see that

$$\begin{aligned}\frac{1}{T}\int_t^{t+T}\eta(s)ds =& \frac{1}{T}\int_t^{t+T}\Big\{\nu\lambda_{m+1} - C(\nu)|Av|^{\frac{4}{3}} - C(\nu)a^8(2\alpha+1)^8||u||^{16\alpha} \\ &- C(\nu)|b|^8(2\beta+1)^8||u||^{16\beta} - \kappa a(2\alpha+1)|Av| \\ &- \widetilde{\kappa}|b|(2\beta+1)|Av|\Big\}ds \\ \ge\ & \nu\lambda_{m+1} - \frac{\widetilde{\kappa}|b|(2\beta+1)+\kappa a(2\alpha+1)+C(\nu)}{\nu^2}\left[(2+\mathcal{A}_1)F^2 + 2\mathcal{A}_1\eta_0\right] \\ &- 2\frac{C(\nu)\left[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\right]}{\nu}\left[(2+\mathcal{A}_1)F^2 + \mathcal{A}_1\eta_0\right] \\ =\ & \nu\lambda_{m+1} - K\frac{F^2}{\nu^2} - \Big\{2C(\nu) + 2\widetilde{\kappa}|b|(2\beta+1) + 2\kappa a(2\alpha+1) \\ &+ 2C(\nu)\left[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\right]\Big\}\frac{\mathcal{A}_1\eta_0}{\nu^2} \\ >\ & 0.\end{aligned}$$

The estimate

$$\lambda_m \sim C\lambda_1 m^{\frac{2}{3}}$$

leads to

$$m \geq C' \Big[K\nu^2 \lambda_1 G^2 + \Big\{ 2C(\nu) + 2\widetilde{\kappa}|b|(2\beta+1) + 2\kappa a(2\alpha+1) + 2C(\nu)\big[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\big] \Big\} \frac{\mathcal{A}_1 \eta_0}{\nu^2 \lambda_1} \Big]^{\frac{3}{2}},$$

where

$$K = \Big\{ C(\nu) + \widetilde{\kappa}|b|(2\beta+1) + \kappa a(2\alpha+1) + 2C(\nu)\big[a^8(2\alpha+1)^8 + |b|^8(2\beta+1)^8\big] \Big\}(2+\mathcal{A}_1).$$

Hence, we conclude that m satisfies inequality (23). The first m modes associated with $P_m u$ are the determining modes for a regular solution of (6).

The proof of Theorem 4 is completed. □

4.4. Some Remarks

In addition, some related topics will be stated in this section.

(I) Consider the following data assimilation algorithm with periodic boundary condition

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + a|u|^{2\alpha}u + b|u|^{2\beta}u + \nabla p = f + \mu(\mathcal{I}_h(u) - \mathcal{I}_h(v)), \\ \nabla\cdot u = 0, \\ u(t=0) = u_0, \end{cases} \tag{26}$$

where μ is a nudging parameter, $\mathcal{I}_h(u)$ is an interpolation operator, and h denotes spatial resolution of the collected measurements. In [43], the following types of interpolation operators are mainly considered. The first operator $\mathcal{I}_h^1 : H^1 \longrightarrow L^2(\Omega)$ is linear satisfying

$$||\phi - \mathcal{I}_h^1(\phi)||^2_{L^2(\Omega)} \leq c_0 h^2 ||\nabla\phi||^2_{L^2(\Omega)}, \tag{27}$$

for every $\phi \in H^1(\Omega)$, where $c_0 > 0$ is a dimensionless constant.

The second operator $\mathcal{I}_h^2 : H^2(\Omega) \longrightarrow L^2(\Omega)$ is also linear satisfying

$$||\phi - \mathcal{I}_h^2(\phi)||^2_{L^2(\Omega)} \leq c_0 h^2 ||\nabla\phi||^2_{L^2(\Omega)} + c_1 h^4 ||\Delta\phi||^2_{L^2(\Omega)}, \tag{28}$$

for every $\phi \in H^2(\Omega)$, where $c_1 > 0$ is a dimensionless constant.

In [43], Titi studied the well-posedness and regularity of weak solutions for different interpolations $\mathcal{I}_h$. Based on well-posedness and regularity of weak solutions, it is possible to study the determining modes for systems (26). The main difficulty lies in dealing with the term of $\mu(\mathcal{I}_h(u) - \mathcal{I}_h(v))$.

(II) The three-dimensional MHD model of electrically conducting fluid in a porous medium with periodic boundary conditions is described as

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u - b\cdot\nabla b + a|u|^{2\alpha}u + \nabla p = 0, \\ \partial_t b + \kappa\nabla\times(\nabla\times b) + u\cdot\nabla b - b\cdot\nabla u = 0, \\ \nabla\cdot u = 0, u(t=0) = u_0, b(t=0) = b_0, \end{cases} \tag{29}$$

where u is the fluid velocity, b is the magnetic field, and p is the pressure. The viscosity ν, resistivity κ and a, α are nonnegative constants. The first equation of (29) is about the porous medium on domain Ω, and the second one is the magnetic field evolution equation. In particular, if $\nabla \cdot b = 0$, it can be concluded that

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u - b \cdot \nabla b + a|u|^{2\alpha} u + \nabla p = 0, \\ \partial_t b - \kappa \Delta u + u \cdot \nabla b - b \cdot \nabla u = 0, \\ \nabla \cdot u = 0, u(t=0) = u_0, b(t=0) = b_0. \end{cases} \tag{30}$$

For system (30), Titi investigated global well-posedness and regularity in [45]. It is possible to study the determining modes for systems (30). However, the system (30) is coupled with two equations, which leads to more difficulty than the Brinkman–Forchheimer extended Darcy model.

5. The Asymptotically Determining for an Autonomous System

Consider the Brinkman–Forchheimer-extended-Darcy equation in the periodic case

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + a|u|^{2\alpha} u + b|u|^{2\beta} u + \nabla p = f(x), \\ \nabla \cdot u = 0, \\ u(t=0) = u_0. \end{cases} \tag{31}$$

By using the Leray projector P_L to (31), the autonomous system (31) is transformed into

$$\begin{cases} \partial_t u + \nu A u + B(u) + F(u) = f(x), \\ \nabla \cdot u = 0, \end{cases} \tag{32}$$

where $f := P_L f(x)$, $A = -\Delta$, $B(u) := P_L(u \cdot \nabla) u$ and $F(u) = P_L(a|u|^{2\alpha} u + b|u|^{2\beta} u)$.

Based on the well-posedness of system (32), the existence of a global attractor can be stated as follows.

Theorem 5. *Assume the hypotheses in Theorems 1 and 2 hold for an autonomous system (32). Then, the problem (32) with a periodic boundary condition possesses a global attractor $\mathcal{A} \in \dot{H}_{per}$ for semigroup $S(t) : \dot{H}_{per} \to \dot{H}_{per}$ generated by a global weak solution.*

Proof. The proof is similar to the procedure in [6], which will be skipped here. □

Consider the system $\mathcal{F} = \{F_1, \cdots, F_m\}$, where F_i are the Fourier modes $F_i = (u, w_i)$ for $u = \sum_{i=1}^{\infty} (u, w_i) w_i$. Then, the asymptotic determination is defined via the functional $\mathcal{F}$ as follows.

Definition 1. *Assume that $u_1(t)$ and $u_2(t)$ are the two trajectories of the global attractor of problem (32). Then, the dynamic system $(\dot{H}_{per}, S(t))$ is called asymptotically determining if there exists a system $\mathcal{F}$ such that*

$$F_n(u_1(t)) - F_n(u_2(t)) = 0 \ \ as \ \ t \to \infty$$

implies that

$$\lim_{t \to \infty} ||u_1(t) - u_2(t)||_{\dot{H}_{per}} = 0 \ \ for \ \ all \ \ n = 1, \cdots, m.$$

Next, the asymptotically determining of trajectories inside a global attractor can be proved via the system $\mathcal{F}$ when m is large enough.

Theorem 6. *Suppose that $m \in N$ satisfies*

$$m \geq C' \left(\frac{\nu \sigma^{\frac{1}{1-\alpha}} + 2|b| \widehat{\kappa}_0 [\frac{1}{\sigma}]^{\frac{\beta}{\alpha - \beta}}}{\nu^2 \lambda_1} \right)^{\frac{3}{2}},$$

where $\sigma = \frac{a\kappa_0}{2|b|\widehat{\kappa}_0 + \frac{1}{\nu}}$ and C' is a positive constant. Then, $\mathcal{F}$ is asymptotically determining for the dynamic system generated by (32) with the periodic boundary.

Proof. Let u and v be two trajectories inside a global attractor. Suppose that $P_m(u(t) - v(t)) = 0$ for $t \in \mathbb{R}^+$. Then, $w(t) = u(t) - v(t)$ satisfies

$$\begin{cases} \partial_t w + \nu A u + B(u,w) + B(w,v) + a(|u|^{2\alpha}u - |v|^{2\alpha}v) + b(|u|^{2\beta}u - |v|^{2\beta}v) = 0, \\ w(t=0) = u_0 - v_0. \end{cases} \tag{33}$$

Take the inner product of (33) with $w(t)$. This leads to

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu||w||^2 + b(w,v,w) + a(|u|^{2\alpha}u - |v|^{2\alpha}v, w) + b(|u|^{2\beta}u - |v|^{2\beta}v, w) = 0. \tag{34}$$

For any $\alpha > 1$, by the Hölder and Young inequalities, we deduce

$$\begin{aligned} |b(w,v,w)| &\le \int_\Omega |v||w||\nabla w|dx \\ &\le \int_\Omega |v||w|^{\frac{1}{\alpha}}|w|^{1-\frac{1}{\alpha}}|\nabla w|dx \\ &\le |||v||w|^{\frac{1}{\alpha}}||_{L^{2\alpha}(\Omega)}|||w|^{1-\frac{1}{\alpha}}||_{L^{\frac{2\alpha}{\alpha-1}}(\Omega)}||w|| \\ &\le \frac{\sigma}{2\nu}||v|^\alpha w|^2 + \frac{\sigma^{\frac{1}{1-\alpha}}}{2\nu}|w|^2 + \frac{\nu}{2}||w||^2 \end{aligned} \tag{35}$$

and

$$\begin{aligned} b\int_\Omega (|u|^{2\beta}u - |v|^{2\beta}v)wdx &\le |b|\widehat{\kappa}_0 \int_\Omega (|u|+|v|)^{2\beta}|w|^{2\frac{\beta}{\alpha}}|w|^{2(1-\frac{\beta}{\alpha})}dx \\ &\le |b|\widehat{\kappa}_0|(|u|+|v|)^\alpha|w||^{2\frac{\beta}{\alpha}}|w|^{2\frac{\alpha-\beta}{\alpha}} \\ &\le \sigma|b|\widehat{\kappa}_0|(|u|+|v|)^\alpha|w||^2 + |b|\widehat{\kappa}_0[\frac{1}{\sigma}]^{\frac{\beta}{\alpha-\beta}}|w|^2 \end{aligned} \tag{36}$$

because of

$$\begin{aligned} a\int_\Omega (|u|^{2\beta}u - |v|^{2\beta}v)wdx &\ge a\kappa_0\int_\Omega |w|^2(|u|+|v|)^{2\alpha}dx \\ &= a\kappa_0|(|u|+|v|)^\alpha|w||^2 \end{aligned} \tag{37}$$

from Lemma 4.

Substituting (35)–(37) into (34), we obtain

$$\frac{d}{dt}|w|^2 + \nu||w||^2 + (2a\kappa_0 - 2\sigma|b|\widehat{\kappa}_0 - \frac{\sigma}{\nu})|(|u|+|v|)^\alpha|w||^2 \le (\frac{\sigma^{\frac{1}{1-\alpha}}}{\nu} + 2|b|\widehat{\kappa}_0[\frac{1}{\sigma}]^{\frac{\beta}{\alpha-\beta}})|w|^2. \tag{38}$$

Choose appropriate $\sigma = \frac{a\kappa_0}{2|b|\widehat{\kappa}_0 + \frac{1}{\nu}}$ such that $2a\kappa_0 - 2\sigma|b|\widehat{\kappa}_0 - \frac{\sigma}{\nu} = a\kappa_0 > 0$. This results in

$$\frac{d}{dt}|w|^2 + \{\nu\lambda_{m+1} - (\frac{\sigma^{\frac{1}{1-\alpha}}}{\nu} + 2|b|\widehat{\kappa}_0[\frac{1}{\sigma}]^{\frac{\beta}{\alpha-\beta}})\}|w|^2 \le 0 \tag{39}$$

from (38).

The Gronwall inequality gives the boundedness

$$|w|^2 \le |w_0|^2 e^{-Lt}, \tag{40}$$

where $L = \nu\lambda_{m+1} - (\frac{\sigma^{\frac{1}{1-\alpha}}}{\nu} + 2|b|\widehat{\kappa}_0[\frac{1}{\sigma}]^{\frac{\beta}{\alpha-\beta}}) > 0$, which lead to $|w|^2 \to 0$ as $t \to \infty$.

Since $\lambda_m \sim C\lambda_1 m^{\frac{2}{3}}$, $\mathcal{F}$ is asymptotically determining for the dynamic system generated by (32) when m satisfies

$$m \geq C'(\frac{\nu\sigma^{\frac{1}{1-\alpha}} + 2|b|\widehat{\kappa}_0[\frac{1}{\sigma}]^{\frac{\beta}{\alpha-\beta}}}{\nu^2\lambda_1})^{\frac{3}{2}},$$

where $\sigma = \frac{a\kappa_0}{2|b|\widehat{\kappa}_0 + \frac{1}{\nu}}$. The poof is completed. $\square$

6. Conclusions and Further Research

This paper has presented the determining modes and asymptotic determination of the 3D incompressible Brinkman–Forchheimer extended Darcy system for non-autonomous and autonomous cases, respectively, under assumptions on generalized Grashof numbers related to modes, which are the basis of theoretical analysis for finite element computation and long-time behavior for the porous medium occupied by fluid flow, especially the tracking property of complete trajectories for the dynamic systems. Furthermore, the determining modes and finite element volumes for the presented model in this paper are more interesting, which need further consideration in research.

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